Once again, we compute the frequency sum using the Fermi distribution function

\[
\frac{1}{\beta} \sum_{\omega_n} g(i \omega_n) = \sum_{\omega_n \text{outside } C} f(\omega_n) \text{Res}(g, \omega_n)
\]  

(3.81)

Here, \(g(i \Omega_n) = G^{00}(k, i \Omega_n) G^{00}(k + q, i \Omega_n + i \omega_n)\).

\[
g(z) = G^{00}(k, z) G^{00}(k + q, z + i \omega_n) = \frac{1}{z - \epsilon(k)} \frac{1}{z - i \omega_n - \epsilon(k + q)}
\]  

(3.82)

Obviously, \(g(z)\) has two singularity points: \(z_1 = \epsilon(k)\) and \(z_2 = \epsilon(k + q) - i \omega_n\). Residue at \(z_1\) can be found using

\[
\text{Res}[g(z), z_1] = \lim_{z \to z_1} (z - z_1) g(z) = \lim_{z \to z_1} \frac{1}{z - z_1} \frac{1}{z - \epsilon(k)} = \frac{1}{z_1 - z_2}
\]  

(3.83)

Similarly, we find that \(\text{Res}[g(z), z_2] = \frac{1}{z_1 - z_2}\), therefore,

\[
\frac{1}{\beta} \sum_{\omega \Omega_n} g(i \Omega_n) = \sum_{\omega_n \text{outside } C} f(\omega_n) \text{Res}(g, \omega_n) = \frac{f(z_1)}{z_1 - z_2} + \frac{f(z_2)}{z_2 - z_1} = \frac{f(z_1) - f(z_2)}{z_1 - z_2}
\]  

(3.84)

So

\[
\Pi^{(0)}(q, i \omega_n) = \frac{1}{2 \pi^2} \int \frac{d k}{(2 \pi)^2} G^{00}(k, i \Omega_n) G^{00}(k + q, i \Omega_n + i \omega_n) = \int \frac{d k}{(2 \pi)^2} f(\epsilon(k)) \epsilon(k) - f(\epsilon(k + q))
\]  

(3.85)

This integral can be computed analytically at \(T = 0\) (see Mahan, page 437). Here, we will only look at the limit of \(\omega \to 0\) and \(q \to 0\). At \(\omega = 0\) and \(q \to 0\)

\[
\Pi^{(0)}(q, \omega = 0) = \int \frac{d k}{(2 \pi)^2} \frac{f(\epsilon(k)) - f(\epsilon(k + q))}{\epsilon(k) - \epsilon(k + q)} = \int \frac{d k}{(2 \pi)^2} \frac{d f(\epsilon)}{d \mu} = - \int \frac{d k}{(2 \pi)^2} \frac{d f(\epsilon)}{d \mu}
\]  

(3.86)

Here, we used the fact that

\[
\frac{d}{d \epsilon} \exp[\beta(\epsilon - \mu)] = \frac{d}{d \mu} \exp[\beta(\epsilon - \mu)]
\]  

(3.87)

Therefore,

\[
\Pi^{(0)}(q, \omega = 0) = - \int \frac{d k}{(2 \pi)^2} \frac{d f(\epsilon)}{d \mu} = \frac{d}{d \mu} \int \frac{d k}{(2 \pi)^2} f(\epsilon) = - \frac{d N}{d \mu} = -N(0)
\]  

(3.88)

Here, \(N = \int \frac{d k}{(2 \pi)^2} f(\epsilon)\) is the total number of particles. \(\frac{d N}{d \mu}\) is the density of states \(N(0)\) at the Fermi level. At \(T = 0\), for a Fermi liquid,

\[
\text{Number of states} = 4 \pi g \int \frac{k^2 dk}{(2 \pi)^3} = \frac{g}{2 \pi^2} \int k^2 dk = \frac{g}{2 \pi^2} \int 2 m \epsilon d \sqrt{2 m \epsilon} = \frac{g}{\sqrt{2} \pi^2} m^{3/2} \int \sqrt{\epsilon} d \epsilon
\]  

(3.89)

