Dynamic Team Decision Theory

EECS 558 Project Report
Shrutivandana Sharma and David Shuman
December, 2005
I. Introduction

While the stochastic control problem features one decision maker acting over time, many complex controlled systems involve multiple decision makers (DMs). In general, these decision makers differ in three ways: 1) they control different decision variables; 2) they base their decisions on different information; and 3) they have different goals or objectives. A team is an organization in which there is a single goal or payoff common to all decision makers. Accordingly, in the general team decision problem, the decision makers differ only in the first two respects described above.

The team decision problem was initially examined by Marschak and Radner in the context of organization theory, and has since found applications in a wide range of engineering and economic systems, including electric power, traffic, and communications systems. Despite its applicability, however, very few general results are known about the team decision problem due to its inherent difficulties. The purpose of this report is to provide a brief introduction to dynamic team decision theory, including an overview of some of the key results. Section II of this paper formulates the team decision problem. Section III summarizes three situations in which a dynamic team problem can be decomposed into several smaller dynamic team problems. Section IV proceeds to describe one example of such decomposition – the one step communication delay problem. Finally, Section V summarizes Witsenhausen’s important proof that all discrete variable and most continuous variable dynamic team problems can be mapped into equivalent static team problems.

II. Formulation of the Dynamic Team Decision Problem

The general dynamic team decision problem is formulated as follows:

- The team consists of N decision makers, $DM_i$, $i=1,2,...,N$
- The decision makers act in a fixed sequence (w.o.l.g. $DM_1$, $DM_2$, ..., $DM_N$). Without this assumption, the problem becomes a non-sequential team problem that falls into a more general (and more difficult) class of problems
- The basic random variables are represented by the vector $\xi$
- $\xi$ has a known probability distribution, $F(\xi)$, and is independent of the control law
- Information $z_i$ available to each $DM_i$ is given by $z_i = \eta_i(\xi, u)$, $i=1,2,...,N$
- $\eta_i = (\eta_{i1}, \eta_{i2}, ..., \eta_{iN})$ is called the information structure of the team
- Admissible decision rules are of the form $u_i = \gamma_i(z_i)$
- The objective is to find a decision rule $\gamma = (\gamma_1, \gamma_2, ..., \gamma_N)$ to minimize $J(\gamma) = E\{w(\xi, u)\}$, where $w(*)$ is the total cost function. For a fixed policy $\gamma$, the above expectation is with respect to the basic random variables $\xi$
- $w(*)$, $\eta(*)$, and $F(\xi)$ are assumed to be common information, known a priori to all decision makers
By way of definition, a team is said to be \textit{static} if \( \eta_i, i=1,2,...,N, \) is independent of \( u, \) i.e., \( z_i = \eta_i(\xi), \) and \textit{dynamic} otherwise.

The model described on page 1558 of [1] with observations made at \( M \) posts and control inputs applied at \( k \) stations at each discrete step of a length \( T \) time horizon can easily be converted into the above formulation by re-labeling the \( T \) controllers \( DM_1, DM_2, ..., DM_N. \) If we take \( k=1 \) and assume perfect recall in the model of [1], then we have the stochastic control problem. In general, the information available to each decision maker can be any subset of previous observations and control inputs. Much of the difficulty of the dynamic team decision problem originates from the functional dependencies between the controllers’ information, and, therefore, analysis of the information structure plays a critical role in solving any team decision problem. For more on specific information structures (alternatively called information patterns) including the classical, n-step delayed sharing, and amnesic patterns, see [1] and [5].

\section*{III. Decomposition of the Dynamic Team Decision Problem}

Due to the inherent difficulty of the team decision problem, a group of authors posed the question “when can a dynamic team problem be decomposed into several smaller dynamic team problems?” Three such scenarios are discussed below. It is worth noting that even when the team decision problem can be decomposed, there is no guarantee that the resulting problems will be tractable. For this reason, these results are best viewed as more philosophical results that can be of help in trying to gain further insight into specific examples within this class of problems.

