2.1. The relaxation time approximation

Here we treat electrons as a free ideal gas (classical).

2.1.1. Totally ignore interactions/scatterings

Under a static electric field \( E \), electrons get a constant force.
\[
\vec{F} = -e \vec{E}
\]  
(2.1)

The second Law of Newton tell us that
\[
\vec{F} = \frac{d\vec{P}}{dt}
\]  
(2.2)

If we have a constant force, \( \vec{P} \) increases linearly as a function of \( t \)
\[
\vec{P}(t) = \vec{P}(t = 0) + F \cdot t
\]  
(2.3)

Conclusion: If we wait long enough, the velocity of an electron will diverge, which is obviously impossible and wrong.

2.1.2. The relaxation time approximation

The reason why we get a wrong answer is because electrons are not really a free gas. There are scatterings/interactions (between electrons, between an electron and an ion, and between an electron and an impurity). Thought these scatterings, electron gives its momentum and kinetic energy to other particles, e.g. ions or impurities or another electron, i.e. dissipation.

Dissipation slows down the flow of electrons, like viscosity in a liquid. To describe this effect, we can introduce a force along the \( -\vec{P} \) direction.
\[
\vec{F} = -e \vec{E} + \vec{F}_{\text{dissipation}}
\]  
(2.4)

Note: this is an approximation, because the real force on an electron is very complicated. It varies with time and it could be pointing at any direction (depending on the locations of other electrons and ions). Here, we take an average value for simplification. We know 2 things about this force:

1. it is along the \( -\vec{P} \) direction (on average)
2. it is zero, if there is no current, i.e. the force is zero if \( \vec{P} = 0 \).

Here, we introduce an extra assumption: the force is an analytic function of \( P \), i.e., we can do a Taylor expansion for the function \( \vec{F}_{\text{dissipation}}(\vec{P}) \).

According to the two facts listed above, this Taylor expansion must take the following form.
The coefficient of the first order term $1/\tau$ has the unit of $1/time$, so $\tau$ has the unit of time. We call it the relaxation time or the collision time. Roughly speaking, this is the average time interval between two collisions.

Now the Newton's second law says
\[
\frac{\dot{\vec{P}}}{\tau} = -e\vec{E} - \frac{\vec{P}}{\tau}
\]
This is the relaxation time approximation

(Read Chapter 1 for the historical approach that people used when they tried to "derive" this approximation)

### 2.1.3. Collision time $\tau$

The relaxation-time approximation doesn’t care about the source of the dissipation. It assumes that wherever it comes from, it can be described by only a single parameter, which is $\tau$. In reality, we know that it comes from mainly three sources:

- collisions between electrons $\tau_{ee}$
- collisions between electrons and phonons: $\tau_{ep}$
- collisions between electrons and impurities: $\tau_i$

If we assume that different scatterings are independent, the total $\tau$ satisfies
\[
\frac{1}{\tau} = \frac{1}{\tau_{ee}} + \frac{1}{\tau_{ep}} + \frac{1}{\tau_i}
\]
At low $T$, impurity scattering dominates. At high $T$ e-p scatterings dominates.

Think about why we add inverse $\tau$ from different sources, instead of adding $\tau$s directions

### 2.1.4. Stable solution

\[
\frac{\dot{\vec{P}}}{\tau} = -e\vec{E} - \frac{\vec{P}}{\tau} = 0
\]
\[
\vec{P} = -e\vec{E}\tau
\]
\[
\vec{v} = -\frac{e\tau}{m}\vec{E}
\]
We know that
\[
\vec{j} = -e n \vec{v}
\]
where $n$ is the density of electron
\[
\vec{j} = e n \frac{e \tau}{m} = \frac{e^2 n \tau}{m} \vec{E}
\]
This is the Ohm’s Law
\[
\vec{j} = \sigma \vec{E}
\]
where the conductivity
\[
\sigma = \frac{e^2 n \tau}{m}
\]
Or say...
\[ \rho = \frac{m}{e^2 n} \frac{1}{\tau} \]  

We know that \( \frac{1}{\tau} = \frac{1}{\tau_{ee}} + \frac{1}{\tau_{ep}} + \frac{1}{\tau_{i}} \), so

\[ \rho = \frac{m}{e^2 n} \left( \frac{1}{\tau_{ee}} + \frac{1}{\tau_{ep}} + \frac{1}{\tau_{i}} \right) = \frac{m}{e^2 n} \frac{1}{\tau_{ee}} + \frac{m}{e^2 n} \frac{1}{\tau_{ep}} + \frac{m}{e^2 n} \frac{1}{\tau_{i}} \]  

