Local public good provision in networks: A Nash implementation mechanism☆

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Abstract

In this paper we study local public goods provision in decentralized information networks. Local public goods are network users' actions that directly affect the utilities of arbitrary subsets of network users. We consider networks where each user knows only that part of the network that either affects it or is affected by it. Furthermore, each user's utility and action space are its private information, and each user is a self utility maximizer. For such a network we formulate a local public goods provision problem in the framework of implementation theory. For this problem we develop a game form that, (i) results in optimum centralized local public goods provision at all Nash equilibria of the induced game (Nash implementation); (ii) leads to voluntary participation by all users (individual rationality); and (iii) results in budget

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Abbreviations: Nash equilibria (NE); Guaranteed display (GD); Base Station (BS); Code Division Multiple Access (CDMA); Minimum Mean Square Error (MMSE); Multi-User Detector (MUD); Quality of Service (QoS); Power Spectral Density (PSD)

balance at all Nash equilibria and off equilibrium.

Keywords:
Network, Local public good, Decentralized information, Resource allocation, Mechanism design, Implementation theory, Nash implementation, Individual rationality, Budget balance

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1. Introduction

In networks individuals' actions often influence the performance of their directly connected neighbors. Such an influence of individuals' actions on their neighbors' performance can propagate in the entire network through various paths. Thus it can affect the performance of the entire network. Examples include several local public good networks. For instance, when a jurisdiction institutes a pollution abatement program, the benefits also accrue to nearby communities. Or, when a university builds a new library, the nearby colleges also benefit from the subscription to the new library. The local public good nature is also seen in the spread of information and innovation in social and research networks. The influence of neighbors can also be observed in online advertising where the utility (users' attention) that an advertiser gets may be increased or decreased by the presence of other advertisers on a webpage.

A local public good network differs from a typical public good system as a local public good (alternatively, the action of an individual) is accessible to and influences the utilities of individuals in a particular locality (neighborhood) within a big network. On the other hand a public good is accessible to and influences the utilities of all individuals in the system (Mas-Colell et al., 2002, Chapter 11). Because of the localized interactions of individuals, in local public good networks (such as ones described above) the information about the network is often localized; i.e., the individuals are usually aware of only their neighborhoods and not the entire network. In many situations the individuals also have some private information about the network or their own characteristics that are not known to anybody else in the network. Furthermore, the individuals may also be selfish, i.e. they care only about their own benefit in the network. Such a decentralized information local public good network with selfish users gives rise to several research issues. In the next section we provide a literature survey on prior research in local public good networks.

1.1. Literature survey

There exists a large literature on local public goods within the context of local public good provision by various municipalities that follows the seminal work of Tiebout (1956). These works mainly consider network formation problems in which individuals choose where to locate based on their knowledge of the revenue and expenditure patterns (on local public goods) of various municipalities. In this paper we consider the problem of determining the levels of local public goods (actions of network users) for a given network; thus, the problem addressed in this paper is distinctly different from those in the above literature. Recently, Bramoull and Kranton (2007) and Yuan (Preprint) analyzed the influence of selfish users' behavior on the provision of local public goods in networks with fixed links among the users. Bramoull and Kranton (2007) study a network model in which each user's payoff equals its benefit from the sum of efforts (treated as local public goods) of its neighbors less a cost for exerting its own effort. For concave benefit and linear costs, the authors analyze Nash equilibria (NE) of the game where each user's strategy is to choose its effort level that maximizes its own payoff from the provision of local public goods. The authors show that at such NE, *specialization* can occur in local public goods provision. Specialization means that only a subset of individuals contribute to the local public goods and others free ride. The authors further show that specialization can benefit the society as a whole because among all NE, the ones that are "specialized" result in the highest social welfare (sum of all users' payoffs). However, Bramoull and Kranton (2007) also show that none of the NE of abovementioned game can result in a local public goods provision that achieves the maximum possible social welfare. In Yuan (Preprint) the work of Bramoull and Kranton (2007) is extended to directed networks where the externality effects of users' efforts on each others' payoffs can be unidirectional or bidirectional. Yuan (Preprint) investigates the relation between the structure of directed networks and the existence and nature of Nash equilibria of users' effort levels in those networks. For that matter they introduce a notion of ergodic stability to study the influence of perturbation of users' equilibrium efforts on the stability of NE. However, none of the NE of the game analyzed in Yuan (Preprint) result in a local public goods provision that achieves optimum social welfare.

In this paper we consider a generalization of the network models investigated in Bramoull and Kranton (2007) and Yuan (Preprint). Specifically, we consider a fixed network where the actions of each user directly affect the utilities of an arbitrary subset of network users. In our model, each user's utility from its neighbors' actions is an arbitrary concave function of its neighbors' action profile. Each user in our model knows only that part of the network that either affects it or is affected by it. Furthermore, each user's utility and action space are its private information, and each user is a self utility maximizer. Even though the network model we consider has similarities with those investigated in Bramoull and Kranton (2007) and Yuan (Preprint), the problem of local public goods provision we formulate in this paper is different from those in both the above works. Specifically, we formulate a problem of local public goods provision in the framework of implementation theory [2] and address questions such as – How should the network users communicate so as to preserve their private information, yet make it possible to determine actions that achieve optimum social welfare? How to provide incentives to the selfish users to take actions that optimize the social welfare? How to make the selfish users voluntarily participate in any action determination mechanism that aims to optimize the social welfare? In a nutshell, our work focuses on *designing a mechanism that can implement the optimum social welfare in NE* and thus it follows the philosophy of Chen (2002); Groves and Ledyard (1977); Hurwicz (1979); Walker (1981) in the context of local public goods provision. To the best of our knowledge the resource allocation problem and its solution that we present in this paper is the first attempt to analyze a local public goods network model in the framework of implementation theory. In the next section we state our contributions in the problem formulation and solution presented in this paper.

[2] Refer to Jackson (2001); (Sharma, 2009, Chapter 3); Sharma and Teneketzis (2010); Stoenescu and Teneketzis (2005) for an introduction to implementation theory.

1.2. Contribution of the paper

The key contributions of this paper are:

- The formulation of a problem of local public goods provision in the framework of implementation theory.

- The specification of a game form [3] (decentralized allocation mechanism) for the above problem that, (i) implements in Nash equilibria [4] the optimal solution of the corresponding centralized local public goods provision problem; (ii) is individually rational; [5] and (iii) results in budget balance at all Nash equilibria and off equilibrium.

The rest of the paper is organized as follows. In Section 2.1 we present the model of a local public good network. In Section 2.2 we discuss two motivating applications. In Section 2.3 we formulate the local public goods provision problem. In Section 3.1 we discuss how the problem formulated in Section 2.3 can be addressed with an implementation theory approach. In Section 3.2 we develop ideas for the construction of a game form for the above problem and follow that with the specification of a game form in Section 3.3. We discuss the properties of the proposed game form in Section 3.4 and we present their proofs in Appendices Appendix A and Appendix B. We

[3]The definition of a game form is given in Section 3.1. See Jackson (2001); (Sharma, 2009, Chapter 3); Sharma and Teneketzis (2010); Stoenescu and Teneketzis (2005) for more detailed definitions.

[4]Refer to (Sharma, 2009, Chapter 3) and Sharma and Teneketzis (2010) for the definition of "implementation in Nash equilibria."

[5]Refer to (Sharma, 2009, Chapter 3) and Sharma and Teneketzis (2010) for the definition of "individual rationality."

comment on implementation aspects of the proposed game form in Section 3.5 and conclude with a discussion on future directions in Section 4.

Before we present the model of local public good network in Section 2, we describe the notation that we will use throughout the paper.

Notation:

We use bold font to represent vectors and normal font for scalars. We use bold uppercase letters to represent matrices. We represent the element of a vector by a subscript on the vector symbol, and the element of a matrix by double subscript on the matrix symbol. To denote the vector whose elements are all x_i such that $i \in \mathcal{S}$ for some set $\mathcal{S}$, we use the notation $(x_i)_{i \in \mathcal{S}}$ and we abbreviate it as $\boldsymbol{x}_{\mathcal{S}}$. We treat bold $\mathbf{0}$ as a zero vector of appropriate size which is determined by the context. We use the notation $(x_i, \boldsymbol{x}^*/i)$ to represent a vector of dimension same as that of $\boldsymbol{x}^*$, whose ith element is x_i and all other elements are the same as the corresponding elements of $\boldsymbol{x}^*$. We represent a diagonal matrix of size $N \times N$ whose diagonal entries are elements of the vector $\boldsymbol{x} \in \mathbb{R}^N$ by $\text{diag}(\boldsymbol{x})$.

2. The network resource allocation problem

In this section we present a model of local public good network and formulate a resource allocation problem for it. We first describe the components of the model and the assumptions we make on the properties of the network. We then present the resource allocation problem and formulate it as an optimization problem.

2.1. The model (M)

We consider a network consisting of N users and one network operator. We denote the set of users by $\mathcal{N} := \{1, 2, \ldots, N\}$. Each user $i \in \mathcal{N}$ has to take an action $a_i \in \mathcal{A}_i$ where $\mathcal{A}_i$ is the set that specifies user i's feasible actions. In a real network, a user's actions can be consumption/generation of resources or decisions regarding various tasks. We assume that,

Assumption 1. *For all $i \in \mathcal{N}$, $\mathcal{A}_i$ is a convex and compact set in $\mathbb{R}$ that includes 0.* [6] *Furthermore, for each user $i \in \mathcal{N}$, the set $\mathcal{A}_i$ is its private information, i.e. $\mathcal{A}_i$ is known only to user i and nobody else in the network.*

Because of the users' interactions in the network, the actions taken by a user directly affect the performance of other users in the network. Thus, the performance of the network is determined by the collective actions of all users. We assume that the network is large-scale, thus, every user's actions directly affect only a subset of network users in $\mathcal{N}$. We depict the above feature by a directed graph as shown in Fig. 1. In the graph, an arrow from j to i indicates that user j affects user i; we represent the same in the text as $j \to i$. We assume that $i \to i$ for all $i \in \mathcal{N}$.

Mathematically, we denote the set of users that affect user i by $\mathcal{R}_i := \{k \in \mathcal{N} \mid k \to i\}$. Similarly, we denote the set of users that are affected by user j by $\mathcal{C}_j := \{k \in \mathcal{N} \mid j \to k\}$. We represent the interactions of all

[6] In this paper we assume the sets $\mathcal{A}_i, i \in \mathcal{N}$, to be in $\mathbb{R}$ for simplicity. However, the decentralized mechanism and the results we present in this paper can be easily generalized to the scenario where for each $i \in \mathcal{N}$, $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ is a convex and compact set in higher dimensional space $\mathbb{R}^{n_i}$. Furthermore, each space $\mathbb{R}^{n_i}$ can be of a different dimension n_i for different $i \in \mathcal{N}$.

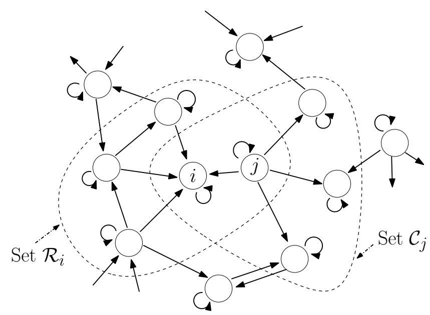

Figure 1: A large scale network depicting the neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_j$ of users i and j respectively.

network users together by a graph matrix $\boldsymbol{G} := [G_{ij}]_{N \times N}$. The matrix $\boldsymbol{G}$ consists of 0's and 1's, where $G_{ij} = 1$ represents that user i is affected by user j, i.e. $j \in \mathcal{R}_i$ and $G_{ij} = 0$ represents no influence of user j on user i, i.e. $j \notin \mathcal{R}_i$. Note that $\boldsymbol{G}$ is not necessarily a symmetric matrix. However, $G_{ii} = 1$ for all $i \in \mathcal{N}$ because $i \to i$. We assume that,

Assumption 2. *The sets $\mathcal{R}_i, \mathcal{C}_i, i \in \mathcal{N}$, are independent of the users' action profile $\boldsymbol{a}_{\mathcal{N}} := (a_k)_{k \in \mathcal{N}} \in \prod_{k \in \mathcal{N}} \mathcal{A}_k$. Furthermore, for each $i \in \mathcal{N}$, $|\mathcal{C}_i| \geq 3$.*

Assumption 2 implies that the graph matrix $\boldsymbol{G}$ does not depend on the users' actions. There are applications (for example see (Jackson, 2008, Chapter 6)) where this assumption does not hold; we do not consider such scenarios in this paper. Examples of applications where Assumption 2 is valid are discussed in Section 2.2. Other such examples can be found in Bramoull and Kranton (2007); (Sharma, 2009, Chapter 5); Yuan (Preprint).

