1.7.8. k, ω-space

If the system have momentum and energy conservation laws, it is typically much easier to compute Green’s functions in the momentum-energy space. In the k, ω-space, we use the same diagrams and in addition, we also:

- Assign momentum and frequency to each line and keep the momentum and energy conservation law at each ending point or crossing points.
- For each loop, there is a pair of unknown q and i Ωn and we need to integrate/sum over them.
- For each fermionic loop, we get an extra factor of (-1), which comes the commutation relation.
- For mth order diagram, we get a factor (-1)m

For example, using the same diagram above, we can compute the two point correlation functions as

\[
G(k, i ω_n) = G^{(0)}(k, i ω_n) + \int \frac{dq}{(2 π)^3} \frac{1}{β} \sum_{Ωn} G^{(0)}(k, i ω_n) G^{(0)}(k, i ω_n) V(0, 0) G^{(0)}(k, i Ω_n) - \\
\int \frac{dq}{(2 π)^3} \frac{1}{β} \sum_{Ωn} G^{(0)}(k, i ω_n) G^{(0)}(k, i ω_n) G^{(0)}(k + q, i ω_n + i Ω_n) V(q, i Ω_n) + ...
\]

(1.195)

1.8. The Dyson’s equation

If we look at the diagrams, there are many repeating structures. These repeating structures can be utilized to simplify the calculation.

1.8.1. The sum of a geometric series

How do we compute the sum of a geometric series:

\[
X = a + a q + a q^2 + a q^3 + ...
\]

(1.196)

First, we notice that

\[
q X = a q + a q^2 + a q^3 + a q^4 + ...
\]

(1.197)

then, we rewrite X as

\[
X = a + a q + a q^2 + a q^3 + ... = a + q X
\]

(1.198)

So,

\[
X - q X = a
\]

(1.199)

\[
X = \frac{a}{1 - q}
\]

(1.200)

We can use the same trick to sum many Feynman diagrams.

1.8.2. example 1: the Hartree approximation

For diagrams, we can use the same trick. For example, the following diagrams can be summed together

\[
\begin{align*}
\text{1’} & \rightarrow 1 ^{+} \\
\text{1} & \rightarrow \text{1’} ^{+} \text{1} \text{1’} ^{+} \text{1} \text{1} ^{+} \text{1} ^{+} \text{1} \text{1} ^{+} \text{…}
\end{align*}
\]

Here we sum over these diagrams and ignore others. This approximation is known as the Hartree approximation.

\[
G(k, i ω_n) = G^{(0)}(k, i ω_n) + G^{(0)}(k, i ω_n) \Sigma_q(k, i ω_n) G^{(0)}(k, i ω_n) + G^{(0)}(k, i ω_n) \Sigma_q(k, i ω_n) G^{(0)}(k, i ω_n) \Sigma_q(k, i ω_n) G^{(0)}(k, i ω_n) + G^{(0)}(k, i ω_n) \Sigma_q(k, i ω_n) G^{(0)}(k, i ω_n) \Sigma_q(k, i ω_n) G^{(0)}(k, i ω_n) \Sigma_q(k, i ω_n) G^{(0)}(k, i ω_n) + ...
\]

(1.201)

where

\[
\Sigma_q(k, i ω_n) = \int \frac{dq}{(2 π)^3} \frac{1}{β} \sum_{i Ω_n} V(0, 0) G^{(0)}(q, i Ω_n)
\]

(1.202)

This term is known as the self-energy correction. It is the same diagram we considered above but with external legs removed. Using the same trick mentioned above, we find
\[ G^{\text{H}}(k, i \omega_n) = G^{(0)}(k, i \omega_n)[1 - \Sigma_{\text{H}}(k, i \omega_n) G^{(0)}(k, i \omega_n)]^{-1} = \frac{1}{i \omega_n - \epsilon(k) - \Sigma_{\text{H}}(k, i \omega_n)} \]  

(1.203)

This technique and this formula is known as the Dyson’s equation. Here, we find that the Green’s function for interacting particles is very similar to the free Green’s function \( G^{(0)} \). The only thing interactions does is to change the single particle energy \( \epsilon(k) \) into \( \epsilon(k) + \Sigma_{\text{H}}(k, i \omega_n) \). In other words, the interactions changes the “energy” of the particle by \( \Sigma_{\text{H}}(k, i \omega_n) \). This is the reason why this term is called a self-energy correction. However, it is important to keep in mind that this term is NOT really a shift in \( \epsilon(k) \), because it is also a function of frequency.

### 1.8.3. example 2: the Hartree-Fock approximation

We can get more accurate results by adding more diagrams into our calculation. For example,

![Diagram](image)

The approximation that sums over these diagrams (and ignore others) are known as the Hartree-Fock approximation.

\[ G(k, i \omega_n) = G^{(0)}(k, i \omega_n) + G^{(0)}(k, i \omega_n) \Sigma_{\text{HF}}(k, i \omega_n) G^{(0)}(k, i \omega_n) + G^{(0)}(k, i \omega_n) \Sigma_{\text{HF}}(k, i \omega_n) G^{(0)}(k, i \omega_n) \Sigma_{\text{HF}}(k, i \omega_n) G^{(0)}(k, i \omega_n) + \ldots \]

