ignored in the Hartree-Fock approximation. (This is why the HF approximation fails).

3.6.1. Interactions between electrons in a solid

For two electrons in vacuum, the Coulomb’s law tells us that the interaction energy between these two electrons is

$$ V(r) = \frac{e^2}{r} $$

(3.63)

Q: What’s the Coulomb’s law inside a solid material?

A: It depends on whether the material is a metal or an insulator.

For an insulator, we know that the Coulomb’s law is

$$ V(r) = \frac{1}{\epsilon} \frac{e^2}{r} $$

(3.64)

where $\epsilon$ is the dielectric constant (permittivity) of the solid. Here, inside an insulator, the solid medium modifies the interaction strength by a factor $1/\epsilon$, but the potential still decays as $1/r$.

3.6.2. Screening

For a metal, the screening effect changes the $1/r$ interaction into a short range interaction which decays exponentially.

Q: What is the $E$ field in a metal? (consider only the static $E$ field)

A: $E = 0$

This is because we have Ohm’s law

$$ j = \sigma E $$

(3.65)

In equilibrium, $j = 0$, so $E = 0$. This means that the electric field is reduced dramatically in a solid, and therefore we expect the interaction strength between two charge particles to become much weaker in a metal. To see this, we consider two test charge with charge $e$ inside a metal. Because electrons can move in a metal, electrons will redistribute after we introduces these two test charge. For a positive test charge, electrons will be attracted to the test charge and thus the total electric charge near the test charge (test charge+electrons) is almost zero. (Same will happen for negative charge). In other words, conducting electrons “screen” the test charge, which reduces the interaction dramatically. This effect can be described as

$$ V(q, \omega) = \frac{V^{(0)}(q)}{\epsilon(q, \omega)} $$

(3.66)

Here, $V(q, \omega)$ is the effective interactions that electrons inside a metal feel in the momentum-frequency space. $V^{(0)}(q) = 4\pi e^2/q^2$ is the Coulomb potential in vacuum. The denominator $\epsilon(q, \omega)$ is the dielectric function. For insulators, the dielectric constant is the $\omega \to 0$ and $q \to 0$ limit of $\epsilon(q, \omega)$. For a metal, we will show below that $\epsilon(q = 0, \omega = 0) = \infty$, which makes the effective potential $V(q, \omega)$ finite at $q = 0$ and $\omega = 0$. If $V(q, \omega)$ is finite at $q = 0$ and $\omega = 0$, the effective interaction $V(r, \omega = 0)$ decays exponentially as a function of $r$ (instead of $1/r$). In other words, in a metal the e-e interactions are short-range interactions (not $1/r$). In H-F, we use the $1/r$ interaction, instead of the correct effective interaction. And this is the reason why the theory fails.

3.6.3. $V(q, \omega_n)$ and Dyson’s equation

Here, we consider the following diagrams

$$
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram1.png}
\end{array}
$$

(3.67)

If we define effective interaction $V(q, \omega)$ as

$$
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram2.png}
\end{array}
$$

(3.68)

and then all the diagrams above can be represented as a simple first order diagram.

$$
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram3.png}
\end{array}
$$

(3.69)

This formula only contains the first order perturbation in terms of the effective interaction strength $V$, but it includes infinite diagrams in terms of $V^{(0)}$. The physical meaning here is, if you substitute the interaction $V^{(0)}$ by the effective interaction $V$, then the Fock diagram contains a lot more higher order contributions. We can further improve the accuracy by including more and more terms in effective interaction $V$. Here, the
strategy is, we treat interaction (dashed line) as a Green's function (for some bosonic particles), and then we plot all higher order corrections to the interaction strength. The sum gives us an effective interaction. When we compute any physical observables, e.g. the Fermion Green’s function, we will not use the bare interaction \( V^{(0)} \), but the effective interaction \( V \). The physical meaning is that inside a metal, electrons will screen the Coulomb interaction, and thus effectively the interaction between electrons is \( V \), instead of \( V^{(0)} \). And please notice that here we didn’t include any more diagram. It is just a way to organize the diagrams to make the calculation more efficient.