The performance of a user that results from actions taken by the users affecting it is quantified by a utility function. We denote the utility of user $i \in \mathcal{N}$ resulting from the action profile $\boldsymbol{a}_{\mathcal{R}_i} := (a_k)_{k \in \mathcal{R}_i}$ by $u_i(\boldsymbol{a}_{\mathcal{R}_i})$. We assume that,

Assumption 3. *For all $i \in \mathcal{N}$, the utility function $u_i : \mathbb{R}^{|\mathcal{R}_i|} \to \mathbb{R}$ is concave in $\boldsymbol{a}_{\mathcal{R}_i}$ and $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = 0$ for $a_i \notin \mathcal{A}_i$.* [7] *The function u_i is user i's private information.*

The assumptions that u_i is concave and is user i's private information are reasonable as evidenced by the applications described in Bramoull and Kranton (2007); (Sharma, 2009, Chapter 5); Yuan (Preprint). The assumption, $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = 0$ for $a_i \notin \mathcal{A}_i$, is made for the following reason. Because $\mathcal{A}_i$ is the set of user i's feasible actions and user i knows this set (Assumption 1), it also knows that any action profile $\boldsymbol{a}_{\mathcal{R}_i}$ in which $a_i \notin \mathcal{A}_i$, is not possible to occur. Therefore, such an action profile $\boldsymbol{a}_{\mathcal{R}_i}$ does not provide any utility to user i.

We assume that,

Assumption 4. *Each network user $i \in \mathcal{N}$ is strategic and non-cooperative/selfish.*

Assumption 4 implies that the users have an incentive to misrepresent their private information, e.g. a user $i \in \mathcal{N}$ may not want to report to other users or to the network operator its true preference for the users' actions, if this results in an action profile in its own favor.

Each user $i \in \mathcal{N}$ pays a tax $t_i \in \mathbb{R}$ to the network operator. This tax can be imposed for the following reasons: (i) For the use of the network by the users. (ii) To provide incentives to the users to take actions that achieve a network-wide performance objective. The tax is set according to the rules

[7] Note that a_i is always an element of $\boldsymbol{a}_{\mathcal{R}_i}$ because $i \to i$ and hence $i \in \mathcal{R}_i$.

specified by a mechanism and it can be either positive or negative for a user. With the flexibility of either charging a user (positive tax) or paying compensation/subsidy (negative tax) to a user, it is possible to induce the users to behave in a way such that a network-wide performance objective is achieved. For example, in a network with limited resources, we can set "positive tax" for the users that receive resources close to the amounts requested by them and we can pay "compensation" to the users that receive resources that are not close to their desirable ones.

Thus, with the available resources, we can satisfy all the users and induce them to behave in a way that leads to a resource allocation that is optimal according to a network-wide performance criterion. We assume that,

Assumption 5. *The network operator does not have any utility associated with the users' actions or taxes. It does not derive any profit from the users' taxes and acts like an accountant that redistributes the tax among the users according to the specifications of the allocation mechanism.*

Assumption 5 implies that the tax is charged in a way such that

$$\sum_{i \in \mathcal{N}} t_i = 0. \tag{1}$$

To describe the "overall satisfaction" of a user from the performance it receives from all users' actions and the tax (subsidy) it pays (receives) for it, we define an "aggregate utility function" $u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) : \mathbb{R}^{|\mathcal{R}_i|+1} \to \mathbb{R} \cup \{-\infty\}$ for each user $i \in \mathcal{N}$ as follows:

$$u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) := \begin{cases} -t_i + u_i(\boldsymbol{a}_{\mathcal{R}_i}), & \text{if } a_i \in \mathcal{A}_i,\ a_j \in \mathbb{R}, j \in \mathcal{R}_i \backslash \{i\}, \\ -\infty, & \text{otherwise.} \end{cases} \tag{2}$$

The definition of u_i^A indicates that an allocation $(\boldsymbol{a}_{\mathcal{R}_i}, t_i)$ is of no significance to user i if $a_i \notin \mathcal{A}_i$. This is because, as mentioned earlier, user i knows that an allocation $(\boldsymbol{a}_{\mathcal{R}_i}, t_i)$ in which $a_i \notin \mathcal{A}_i$ is not possible to occur as i cannot take an action outside $\mathcal{A}_i$. Because u_i and $\mathcal{A}_i$ are user i's private information (Assumptions 1 and 3), the aggregate utility u_i^A is also user i's private information. As stated in Assumption 4, users are non-cooperative and selfish. Therefore, *the users are self aggregate utility maximizers.*

In this paper we restrict attention to static problems. Specifically, we make the following assumption:

Assumption 6. *The set $\mathcal{N}$ of users, the graph $\boldsymbol{G}$, users' action spaces $\mathcal{A}_i, i \in \mathcal{N}$, and their utility functions $u_i, i \in \mathcal{N}$, are fixed in advance and they do not change during the time period of interest.*

Assumption 6 is restrictive. Ideally, we would like to address dynamic problems where $\mathcal{N}$, $\boldsymbol{G}$, $\mathcal{A}_i, i \in \mathcal{N}$, and $u_i, i \in \mathcal{N}$, change over time. At this point we are unable to handle dynamic problems, and for this reason we restrict attention to static problems.

We also assume that,

Assumption 7. *Every user $i \in \mathcal{N}$ knows the set $\mathcal{R}_i$ of users that affect it as well as the set $\mathcal{C}_i$ of users that are affected by it. The network operator knows $\mathcal{R}_i$ and $\mathcal{C}_i$ for all $i \in \mathcal{N}$.*

In networks where the sets $\mathcal{R}_i$ and $\mathcal{C}_i$ are not known to the users beforehand, Assumption 7 is still reasonable because of the following reason. As the graph $\boldsymbol{G}$ does not change during the time period of interest (Assumption 6), the information about the neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_i, i \in \mathcal{N}$, can be passed to the respective users by the network operator before the users determine their actions. Alternatively, the users can themselves determine the set of their neighbors before determining their actions. [8]

Thus, Assumption 7 can hold true for the rest of the action determination process.

In the next section we present some applications that motivate Model (M).

2.2. Applications

2.2.1. Application A: Online advertising

Consider an online guaranteed display (GD) ad system. In existing GD systems, individual advertisers sign contracts with web-publishers in which web-publishers agree to serve (within some given time period) a fixed number of impressions [9] of each ad for a lump sum payment. Here we consider an extension of current GD systems in which multiple advertisers can form clusters as shown in Fig. 2 and contracts can be signed for the number of impressions of each cluster. Figure 2 shows three display ad clusters. In each

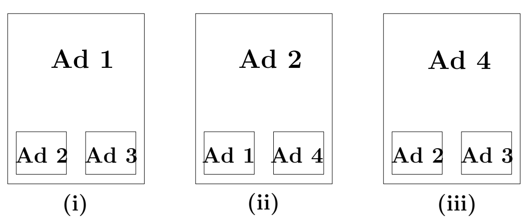

Figure 2: Three display ad clusters, each consisting of one main ad and two sub ads.

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[8]The exact method by which the users get information about their neighbor sets in a real network depends on the network characteristics.

[9]A display instance of an ad

ad cluster there is one main ad and two sub ads. For example in Fig. 2-(i), ad 1 is the main ad and ad 2 and ad 3 are the two sub ads. Suppose that the ad clusters are formed in a way so that in each cluster, the main ad creates a positive externality [10] to the sub ads. For example, ad 1 can be a Honda ad, whereas ad 2 and ad 3 can be ads of local Honda dealer and local Honda mechanic. We call the ad cluster in which ad i appears as the main ad as cluster i. [11] The arrangements of ads in clusters can be described by a graph similar to one shown in Fig. 1. In this graph an arrow from j to i would represent that ad i appears in cluster j. For each impression of cluster i, $i \in \{1, 2, 4\}$, advertiser i pays some fixed prespecified amount of money (bid) $b_i \in \mathbb{R}_+$ to the web publisher. Suppose the bid b_i is known only to advertiser i and the web publisher. Let $a_i, i \in \{1, 2, 4\}$, denote the number of impressions of clusters i delivered to the users. Furthermore, let A_i^{max} be the maximum number of impressions advertiser i can request for cluster i, i.e. $0 \leq a_i \leq A_i^{max}$. The constraint A_i^{max} may arise due to the budget constraint of advertiser i, or due to the restrictions imposed by the web publisher. For these reasons A_i^{max} may be private information of advertiser i (similar to Assumption 1) or private knowledge between advertiser i and the web publisher. Note that in the ad network, the number of impressions $a_i, i \in \{1, 2, 4\}$, can take only natural number values; therefore, the assumption of convex action sets $\mathcal{A}_i, i \in \mathcal{N}$, in Assumption 1 can be thought of as an approximation to this case. Note also that in the above ad network example,

[10]See (Mas-Colell et al., 2002, Chapter 11) for the definition of externality.

[11]We assume that there is at most one ad cluster in which ad i appears as the main ad; hence such a notation is well defined.

no cluster is associated with advertiser 3 or the web publisher (i.e. there is no cluster with main ad 3 or an ad of the web publisher). Such a scenario can be captured by Model (M) by associating dummy action variables a_3 and a_0 with advertiser 3 and web publisher respectively, and assuming $A_3^{max} = A_0^{max} = 0$.

Because of the way clusters are formed, each advertiser obtains a non-negative utility from the impressions of the clusters that it is part of. Thus we can represent the utilities of the four advertisers in the ad network of Fig. 2 as follows:

$$\begin{aligned} u_1(a_1, a_2) &= c_{11}a_1 + c_{12}a_2 - b_1a_1 \\ u_2(a_1, a_2, a_4) &= c_{21}a_1 + c_{22}a_2 + c_{24}a_4 - b_2a_2 \\ u_3(a_1, a_4) &= c_{31}a_1 + c_{34}a_4 \\ u_4(a_2, a_4) &= c_{42}a_2 + c_{44}a_4 - b_4a_4. \end{aligned} \tag{3}$$

In (3) $c_{ij} \in \mathbb{R}_+, i, j \in \{1, 2, 3, 4\}$ are non negative real valued constants. The constant c_{ij} represents the value obtained by advertiser i from each impression of cluster j. Suppose that for each j, c_{ij} is advertiser i's private information. The term $-b_ia_i, i \in \{1, 2, 4\}$ represents the loss in utility/monetary value incurred by advertiser i due to the prespecified payment it makes to the web publisher. Because the web publisher receives payments from the advertisers, it also obtains a utility as follows:

$$u_0(a_1, a_2, a_4) = b_1a_1 + b_2a_2 + b_4a_4. \tag{4}$$

Since each bid $b_i, i \in \{1, 2, 4\}$, is known to the web publisher, and none of the advertisers know all of these bids, the utility function u_0 is web publisher's private information. Similarly, since c_{ij} for each j and the bid b_i are private information of advertiser i, for each $i \in \{1, 2, 4\}$ the utility function u_i is advertiser i's private information. These properties of utility functions along with their linearity given by (3) and (4) are modeled by Assumption 3 in Model (M). If we assume that the arrangements of ads in clusters and the bids $b_i, i \in \{1, 2, 4\}$, of advertisers are predetermined, and do not change with any decision regarding the impressions delivery of various clusters, then this leads to assumptions 2 and 6 in Model (M).

As represented by (3), each advertiser benefits from a number of other advertisers by being part of their ad clusters. For this reason, in addition to making a direct payment to the web publisher, each advertiser should also make a payment to those advertisers that create positive externalities to it. Furthermore, because of their mutual payments the web publisher may offer discounts to the advertisers in their direct payments to her. These discounts can indirectly encourage advertisers to participate in the clustered GD ad scheme. However, one problem in the implementation of above type of payments/discounts is that in a big network, the advertisers may not have direct contracts with all their cluster sharing advertisers. Furthermore, if the advertisers and the web publisher are strategic and self utility maximizers, they would try to negotiate payments so as to get cluster impressions that maximize their respective utilities. In such a scenario, the abovementioned distribution of money among advertisers and the web publisher can be facilitated through a third party ad agency to which all advertisers and the web publisher can subscribe. The role of an ad agency can be mapped to that of the network operator in Model (M) described by Assumption 5.

