Here, we set the Planck constant \( \hbar \) to unity for simplicity.

### 1.4.1. Equation of motion for annihilation operators

For the Hamiltonian

\[
H = \int d\vec{r} \frac{\nabla \psi^*(\vec{r}) \nabla \psi(\vec{r})}{2m} + \frac{i}{2} \int d\vec{r} d\vec{r}' V(\vec{r} - \vec{r}') \psi^*(\vec{r}) \psi^*(\vec{r}') \psi(\vec{r}') \psi(\vec{r})
\]

the equation of motion for the annihilation operator \( \psi \) is

\[
i \frac{\partial \psi(\vec{r}_0, t)}{\partial t} = \left[ \psi(\vec{r}_0), H \right] = \int d\vec{r} \frac{\nabla \delta(\vec{r} - \vec{r}_0) \nabla \psi(\vec{r})}{2m} + \frac{1}{2} \int d\vec{r} d\vec{r}' V(\vec{r} - \vec{r}') \psi^*(\vec{r}) \psi^*(\vec{r}') \psi(\vec{r}') \psi(\vec{r}) + \frac{1}{2} \int d\vec{r} d\vec{r}' V(\vec{r} - \vec{r}') \delta(\vec{r} - \vec{r}_0) \psi^*(\vec{r}) \psi(\vec{r}') \psi(\vec{r}) \psi(\vec{r})
\]

\[
= -\frac{\nabla^2 \psi(\vec{r}_0, t)}{2m} + \int d\vec{r} V(\vec{r}_0 - \vec{r}) \psi^*(\vec{r}) \psi(\vec{r})
\]

Similarly, for the conjugate operator \( \psi^\dagger \), we have

\[
i \frac{\partial \psi^\dagger(\vec{r}_0, t)}{\partial t} = -\frac{\nabla^2 \psi^\dagger(\vec{r}_0, t)}{2m} + \psi^\dagger(\vec{r}_0) \int d\vec{r} V(\vec{r}_0 - \vec{r}) \psi(\vec{r})
\]

For noninteracting particles \( (V = 0) \), this equation is very similar to the Schrodinger equation. But please keep in mind that \( \psi \) and \( \psi^\dagger \) here are operators, instead of wavefunctions.

### 1.4.2. Equation of motion for correlation functions

Define two point correlation functions

\[
G^>(\vec{r}_1, t_1; \vec{r}_2, t_2) = \frac{1}{2} \left\{ \psi(\vec{r}_1, t_1) \psi^*(\vec{r}_2, t_2) \right\}
\]

Here, the factor \( 1/2 \) is introduced for historical reason. For systems with translational symmetry in space and time, \( G^>(\vec{r}_1, t_1; \vec{r}_2, t_2) \) only depends the time and position difference between 1 and 2

\[
G^>(\vec{r}_1, t_1; \vec{r}_2, t_2) = G^>(\vec{r}_1 - \vec{r}_2, t_1 - t_2) = G^>(\vec{r}, t)
\]

Here, we define \( \vec{r} = \vec{r}_1 - \vec{r}_2 \) and \( t = t_1 - t_2 \)

**Q:** What is the equations of motion for \( G^>(\vec{r}, t) \)?

\[
\partial_t G^>(\vec{r}, t) = \partial_t G^>(\vec{r}, t_1 - t_2) = \partial_t \left[ \frac{1}{2} \left\{ \psi(\vec{r}_1, t_1) \psi^*(\vec{r}_2, t_2) \right\} \right] = \left\{ \frac{-i}{2} \left[ \partial_t, \psi(\vec{r}_1, t_1) \right] \psi^*(\vec{r}_2, t_2) \right\}
\]

\[
= \left\{ \left[ -\frac{\nabla^2 \psi(\vec{r}_1, t_1)}{2m} + V(\vec{r}_0 - \vec{r}) \psi^*(\vec{r}, t_1) \psi(\vec{r}, t_1) \psi(\vec{r}_1, t_1) \psi(\vec{r}_2, t_2) \right] \right\}
\]

\[
= \frac{1}{2m} \nabla^2 \psi(\vec{r}_1, t_1) \psi^*(\vec{r}, t_2) - V(\psi^\dagger \psi \psi) = \frac{i}{2m} \nabla^2 G^>(r, t) - V <\psi^\dagger \psi \psi>
\]

If \( V = 0 \) (free systems), we have a closed partial differential equation for two fermion correlation functions.

\[
i \partial_t G_0^>(r, t) + \frac{\nabla^2}{2m} G_0^>(r, t) = 0
\]