(1.204)

here the Hartree-Fock self-energy correction \( \Sigma_{\text{HF}}(k, i \omega_n) \) is

\[ \Sigma_{\text{HF}}(k, i \omega_n) = \int \frac{d \mathbf{q}}{(2 \pi)^3} \frac{1}{\beta} \sum_{\Omega} V(q, 0) G^{(0)}(q, i \Omega_n) - \int \frac{d \mathbf{q}}{(2 \pi)^3} \frac{1}{\beta} \sum_{\Omega} G^{(0)}(q + \mathbf{k}, i \omega_n + i \Omega_n) V(q, i \Omega_n) \]

(1.205)

Using the same trick, (the Dyson’s equation), we find that

\[ G^{\text{Hf}}(k, i \omega_n) = G^{(0)}(k, i \omega_n)[1 - \Sigma_{\text{HF}}(k, i \omega_n) G^{(0)}(k, i \omega_n)]^{-1} = \frac{1}{i \omega_n - \epsilon(k) - \Sigma_{\text{Hf}}(k, i \omega_n)} \]

(1.206)

The final result is almost the same as the Hartree approximation. We just need to change \( \Sigma_{\text{H}}(k, i \omega_n) \) into \( \Sigma_{\text{HF}}(k, i \omega_n) \).

### 1.8.4. example 3: include more diagrams

If we include more diagrams, the Green’s function still takes the same structure

\[ G(k, i \omega_n) = G^{(0)}(k, i \omega_n) + G^{(0)}(k, i \omega_n) \Sigma(k, i \omega_n) G^{(0)}(k, i \omega_n) + G^{(0)}(k, i \omega_n) \Sigma(k, i \omega_n) G^{(0)}(k, i \omega_n) \Sigma(k, i \omega_n) G^{(0)}(k, i \omega_n) + \ldots \]

(1.207)

Here, the self-energy correction contains more diagrams

\[ \Sigma(k, i \omega_n) = \int \frac{d \mathbf{q}}{(2 \pi)^3} \frac{1}{\beta} \sum_{\Omega} V(q, 0) G^{(0)}(q, i \Omega_n) - \int \frac{d \mathbf{q}}{(2 \pi)^3} \frac{1}{\beta} \sum_{\Omega} G^{(0)}(q + \mathbf{k}, i \omega_n + i \Omega_n) V(q, i \Omega_n) \]

(1.208)

Using Dyson’s equation, we find that

\[ G(k, i \omega_n) = \frac{G^{(0)}(k, i \omega_n)}{1 - \Sigma(k, i \omega_n) G^{(0)}(k, i \omega_n)} = \frac{1}{i \omega_n - \epsilon(k) - \Sigma(k, i \omega_n)} \]

(1.209)

### 1.8.5. One-particle irreducible diagrams

If we use the Dyson’s equation, the key is to avoid double counting. For example, the second order diagram

![Diagram](image)

has been
included in the Hartree approximation when we include \( \psi^\dagger \) in the self-energy. Therefore, when we compute second order self-energy corrections, we should not include this diagram again. The rule to avoid double counting is to use 1-particle irreducible diagrams in the self-energy \( \Sigma \). One-particle irreducible diagrams means that if we cut one internal link (solid line), the diagram is still connected. For example, the following diagrams are 1-particle irreducible diagrams, and thus they should be include din the self-energy:

This diagram is not 1-particle irreducible, and thus we should not include them in the self-energy correction (it has already been taken care of by the first order term).

1.8.6. Summary

- Draw all possible 1-particle-irreducible diagrams.
- Remove external legs to get the self energy \( \Sigma \) (note: each fermion loop contributes a factor \(-1\). For nth order diagram, we need an factor \((-1)^n\) in the momentum space formula).
- Use Dyson’s equation to get the full Green’s function. \( G(i\omega_n) = \frac{1}{i\omega_n - \Sigma(k,i\omega_n)} \)
- Other particles can be treated using the same approach (photons, phonons, etc.)

1.9. Physical meaning (fermions)

The physical meaning of the spectral function \( A(k, \omega) \) is: if we have a particle with momentum \( k \), \( A(k, \omega)/2\pi \) is the probability for this particle to have energy \( \omega \).

1.9.1. \( A(k, \omega) \geq 0 \)

The proof can be found in the book of Mahan (page 151). We will not show it here.

1.9.2. \( \int\frac{d\omega}{2\pi} A(k, \omega) = 1 \)

\[ A(k, \omega) = G^\ast(k, \omega) + G^\dagger(k, \omega) \]  
(1.210)

Integrate over all \( \omega \)

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(k, \omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^\ast(k, \omega) + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^\dagger(k, \omega) \]  
(1.211)

Notice that

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} G^\ast(k, \omega) = i G^\ast(k, t) = \langle \psi(k, t_0 + t) \psi^\dagger(k, t_0) \rangle \]  
(1.212)

If we set \( \tau = 0 \), we find that by integrate over all \( \omega \), we get the equal-time correlation function

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^\ast(k, \omega) = \langle \psi(k, t_0) \psi^\dagger(k, t_0) \rangle \]  
(1.213)

Similarly,

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^\dagger(k, \omega) = \langle \psi^\dagger(k, t_0) \psi(k, t_0) \rangle \]  
(1.214)

Therefore,

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(k, \omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^\ast(k, \omega) + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^\dagger(k, \omega) = \langle \psi(k, t_0) \psi^\dagger(k, t_0) + \psi^\dagger(k, t_0) \psi(k, t_0) \rangle = (1) = 1 \]  
(1.215)