2.2.2. Application B: Power allocation in cellular networks

Consider a single cell downlink wireless data network consisting of a Base Station (BS) and N mobile users as shown in Fig. 3. The BS uses Code

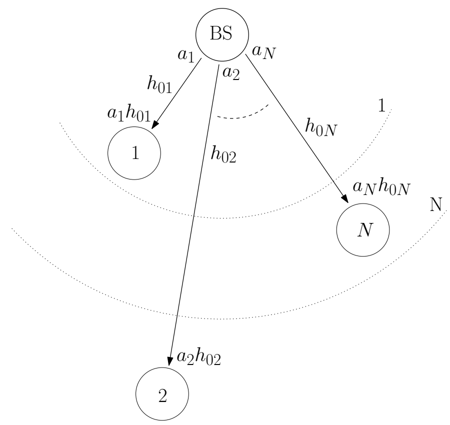

Figure 3: A downlink network with N mobile users and one base station

Division Multiple Access technology (CDMA) to transmit data to the users and each mobile user uses Minimum Mean Square Error Multi-User Detector (MMSE-MUD) receiver to decode its data. The signature codes used by the BS are not completely orthogonal as this helps increase the capacity of the network. Because of non-orthogonal codes, each user experiences interference due to the BS transmissions intended for other users. However, as the users in the cell are at different distances from the BS, and the power transmitted by the BS undergoes propagation loss, not all transmissions by the BS create interference to every user. For example, let us look at arcs 1 and N shown in Fig. 3 that are centered at the BS. Suppose the radius of arc 1 is much smaller than that of arc N. Then, the signal transmitted by the BS for users inside circle 1 (that corresponds to arc 1) will become negligible when it reaches outside users such as user N or user 2. On the other hand, the BS signals transmitted for user N and user 2 will be received with significant power by the users inside circle 1. This asymmetric interference relation between the mobile users can be depicted in a graph similar to one shown in Fig. 1. In the graph an arrow from j to i would represent that the signal transmitted for user j also affects user i. Note that since the signal transmitted for user i must reach i, the assumption $i \to i$ made in Section 2.1 holds in this case. If the users do not move very fast in the network, the network topology can be assumed to be fixed for small time periods. Therefore, if the BS transmits some pilot signals to all network users, the users can figure out which signals are creating interference to their signal reception. Thus, each user would know its (interfering) neighbor set as assumed in Assumption 7. Note that if the power transmitted by the BS to the users change, it may result in a change in the set of interfering neighbors of each user. This is different from Assumption 2 in Model (M). However, if the transmission power fluctuations resulting from a power allocation mechanism are not large, the set of interfering neighbors can be assumed to be fixed, and this can be approximated by Assumption 2.

The Quality of Service (QoS) that a user receives from decoding its data is quantified by a utility function. Due to interference the utility $u_i(\cdot)$ of user $i, i \in \mathcal{N}$, is a function of the vector $\boldsymbol{a}_{\mathcal{R}_i}$, where a_j is the transmission power used by the BS to transmit signals to user $j, j \in \mathcal{N}$, and $\mathcal{R}_i$ is the set of users such that the signals transmitted by the BS to users in $\mathcal{R}_i$ also reach user i. Note that in this case all transmissions, in other words the actions $a_i, i \in \mathcal{N}$, are carried out by the BS unlike Model (M) where each user $i \in \mathcal{N}$ takes its own action a_i. However, as we discuss below, the BS is only an agent which executes the outcome of the mechanism that determines these transmission powers. Thus, we can embed the downlink network scenario into Model (M) by treating each a_i as a decision "corresponding" to user $i, i \in \mathcal{N}$, which is executed by the BS for i. Since each user uses an MMSE-MUD receiver, a measure of user i's ($i \in \mathcal{N}$) utility can be the negative of the MMSE at the output of its receiver, [12] i.e.,

$$
\begin{aligned}
u_i(\boldsymbol{a}_{\mathcal{R}_i}) &= -MMSE_i \\
&= -\min_{\boldsymbol{z_i}^T \in \mathbb{R}^{1\times N}} E[\|b_i - \boldsymbol{z_i}^T \boldsymbol{y_i}\|^2] \\
&= -\big[(\boldsymbol{I} + \frac{2}{N_{0i}} \boldsymbol{S}_i \boldsymbol{X}_{\mathcal{R}_i} \boldsymbol{S}_i)^{-1}\big]_{ii}, \quad i \in \mathcal{N}.
\end{aligned}
\tag{5}
$$

In (5) b_i is the transmitted data symbol for user i, $\boldsymbol{y_i}$ is the output of user i's matched filter generated from its received data, $\boldsymbol{I}$ is the identity matrix of size $N \times N$, $N_{0i}/2$ is the two sided power spectral density (PSD) of thermal noise, $\boldsymbol{X}_{\mathcal{R}_i}$ is the cross-correlation matrix of signature waveforms corresponding to the users $j \in \mathcal{R}_i$, and $\boldsymbol{S}_i := \text{diag}((S_{ij})_{j\in\mathcal{R}_i})$ is the diagonal matrix consisting of the signal amplitudes $S_{ij}, j \in \mathcal{R}_i$, received by user i. S_{ij} is related to a_j as $S_{ij}^2 = a_j h_{0i}$, $j \in \mathcal{R}_i$, where h_{0i} is the channel gain from the BS to user i which represents the power loss along this path. As shown in Sharma and Teneketzis (2009, 2010), the utility function given by (5) is close to concave in $\boldsymbol{a}_{\mathcal{R}_i}$. Thus, Assumption 3 in Model (M) can be thought of as an approximation to the downlink network scenario.

Note that to compute user i's utility given in (5), knowledge of N_{0i}, $\boldsymbol{X}_{\mathcal{R}_i}$, and h_{0i} is required. The BS knows $\boldsymbol{X}_{\mathcal{R}_i}$ for each $i \in \mathcal{N}$ as it selects the signature waveform for each user. On the other hand, user $i, i \in \mathcal{N}$,

[12]See Verdu (2003) for the derivation of (5).

knows the PSD N_{0i} of thermal noise and the channel gain h_{0i} as these can be measured only at the respective receiver. Consider a network where the mobile users are selfish and non cooperative. Then, these users may not want to reveal their measured values N_{0i} and h_{0i}. On the other hand if the network operator that owns the BS does not have a utility and is not selfish, then, the BS can announce the signature waveforms it uses for each user. Thus, each user $i \in \mathcal{N}$ would know its corresponding cross correlation matrix $\boldsymbol{X}_{\mathcal{R}_i}$ and consequently, its utility function u_i. However, since N_{0i} and h_{0i} are user i's private information, the utility function u_i is private information of i which is similar to Assumption 3 in Model (M). If the wireless channel conditions vary slowly compared to the time period of interest, the channel gains and hence the users' utility functions can be assumed to be fixed. As mentioned earlier, for slowly moving users the network topology and hence the set of interfering neighbors can also be assumed to be fixed. These features are captured by Assumption 6 in Model (M).

In the presence of limited resources, the provision of desired QoS to all network users may not be possible. To manage the provision of QoS under such a situation the network operator (BS) can charge tax to the users and offer them the following tradeoff. It charges positive tax to the users that obtain a QoS close to their desirable one, and compensates the loss in the QoS of other users by providing a subsidy to them. Such a redistribution of money among users through the BS is possible under Assumption 5 in Model (M).

Having discussed various applications that motivate Model (M), we now go back to the generic model (M) and formulate a resource allocation problem for it.

2.3. The resource allocation problem (P_D)

For the network model (M) we wish to develop a mechanism to determine the users' action profile $\boldsymbol{a}_{\mathcal{N}} := (a_1, a_2, \ldots, a_N)$ and tax profile $\boldsymbol{t}_{\mathcal{N}} := (t_1, t_2, \ldots, t_N)$. We want the mechanism to work under the decentralized information constraints imposed by the model and to lead to a solution to the following centralized problem.

Problem ($\boldsymbol{P_C}$)

$$\begin{aligned} \max_{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}})} \quad & \sum_{i \in \mathcal{N}} u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) \\ \text{s.t.} \quad & \sum_{i \in \mathcal{N}} t_i = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} \equiv \max_{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \in \mathcal{D}} \quad & \sum_{i \in \mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}) \\ \text{where,} \quad & \mathcal{D} := \{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \in \mathbb{R}^{2N} \mid a_i \in \mathcal{A}_i \ \forall\, i \in \mathcal{N}; \sum_{i \in \mathcal{N}} t_i = 0\} \end{aligned} \tag{7}$$

The centralized optimization problem (6) is equivalent to (7) because for $(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \notin \mathcal{D}$, the objective function in (6) is negative infinity by (2). Thus $\mathcal{D}$ is the set of feasible solutions of Problem (P_C). Since by Assumption 3, the objective function in (7) is concave in $\boldsymbol{a}_{\mathcal{N}}$ and the sets $\mathcal{A}_i, i \in \mathcal{N}$, are convex and compact, there exists an optimal action profile $\boldsymbol{a}_{\mathcal{N}}^*$ for Problem (P_C). Furthermore, since the objective function in (7) does not explicitly depend on $\boldsymbol{t}_{\mathcal{N}}$, an optimal solution of Problem (P_C) must be of the form $(\boldsymbol{a}_{\mathcal{N}}^*, \boldsymbol{t}_{\mathcal{N}})$, where $\boldsymbol{t}_{\mathcal{N}}$ is any feasible tax profile for Problem (P_C), i.e. a tax profile that satisfies (1).

The solutions of Problem (P_C) are ideal action and tax profiles that we would like to obtain. If there exists an entity that has centralized information about the network, i.e. it knows all the utility functions $u_i, i \in \mathcal{N}$, and all action spaces $\mathcal{A}_i, i \in \mathcal{N}$, then that entity can compute the above ideal profiles by solving Problem (P_C). Therefore, we call the solutions of Problem (P_C) optimal centralized allocations. In the network described by Model (M), there is no entity that knows perfectly all the parameters that describe Problem (P_C) (Assumptions 1 and 3). Therefore, we need to develop a mechanism that allows the network users to communicate with one another and that leads to optimal solutions of Problem (P_C). Since a key assumption in Model (M) is that the users are strategic and non-cooperative, the mechanism we develop must take into account the users' strategic behavior in their communication with one another. To address all of these issues we take the approach of implementation theory Jackson (2001) for the solution of the decentralized allocation problem for Model (M). Henceforth we call the above decentralized resource allocation problem for Model (M) as Problem (P_D). In the next section we formulate Problem (P_D) in the framework of implementation theory, and present a decentralized resource allocation mechanism (game form) that works under the constraints imposed by Model (M) and achieves optimal centralized allocations.

3. A decentralized resource allocation mechanism

We begin this section by stating Problem (P_D) in the language of implementation theory. We then discuss an approach on how to construct a game form for this problem and follow that discussion with the specification of the proposed game form. We conclude the section by stating the properties of the proposed game form. These properties are summarized in Theorems 1 and 2 the proofs of which appear in the appendices.

3.1. Embedding Problem (P_D) in implementation theory framework

In the implementation theory framework a resource allocation problem is described by specifying a triple $(\mathcal{E}, \mathcal{D}, \gamma)$. [13] The *environment space* $\mathcal{E}$ and the *action space* $\mathcal{D}$ characterize the problem model, and the goal correspondence $\gamma : \mathcal{E} \to \mathcal{D}$ characterizes the desirable centralized allocations for the problem.

There are N users in the network model (M); therefore the environment space $\mathcal{E}$ of Problem (P_D) is a product space of N environment spaces $(\mathcal{E}_i, i \in \mathcal{N})$, one corresponding to each user. The environment $\boldsymbol{e}_i$ of user $i, i \in \mathcal{N}$, consists of the set $\mathcal{A}_i \times \mathbb{R}$ of its feasible actions and taxes, its utility function u_i, its information about its neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_i$, and its (common) knowledge [14] about the facts described by Assumptions 2, 4, 5, 6 and 7. The environment space $\mathcal{E}_i$ of user i is the space of all possible environments $\boldsymbol{e}_i$, i.e., it consists of the following: the space of all sets $\mathcal{A}_i \times \mathbb{R} \subset \mathbb{R}^2$ such that $\mathcal{A}_i \subset \mathbb{R}$ is convex and compact and $0 \in \mathcal{A}_i$, the space of all concave functions $u_i : \mathbb{R}^{|\mathcal{R}_i|} \to \mathbb{R}$ such that $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = 0$ for $a_i \notin \mathcal{A}_i$, the space of all finite subsets $\mathcal{R}_i$ and $\mathcal{C}_i$ of the set of natural numbers, and the common knowledge mentioned above.

The action space $\mathcal{D}$ of Problem (P_D) is the space of all feasible action and tax profiles $(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}})$ as defined in (7).

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[13] Refer to (Sharma, 2009, Chapter 3) and Sharma and Teneketzis (2010).

[14] See Aumann (1976); Washburn and Teneketzis (1984) for the definition of common knowledge.

The goal correspondence γ for Problem (P_D) maps each environment $\boldsymbol{e} \in \mathcal{E}$ to the set of action and tax profiles $(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \in \mathcal{D}$ that are solutions to Problem (P_C).

Having described Problem (P_D) in the framework of implementation theory, we now look at the specification of a decentralized mechanism from the implementation theory perspective. In implementation theory a decentralized resource allocation mechanism is specified in terms of a game form $(\mathcal{M}, f)$, where $\mathcal{M} := \prod_{i \in \mathcal{N}} \mathcal{M}_i$ is the message/strategy space and $f : \mathcal{M} \to \mathcal{D}$ is the outcome function.