The sub-index 0 here implies that we are considering a non-interacting system without interaction. We can solve this partial differential equation (with proper initial conditions and boundary conditions), and obtain the correlation function \( G_0^> \)

For interacting systems \( (V \neq 0) \), the story is not as simple.

\[
i \partial_t G^>(r, t) + \frac{\nabla^2}{2m} G^>(r, t) = -i V <\psi^\dagger \psi \psi >
\]

So we have an **inhomogeneous partial differential equation.**
\[ i \partial_t G^r(r_1, t) + \frac{\nabla^2}{2m} G^r(r_1, t) = f(r, t) \]  

(1.79)

The terminology **inhomogeneous** equation means that the r.h.s. of the equation is nonzero. Inhomogeneous equation may look complicated, because for a different \( f(r, t) \), it seems that we will need to solve a different equation. However, this is not the case. We just need to solve one question and then for any \( f(r, t) \), we can get the solution directly.

**Q:** How do we solve an inhomogeneous partial differential equation?

**A:** Let’s look at the E&M textbook. The Green’s function method.

### 1.4.3. E&M: electric potential \( \phi(r) \) for charge distribution \( \rho(r) \)

**Gauss’s law**

\[ \nabla \cdot \vec{E} (\vec{r}) = \rho(\vec{r}) \]  

(80)

We know that

\[ \vec{E} (\vec{r}) = -\nabla \phi (\vec{r}) \]  

(81)

so

\[ \nabla^2 \phi (\vec{r}) = -\rho (\vec{r}) \]  

(82)

This is an inhomogeneous equation. How do we solve it? We first solve a different equation:

\[ \nabla^2 G (\vec{r}_1, \vec{r}_2) = -\delta (\vec{r}_1 - \vec{r}_2) \]  

(83)

\( G(\vec{r}_1, \vec{r}_2) \) is the electric potential at point \( \vec{r}_1 \) induced by a point charge located at position \( \vec{r}_2 \). We know the solution of this equation, which is just the Coulomb’s law

\[ G(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi} \frac{1}{|\vec{r}_1 - \vec{r}_2|} \]  

(84)

After we find \( G(\vec{r}_1, \vec{r}_2) \), the solution for the inhomogeneous equation \( \nabla^2 \phi (\vec{r}) = \rho (\vec{r}) \) can be obtained easily as

\[ \phi (\vec{r}) = \int d\vec{r}_0 \ G(\vec{r}, \vec{r}_0) \rho (\vec{r}_0) = \int d\vec{r}_0 \ \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}_0|} \rho (\vec{r}_0) \]  

(85)

Mathematicians call \( G(\vec{r}_1, \vec{r}_2) \) the Green’s function. And this methods of solving inhomogeneous PDEs are known as the Green’s function approach. In general, one first substitute the inhomogeneous part with a delta function. Then, one solve this new PDE, whose solution is the Green’s function. Once the Green’s function is obtained, one can write down the solution of the inhomogeneous PDE very easily using an integral. For the correlation function \( G^r \), the equation of motion is

\[ i \partial_t G^r(r_1, t) + \frac{\nabla^2}{2m} G^r(r_1, t) = -i V < \psi^\dagger \psi \psi > \]  

(86)

What we will need to do here is to substitute the r.h.s. by a delta function

\[ \left\{ i \partial_t + \frac{\nabla^2}{2m} \right\} G(r, t) = \delta (r) \delta (t) \]  

(87)

and \( G(r, t) \) is our Green’s function.

### 1.4.4. Time ordering: a trick to get the delta function

Define time-ordered correlation functions (the Green’s functions)

\[ G(\vec{r}_1, t_1; \vec{r}_2, t_2) = \frac{1}{i} \langle T \psi (\vec{r}_1, t_1) \psi (\vec{r}_2, t_2) \rangle \]  

(88)

Here, \( T \psi (\vec{r}_1, t_1) \psi (\vec{r}_2, t_2) \) is known as the time-ordered product.

\[ T \psi (\vec{r}_1, t_1) \psi (\vec{r}_2, t_2) = \begin{cases} \psi (\vec{r}_1, t_1) \psi (\vec{r}_2, t_2) & \text{if } t_1 > t_2 \\ \pm \psi (\vec{r}_2, t_2) \psi (\vec{r}_1, t_1) & \text{if } t_1 < t_2 \end{cases} \]  

(89)

For bosons, we use the + sign and for fermions we use the - sign. Another way to write down the same product is