In QM I, we treat the spin of an electron as an independent quantity, independent from the orbit angular momentum. For example in a hydrogen atom, for any eigenwavefunction $\psi_{nlm}(x, y, z)$, it actually means two degenerate states: (1) one electron with wavefunction $\psi_{nlm}(x, y, z)$ and spin up and (2) one electron with wavefunction $\psi_{nlm}(x, y, z)$ and spin down. This conclusion remains the same after we take into account the relativistic correction (now the eigenenergy depends on both $n$ and $l$, but still for every eigenwavefunction, it means two degenerate state when we take into account the spins).

In this section, we will consider one more effect, which was ignored previously and this effect will tell us the spin of an electron and its orbital motion are coupled together.

**Warning:** you may find that the derivation in this section very disturbing. Because for multiple times, after some derivations, we will say the following without providing much justification, “by the way, this result is in fact not quite right, and we will need to throw in an extra factor of 2 to get the correct answer.” The reason for these extra factors of 2 is because this section is NOT treating spin-orbit coupling in the rigorous and correct way, which requires Dirac’s equation. Instead, what we are trying to do here is to use various tricks trying to recover Dirac’s finally conclusion without using Dirac’s equation. These tricks (they are not rigorous at all) get some part of the story right, but in many cases, they lead to wrong results. Because we already know the right answer from Dirac’s equation, whenever we find that these tricks fail to get the correct answer, we will correct it by adding some extra factor. Within our deviation, these extra factor looks totally unreasonable and weird, but if one start from Dirac’s equation, the results are all very natural and straightforward. *Bottom line, please don’t take these derivations very seriously, because they are not supposed to give (fully) correct description after all. But the physics, at the end of the day, is correct.*

**Magnetic dipole of an electron**

If a charge particle moves in circles, it creates circular current, and the circular current will result in an magnetic dipole moment (according to E&M). To see this, we use a simple model to demonstrate this physics. Assuming that we have a ring, and there is a charged particle (with charge $q$) moving around the ring. The dipole moment is

$$\vec{\mu} = \frac{1}{2} q \vec{r} \times \vec{v}$$

where $q$ is the charge of the particle. $\vec{r}$ and $\vec{v}$ are the location and velocity of the particle.

$$\vec{\mu} = \frac{1}{2} q \vec{r} \times \vec{v} = \frac{q}{2m} (m \vec{r} \times \vec{v}) = \frac{q}{2m} \vec{L}$$

where $\vec{L} = m \vec{r} \times \vec{v}$ is the angular momentum.

Similarly, if we have a spinning charged particle, the angular momentum from the spin will also result in a magnetic dipole. The dipole moment from spins is also proportional to the angular momentum of the spin, but with an extra factor known as the $g$-factor

$$\vec{\mu} = g \frac{q}{2m} \vec{S}$$

The charge of an electron is $-e$ (negative charge), so

$$\vec{\mu} = -e \frac{g}{2m} \vec{S}$$

and $g$ is a number, whose value is really close to 2. **In this course, we will say that $g = 2$ for simplicity**, but in reality, $g = 2.00231930436182$. In Dirac theory, $g$ is exactly 2. The reason the real value of $g$ is a little bit larger than 2 is due to interactions between electrons and photons (light), which wasn’t considered in Dirac’s equation.

**Note:** in many cases, people absorb the minus sign into the definition of $g$, 

$$\vec{\mu} = g \frac{e}{2m} \vec{S}$$

where $g = -2$. But no matter what convention one adopts,

$$\vec{\mu} = -\frac{e}{m} \vec{S}$$

We can define the Bohr magneton, which is a fundamental physics constant
\[ \mu_B = \frac{e \hbar}{2m} = 9.27400968 \times 10^{-24} \text{J/T} \] (2.327)

and

\[ \bar{\mu} = g \frac{e}{2m} \frac{\hbar}{h} = g \frac{e \hbar}{2m \hbar} = g \mu_B \frac{\hbar}{h} \] (2.328)

For an electron, the magnetic dipole is \( \pm \mu_B \). We demonstrate this by considering the dipole moment along the z direction

\[ \mu_z = g \mu_B S_z \frac{\hbar}{h} \] (2.329)

The spin operator \( S_z \) has eigenvalues \( \pm \hbar / 2 \), and \( g = -2 \). For an eigenstate of \( S_z \),

\[ \mu_z = \begin{cases} -\mu_B & \text{if } S_z \text{ eigenvalue is } +\hbar / 2, \text{ i.e. spin up} \\ +\mu_B & \text{if } S_z \text{ eigenvalue is } -\hbar / 2, \text{ i.e. spin down} \end{cases} \] (2.330)