Therefore, our objective of designing a decentralized allocation mechanism for model (M) transforms into designing a game form. For our problem, we want to develop a game form $(\mathcal{M}, f)$ that is *individually rational*, *budget balanced*, and that *implements in Nash equilibria*

the goal correspondence γ. Individual rationality guarantees voluntary participation of the users in the allocation process specified by the game form, budget balance guarantees that there is no money left unclaimed/not allocated at the end of the allocation process (i.e. it ensures (1)), and implementation in NE guarantees that the allocations corresponding to the set of NE of the game $(\mathcal{M}, f, \{u_i^A\}_{i \in \mathcal{N}})$ are a subset of the optimal centralized allocations (solutions of Problem (P_C)).

We would like to clarify at this point the definition of individual rationality/voluntary participation condition in the context of our problem. Note that in network model (M), the participation/non-participation of each user determines the network structure and the set of local public goods (users' actions) accessible to the participating users. To define individual rationality in this setting we consider our mechanism to be consisting of two stages as discussed in (Fudenberg and Tirole, 1991, Chapter 7). In the first stage, knowing the game form, each user makes a decision whether to participate in the game form or not. The users who decide not to participate are considered out of the system. Those who decide to participate follow the game form to determine the levels of local public goods in the network formed by them. [15] In such a two stage mechanism, individual rationality implies the following. If the network formed by the participating users satisfies all the properties of Model (M), [16] then, at all NE of the game (among the participating users) induced by the game form, the utility of each participating user will be at least as much as the utility it obtains without participation (i.e. if the user is out of the system). This in turn implies that, if there are at least two other participating users that are affected by the actions of a user, then such a user voluntarily participates in the game form.

We would also like to clarify the rationale behind choosing NE as the solution concept for our problem. Note that because of assumptions 1 and 3 in Model (M), the environment of our problem is one of incomplete information. Therefore one may speculate the use of Bayesian Nash or dominant strategy as appropriate solution concepts for our problem. However, since the users

[15] This network is a subgraph obtained by removing the nodes corresponding to non-participating users from the original graph (directed network) constructed by all the users in the system.

[16] In particular, the network formed by the participating users must satisfy Assumption 2 that there are at least three users affected by each local public good in this network. Note that all other assumptions of Model (M) automatically carry over to the network formed by any subset of the users in Model (M).

in Model (M) do not possess any prior beliefs about the utility functions and action sets of other users, we cannot use Bayesian Nash as a solution concept for Model (M). Furthermore, because of impossibility results for the existence of non-parametric efficient dominant strategy mechanisms in classical public good environments Groves and Ledyard (1987), we do not know if it is possible to design such mechanisms for the local public good environment of Model (M). The well known Vickrey-Clarke-Groves (VCG) mechanisms that achieve incentive compatibility and efficiency with respect to non-numeraire goods, do not guarantee budget balance Groves and Ledyard (1987). Hence they are inappropriate for our problem as budget balance is one of the desirable properties in our problem. VCG mechanisms are also unsuitable for our problem because they are direct mechanisms and any direct mechanism would require infinite message space to communicate the generic continuous (and concave) utility functions of users in Model (M). Because of all of above reasons, and the known existence results for non-parametric, individually rational, budget-balanced Nash implementation mechanisms for classical private and public goods environments Groves and Ledyard (1987), we choose Nash as the solution concept for our problem. We interpret the complete information Nash game equilibria in the incomplete information environment of Model (M) in the same way as suggested by (Groves and Ledyard, 1987, Section 4, page 69) and (Reichelstein and Reiter, 1988, page 664). We present the definition and detailed interpretation of NE for our current problem at the end of Section 3.3. In the next section we construct a game form for the resource allocation problem (P_D) that achieves the abovementioned desirable properties – Nash implementation, individual rationality, and budget balance.

3.2. Constructing game form for Problem (P_D)

As discussed in the previous section, we are interested in determining a game form that has the following properties: (i) It implements in NE the optimal solution of Problem (P_C); (ii) It is individually rational; and (iii) It is budget balanced. In this section we first develop a conceptual framework that must guide the construction of game forms which possess the above properties. We then present a game form that is designed within the developed framework.

We begin with a discussion on the construction of the message space. Since an allocation for Problem (P_D) consists of the action profile and the tax profile of the users, the message exchange among the users should contain information that is helpful in determining the optimal values of these profiles. Since each user's utility is affected by the actions of a subset of network users, each user should have a contribution in determining the actions of all its neighbors that affect its utility. Furthermore, a user should make a payment for the actions of all these neighbors because they all contribute to its utility. Since each neighbor's action makes a different contribution to the user's utility, the user may make different payments for each neighbor's actions. One way to take into account the above two factors is to let each user communicate as its message/strategy a proposal that consists of two components: one that indicates what actions the user wants its neighbors to take; and the other that indicates the price the user wants to pay for the actions of each of its neighbors.

We next discuss the construction of the outcome function. The specification of the outcome function is arguably the most important and challenging task in the construction of a game form/decentralized resource allocation mechanism. Since the designer of the mechanism cannot alter the users' utility functions $u_i, i \in \mathcal{N}$, the only way it can achieve the objectives of Nash implementation, individual rationality, and budget balance is through the provision of appropriate tax functions/incentives that induce strategic users to follow the mechanism's operational rules. Below we develop the guidelines for the construction of outcome functions that achieve each of the above objectives.

To achieve implementation in NE, the outcome function must make sure that all NE of the message exchange (that is done according to the discussion presented above) lead to optimal centralized allocations. This suggests that the outcome function must induce price taking behavior for all users at all NE. If price taking behavior is achieved, then, through NE price control, the mechanism can induce users to take actions that are optimal for their own objective and for the centralized problem (P_C). As discussed in the previous paragraph, a user should make a payment for the actions of each of its neighbors that affect its utility. In order for the mechanism to induce price taking behavior, the NE price that a user $i \in \mathcal{N}$ pays for its neighbors' actions must depend only on the messages/proposals of users other than i. Thus, the NE tax of user $i, i \in \mathcal{N}$, must be of the form $\sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_j^*$ where $\hat{a}_j^*$ is the NE action of user j and l_{ij}^* is the NE price of this action for user i that is independent of user i's message. With the NE tax form $\sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_j^*$, each user $i \in \mathcal{N}$ can influence its NE aggregate utility only through the actions $\hat{a}_j^*, j \in \mathcal{R}_i$. Since each user's utility is its private information, the utility maximizing actions of a user are known only to that user. Therefore, to allow each user to obtain its utility maximizing actions at given NE prices, the outcome function must provide each user $i \in \mathcal{N}$ an independent control, through its action proposal, over each of the actions $\hat{a}_j^*, j \in \mathcal{R}_i$. In other words, each action $\hat{a}_j^*, j \in \mathcal{N}$, must be independently controlled by each of the users $i \in \mathcal{C}_j$ and this fact should be reflected in the form of the outcome function.

To achieve budget balance, the NE prices $l_{ij}^*, j \in \mathcal{R}_i, i \in \mathcal{N}$, must satisfy

$$\sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_j^* = 0,$$

or, equivalently, [17]

$$\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{C}_j} l_{ij}^* \hat{a}_j^* = 0. \tag{8}$$

One way to satisfy the requirement in (8) is to set for each $j \in \mathcal{N}$, $\sum_{i \in \mathcal{C}_j} l_{ij}^* = 0$.

The features of the outcome function discussed so far could lead to price taking behavior and budget balance. However, the construction of an outcome function with the above features only may lead to the following difficulty. Since each user knows that its price proposal does not affect its own tax and hence, its aggregate utility, it may propose arbitrary prices for its neighbors in its price proposal. One way to overcome this difficulty without altering price taking behavior and budget balance is to add a penalty to the tax form of each user. To preserve the price taking behavior of the

[17] From the construction of the graph matrix $\mathcal{G}$ and the sets $\mathcal{R}_i$ and $\mathcal{C}_j$, $i, j \in \mathcal{N}$, the sum $\sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{R}_i} (\cdot)$ is equivalent to the sum $\sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{C}_j} (\cdot)$.

users at NE, this penalty should be imposed only at off NE messages. The penalty should depend on each user's own price proposal and it should increase with the user's price proposal. However, to avoid unnecessary penalties, the penalty of a user should be reduced if its action proposal for its neighbors is in agreement with other users' action proposals. Adding to the tax form a penalty term with the above characteristics may result in an unbalanced budget. To preserve budget balance a third term should be added to the tax of each user. This term must balance the net flow of the money due to the penalty term. Since the penalty is imposed on the users only at off NE messages, this balancing term should be included in the users' tax only at off NE messages. To prevent the balancing term from altering a user's strategic behavior that is governed by the first two terms in the user's tax, the balancing term should be independent of the user's own message.

To achieve individual rationality the outcome function must make sure that at all NE, the utility of each user is at least as much as its initial utility. This property is achieved if the outcome function has the following features discussed earlier in this section: (i) It induces price taking behavior; and (ii) It gives each user an independent control over the actions that affect its utility. Since each user can control the actions that affect its NE utility, for any set of NE prices $l_{ij}^*, j \in \mathcal{R}_i$, a user $i \in \mathcal{N}$ can force all the actions $\hat{a}_j^*, j \in \mathcal{R}_i$, to be 0, thereby also making its NE payment $\sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_j^* = 0$. Thus, with the above features of the outcome function, each user can independently guarantee a minimum of zero utility for itself which is its initial utility.

With the guidelines developed above, we proceed with the construction of a game form in the next section.

3.3. A game form

In this section we present a game form for the resource allocation problem presented in Section 2.3. We provide explicit expressions of each of the components of the game form, the message space and the outcome function. The construction of these components is motivated by the arguments presented in the previous section. We assume that the game form is common knowledge among the users and the network operator.

The message space:

We let each user $i \in \mathcal{N}$ send to the network operator a message $\boldsymbol{m}_i \in \mathcal{M}_i := \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}$ that has the following form:

$$\boldsymbol{m}_i := ({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}); \quad {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|},\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}, \tag{9}$$

where,

$${}^i\boldsymbol{a}_{\mathcal{R}_i} := ({}^ia_k)_{k\in\mathcal{R}_i} \text{ and } {}^i\boldsymbol{\pi}_{\mathcal{R}_i} := ({}^i\pi_k)_{k\in\mathcal{R}_i}, \quad i \in \mathcal{N}. \tag{10}$$

User i also sends the component $({}^ia_k, {}^i\pi_k), k \in \mathcal{R}_i$, of its message to its neighbor $k \in \mathcal{R}_i$. In this message, ${}^ia_k$ is the action proposal for user $k, k \in \mathcal{R}_i$, by user $i, i \in \mathcal{N}$. Similarly, ${}^i\pi_k$ is the price that user $i, i \in \mathcal{N}$, proposes to pay for the action of user $k, k \in \mathcal{R}_i$. A detailed interpretation of these message elements is given in Section 3.4.

The outcome function:

After the users communicate their messages to the network operator, their actions and taxes are determined as follows. For each user $i \in \mathcal{N}$, the network operator determines the action $\hat{a}_i$ of user i from the messages communicated by its neighbors that are affected by it (set $\mathcal{C}_i$), i.e. from the message profile $\boldsymbol{m}_{\mathcal{C}_i} := (\boldsymbol{m}_k)_{k\in\mathcal{C}_i}$:

$$\hat{a}_i(\boldsymbol{m}_{\mathcal{C}_i}) = \frac{1}{|\mathcal{C}_i|}\sum_{k\in\mathcal{C}_i} {}^k a_i, \quad i \in \mathcal{N}. \tag{11}$$

To determine the users' taxes the network operator considers each set $\mathcal{C}_j$, $j \in \mathcal{N}$, and assigns indices $1, 2, \ldots, |\mathcal{C}_j|$ in a cyclic order to the users in $\mathcal{C}_j$. Each index $1, 2, \ldots, |\mathcal{C}_j|$ is assigned to an arbitrary but unique user $i \in \mathcal{C}_j$. Once the indices are assigned to the users in each set $\mathcal{C}_j, j \in \mathcal{N}$, they remain fixed throughout the time period of interest. We denote the index of user $i \in \mathcal{N}$ associated with set $\mathcal{C}_j$, $j \in \mathcal{N}$, by $\mathcal{I}_{ij}$. The index $\mathcal{I}_{ij} \in \{1, 2, \ldots, |\mathcal{C}_j|\}$ if $i \in \mathcal{C}_j$, and $\mathcal{I}_{ij} = 0$ if $i \notin \mathcal{C}_j$. Since each index $1, 2, \ldots, |\mathcal{C}_j|$ is assigned to a unique user $i \in \mathcal{C}_j$, for each $\mathcal{C}_j, j \in \mathcal{N}$, $\mathcal{I}_{ij} \neq \mathcal{I}_{kj} \;\; \forall\, i, k \in \mathcal{C}_j;\; i \neq k$. Note also that for any user $i \in \mathcal{N}$, and any $j, k \in \mathcal{R}_i$, the indices $\mathcal{I}_{ij}$ and $\mathcal{I}_{ik}$ are not necessarily the same and are independent of each other. We denote the user with index $k \in \{1, 2, \ldots, |\mathcal{C}_j|\}$ in set $\mathcal{C}_j$ by $\mathcal{C}_{j(k)}$. Thus, $\mathcal{C}_{j(\mathcal{I}_{ij})} = i$ for $i \in \mathcal{C}_j$, $j \in \mathcal{N}$. The cyclic order indexing means that, if $\mathcal{I}_{ij} = |\mathcal{C}_j|$, then $\mathcal{C}_{j(\mathcal{I}_{ij}+1)} = \mathcal{C}_{j(1)}$, $\mathcal{C}_{j(\mathcal{I}_{ij}+2)} = \mathcal{C}_{j(2)}$, and so on. In Fig. 4 we illustrate the above indexing rule for the set $\mathcal{C}_j$ shown in Fig. 1.