\subsection*{Preliminary: Precedence Diagrams to Represent Information Structures}

In [4], Yoshikawa refines slightly the notation of precedence diagrams, originally presented in [2] by Ho and Chu. See these papers for examples of precedence diagrams. The three important definitions follow:

- \( u_i \) \textit{affects} \( z_j, \) written \( iRj, \) if there exist \( \xi, u^a := [u_1, u_2, ..., u_{i-1}, u_i^a, u_{i+1}, ..., u_N]^T \), and \( u^b := [u_1, u_2, ..., u_{i-1}, u_i^b, u_{i+1}, ..., u_N]^T \) such that \( \eta_j(\xi, u^a) \neq \eta_j(\xi, u^b) \)

- \( DM_i \) is a \textit{precedent} of \( DM_j, \) written \( DM_i \rightarrow DM_j, \) if 1) \( iRj, \) or 2) there exist \( r, s, ..., t \) such that \( iRr, rRs, ..., tRj \)

- Information \( z_i \) of \( DM_i \) is \textit{nested} in information \( z_j \) of \( DM_j, \) written \( DM_i \longrightarrow DM_j, \) if there exists a measurable function \( f_{ji} \) such that for any \( \xi, z_i = f_{ji}^T(z_j) \)

Note that for causality to hold, if \( DM_i \) is a precedent of \( DM_j, \) then \( DM_j \) cannot be a precedent of \( DM_i. \) Yet, it is possible for \( DM_i \) to be a precedent of \( DM_j, \) and for the information of \( DM_j \) to be nested in the information of \( DM_i \) at the same time. See Example 2 in [4] for such a system.

\subsection*{Decomposition Scenario I: Independent Partition}

Informally, Yoshikawa [4] defines an independent partition to be “a partition for which there is no precedence relation between any pair of groups and the total cost function is given by the sum of
the cost function for each group.” More formally, a partition \((H_1,H_2,\ldots,H_K)\) is an **i-partition** of \(H=\{DM_i: i=1,2,\ldots,N\}\) if:

(i) \(\forall i,j \in \{1,2,\ldots,K\}, i \neq j : DM_i \in H_i \text{ and } DM_j \in H_j \Rightarrow DM_i \not\rightarrow DM_j\).

(ii) \(w(\xi, u) = \sum_{i=1}^{K} w_i(\xi, u_{H_i})\)

Theorem 1 of [4] essentially says that if a team \(H\) has an independent partition, then the optimal solution can be found by optimizing each sub-team according to its sub-cost function \(w_i(\xi, u_{H_i})\). Combining these optimal sub-solutions will result in the optimal solution of the original problem. In [4] Eq. 14, Yoshikawa also relaxes condition (ii) above to allow for monotonically increasing cost functions with respect to every \(w_i\).

**Decomposition Scenario II: Sequential Partition**

A sequential partition (s-partition) is also defined by two conditions. [4] First, the decisions of a group do not affect the information of any previous group. Second, the information of all members of a group is nested in the information of the following group. More formally, a partition \((H_1,H_2,\ldots,H_K)\) is an **s-partition** of \(H\) if:

(i) \(\forall i \in \{1,2,\ldots,K-1\} \forall j \in \{i+1,\ldots,K\} \forall DM_i \in H_i \forall DM_j \in H_j : DM_i \rightarrow DM_j\)

(ii) \(\forall i \in \{1,2,\ldots,K-1\} \forall DM_i \in H_i \forall DM_j \in H_{i+1} : DM_i \not\rightarrow DM_j\)

Note that both conditions above can be determined from a precedence diagram alone. The sequential nestedness of information leads to Theorem 2 of [4], which basically says that the dynamic programming technique can be applied to a team with an s-partition to find the optimal solution. For example, consider the precedence diagram in Figure 1 below. Suppose we fix the actions for all but the last group, \(H_3\). Since the actions of the last group do not affect that of any of the previous groups, the actions of \(H_3\) can be optimized independently of the previous groups. With knowledge of \(H_3\)'s optimal actions corresponding to every possible set of previous actions, we are left with a new two group s-partition and an updated cost function that depends only on the actions of \(H_1\) and \(H_2\). The actions of \(H_2\) can then be optimized, and so forth. Once the first group’s optimal policy is determined, the following groups’ optimal policies can be chosen accordingly based on the complete description of the optimal policies under all fixed but arbitrary previous actions, and this completes the determination of the team’s optimal policy.