Here we compute \( V \) by considering diagrams shown above. This approximation (ignoring all other diagrams) are called the Random Phase Approximation (RPA).

\[
V^{\text{RPA}}(q, i \omega_n) = V^{(0)}(q, i \omega_n) + V^{(0)}(q, i \omega_n) \Pi^{(0)}(q, i \omega_n) V^{(0)}(q, i \omega_n) + V^{(0)}(q, i \omega_n) \Pi^{(0)}(q, i \omega_n) V^{(0)}(q, i \omega_n) V^{(0)}(q, i \omega_n) \ldots
\] (3.70)

Here, the leading order correction is \( \Pi^{(0)} \) is called a bubble. This name comes from the fact that the Feynman diagram for \( \Pi^{(0)} \) looks like a bubble.

\[
\Pi^{(0)}(q, i \omega_n) = \int \frac{d k}{(2 \pi)^d} \frac{1}{\beta} \sum n_k \, G^{(0)}(k, i \Omega_k) \, G^{(0)}(k + q, i \Omega_n + i \omega_n)
\] (3.71)

If we include all other diagram, we can get more accurate \( \Pi(q, i \omega_n) \).

\[
V(q, i \omega_n) = V^{(0)}(q, i \omega_n) + V^{(0)}(q, i \omega_n) \Pi(q, i \omega_n) V^{(0)}(q, i \omega_n) + V^{(0)}(q, i \omega_n) \Pi(q, i \omega_n) V^{(0)}(q, i \omega_n) + \ldots
\] (3.72)

where

\[
\Pi(q, i \omega_n) = \int \frac{d k}{(2 \pi)^d} \frac{1}{\beta} \sum n_k \, G^{(0)}(k, i \Omega_k) \, G^{(0)}(k + q, i \Omega_n + i \omega_n)
\] (3.73)

Using Dyson’s equation, we find

\[
V(q, i \omega_n) = V^{(0)}(q, i \omega_n) \frac{1}{1 - \epsilon(q, i \omega_n)} = \frac{1}{V^{(0)}(q, i \omega_n) \epsilon(q, i \omega_n)}
\] (3.74)

Notice that

\[
V(q, i \omega_n) = V^{(0)}(q, i \omega_n) \frac{1}{1 - \epsilon(q, i \omega_n)} = \frac{V^{(0)}(q, i \omega_n)}{\epsilon(q, i \omega_n)}
\] (3.75)

This tells us that the dielectric function for a metal is

\[
\epsilon(q, i \omega_n) = 1 - \Pi(q, i \omega_n) V^{(0)}(q, i \omega_n) = 1 - \frac{4 \pi e^2}{q^2} \Pi(q, i \omega_n).
\] (3.76)

As will be shown later, in the limit of \( \omega \rightarrow 0 \) and \( q \rightarrow 0 \), \( \Pi(q, \omega) \) of a metal goes to a finite constant value

\[
\lim_{q \rightarrow 0} \epsilon(q, \omega = 0) = O\left(\frac{1}{q^2}\right) = \infty
\] (3.77)

i.e. the dielectric constant in a metal is infinite. Since both \( V^{(0)}(q) \) and \( \epsilon(q, \omega = 0) \) diverge as \( 1/q^2 \) in the small \( q \), the effective interaction is NOT singular at \( \omega = 0 \) and \( q = 0 \)

\[
V(q, \omega = 0) = \frac{V^{(0)}(q)}{\epsilon(q, \omega = 0)} = O\left(1/q^2\right) = \text{finite at } q \rightarrow 0
\] (3.78)

This tell us that screening effect helps us remove the singularity in the interaction strength at small \( q \). This will cure all the problem we had in the HF approximation.

### 3.6.4. The random phase approximation

Here we use the RPA, where we ignore all higher order term in \( \Pi(q, i \omega_n) \) beyond the bubble diagram.

\[
\Pi(q, i \omega_n) = \Pi^{(0)}(q, i \omega_n) + O(V^2)
\] (3.79)

\[
\Pi^{(0)}(q, i \omega_n) = \int \frac{d k}{(2 \pi)^d} \frac{1}{\beta} \sum \epsilon n_k \, G^{(0)}(k, i \Omega_k) \, G^{(0)}(k + q, i \Omega_n + i \omega_n)
\] (3.80)