Based on the indexing described above, the users' taxes are determined as follows. For each $i \in \mathcal{N}$, the tax $\hat{t}_i$ is determined from the message profile $(\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}$ as,

$$\begin{aligned}\hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = &\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\; \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}) + \sum_{j\in\mathcal{R}_i} {}^i\pi_j \left({}^i a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a_j\right)^2 \\ &- \sum_{j\in\mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}} a_j\right)^2, \qquad i \in \mathcal{N},\end{aligned} \tag{12}$$

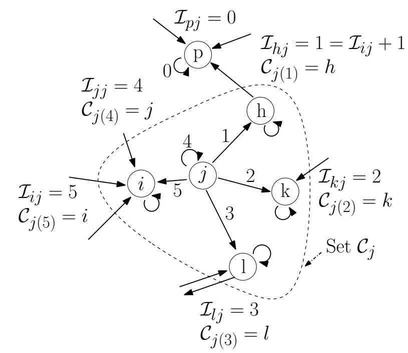

Figure 4: Illustration of indexing rule for set $\mathcal{C}_j$ shown in Fig. 1. The index $\mathcal{I}_{rj}$ of each user $r \in \mathcal{C}_j$ is indicated on the arrow directed from user j to user r. The notation to denote these indices and to denote the user with a particular index is shown outside the dashed boundary demarcating the set $\mathcal{C}_j$.

where,

$$l_{ij}(\boldsymbol{m}_{\mathcal{C}_j}) = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j, \qquad j \in \mathcal{R}_i,\ i \in \mathcal{N}. \tag{13}$$

The game form given by (9)–(13) and the users' aggregate utility functions in (2) induce a game $(\mathcal{M}, f, \{u_i^A\}_{i\in\mathcal{N}})$. We define a NE of this game as a message profile $\boldsymbol{m}^*_{\mathcal{N}}$ that has the following property:

$$\begin{aligned} &u_i^A\Big(\big(\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})\big)_{j\in\mathcal{R}_i},\ \hat{t}_i\big((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}\big)\Big) \geq \\ &u_i^A\Big(\big(\hat{a}_j(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_j}/i)\big)_{j\in\mathcal{R}_i},\ \hat{t}_i\big((\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big), \quad \forall\, \boldsymbol{m}_i \in \mathcal{M}_i,\ \forall\, i \in \mathcal{N}. \end{aligned} \tag{14}$$

We interpret the NE defined in (14) in the way of (Groves and Ledyard, 1987, Section 4, page 69) and (Reichelstein and Reiter, 1988, page 664) as described below. NE in general describe strategic behavior of users in games of complete information. This can be seen from (14) where, to define the NE, it requires complete information of all users' aggregate utility functions.

However, the users in Model (M) do not know each other's utilities; therefore, the game $(\mathcal{M}, f, \{u_i^A\}_{i \in \mathcal{N}})$ induced by the game form given in (9)–(13) and the users' aggregate utility functions in (2) is not one of complete information. We can create a game of complete information by increasing the message/strategy space following Maskin's approach Maskin. However, such an approach would result in an infinite dimensional message/strategy space for the corresponding game. We do not follow Maskin's approach; instead, we adopt the interpretation of Groves and Ledyard (1987) and Reichelstein and Reiter (1988). Specifically, by quoting Reichelstein and Reiter (1988), "we interpret our analysis as applying to an unspecified (message exchange) process in which users grope their way to a stationary message and in which the Nash property (14) is a necessary condition for stationarity." Alternatively, by quoting Groves and Ledyard (1987), "we do not suggest that each user knows all of system environment when it computes its message. We do suggest, however, that the complete information Nash game-theoretic equilibrium messages may be the possible stationary messages of some unspecified dynamic message exchange process".

In the next section we show that the allocations obtained by the game form presented in (9)–(13) at all NE message profiles (satisfying (14)), are optimal centralized allocations.

3.4. Properties of the game form

We begin this section with an *intuitive discussion* on how the game form presented in Section 3.3 achieves optimal centralized allocations. We then formalize the results in Theorems 1 and 2.

To understand how the proposed game form achieves optimal centralized allocations, let us look at the properties of NE allocations corresponding to this game form. A NE of the game induced by the game form (9)–(13) and the users' utility functions (2) can be interpreted as follows: Given the users' messages $\boldsymbol{m}_k, k \in \mathcal{C}_i$, the outcome function computes user i's action as $1/|\mathcal{C}_i|\big(\sum_{k\in\mathcal{C}_i} {}^k a_i\big)$. Therefore, user i's action proposal ${}^i a_i$ can be interpreted as the increment over the sum of other users' action proposals for i that i desires so as to bring its allocated action $\hat{a}_i$ to its own desired value. Thus, if the computed action for i based on the neighbors' proposals does not lie in $\mathcal{A}_i$, user i can propose an appropriate action ${}^i a_i$ and bring its allocated action within $\mathcal{A}_i$. The flexibility of proposing any action ${}^i a_i \in \mathbb{R}$ gives each user $i \in \mathcal{N}$ the capability to bring its allocation $\hat{a}_i$ within its feasible set $\mathcal{A}_i$ by unilateral deviation. Therefore, at any NE, $\hat{a}_i \in \mathcal{A}_i,\ \forall\, i \in \mathcal{N}$. By taking the sum of taxes in (12) it can further be seen, after some computations, that the allocated tax profile $(\hat{t}_i)_{i\in\mathcal{N}}$ satisfies (1) (even at off-NE messages). [18] Thus, all NE allocations $\Big((\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}))_{i\in\mathcal{N}},\ (\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}))_{i\in\mathcal{N}}\Big)$ lie in $\mathcal{D}$ and hence are feasible solutions of Problem (P_C).

To see further properties of NE allocations, let us look at the tax function in (12). The tax of user i consists of three types of terms. The type-1 term is $\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j})$; it depends on all action proposals for each of user i's neighbors $j \in \mathcal{R}_i$, and the price proposals for each of these neighbors by users other than user i. The type-2 term is $\sum_{j\in\mathcal{R}_i} {}^i\pi_j \left({}^i a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2$; this term depends on ${}^i\boldsymbol{a}_{\mathcal{R}_i}$ as well as ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$. Finally, the type-3 term is the following: $-\sum_{j\in\mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2$; this term depends only on the

[18] For details refer to Appendix Appendix A.

messages of users other than i. Since ${}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}$ does not affect the determination of user i's action, and affects only the type-2 term in $\hat{t}_i$, the NE strategy of user $i, i \in \mathcal{N}$, that minimizes its tax is to propose for each $j \in \mathcal{R}_i$, ${}^{i}\boldsymbol{\pi}_j = 0$ unless at the NE, ${}^{i}a_j = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j$. Since the type-2 and type-3 terms in the users' tax are similar across users, for each $i \in \mathcal{N}$ and $j \in \mathcal{R}_i$, all the users $k \in \mathcal{C}_j$ choose the above strategy at NE. Therefore, the type-2 and type-3 terms vanish from every users' tax $\hat{t}_i, i \in \mathcal{N}$, at all NE. Thus, the tax that each user $i \in \mathcal{N}$ pays at a NE $\boldsymbol{m}^*_{\mathcal{N}}$ is of the form $\sum_{j \in \mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$. The NE term $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}), i \in \mathcal{N}, j \in \mathcal{R}_i$, can therefore be interpreted as the "personalized price" for user i for the NE action $\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$ of its neighbor j. Note that at a NE, the personalized price for user i is not controlled by i's own message. The reduction of the users' NE taxes into the form $\sum_{j \in \mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$ implies that at a NE, each user $i \in \mathcal{N}$ has a control over its aggregate utility only through its action proposal. [19] If all other users' messages are fixed, each user has the capability of shifting the allocated action profile $\hat{\boldsymbol{a}}_{\mathcal{R}_i}$ to its desired value by proposing an appropriate ${}^{i}\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$ (See the discussion in the previous paragraph). Therefore, the NE strategy of each user $i \in \mathcal{N}$ is to propose an action profile ${}^{i}\boldsymbol{a}_{\mathcal{R}_i}$ that results in an allocation $\hat{\boldsymbol{a}}_{\mathcal{R}_i}$ that maximizes its aggregate utility. Thus, at a NE, each user maximizes its aggregate utility for its given personalized prices. By the construction of the tax function, the sum of the users' tax is zero at all NE and off equilibria. Thus, the individual aggregate utility maximization of the users also result in the maximization of the sum of users' aggregate utilities subject to the

[19] Note that user i's action proposal determines the actions of all the users $j \in \mathcal{R}_i$; thus, it affects user i's utility $u_i\Big(\big(\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})\big)_{j \in \mathcal{R}_i}\Big)$ as well as its tax $\sum_{j \in \mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$.

budget balance constraint which is the objective of Problem (P_C).

It is worth mentioning at this point the significance of type-2 and type-3 terms in the users' tax. As explained above, these two terms vanish at NE. However, if for some user $i \in \mathcal{N}$ these terms are not present in its tax $\hat{t}_i$, then, the price proposal ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ of user i will not affect its tax and hence, its aggregate utility. In such a case, user i can propose arbitrary prices ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ because they would affect only other users' NE prices. The presence of type-2 and type-3 terms in user i's tax prevent such a behavior as they impose a penalty on user i if it proposes a high value of ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ or if its action proposal for its neighbors deviates too much from other users' proposals. Even though the presence of type-2 and type-3 terms in user i's tax is necessary as explained above, it is important that the NE price $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}), j \in \mathcal{R}_i$ of user $i \in \mathcal{N}$ is not affected by i's own proposal ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$. This is because, in such a case, user i may influence its own NE price in an unfair manner and may not behave as a price taker. To avoid such a situation, the type-2 and type-3 terms are designed in a way so that they vanish at NE. Thus, this construction induces price taking behavior in the users at NE and leads to optimal allocations.

From all of above discussion it can be seen that the proposed message space, the action function, and the tax function (with three types of terms) satisfy the features, discussed in Section 3.2, that are required to achieve the properties of Nash implementation, individual rationality, and budget balance.

The *results* that formally establish the above properties of the game form are summarized in Theorems 1 and 2 below.

Theorem 1. *Let $\boldsymbol{m}^*_{\mathcal{N}}$ be a NE of the game induced by the game form pre-*

sented in Section 3.3 and the users' utility functions (2). Let $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) := (\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}), \hat{\boldsymbol{t}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}})) := \Big((\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}))_{i\in\mathcal{N}},\ (\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}))_{i\in\mathcal{N}} \Big)$ *be the action and tax profiles at* $\boldsymbol{m}^*_{\mathcal{N}}$ *determined by the game form. Then,*

(a) Each user $i \in \mathcal{N}$ *weakly prefers its allocation* $(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i)$ *to the initial allocation* $(\mathbf{0}, 0)$*. Mathematically,*

$$u^A_i\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big) \geq u^A_i\Big(\mathbf{0},\ 0\Big), \qquad \forall\, i \in \mathcal{N}.$$

(b) $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ *is an optimal solution of Problem* (P_C)*.*

⊠

Theorem 2. *Let* $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ *be an optimum action profile corresponding to Problem* (P_C)*. Then,*

(a) There exist a set of personalized prices $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$*, such that*

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} = \underset{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i \setminus \{i\}}}{\arg\max} \ -\sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \quad \forall\, i \in \mathcal{N}.$$

(b) There exists at least one NE $\boldsymbol{m}^*_{\mathcal{N}}$ *of the game induced by the game form presented in Section 3.3 and the users' utility functions (2) such that,* $\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}) = \hat{\boldsymbol{a}}^*_{\mathcal{N}}$*. Furthermore, if* $\hat{t}^*_i := \sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j, i \in \mathcal{N}$*, the set of all NE* $\boldsymbol{m}^*_{\mathcal{N}} = (\boldsymbol{m}^*_i)_{i\in\mathcal{N}} = ({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i})$ *that result in* $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ *is characterized by the solution of the following set of conditions:*

$$\begin{aligned}
\frac{1}{|\mathcal{C}_i|}\sum_{k\in\mathcal{C}_i} {}^k a^*_i &= \hat{a}^*_i, \quad i \in \mathcal{N}, \\
{}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi^*_j &= l^*_{ij}, \quad j \in \mathcal{R}_i,\ i \in \mathcal{N}, \\
{}^i\pi^*_j \left({}^i a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a^*_j\right)^2 &= 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}, \\
{}^i\pi^*_j &\geq 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}.
\end{aligned}$$

⊠

Because Theorem 1 is stated for an arbitrary NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by the game form presented in Section 3.3 and the users' utility functions (2), the assertion of the theorem holds for all NE of this game.