![Figure 1. Example of an S-Partition](image-url)
Decomposition Scenario III: Partial Nestedness

We have described two forms of information structures for which the optimal solutions, if they exist, can be found by solving subproblems of the original problem. In [2] Ho and Chu discuss a third form of information structure, partial nestedness, in combination with a set of additional assumptions. Under this scenario, a unique optimal solution is guaranteed, and its form is known to be affine in \( z_i \); however, due to the additional assumptions, the domain of applicability is more limited.

In [2], Ho and Chu assume the basic random variables are independent, zero mean, and jointly Gaussian; the cost function is quadratic, and the information structure takes on the linear form:

\[
z_i = H_i \xi + \sum_j D_{ij} u_j
\]

where the matrices H and D are known to all DMs. (A1)

[2] leverages the theorem of Radner that for a static team with the set of assumptions (A1), the optimal control law is given by:

\[
u_i = \gamma_i(z_i) = A_i z_i + b_i \quad \forall i, \quad (b_{i1}^T, b_{i2}^T, \ldots, b_{in}^T) = c^T Q^{-1}, \quad \sum_j Q_j A_j (H_i X H_i^T) = -S_i X H_i^T
\]

An information structure is said to be partially nested if \( DM_i \rightarrow DM_j \) implies \( DM_i \rightarrow \rightarrow DM_j \), i.e. the follower can always deduce the actions of its precedents for a given policy, \( \gamma \).

The key result in [2] is for a dynamic team satisfying (A1) and having partially nested information structure

\[
z_i = H_i \xi + \sum_{DM_j \rightarrow DM_i} D_{ij} u_j
\]

It is shown that the above information structure is equivalent to an information structure in static form for any fixed set of control laws. It then follows from Radner’s result that the optimal control law for each member of such a dynamic team exists, is unique, and is linear in \( z_i \).

As a final remark on partial nestedness, note that while an s-partition is similar conceptually to partial nestedness on the group level, partial nestedness is neither a necessary nor sufficient condition for the existence of an s-partition.

IV. The One Step Communication Delay Problem

The decomposition methods described in Section III have found applications in some important decentralized control problems. One such application is the one step communication delay problem. This problem features a discrete time stochastic system with two controllers and the following system dynamics:

\[
X_{t+1} = f_t(X_t, U_t^1, U_t^2, N_t), \quad t = 1, 2, \ldots, T
\]

\[
Y_t^j = h_t^j(X_t, M_t), \quad j = 1, 2
\]
where $X_t, N_1, N_2, \ldots, N_T, M_1, M_2, \ldots, M_T$ are mutually independent. Additionally, the information structure is one step delay sharing, i.e.

$$Z_t^i = (Y_i, Y_{i-1}, Y_t^i, U_1, U_2, \ldots, U_{t-1}), \ i = 1, 2$$

The cost function is described by

$$w = \sum_{t=1}^T c_t(X_{t+1}, X_t, U_t)$$

and the precedence diagram for this system is shown below in Figure 2. It is easy to see from this diagram that $H = \{H_1, H_2, \ldots, H_T\}$ represents a sequential partition, and the information structure is also partially nested. In the case that the system is linear, the noise is Gaussian, and the cost function is quadratic (LQG), Theorem 2 of Ho and Chu [2] for teams with partially nested information structure can be invoked to show that the optimal solution is affine in the information available to each controller.