If we define

\[ \rho_{ee} = \frac{m}{e^2 n} \frac{1}{\tau_{ee}} \quad \text{and} \quad \rho_{ep} = \frac{m}{e^2 n} \frac{1}{\tau_{ep}} \quad \text{and} \quad \rho_{i} = \frac{m}{e^2 n} \frac{1}{\tau_{i}} \]

we will find that

\[ \rho = \rho_{ee} + \rho_{ep} + \rho_{i} \]  

2.1.5. Resistivity and temperature-dependence

In a real metal \( \rho \) is a function of \( T \). Typically, \( \rho \) decreases as \( T \) goes down. At \( T = 0 \), \( \rho \) goes to a finite value \( \rho(T = 0) \), which is known as the residue resistivity.

At low \( T \), \( \rho_{ee} \) and \( \rho_{ep} \) decreases to 0 as \( T \) is reduced down to 0, while \( \rho_{i} \) is \( T \) independent. Very typically, \( \rho_{ee} \propto T^2 \), \( \rho_{ep} \propto T^5 \) and \( \rho_{i} \propto \text{constant} \)

\[ \rho = \rho_{i} + A_{ee} T^2 + A_{ep} T^5 \]  

At low temperature, \( \rho_{i} \gg \rho_{ee} \gg \rho_{ep} \), so impurity scattering is the dominate contribution for \( \rho \).

At high temperature, \( \rho \) is dominated by phonons (scattering between electrons and lattices) and typically it is a linear function of \( T \) (Page 524)

\[ \rho = A T \]  

This is because the number of phonons is proportional to \( T \) at high temperature. And thus the scattering probability is proportional to \( T \).

Q: why the number of phonons is proportional to \( T \) at high temperature?

A: The equipartition theorem in classical statistical physics: in a classical system, each degree of freedom contributes \( k_B T/2 \) to the average total energy.

At high temperature, quantum effects becomes negligible, and we always recover the classical limit. Here, the equipartition theorem tells us that the total energy in elastic waves must be \( \propto T \).

Particle-wave duality in quantum mechanics tells us that waves are particles. Each particle has energy \( \hbar \omega \). Total number of particles is about \( \frac{\text{total energy}}{\hbar \omega} \). Because the total energy is \( \propto T \), the total particle number must also be linear in \( T \).

\[ n = \frac{1}{\exp\left(\frac{\hbar \omega}{k_B T}\right) - 1} = \frac{1}{\left(1 + \frac{\hbar \omega}{k_B T} + O(T^{-2})\right) - 1} = \frac{k_B T}{\hbar \omega} \]

2.2. The Hall effect

If we applying uniform a magnetic field along \( z \), then by applying an electric current along \( x \), an electric field will be generated along \( y \) direction.
2.2.1. treatment #1: balancing the force along y

The Lorentz force along y is balanced by the electric field along y

\[ e v_x B_z - e E_y = 0 \tag{2.22} \]

\[ v_x = \frac{E_y}{B_z} \tag{2.23} \]

\[ j_x = -e n v_x = -\frac{e}{B_z} E_y \tag{2.24} \]

Resistivity

\[ \rho_{xx} = \frac{E_x}{j_x} = ?? \tag{2.25} \]

The value of \( \rho_{xx} \) is not universal. It is very sensitive to scattering time (\( \tau \)), which depends on temperature, impurity density, etc.

The Hall resistivity

\[ \rho_{xy} = \frac{E_y}{j_x} = -\frac{1}{e n} B_z \tag{2.26} \]

The Hall resistivity is a linear function of \( B \). The coefficient is known as the Hall coefficient

\[ R_H = -\frac{1}{e n} \tag{2.27} \]

The Hall coefficient directly measures the carrier density (electron density).

NOTE:

- If you use the Gaussian units, there will be an extra factor \( c \) in \( R_H \)
- The sign of \( R_H \) is very important. If the electric current is carried by particles with a negative charge (electrons), \( R_H < 0 \). If particles that carries the current has positive charge (holes), \( R_H > 0 \). Materials with \( R_H < 0 \) is called electron-like and for \( R_H > 0 \), they are called hole-like.
- The electron density measured from this formula is in good agreement with the total number of valence electrons for material with low electron density (e.g \( Z = 1 \)), but the agreement is bad for \( Z = 2 \) or 3 (even the sign is wrong). See Table 1.4 on Page 15.
- Be careful about Hall resistance and Hall resistivity.