Part (a) of Theorem 1 establishes that the game form presented in Section 3.3 is *individually rational*, i.e., at any NE allocation, the aggregate utility of each user is at least as much as its aggregate utility before participating in the game/allocation process. Because of this property of the game form, each user voluntarily participates in the allocation process.

Part (b) of Theorem 1 asserts that all NE of the game induced by the game form presented in Section 3.3 and the users' utility functions (2) result in optimal centralized allocations (solutions of Problem (P_C)). Thus the set of NE allocations is a subset of the set of centralized allocations. This establishes that the game form presented in Section 3.3 *implements in NE* the goal correspondence γ defined by Problem (P_C) (see Section 3.1). Because of this property, the game form guarantees to provide a centralized allocation irrespective of which NE is achieved in the game induced by the game form.

The assertion of Theorem 1 that establishes the above two properties of the game form is based on the assumption that there exists a NE of the game induced by the game form of Section 3.3 and the users' utility functions (2). However, Theorem 1 does not say anything about the existence of NE.

Theorem 2 asserts that NE exist in the above game, and provides conditions that characterize the set of all NE that result in optimal centralized allocations of the form $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) = (\hat{\boldsymbol{a}}^*_{\mathcal{N}}, (\sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j)_{i\in\mathcal{N}})$, where $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ is any optimal centralized action profile.

Theorem 2 also establishes the following property of the game form. Since the optimal action profile $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$ in the statement of Theorem 2 is arbitrary, the theorem implies that the game form of Section 3.3 can obtain each of the optimum action profiles of Problem (P_C) through at least one of the NE of the induced game. This establishes that the above game form is not biased towards any particular optimal centralized action profile.

We present the proofs of Theorem 1 and Theorem 2 in Appendices Appendix A and Appendix B respectively.

In the next section we present a discussion on how the game form of Section 3.3 can be implemented in a real system and we also discuss the limitations associated with it.

3.5. Implementation of the game form

In this section we discuss two aspects of implementation of the decentralized mechanism specified by the game form of Section 3.3. First we discuss how the game form itself can be implemented, i.e., how the message communication and the determination of allocations specified by the game form can be carried out in a real system. We then discuss how NE can be achieved in the game induced by the above game form.

We will show below that the presence of a network operator is important for the implementation of the game form. To see this let us first suppose that the network operator is not present in the network. As discussed in Section 3.3 the outcome function specifies the allocation $(\hat{a}_i, \hat{t}_i)$ for a user $i \in \mathcal{N}$ based on its neighbors' messages. Since the game form is common knowledge among the users, if each user announces its messages to all its neighbors, every user can have the required set of messages to compute its own allocations. However, with this kind of local communication, the messages required to compute user i's allocation are not necessarily known to users other than i. Therefore, even though the other users know the outcome function for user i, no other user can check if the allocation determined by user i corresponds to its neighbors' messages. Since each user $i \in \mathcal{N}$ is selfish, it cannot be relied upon for the determination of its allocation. Therefore, in large-scale systems such as one represented by Model (M), where each user does not hear all other users' messages, the presence of a network operator is extremely important. The network operator's role is twofold. First, according to the specification of the game form (of Section 3.3) each user announces its messages to its neighbors as well as to the network operator. The network operator knows the network structure (Assumption 7) and the outcome function for each user. Thus, it can compute all the allocations based on the messages it receives, and then it can tell each user its corresponding allocation (or it can check whether the allocation (a_i^*, t_i^*) implemented by user $i, i \in \mathcal{N}$, is the same as that specified by the mechanism). The other role of the network operator that facilitates implementation of the game form is the following. Note that the game form specifies redistribution of money among the users by charging each user an appropriate positive or negative tax (see (1)). This means that the tax money must go from one subset of the users to the other subset of users. Since the users do not have complete network information, nor do they know the allocations of other users in the network, they cannot determine the appropriate flow of money in the network. The network operator implements this redistribution of money by acting as an accountant that collects money from the users who must pay positive tax according to the game form and gives the money back to the users who must receive subsidies (negative tax).

The discussion presented above shows how the game form of Section 3.3 can be implemented in the presence of a network operator. However, to achieve the properties of the game form described by Theorems 1 and 2, we need a method to obtain NE of the game induced by this game form. Even though the above game form achieves implementation in NE, at present we do not have an algorithm for the computation of these equilibria. For our problem, best response dynamics do not guarantee convergence to NE because the games induced by the proposed game form are not, in general, supermodular. For this reason, in this paper we restricted our focus to equilibrium analysis of the proposed mechanism. In Section 4 we discuss a few approaches for the development of efficient mechanisms that can compute NE.

4. Future directions

The problem formulation and the solution of the resource allocation problem for local public good networks presented in this paper open up several new directions for future research. First, as discussed in the previous section, the development of efficient mechanisms that can compute NE is an important open problem. To address this problem there can be two different directions for future research. (i) The development of algorithms that converge to NE of game forms that implement the social welfare correspondence of the local public goods provision problem. (ii) The development of alternative mechanisms/game forms that lead to supermodular games. Second, the network model we studied in this paper assumed a given set of users and a given network topology. In many local public good networks such as social or research networks, the set of network users and the network topology must be determined as part of network objective maximization. These situations give rise to interesting admission control and network formation problems many of which are open research problems. Finally, in this paper we focused on static resource allocation problem where the system characteristics do not change with time. The development of implementation mechanisms for local public good networks under dynamic situations, where the system characteristics change during the determination of resource allocation, are important research problems. Resource allocation mechanisms for these systems must take into account the dynamics of the system and can be addressed using dynamic game theory and dynamic mechanism design.

In the appendices that follow, we present the proof of Theorems 1 and 2. We divide the proof into several claims to organize the presentation.

Appendix A. Proof of Theorem 1

We prove Theorem 1 in four claims. In Claims 2 and 3 we show that all users weakly prefer a NE allocation (corresponding to the game form presented in Section 3.3) to their initial allocations; these claims prove part (a) of Theorem 1. In Claim 1 we show that a NE allocation is a feasible solution of Problem (P_C). In Claim 4 we show that a NE action profile is an optimal action profile for Problem (P_C). Thus, Claim 1 and Claim 4 establish that a NE allocation is an optimal solution of Problem (P_C) and prove part (b) of Theorem 1.

Claim 1. *If $\boldsymbol{m}^*_{\mathcal{N}}$ is a NE of the game induced by the game form presented in Section 3.3 and the users' utility functions (2), then the action and tax profile $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) := (\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}), \hat{\boldsymbol{t}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}))$ is a feasible solution of Problem (P_C), i.e. $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) \in \mathcal{D}$.*

Proof:

We prove the feasibility of the NE action and tax profiles in two steps. First we prove the feasibility of the NE tax profile, then we prove the feasibility of the NE action profile.

To prove the feasibility of NE tax profile, we need to show that it satisfies (1). For this, we first take the sum of second and third terms on the Right Hand Side (RHS) of (12) over all $i \in \mathcal{N}$, i.e.

$$\sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} \left[{}^{i}\pi_j \left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j \right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j \right)^2 \right]. \tag{A.1}$$

From the construction of the graph matrix $\mathcal{G}$ and the sets $\mathcal{R}_i$ and $\mathcal{C}_j$, $i, j \in \mathcal{N}$, the sum $\sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i}(\cdot)$ is equal to the sum $\sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j}(\cdot)$. Therefore, we can rewrite (A.1) as

$$\sum_{j\in\mathcal{N}} \left[\sum_{i\in\mathcal{C}_j} {}^{i}\pi_j \left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j \right)^2 - \sum_{i\in\mathcal{C}_j} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j \right)^2 \right]. \tag{A.2}$$

Note that both the sums inside the square brackets in (A.2) are over all $i \in \mathcal{C}_j$. Because of the cyclic indexing of the users in each set $\mathcal{C}_j$, $j \in \mathcal{N}$, these two sums are equal. Therefore the overall sum in (A.2) evaluates to zero. Thus, the sum of taxes in (12) reduces to

$$\sum_{i\in\mathcal{N}} \hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\, \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}). \tag{A.3}$$

Combining (13) and (A.3) we obtain

$$\sum_{i\in\mathcal{N}} \hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \sum_{j\in\mathcal{N}} \left[\sum_{i\in\mathcal{C}_j} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j - \sum_{i\in\mathcal{C}_j} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j \right] \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}) = 0. \quad \text{(A.4)}$$

The second equality in (A.4) follows because of the cyclic indexing of the users in each set $\mathcal{C}_j$, $j \in \mathcal{N}$, which makes the two sums inside the square brackets in (A.4) equal. Because (A.4) holds for any arbitrary message profile $\boldsymbol{m}_{\mathcal{N}}$, it follows that at NE $\boldsymbol{m}^*_{\mathcal{N}}$,

$$\sum_{i\in\mathcal{N}} \hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = 0. \quad \text{(A.5)}$$

To complete the proof of Claim 1, we have to prove that for all $i \in \mathcal{N}$, $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \in \mathcal{A}_i$. We prove this by contradiction. Suppose $\hat{a}^*_i \notin \mathcal{A}_i$ for some $i \in \mathcal{N}$. Then, from (2), $u^A_i(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i) = -\infty$. Consider $\widetilde{\boldsymbol{m}}_i = (({}^i\widetilde{a}_i, {}^i\boldsymbol{a}^*_{\mathcal{R}_i}/i), {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i})$ where ${}^ia^*_k, k \in \mathcal{R}_i\backslash\{i\}$, and ${}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}$ are respectively the NE action and price proposals of user i and ${}^i\widetilde{a}_i$ is such that

$$\hat{a}_i(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_i}/i) = \frac{1}{|\mathcal{C}_i|}\Big({}^i\widetilde{\boldsymbol{a}}_i + \sum_{\substack{k\in\mathcal{C}_i \\ k\neq i}} {}^ka^*_i \Big) \in \mathcal{A}_i. \quad \text{(A.6)}$$

Note that the flexibility of user i in choosing any message ${}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$ (see (9)) allows it to choose an appropriate ${}^i\widetilde{a}_i$ that satisfies the condition in (A.6). For the message $\widetilde{\boldsymbol{m}}_i$ constructed above,

$$\begin{aligned} &u^A_i\Big(\big(\hat{a}_k(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\ \ \hat{t}_i\big((\widetilde{\boldsymbol{m}}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \\ =&-\hat{t}_i\big((\widetilde{\boldsymbol{m}}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big) + u_i\Big(\big(\hat{a}_k(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i}\Big) \qquad \text{(A.7)} \\ >&-\infty = u^A_i(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i). \end{aligned}$$

Thus if $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \notin \mathcal{A}_i$ user i finds it profitable to deviate to $\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}$. Inequality (A.7) implies that $\boldsymbol{m}^*_{\mathcal{N}}$ cannot be a NE, which is a contradiction. Therefore, at any NE $\boldsymbol{m}_{\mathcal{N}}^*$, we must have $\hat{a}_i(\boldsymbol{m}_{\mathcal{C}_i}^*) \in \mathcal{A}_i \; \forall \; i \in \mathcal{N}$. This along with (A.5) implies that, $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*) \in \mathcal{D}$. □

Claim 2. *If $\boldsymbol{m}_{\mathcal{N}}^*$ is a NE of the game induced by the game form presented in Section 3.3 and the users' utility functions (2), then, the tax $\hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j}^*)_{j\in\mathcal{R}_i}) =: \hat{t}_i^*$ paid by user $i, i \in \mathcal{N}$, at the NE $\boldsymbol{m}_{\mathcal{N}}^*$ is of the form $\hat{t}_i^* = \sum_{j\in\mathcal{R}_i} l_{ij}^* \; \hat{a}_j^*$, where $l_{ij}^* = l_{ij}(\boldsymbol{m}_{\mathcal{C}_j}^*)$ and $\hat{a}_j^* = \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}^*)$.*

Proof:

Let $\boldsymbol{m}_{\mathcal{N}}^*$ be the NE specified in the statement of Claim 2. Then, for each $i \in \mathcal{N}$,

$$u_i^A\Big(\big(\hat{a}_k(\boldsymbol{m}_i, \boldsymbol{m}_{\mathcal{C}_k}^*/i)\big)_{k\in\mathcal{R}_i},\; \hat{t}_i\big((\boldsymbol{m}_i,\, \boldsymbol{m}_{\mathcal{C}_j}^*/i)_{j\in\mathcal{R}_i}\big)\Big) \le\; u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*, \hat{t}_i^*\Big),\;\; \forall\; \boldsymbol{m}_i \in \mathcal{M}_i. \tag{A.8}$$