Effective B field from the nucleon

If we stand on an electron (using the electron as our reference frame), we will find that the nucleon is moving around us in a circle. Because the nucleon has positive charge \( +e \), when it moves around us, it generates a circular current and thus leads to a magnetic field. According to the "Biot–Savart law" in E&M, the B field generated by a wire with current \( I \) is

\[ \vec{B} = \frac{\mu_0}{4\pi} \int \frac{d\vec{l} \times \vec{r}}{r^3} \] (2.331)

where \( d\vec{l} \) is a small section of the wire and the direction is parallel to the wire. \( \vec{r} \) is the distance between the wire and the place at which we want to measure the B field. For a circular motion of the nucleon, the wire here is a circle and we want to know the B field at the center of the circle

\[ B = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{2\pi r dr}{r^3} = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{2\pi r}{2\pi r} = \frac{\mu_0}{2} \] (2.332)

The current \( I \) here is

\[ I = \frac{e}{T} \] (2.333)

where \( e \) is (the absolute value of) the charge of an electron (remember that the nucleon in a hydrogen atom is +e). \( T \) is the time it takes for the nucleon to go around a circle.

\[ I = \frac{e}{T} = \frac{e}{2\pi \omega} = \frac{e \omega}{2\pi} \] (2.334)

Here, \( \omega \) is the angular velocity. Notice that the angular velocity here is the same as the angular velocity of the electron \( \omega \) (in the rest frame).

\[ I = \frac{e \omega}{2\pi} = \frac{eL}{2\pi m r^2} \] (2.335)

So

\[ B = \frac{\mu_0 I}{2r} = \frac{\mu_0}{2} \frac{eL}{2\pi m r^3} = \epsilon_0 \mu_0 \frac{eL}{4\pi \epsilon_0 m r^3} \] (2.336)

Notice that \( \epsilon_0 \mu_0 = 1/c^2 \) and in addition, it is easy to realize that \( \vec{B} \parallel \vec{L} \), so we get

\[ \vec{B} = \epsilon_0 \mu_0 \frac{eL}{4\pi \epsilon_0 m r^3} = \frac{e}{4\pi \epsilon_0 m c^2 r^3} \] (2.337)

Magnetic dipole in an B field

We proved above that using the frame of the electron, the electron feels a B field, which is generated by the nucleon
We have a dipole in a $B$ field, we shall have energy

$$H' = -\vec{\mu} \cdot \vec{B} = \frac{e^2}{4\pi \varepsilon_0} \frac{1}{m^2 c^2 r^3} \vec{S} \cdot \vec{L}$$

(2.340)

Here, our naive tricks miss a factor of 1/2 in comparison to the correct result (from Dirac’s equation). The right result should be

$$H_{SO} = \frac{e^2}{8\pi \varepsilon_0} \frac{1}{m^2 c^2 r^3} \vec{S} \cdot \vec{L}$$

(2.341)

Therefore, we shall add one extra term to the Hamiltonian

$$H = H_0 + H_{SO}$$

(2.342)

Here, $H_0$ is what we learned in QM1. And $H_{SO}$ is this new term. Here, we treat $H_0$ as unperturbed Hamiltonian, and treat $H_{SO}$ as a small perturbation.

**Basis without $H_{SO}$**

For the commutation relations for the orbital angular momentum, we know that

$$[L_x, L_y] = i \hbar L_z$$

(2.343)

$$[L_y, L_z] = i \hbar L_x$$

(2.344)

$$[L_z, L_x] = i \hbar L_y$$

(2.345)

or we can write the same formular as

$$L_i L_j = i \hbar \epsilon_{i,j,k} L_k$$

(2.346)

where $\epsilon_{i,j,k}$ is the Levi-Civita symbol.

For $L$, we know that we can define the operator $L^2$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

(2.347)

And we know that it commute with $L_z$, $[L^2, L_z] = 0$. As a result, we cannot measure all the three components of the angular momentum due to the uncertainly principle. But, we can measure $L^2$ and $L_z$ at the same time, by defining common eigenstates for these two operators

$$L^2 \psi_{l,m}(x, y, z) = l(l+1) \hbar^2 \psi_{l,m}(x, y, z)$$

(2.348)

$$L_z \psi_{l,m}(x, y, z) = m \hbar \psi_{l,m}(x, y, z)$$

(2.349)

Here, $l$ is a non-negative integer, $l = 0, 1, 2, \ldots$, and $m$ is an integer between $+l$ and $-l$. The eigenwavefunctions is very easy to write down in spherical corrediate

$$\psi_{l,m}(r, \theta, \phi) = R(r) Y_{l,m}(\theta, \phi)$$

(2.350)

where $R(r)$ is an arbitrary function of $r$ (the function doesn’t depend on $\theta$ or $\phi$), and $Y_{l,m}(\theta, \phi)$ are a set of special functions known as the spherical harmonics.

For spins, we have the same commutation relation,

$$[S_x, S_y] = i \hbar S_z$$

(2.351)

$$[S_y, S_z] = i \hbar S_x$$

(2.352)

$$[S_z, S_x] = i \hbar S_y$$

(2.353)