So

\[
N(0) = \frac{g}{\sqrt{2} \pi^2} m^{3/2} \sqrt{\epsilon_F}
\]  

(3.90)

And thus the dielectric function

\[
e^{RPA}(q, i \omega_n) = 1 - \frac{4 \pi e^2}{q^2} \Pi^{RPA}(q, i \omega_n)
\]  

(3.91)

In the \(\omega \to 0\) and \(q \to 0\) limit

\[
\lim_{q \to 0} e^{RPA}(q, \omega = 0) = 1 + \frac{4 \pi e^2}{q^2} N(0)
\]  

(3.92)

One can define the Thomas-Fermi screening wave vector \(q_{TF}\)
\[ q^2_{TF} = \frac{6\pi e^2 n}{\epsilon_F} \]  

Here \( n \) is the density of electron and \( n = \frac{2}{3}\epsilon_F N(0) \)

\[
n = \int_0^\infty d\epsilon N(\epsilon) = \frac{g}{\sqrt{2\pi}} m^{3/2} \int d\epsilon \sqrt{\epsilon} = \frac{g}{\sqrt{2\pi}} \frac{2}{3} (m\epsilon_F)^{3/2} = \frac{2}{3} \epsilon_F \frac{g}{\sqrt{2\pi}} m^{3/2} \sqrt{\epsilon_F} = \frac{2}{3} \epsilon_F N(0)
\]

Therefore,

\[
q^2_{TF} = \frac{6\pi e^2 n}{\epsilon_F} = \frac{6\pi e^2}{\epsilon_F} - \epsilon_F N(0) = 4\pi e^2 N(0)
\]

So

\[
\lim_{q\to0} \epsilon_{RPA}(q, \omega = 0) = 1 + \frac{q_{TF}^2}{q^2}
\]

In other words, the dielectric constant for a metal is \( \infty \), which diverges as \( 1/q^2 \) in the \( q \to 0 \) limit. As for the effective interaction, \( V(q, \omega) = \frac{4\pi e^2}{q^2} \) for \( q \to 0 \).

At \( \omega = 0 \), because both denominator and numerator diverge as \( 1/q^2 \) in the small \( q \) limit, \( V(q = 0, \omega = 0) \) is finite. This is very different from the bare interaction \( V^{(0)}(q) \), which is singular at \( q = 0 \).

\[
\lim_{q\to0} V(q, \omega = 0) = \frac{4\pi e^2}{1 + q_{TF}^2} \frac{q^2 + q_{TF}^2}{q^2}
\]

If we convert this formula into the real space, the small \( q \) limit of \( V(q) \) give us the large \( r \) limit of \( V(r) \).

\[
V(r, \omega = 0) = \int \frac{d^d q}{(2\pi)^d} \exp(i \mathbf{q} \cdot \mathbf{r}) V(q, \omega = 0)
\]

When \( r \) is large enough, the small \( q \) part of the integral dominates the whole integral (large \( q \) and large \( r \) means very rapid oscillation in \( \exp(i \mathbf{q} \cdot \mathbf{r}) \), which is zero after integral). So at long distance,

\[
V(r, \omega = 0) \approx \int \frac{d^d q}{(2\pi)^d} \exp(i \mathbf{q} \cdot \mathbf{r}) \frac{4\pi e^2}{q^2 + q_{TF}^2} = \frac{1}{4\pi^2} \int_0^\infty q^2 dq \int_0^\pi d\theta \sin\theta \exp(i qr \cos\theta) \frac{4\pi e^2}{q^2 + q_{TF}^2} = \frac{e^2}{i \pi r} \left[ \int_0^\infty q \exp(i q r) dq - \int_0^\infty q \exp(-i q r) dq \right] = \frac{e^2}{i \pi r} \left[ \int_0^\infty q \exp(i q r) dq \right] = \frac{e^2}{i \pi r} \left[ \frac{\pi}{q^2 + q_{TF}^2} q, q = i q_{TF} \right] = \frac{e^2}{r} \exp(-q_{TF} r)
\]

The interaction decays exponential at long distance with decay rate \( q_{TF} \).