Varaiya and Walrand prove in [3] that even in the one step communication delay problem without the LQG assumptions, there is no loss of optimality in restricting the search for the optimal $\gamma_t^*$ to separated control laws of the form $\phi_t^i(Y_t^i, F_t)$, where $F_t$ is the conditional distribution of $X_{t-1}$ given $(Y_1, \ldots, Y_{t-1}, U_1, \ldots, U_{t-1})$. As in the stochastic control problem with imperfect information, $F_t$, the result of a filtering problem, is independent of the control law and the cost function.

It was originally conjectured in [1], Assertions 8 and 9, that analogous separation results [with and without LQG assumptions] hold in the case of $n$-step delay sharing. However, Varaiya and Walrand also show in [3] that this is not the case by providing an LQG counterexample that has an optimal solution outside the space of separated laws. Finally, [2] and [4] also discuss the application of decomposition methods to other classes of decentralized control problems including periodic sharing patterns and hierarchical control systems.
V. Equivalence of Static and Dynamic Teams

As we noted earlier, much of the difficulty in dealing with dynamic teams is due to the interdependencies among the system variables. Because of this fact, it was hypothesized for a long time that dynamic team problems are more difficult than static team problems. However, in a major breakthrough from 1988, Witsenhausen [5] shows that all discrete variable and most continuous variable dynamic team problems can be mapped into equivalent static team problems.

Witsenhausen’s basic methodology is to shift the dependence amongst the decision makers’ information variables into the cost function. The problem does not become any less complicated in doing so, but the result is a static team problem. In the following, we present a summary of the key ideas used to prove the equivalence of dynamic and static teams.

Alternative Description of the Intrinsic Model

The dynamic team model discussed in previous sections is known as the ‘intrinsic model’ [with a sequential ordering]. To facilitate the proof, Witsenhausen defines a new model for the dynamic team problem, which he shows to be equivalent to the intrinsic model. The key elements of Witsenhausen’s model follow:

- The team consists of N decision makers, $DM_i$, $i = 1, 2, ..., N$ which are assumed to take actions sequentially in time without violating causality.
- The basic random variables are
  \[ \omega = (\omega_0, \omega_1, ..., \omega_N) \]
  They are generated from the underlying probability space
  \[ (\Omega, \mathcal{B}, P) \]
  and assumed to be independent; i.e.
  \[ dP(\omega) = \prod_{i=0}^{N} dP_i(\omega_i) \]
- Observation at time $i$ is generated as
  \[ y_i = g_i(\omega_0, \omega_i, u_1, u_2, ..., u_{i-1}), 1 \leq i \leq N \]
- For each $i = 1, 2, ..., N$, the observations available to $DM_i$ are described by
  \[ k_i \subset \{1, 2, ..., i\} \]
- Admissible decision rules are of the form
  \[ u_i = \gamma_i(\{y_{r_k}\}_{r_k}) \]
  where
  \[ \gamma_i \in \Gamma_i^e \]
  and $\Gamma_i^e$ is the subset of all measurable functions from
  \[ \prod_{r_k}(Y_\mathcal{F}) \to (U_i, \mathcal{U}) \]
The common objective is to find a decision rule
\[ \gamma = [\gamma_1, \gamma_2, \ldots, \gamma_N] \]
to minimize the expected total cost
\[ J(\gamma) = E^\gamma \{V(\omega_0, y_1, y_2, \ldots, y_N, u_1, u_2, \ldots, u_N)\} \]

All decision makers know \textit{a priori}
\[ V(\cdot), g_i(\cdot), k_i \quad \text{and} \quad dP(\omega) \]

\textbf{Key Observations Leading to Static Reduction}

We now observe some key features inherent in this formulation that lead towards an approach for static reduction:

- For a given policy \( \gamma \), the joint pdf of all the random variables in the above formulation is completely determined by the joint pdf \( dP(\omega) \) of \( \omega \)
- \((\omega_1, \omega_2, \ldots, \omega_N)\) do not explicitly appear in \( V(\omega_0, y_1, y_2, \ldots, y_N, u_1, u_2, \ldots, u_N) \)
- \((u_1, u_2, \ldots, u_N)\) are completely determined by \((y_1, y_2, \ldots, y_N)\)
- Therefore, \( J(\gamma) \) is completely determined by the joint pdf \( \omega_0, y_1, y_2, \ldots, y_N \).