2.2.2. Hall Resistance:

In most experiments, the raw data is Hall resistance.

\[ R_{xy} = V_y / I_x = \frac{E_y L_y}{j_x L_x L_z} = \frac{\rho_{xy}}{L_z} = -\frac{1}{e (n L_z)} B_z = -\frac{1}{e n_2D} B_z \tag{2.28} \]

Hall resistance measures the 2D carrier density, \( n_{2D} \), which is the number of particle per 2D area. For a 2D material, this is the 2D carrier density. For a 3D material, the 3D carrier density is \( n_{2D} \) divided by the thickness.

2.2.3. treatment #2: Drude theory

In the calculations above, we didn’t consider the electron collisions. If we treat the collisions using the Drude approximation (assuming that the collisions are described by a single parameter \( \tau \), the collision time), the force on an electron is

\[ \vec{F} = -e \left( \vec{E} + \vec{v} \times \vec{B} \right) - \frac{m}{\tau} \vec{v} \tag{2.29} \]

Using the second law of Newton, we find that

\[ m \frac{d\vec{v}}{dt} = -e \left( \vec{E} + \vec{v} \times \vec{B} \right) - \frac{m}{\tau} \vec{v} \tag{2.30} \]

So we have two equations of motion
\[
\begin{align*}
\frac{d v_x}{dt} + \frac{v_x}{\tau} + e v_y B &= -e E_x \\
\frac{d v_y}{dt} + \frac{v_y}{\tau} - e v_x B &= -e E_y
\end{align*}
\]  
\hspace{0.5cm} (2.31)

The static solution \((dv/dt = 0)\)

\[
\begin{align*}
\frac{v_x}{\tau} + e v_y B &= -e E_x \\
\frac{v_y}{\tau} - e v_x B &= -e E_y
\end{align*}
\]  
\hspace{0.5cm} (2.33)

\hspace{0.5cm} (2.34)

which implies that

\[
\begin{align*}
v_x &= \frac{e \tau}{m} \frac{E_x - \omega_c \tau E_y}{1 + \omega_c^2 \tau^2} \\
v_y &= \frac{e \tau}{m} \frac{E_y + \omega_c \tau E_x}{1 + \omega_c^2 \tau^2}
\end{align*}
\]  
\hspace{0.5cm} (2.35)

\hspace{0.5cm} (2.36)

Here, \(\omega_c = eB/m\). If the current is along \(v_y = 0\), we find that

\[
E_y = -\omega_c \tau E_x
\]  
\hspace{0.5cm} (2.37)

And

\[
\begin{align*}
v_x &= \frac{e \tau}{m} \frac{E_x - \omega_c \tau E_y}{1 + \omega_c^2 \tau^2} = \frac{e \tau}{m} \frac{E_x}{1 + \omega_c^2 \tau^2} = -e E_x
\end{align*}
\]  
\hspace{0.5cm} (2.38)

Thus

\[
\begin{align*}
j_x &= -e n v_x = \frac{e^2 n \tau}{m} E_x
\end{align*}
\]  
\hspace{0.5cm} (2.39)

We recovered the resistivity \(\rho_{xx} = E_x/j_x = m/e^2 n \tau\)

For the Hall resistivity

\[
\rho_{xy} = \frac{E_y}{j_x} = \frac{E_y}{e^2 n \tau E_x} = \frac{m}{e^2 n \tau} \frac{E_y}{E_x} = \frac{m}{e^2 n \tau} \frac{E_y}{e^2 n \tau} \times (\omega_c \tau) = -m \omega_c \tau = -m \frac{e B}{e^2 n m} = \frac{1}{e n} B
\]  
\hspace{0.5cm} (2.40)

\textbf{Warning: we used resistivity }\rho_{xx}\textbf{ and the Hall resistivity }\rho_{xy}. \textbf{We didn't mention anything about conductivity }\sigma_{xx}\textbf{ or the Hall conductivity }\sigma_{xy}. \textbf{There is a reason for that. In fact, it is important to keep in mind that for the Hall bar geometry, }\rho_{xx} = E_x/j_x

\hspace{0.5cm} (2.41)

\hspace{0.5cm} but

\[
\sigma_{xx} \neq j_x/E_x
\]  
\hspace{0.5cm} (2.42)

\textbf{Similarly,}

\[
\rho_{xy} = E_y/j_x
\]  
\hspace{0.5cm} (2.43)

\hspace{0.5cm} but

\[
\sigma_{xy} \neq j_x/E_y
\]  
\hspace{0.5cm} (2.44)

\textbf{2.2.4. Resistivity and conductivity}

\textbf{In a solid, the relation between the resistivity and the conductivity could be more complicated than a simple number inverse. Here are two examples}