Substituting $\boldsymbol{m}_i = ({}^i\boldsymbol{a}_{\mathcal{R}_i}^*, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\; {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}$, in (A.8) and using (11) implies that

$$u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*,\;\; \hat{t}_i\big((({}^i\boldsymbol{a}_{\mathcal{R}_i}^*, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\, \boldsymbol{m}_{\mathcal{C}_j}^*/i)_{j\in\mathcal{R}_i}\big)\Big) \le\; u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*, \hat{t}_i^*\Big), \quad \forall\; {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{A.9}$$

Since u_i^A decreases in t_i (see (2)), (A.9) implies that

$$\hat{t}_i\Big((({}^i\boldsymbol{a}_{\mathcal{R}_i}^*, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\, \boldsymbol{m}_{\mathcal{C}_j}^*/i\big)_{j\in\mathcal{R}_i}\Big) \ge \hat{t}_i^*, \quad \forall\; {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{A.10}$$

Substituting (12) in (A.10) results in

$$\sum_{j\in\mathcal{R}_i}\left[l_{ij}^*\hat{a}_j^* + {}^i\pi_j\left({}^ia_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^*\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2\right]$$
$$\ge \sum_{j\in\mathcal{R}_i}\left[l_{ij}^*\hat{a}_j^* + {}^i\pi_j^*\left({}^ia_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^*\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2\right],$$
$$\forall\; {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{A.11}$$

Canceling the common terms in (A.11) gives

$$\sum_{j \in \mathcal{R}_i} ({}^{i}\pi_j - {}^{i}\pi_j^*) \left({}^{i}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 \geq 0, \qquad \forall\ {}^{i}\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{A.12}$$

Since (A.12) must hold for all ${}^{i}\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}$, we must have that

$$\text{for each } j \in \mathcal{R}_i,\ \text{either } {}^{i}\pi_j^* = 0 \text{ or } {}^{i}a_j^* = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*. \tag{A.13}$$

From (A.13) it follows that at any NE $\boldsymbol{m}_{\mathcal{N}}^*$,

$${}^{i}\pi_j^* \left({}^{i}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 = 0, \quad \forall\ j \in \mathcal{R}_i, \quad \forall\ i \in \mathcal{N}. \tag{A.14}$$

Note that (A.14) also implies that $\forall\ i \in \mathcal{N}$ and $\forall\ j \in \mathcal{R}_i$,

$${}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2 = 0. \tag{A.15}$$

(A.15) follows from (A.14) because for each $i \in \mathcal{N}$, $j \in \mathcal{R}_i$ also implies that $j \in \mathcal{R}_{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}$. Using (A.14) and (A.15) in (12) we obtain that any NE tax profile must be of the form

$$\hat{t}_i^* = \sum_{j \in \mathcal{R}_i} l_{ij}^*\ \hat{a}_j^*, \quad \forall\ i \in \mathcal{N}. \tag{A.16}$$

□

Claim 3. *The game form given in Section 3.3 is individually rational, i.e. at every NE $\boldsymbol{m}_{\mathcal{N}}^*$ of the game induced by this game form and the users' utilities in (2), each user $i \in \mathcal{N}$ weakly prefers the allocation $(\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*, \hat{t}_i^*)$ to the initial allocation $(\mathbf{0}, 0)$. Mathematically,*

$$u_i^A\Big(\mathbf{0},\ 0\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*, \hat{t}_i^*\Big), \qquad \forall\ i \in \mathcal{N}. \tag{A.17}$$

Proof:

Suppose $\boldsymbol{m}^*_{\mathcal{N}}$ is a NE of the game induced by the game form presented in Section 3.3 and the users' utility functions (2). From Claim 2 we know the form of users' tax at $\boldsymbol{m}^*_{\mathcal{N}}$. Substituting that from (A.16) into (A.8) we obtain that for each $i \in \mathcal{N}$,

$$u_i^A\Big(\big(\hat{a}_k(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\ \ \hat{t}_i\big((\boldsymbol{m}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \leq u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big),$$
$$\forall\ \boldsymbol{m}_i = ({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}) \in \mathcal{M}_i. \tag{A.18}$$

Substituting for $\hat{t}_i$ in (A.18) from (12) and using (A.15) we obtain,

$$\begin{aligned} &u_i^A\Bigg(\Big(\hat{a}_k\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_k}/i\big)\Big)_{k\in\mathcal{R}_i}, \\ &\qquad \sum_{j\in\mathcal{R}_i}\Big(l^*_{ij}\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) + {}^i\pi_j\big({}^ia_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\big)^2\Big)\Bigg) \\ &\leq u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big), \qquad \forall\ {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|},\ \forall\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \end{aligned} \tag{A.19}$$

In particular, ${}^i\boldsymbol{\pi}_{\mathcal{R}_i} = \boldsymbol{0}$ in (A.19) implies that

$$\begin{aligned} &u_i^A\Bigg(\Big(\hat{a}_k\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, \boldsymbol{0}), \boldsymbol{m}^*_{\mathcal{C}_k}/i\big)\Big)_{k\in\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i}\Big(l^*_{ij}\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, \boldsymbol{0}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)\Bigg) \\ &\leq u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big), \qquad \forall\ {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}. \end{aligned} \tag{A.20}$$

Since (A.20) holds for all ${}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$, substituting $\frac{1}{|\mathcal{C}_j|}({}^ia_j + \sum_{k\in\mathcal{C}_j\setminus\{i\}} {}^ka_j) = \overline{a}_j$ for all $j \in \mathcal{R}_i$ in (A.20) gives

$$u_i^A\Big(\big(\overline{a}_j\big)_{j\in\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i}\big(l^*_{ij}\ \overline{a}_j\big)\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big),\ \ \forall\ \overline{a}_{\mathcal{R}_i} := (\overline{a}_j)_{j\in\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}. \tag{A.21}$$

For $\overline{a}_{\mathcal{R}_i} = \mathbf{0}$, (A.21) implies further that

$$u_i^A\Big(\mathbf{0},\ 0\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j \in \mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big), \qquad \forall\ i \in \mathcal{N}. \tag{A.22}$$

□

Claim 4. *A NE allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is an optimal solution of the centralized problem (P_C).*

Proof:

For each $i \in \mathcal{N}$, (A.21) can be equivalently written as

$$\begin{aligned} \hat{\boldsymbol{a}}^*_{\mathcal{R}_i} &\in \underset{\overline{\boldsymbol{a}}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\, u_i^A\Big(\overline{\boldsymbol{a}}_{\mathcal{R}_i},\ \sum_{j \in \mathcal{R}_i} l^*_{ij}\ \overline{a}_j\Big) \\ &= \underset{\substack{\overline{a}_i \in \mathcal{A}_i \\ \overline{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i \backslash \{i\}}}{\arg\max} \left\{ -\sum_{j \in \mathcal{R}_i} l^*_{ij}\ \overline{a}_j + u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i}) \right\} \end{aligned} \tag{A.23}$$

Let for each $i \in \mathcal{N}$, $f_{\mathcal{A}_i}(a_i)$ be a convex function that characterizes the set $\mathcal{A}_i$ as, $a_i \in \mathcal{A}_i \Leftrightarrow f_{\mathcal{A}_i}(a_i) \leq 0$. [20]

Since for each $i \in \mathcal{N}$, $u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})$ is assumed to be concave in $\overline{\boldsymbol{a}}_{\mathcal{R}_i}$ and the set $\mathcal{A}_i$ is convex, the Karush Kuhn Tucker (KKT) conditions (Boyd and Vandenberghe, 2004, Chapter 11) are necessary and sufficient for $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ to be a maximizer in (A.23). Thus, for each $i \in \mathcal{N}$ $\exists$ $\lambda_i \in \mathbb{R}_+$ such that, $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ and λ_i satisfy the KKT conditions given below:

$$\begin{aligned} \forall\ j \in \mathcal{R}_i \backslash \{i\}, \quad l^*_{ij} - \nabla_{\overline{a}_j} u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})\ |_{\overline{\boldsymbol{a}}_{\mathcal{R}_i} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} &= 0, \\ l^*_{ii} - \nabla_{\overline{a}_i} u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})\ |_{\overline{\boldsymbol{a}}_{\mathcal{R}_i} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} + \lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i)\ |_{\overline{a}_i = \hat{a}^*_i} &= 0, \\ \lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) &= 0. \end{aligned} \tag{A.24}$$

[20]By Boyd and Vandenberghe (2004) we can always find a convex function that characterizes a convex set.

For each $i \in \mathcal{N}$, adding the KKT condition equations in (A.24) over $k \in \mathcal{C}_i$ results in

$$\sum_{k\in\mathcal{C}_i} l^*_{ki} - \nabla_{\overline{a}_i} \sum_{k\in\mathcal{C}_i} u_k(\overline{\boldsymbol{a}}_{\mathcal{R}_k}) \mid_{\overline{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} +\lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i) \mid_{\overline{a}_i=\hat{a}^*_i} = 0. \tag{A.25}$$

From (13) we have,

$$\sum_{k\in\mathcal{C}_i} l^*_{ki} = \sum_{k\in\mathcal{C}_i} \left({}^{\mathcal{C}_{i(\mathcal{I}_{ki}+1)}}\pi^*_i - {}^{\mathcal{C}_{i(\mathcal{I}_{ki}+2)}}\pi^*_i \right) = 0. \tag{A.26}$$

Substituting (A.26) in (A.25) we obtain [21] $\forall\, i \in \mathcal{N}$,

$$\begin{aligned} -\nabla_{\overline{a}_i} \sum_{k\in\mathcal{C}_i} u_k(\overline{\boldsymbol{a}}_{\mathcal{R}_k}) \mid_{\overline{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} +\lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i) \mid_{\overline{a}_i=\hat{a}^*_i} &= 0, \\ \lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) &= 0. \end{aligned} \tag{A.27}$$

The conditions in (A.27) along with the non-negativity of $\lambda_i, i \in \mathcal{N}$, specify the KKT conditions (for variable $\hat{\boldsymbol{a}}_{\mathcal{N}}$) for Problem (P_C). Since (P_C) is a concave optimization problem, KKT conditions are necessary and sufficient for optimality. As shown in (A.27), the action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ satisfies these optimality conditions. Furthermore, the tax profile $\hat{\boldsymbol{t}}^*_{\mathcal{N}}$ satisfies, by its definition, $\sum_{i\in\mathcal{N}} \hat{t}^*_i = 0$. Therefore, the NE allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is an optimal solution of Problem (P_C). This completes the proof of Claim 4 and hence, the proof of Theorem 1. □

Claims 1–4 (Theorem 1) establish the properties of NE allocations based on the assumption that there exists a NE of the game induced by the game form of Section 3.3 and users' utility functions (2). However, these claims do not guarantee the existence of a NE. This is guaranteed by Theorem 2 which is proved next in Claims 5 and 6.

[21] The second equality in (A.27) is one of the KKT conditions from (A.24).

Appendix B. Proof of Theorem 2

We prove Theorem 2 in two steps. In the first step we show that if the centralized problem (P_C) has an optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$, there exist a set of personalized prices, one for each user $i \in \mathcal{N}$, such that when each $i \in \mathcal{N}$ individually maximizes its own utility taking these prices as given, it obtains $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ as an optimal action profile. In the second step we show that the optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ and the corresponding personalized prices can be used to construct message profiles that are NE of the game induced by the game form of Section 3.3 and users' utility functions in (2).