We now consider the difference between a static and a dynamic team. In a dynamic team, \( \{y_i\}_{i=1}^N \) are dependent on each other, and this dependence is reflected in their joint pdf. (1)

In a static team, \( y_i \) depends only on \( \omega_0 \) and \( \omega_i \), and hence, the dependence amongst the \( y \)'s is much weaker. Further, if for every \( i, y_i \) depends only on \( \omega_i \), then the \( y \)'s are independent and their joint pdf can be written as the product of the marginal pdfs. (2)

Is there a way to separate the dependence amongst the \( y \)'s from the dynamic team probability measure (1) to get a probability measure of the form of a static team (2)? (3)

The answer is given by the ‘\textit{Common denominator condition}’ [5], Eq. (4.2), under which a \textit{conditional probability measure} on some space can be written as a product of an \textit{unconditional probability measure} on the same space, and a \textit{measurable function} of the variables involved, whereby the function absorbs the dependencies from the original probability measure. Thus, if \( f_i(\cdot) \) denotes the measurable function and \( Q_i(dy_i) \) denotes the unconditional probability measure, one can write, for all \( i \),
\[
P(y_i \in S \mid \omega_0, u_1, u_2, \ldots, u_{i-1}) = \int_S f_i(y_i, \omega_0, u_1, u_2, \ldots, u_{i-1})Q_i(dy_i) \quad (4)\]
Example 1: Gaussian Signaling

To illustrate this condition, we consider the ‘Gaussian Signaling’ example discussed in [5]. The system consists of two agents \((N=2)\) controlling decision variables \(u_1\) and \(u_2\). Their respective observations are given by \(y_1 = \omega_1\) and \(y_2 = \omega_2 + u_1\), where \(\omega_0\) is degenerate; \(\omega_1 \sim N(0,1)\); \(\omega_2 \sim N(0,1)\); and \(\omega_0, \omega_1, \omega_2\) are independent. The cost function is \(V = (u_2 - y_1)^2 + k^2(u_1^2)\). The decision rules to be considered are of the form, \(u_1 = \gamma_1(y_1)\) and \(u_2 = \gamma_2(y_2)\). Thus we have,

\[
P(dy_1 | \omega_0) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{y_1^2}{2}\right) dy_1 \quad \text{and,}
\]

\[
P(dy_2 | \omega_0, u_1) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(y_2 - u_1)^2}{2}\right) dy_2 = \exp \left(-\frac{(u_1^2 - 2y_2u_1)}{2}\right) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{y_2^2}{2}\right) dy_2,
\]

from which one can identify the functions \(f_i(\cdot)\) and \(Q_i(dy_i)\), \(i=1,2\) as,

\[
f_1(y_1, \omega_0) = 1,
\]

\[
Q_1(dy_1) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{y_1^2}{2}\right) dy_1 \quad \text{and,}
\]

\[
f_2(y_2, \omega_0, u_1) = \exp \left(-\frac{(u_1^2 - 2y_2u_1)}{2}\right),
\]

\[
Q_2(dy_2) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{y_2^2}{2}\right) dy_2.
\]

We will return to this example below as we discuss the static reduction.

The Static Reduction

Having illustrated the common denominator condition, we will now see how it helps in answering the question in (3). It is proved in [5], Lemma 5.1, that for distributions satisfying the common denominator condition, the joint distribution of \(\omega_0, y_1, y_2, \ldots, y_N\) can be written as

\[
\prod_{i=1}^{N} f_i(y_i, \omega_0, u_1, u_1, \ldots, u_{i-1}) \times \prod_{i=1}^{N} Q_i(dy_i) \times P_0(d \omega_0)
\]