3.6.5. Fermion self-energy within the random phase approximation

For a Fermi liquid, the RPA shows pretty good agreement with experiments and is the minimum approximation to describe an interacting Fermi liquid.

\[
G(k, i\omega_n) = \frac{1}{i\omega_n - \epsilon(k) - \Sigma^{RPA}(k, i\omega_n)}
\]

where
\[ \Sigma^{\text{RPA}}(k, \imath \omega_n) = \text{double dashed line} \]

and the double dashed line is the effective interaction computed using the RPA approximation. We will not go into the detailed calculation. Instead, we just present the result here. After analytic continuation, we get

\[ G(k, \omega) = \frac{1}{\omega - \epsilon(k) - \Sigma^{\text{RPA}}(k, \omega)} \]

(3.103)

For a Fermi liquid at T=0 K, we focus on \( k \approx k_F \) and \( \omega \approx 0 \).

Near \( k_F \), \( \epsilon(k) = \mu + v_F (k - k_F) + O(k - k_F)^2 \).

\[ G^{(0)}(k, \omega) = \frac{1}{\omega - v_F (k - k_F) - \mu + O(k - k_F)^2} \]

(3.104)

Using this \( G^{(0)} \), we compute \( \Sigma^{\text{RPA}} \). For \( \Sigma^{\text{RPA}}(k, \omega) \), we expand it around \( k \approx k_F \) and \( \omega \approx 0 \).

\[ G(k_F, \omega) = \frac{1}{\omega - \nu(k - k_F) - \mu - \Sigma^{\text{RPA}}(k_F, 0) - \delta(k - k_F) + \beta \omega + O(k - k_F)^2 + O(\omega^2)} \]

(3.105)

It is important to notice that \( b \) is real and positive (the \( \omega^2 \) term has a complex coefficient). The imaginary part of \( \Sigma^{\text{RPA}} \) will only appear at order \( \omega^2 \) and above. Therefore, in the limit \( k \approx k_F \) and \( \omega \approx 0 \), the imaginary part of the self-energy is much smaller than the real part, and thus we can ignore the imaginary part in the low-energy limit \( \omega \to 0 \). Therefore,

\[ G^{\text{RPA}}(k_F, \omega) = \frac{Z}{\omega - \nu(k - k_F) - \mu} + O(k - k_F)^2 + O(\omega^2) \]

(3.106)

here \( Z = 1/1 + b \), \( \nu = \frac{\nu + b}{1 + b} \), and \( \mu = \frac{\mu + \Sigma^{\text{RPA}}(k_F, 0)}{1 + b} \). Notice that in the limit of \( k \to k_F \) and \( \omega \to 0 \), \( G^{\text{RPA}}(k_F, \omega) \) looks just like the Green’s function for a free fermion, with a renormalized Fermi velocity and a renormalized chemical potential. This is why we can treat electrons as “free” particles in a Fermi liquid, although the interaction is pretty strong. In addition, we have an extra factor \( Z \), which is known as the Fermi residue

\[ G^{\text{RPA}}(k_F, \omega) = Z \tilde{G}^{\text{Fre}}(k_F, \omega) \]

(3.107)

This extra factor is \( 0 < Z < 1 \). The physical meaning is that now at \( k_F \) the jump in occupation number \( n(k_F - \delta) - n(k_F + \delta) = Z \) not 1.

The imaginary part of the self-energy (inverse life-time of our particle) is proportional to \( \omega^2 \), which means that at zero frequency and at \( k = k_F \), our particles has infinite life time.

### 3.6.6. Summary

In Landau’s Fermi liquid theory, we can consider the low-energy excitations (near the Fermi surface) in an interacting Fermi liquid as quasi-particles. In the limit \( \omega \to 0 \), the life time of our quasi-particles diverge and they behave just like free particles.

These quasi-particles carry the same charge and spin as an electron. However, the dispersion (Fermi velocity) and the chemical potential of these quasi-particles are renormalized by interactions. In addition, interaction reduces the Fermi residue from 1 to some finite value \( 0 < Z < 1 \).

This is why we can treat metals as if the electrons are free particles, although the interaction is in fact pretty strong.