Claim 5. *If Problem (P_C) has an optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$, there exist a set of personalized prices $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$, such that*

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} \in \underset{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i \setminus \{i\}}}{\arg\max} \; -\sum_{j \in \mathcal{R}_i} l^*_{ij}\, \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \qquad \forall\, i \in \mathcal{N}. \tag{B.1}$$

Proof:

Suppose $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ is an optimal action profile corresponding to Problem (P_C). Writing the optimization problem (P_C) only in terms of variable $\hat{\boldsymbol{a}}_{\mathcal{N}}$ gives

$$\begin{aligned} \hat{\boldsymbol{a}}^*_{\mathcal{N}} \in \underset{\hat{\boldsymbol{a}}_{\mathcal{N}}}{\arg\max} \; & \sum_{i \in \mathcal{N}} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \\ \text{s.t.} \;\; & \hat{a}_i \in \mathcal{A}_i, \; \forall\, i \in \mathcal{N}. \end{aligned} \tag{B.2}$$

As stated earlier, an optimal solution of Problem (P_C) is of the form $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}_{\mathcal{N}})$, where $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ is a solution of (B.2) and $\hat{\boldsymbol{t}}_{\mathcal{N}} \in \mathbb{R}^N$ is any tax profile that satisfies (1). Because KKT conditions are necessary for optimality, the optimal solution in (B.2) must satisfy the KKT conditions. This implies that there exist

$\lambda_i \in \mathbb{R}_+, i \in \mathcal{N}$, such that for each $i \in \mathcal{N}$, λ_i and $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ satisfy

$$\begin{aligned} -\nabla_{\hat{a}_i} \sum_{k \in \mathcal{C}_i} u_k(\hat{\boldsymbol{a}}_{\mathcal{R}_k}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_k} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} + \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \mid_{\hat{a}_i = \hat{a}^*_i} = 0, \\ \lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) = 0, \end{aligned} \tag{B.3}$$

where $f_{\mathcal{A}_i}(\cdot)$ is the convex function defined in Claim 4. Defining for each $i \in \mathcal{N}$,

$$\begin{aligned} l^*_{ij} &:= \nabla_{\hat{a}_j} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_i}}, \qquad j \in \mathcal{R}_i \backslash \{i\}, \\ l^*_{ii} &:= \nabla_{\hat{a}_i} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} - \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \mid_{\hat{a}_i = \hat{a}^*_i}, \end{aligned} \tag{B.4}$$

we get $\forall\, i \in \mathcal{N}$,

$$\sum_{k \in \mathcal{C}_i} l^*_{ki} = \nabla_{\hat{a}_i} \sum_{k \in \mathcal{C}_i} u_k(\hat{\boldsymbol{a}}_{\mathcal{R}_k}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_k} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} - \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \mid_{\hat{a}_i = \hat{a}^*_i} = 0. \tag{B.5}$$

The second equality in (B.5) follows from (B.3). Furthermore, (B.4) implies that $\forall\, i \in \mathcal{N}$,

$$\begin{aligned} \forall\, j \in \mathcal{R}_i \backslash \{i\}, \quad l^*_{ij} - \nabla_{\hat{a}_j} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} = 0, \\ l^*_{ii} - \nabla_{\hat{a}_i} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i} = \hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} + \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \mid_{\hat{a}_i = \hat{a}^*_i} = 0. \end{aligned} \tag{B.6}$$

The equations in (B.6) along with the second equality in (B.3) imply that for each $i \in \mathcal{N}$, $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ and λ_i satisfy the KKT conditions for the following maximization problem:

$$\max_{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R}, j \in \mathcal{R}_i \backslash \{i\}}} - \sum_{j \in \mathcal{R}_i} l^*_{ij}\ \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \tag{B.7}$$

Because the objective function in (B.7) is concave (Assumption 3), KKT conditions are necessary and sufficient for optimality. Therefore, we conclude from (B.6) and (B.3) that,

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} \in \arg\max_{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R}, j \in \mathcal{R}_i \backslash \{i\}}} - \sum_{j \in \mathcal{R}_i} l^*_{ij}\ \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \qquad \forall\, i \in \mathcal{N}.$$

$\square$

Claim 6. *Let $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ be an optimal action profile for Problem (P_C), let $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$, be the personalized prices corresponding to $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ as defined in Claim 5, and let $\hat{t}^*_i := \sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j, i \in \mathcal{N}$. Let $\boldsymbol{m}^*_i := ({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}), i \in \mathcal{N}$, be a solution to the following set of relations:*

$$\begin{aligned} \frac{1}{|\mathcal{C}_i|}\sum_{k\in\mathcal{C}_i} {}^k a^*_i &= \hat{a}^*_i, \quad i \in \mathcal{N}, && \text{(B.8)} \\ {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi^*_j &= l^*_{ij}, \quad j \in \mathcal{R}_i,\ i \in \mathcal{N}, && \text{(B.9)} \\ {}^i\pi^*_j \left({}^i a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a^*_j\right)^2 &= 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}, && \text{(B.10)} \\ {}^i\pi^*_j &\geq 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}. && \text{(B.11)} \end{aligned}$$

Then, $\boldsymbol{m}^*_{\mathcal{N}} := (\boldsymbol{m}^*_1, \boldsymbol{m}^*_2, \dots, \boldsymbol{m}^*_N)$ is a NE of the game induced by the game form of Section 3.3 and the users' utility functions (2). Furthermore, for each $i \in \mathcal{N}$, $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) = \hat{a}^*_i$, $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}) = l^*_{ij}, j \in \mathcal{R}_i$, and $\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \hat{t}^*_i$.

Proof:

Note that, the conditions in (B.8)–(B.11) are necessary for any NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by the game form of Section 3.3 and users' utilities (2), to result in the allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ (see (11), (13) and (A.14)). Therefore, the set of solutions of (B.8)–(B.11), if such a set exists, is a superset of the set of all NE corresponding to the above game that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$. Below we show that the solution set of (B.8)–(B.11) is in fact exactly the set of all NE that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$.

To prove this, we first show that the set of relations in (B.8)–(B.11) do have a solution. Notice that (B.8) and (B.10) are satisfied by setting for each $i \in \mathcal{N}$, ${}^k a^*_i = \hat{a}^*_i \ \forall\ k \in \mathcal{C}_i$. Notice also that for each $j \in \mathcal{N}$, the sum over

$i \in \mathcal{C}_j$ of the right hand side of (B.9) is 0. Therefore, for each $j \in \mathcal{N}$, (B.9) has a solution in ${}^i\pi_j^*, i \in \mathcal{C}_j$. Furthermore, for any solution ${}^i\pi_j^*, i \in \mathcal{C}_j, j \in \mathcal{N}$, of (B.9), ${}^i\pi_j^* + c, i \in \mathcal{C}_j, j \in \mathcal{N}$, where c is some constant, is also a solution of (B.9). Consequently, by appropriately choosing c, we can select a solution of (B.9) such that (B.11) is satisfied.

It is clear from the above discussion that (B.8)–(B.11) have multiple solutions. We now show that the set of solutions $\boldsymbol{m}_{\mathcal{N}}^*$ of (B.8)–(B.11) is the set of NE that result in $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*)$. From Claim 5, (B.1) can be equivalently written as

$$\hat{\boldsymbol{a}}_{\mathcal{R}_i}^* \in \operatorname*{arg\,max}_{\hat{\boldsymbol{a}}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}} u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}, \sum_{j \in \mathcal{R}_i} l_{ij}^* \ \hat{a}_j\Big), \qquad i \in \mathcal{N}. \tag{B.12}$$

Substituting $\hat{a}_j|\mathcal{C}_j| - \sum_{k \in \mathcal{C}_j \setminus \{i\}} {}^k a_j^* = {}^i a_j$ for each $j \in \mathcal{R}_i$, $i \in \mathcal{N}$, in (B.12) we obtain

$$\begin{aligned} {}^i\boldsymbol{a}_{\mathcal{R}_i}^* \in \operatorname*{arg\,max}_{{}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}} u_i^A\bigg(\Big(\frac{1}{|\mathcal{C}_j|}\big({}^i a_j + \sum_{k \in \mathcal{C}_j \setminus \{i\}} {}^k a_j^*\big)\Big)_{j \in \mathcal{R}_i}, \\ \sum_{j \in \mathcal{R}_i} l_{ij}^* \ \frac{1}{|\mathcal{C}_j|}\big({}^i a_j + \sum_{k \in \mathcal{C}_j \setminus \{i\}} {}^k a_j^*\big)\bigg), \qquad i \in \mathcal{N}. \end{aligned} \tag{B.13}$$

Because of (B.10), (B.13) also implies that

$$\begin{aligned} ({}^i\boldsymbol{a}_{\mathcal{R}_i}^*, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}^*) \in \operatorname*{arg\,max}_{({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}) \in \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}} u_i^A\bigg(\Big(\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\ \boldsymbol{m}_{\mathcal{C}_j}^*/i\big)\Big)_{j \in \mathcal{R}_i}, \\ \sum_{j \in \mathcal{R}_i} l_{ij}^* \ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\ \boldsymbol{m}_{\mathcal{C}_j}^*/i\big) - \sum_{j \in \mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2\bigg), \\ i \in \mathcal{N}. \end{aligned} \tag{B.14}$$

Furthermore, since u_i^A is strictly decreasing in the tax (see (2)), (B.14) also implies the following:

$$\begin{aligned}({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}) \in \underset{({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}) \in \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}}{\arg\max} & u_i^A\Bigg(\Big(\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)_{j\in\mathcal{R}_i}, \\ & \sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) + \sum_{j\in\mathcal{R}_i} {}^i\pi_j \left({}^ia_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j\right)^2 \\ & - \sum_{j\in\mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a^*_j\right)^2\Bigg), \qquad i \in \mathcal{N}.\end{aligned} \tag{B.15}$$

Eq. (B.15) implies that, if the message exchange and allocation is done according to the game form presented in Section 3.3, then user $i, i \in \mathcal{N}$, maximizes its utility at $\boldsymbol{m}^*_i$ when all other users $j \in \mathcal{N}\backslash\{i\}$ choose their respective messages $\boldsymbol{m}^*_j, j \in \mathcal{N}\backslash\{i\}$. This, in turn, implies that a message profile $\boldsymbol{m}^*_{\mathcal{N}}$ that is a solution to (B.8)–(B.11) is a NE of the game induced by the aforementioned game form and the users' utilities (2). Furthermore, it follows from (B.8)–(B.11) that the allocation at $\boldsymbol{m}^*_{\mathcal{N}}$ is

$$\begin{aligned}\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) &= \frac{1}{|\mathcal{C}_i|}\sum_{k\in\mathcal{C}_i} {}^ka^*_i \;=\; \hat{a}^*_i, && i \in \mathcal{N}, \\ l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}) &= {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi^*_j \;=\; l^*_{ij}, && j \in \mathcal{R}_i,\; i \in \mathcal{N}, \\ \hat{t}_i\big((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}\big) &= \sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j}) + {}^i\pi^*_j\left({}^ia^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j\right)^2 \\ &\quad - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a^*_j\right)^2 \\ &= \sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_i \;=\; \hat{t}^*_i, && i \in \mathcal{N}.\end{aligned} \tag{B.16}$$

From (B.16) it follows that the set of solutions $\boldsymbol{m}^*_{\mathcal{N}}$ of (B.8)–(B.11) is exactly the set of NE that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$. This completes the proof of Claim 6 and hence the proof of Theorem 2. □

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References

Aumann, R., 1976. Agreeing to disagree. Annals of Statistics 4, 1236–1239.

Boyd, S., Vandenberghe, L., 2004. Convex optimization. Cambridge university press.

Bramoullé, Y., Kranton, R., 2007. Public goods in networks. Journal of Economic Theory 135, 478–494.

Chen, Y., 2002. A family of supermodular mechanisms implementing Lindahl allocations for quasilinear environments. Economic Theory 19, 773–790.

Fudenberg, D., Tirole, J., 1991. Game theory. The MIT Press.

Groves, T., Ledyard, J., 1977. Optimal allocation of public goods: A solution to the 'free rider' problem. Econometrica 45, 783–809.

Groves, T., Ledyard, J., 1987. Incentive compatibility since 1972, In T. Groves, R. Radner and S. Reiter, editors, *Information, Incentives, and Economic Mechanisms: Essays in Honor of Leonid Hurwicz*, University of Minnesota Press, Minneapolis. pp. 48–109.

Hurwicz, L., 1979. Outcome functions yielding Walrasian and Lindahl allocations at Nash equilibrium points. Review of Economic studies 46, 217–225.

Jackson, M., 2001. A crash course in implementation theory, in: Social choice and welfare, pp. 655–708.

Jackson, M.O., 2008. Social and economic networks. Princeton University Press.

Mas-Colell, A., Whinston, M.D., Green, J.R., 2002. Microeconomic theory. Oxford University Press.

Maskin, E., . The theory of implementation in Nash equilibrium: A survey, in: Hurwicz, L., Schmeidler, D., Sonnenschein, H. (Eds.), Social goals and social organization, Cambridge university press.

Reichelstein, S., Reiter, S., 1988. Game forms with minimal message space. Econometrica 56, 661–692.

Sharma, S., 2009. A mechanism design approach to decentralized resource allocation in wireless and large-scale networks: Realization and implementation. Ph.D. thesis. University of Michigan, Ann Arbor.

Sharma, S., Teneketzis, D., 2009. An externalities-based decentralized optimal power allocation algorithm for wireless networks. IEEE/ACM Transactions on Networking 17, 1819–1831.

Sharma, S., Teneketzis, D., 2010. A game-theoretic approach to decentralized optimal power allocation for cellular networks. Telecommunication Systems URL: http://dx.doi.org/10.1007/s11235-010-9302-6.

Stoenescu, T., Teneketzis, D., 2005. Decentralized resource allocation mechanisms in networks: realization and implementation, in: Birkhr̈user, E.H.A. (Ed.), Advances in Control, Communiation Networks, and Transportation Systems, pp. 225–263. In honor of Pravin Varaiya.

Tiebout, C.M., 1956. A pure theory of local expenditures. The Journal of Political Economy 64, 416–424.

Verdu, S., 2003. Multiuser detection. Cambridge university press.

Walker, M., 1981. A simple incentive compatible scheme for attaining Lindahl allocations. Econometrica 49, 65–71.

Washburn, R., Teneketzis, D., 1984. Asymptotic agreement among communicating decision makers. Stochastics 13, 103–129.

Yuan, K., Preprint. Public goods in directed networks.