As can be seen, the term in the square bracket defines a probability distribution on \(\omega_0, y_1, y_2, \ldots, y_N\) which is the product of individual distributions, and thus, of the form expected in (2). Therefore, in the present form, the distribution does correspond to a static team. However, the joint distribution still contains a term (the first term) that reflects the dependencies among the system variables. It is important to note at this stage that the goal is to design a policy \(\gamma\) that minimizes the objective function \(J(\gamma)\) and hence to a designer, two systems will be equivalent if they achieve the same goal via the same set of designs. Thus, in the present context, the equivalence between a dynamic and a static team requires that we can find a static system that achieves the same objective with the probability distribution of observation variables satisfying the conditions stated in (2). The detailed conditions defining the equivalence of the two systems are given in...
With this concept of equivalence in mind, let us now see what impact the new form of probability distribution (5) has on the objective function. We can write the original objective as,

\[ J(\gamma) = \prod_{y_1,\ldots,y_N} \cdots \prod_{y_N} V(\omega_0, y_1, y_2, \ldots, y_N, u_1, u_2, \ldots, u_N) \prod_{i=1}^{N} f_i(y_i, \omega_0, u_i, u_{i-1}) P_0(\omega_0) \prod_{i=1}^{N} Q_i(dy) \]  

(6)

\[ = \prod_{y_1,\ldots,y_N} \int \left\{ V(\omega_0, y_1, y_2, \ldots, y_N, u_1, u_2, \ldots, u_N) \prod_{i=1}^{N} f_i(y_i, \omega_0, u_i, u_{i-1}) \right\} P_0(\omega_0) \prod_{i=1}^{N} Q_i(dy) \]

(7)

As can be seen, by absorbing the dependency part into the cost function, the probability distribution has been brought exactly into the form of an independent set of observations. Thus, if a new system is defined in which the primitive random variables \( \{\omega_i^*\}_{i=1}^N \) are generated according to \( \prod_{i=1}^{N} (Y, \gamma, Q) \), and the observations are generated as \( y_i^* = g_i^* (\omega_0, \omega_i^*, u_i^*, u_{i-1}^*) = \omega_i^*, 1 \leq i \leq N \), then the system in (7) will correspond to a static team. The two systems in (6) and (7) achieve exactly the same goals through exactly the same set of decisions, and are therefore equivalent. The difference is that in the first system, the dependence lies amongst the system variables and the complexity is contained in the probability distribution; in the second system, however, the dependence is absorbed in the cost function and the distribution has a simpler form. The above proof by Witsenhausen establishes that to every dynamic team problem that satisfies the common denominator condition, there corresponds a static team problem with the same complexity as the original dynamic team problem.

To illustrate how the complexity shifts to the cost function in the static reduction, we will now complete Example 1. Since \( \omega_0 \) is degenerate, we can find the new cost function corresponding to the static team directly from the functions found in Example 1. Thus,

\[ \hat{V}(y_1, y_2, u_1, u_2) = V(\omega_0, y_1, y_2, u_1, u_2) \prod_{i=1}^{2} f_i(y_i, \omega_0, u_i, u_{i-1}) \]

\[ = \left[ (u_2 - y_2)^2 + k^2(u_1^2) \right] \exp \left( \frac{-(u_1^2 - 2y_2u_2)}{2} \right) \]

It is clear that the new cost function is much more complicated than the original one and hence the new system is no easier to analyze than the original system.

In summary, it is important to note that by establishing the equivalence between dynamic and static teams, Witsenhausen has not proved that the reduction of dynamic team problems to static team problems makes them easier to solve. Rather, the contribution of the paper is in disproving the conjecture that static teams are less complicated than dynamic teams. Thus, the solution of complex team problems, whether dynamic or static, still remains an active area of investigation.
VI. Conclusion

The decentralized nature of information in the dynamic team theory problem poses quite a challenge in finding optimal control laws. This paper has presented an overview of two of the more general results available regarding this class of problems: scenarios under which the problem can be decomposed into several smaller problems, and the equivalence of dynamic and static teams. While these results may provide some insight into specific problems, it seems to us that even after decades of research in the area, the team theory landscape remains surprisingly unplowed considering the widespread applicability of the framework. It is our hope that further progress on such team theoretic problems will be forthcoming.

VII. References


