8

Phase transition and spontaneous symmetry breaking

8.1. Questions:

Q1: Symmetry: if a the Hamiltonian of a system has certain symmetry, can the system have a lower symmetry?

Q2: Analyticity: if we have only analytic functions, can we get something non-analytic?

As will discussed below, the answers are yes and yes. The phenomenon, where the system has a lower symmetry than the Hamiltonian is called *spontaneous symmetry breaking.*

8.2. One example: ferromagnetic transition in the Ising model

Consider a spin lattice where each site has spin-1/2 (say a square lattice). Assuming that the spin want to point in the same direction along the z-axis,

$$H = -J \sum_{\langle i,j\rangle} S_i{}^z S_j{}^z \tag{8.1}$$

where $\langle i,\ j\rangle$ represents the nearest-neighbors and the coupling constant $J > 0$. This system has the Ising symmetry $S^z \to -S^z$ for all the spins, but the state may not have this symmetry.

At high temperature, when entropy dominates, the spins are pointing in random directions, which maximizes the entropy. At low temperature, when entropy contribution is small, the spins wants to align in the same direction (either $+z$ or $-z$) to minimize the energy. We can use the average value of S^z, $\langle S^z\rangle$ as our *order parameter* (assuming all sites have the same $\langle S^z\rangle$). This order parameter shows two different phases. At high temperature, $\langle S^z\rangle = 0$, which is the paramagnetic phase. At low temperature, $\langle S_z\rangle \neq 0$, which is the ferromagnetic phase. The temperature where $\langle S^z\rangle$ turns from zero to nonzero is called the transition point (for a second order phase transition, it is also called the critical point, T_c).

In the low temperature phase, $\langle S^z\rangle$ is nonzero. In this ferromagnetic phase, the state is not invariant under $S^z \to -S^z$, i.e. the system doesn't have the Ising symmetry.

Statistical physics tells us that

$$\langle S_i{}^z\rangle = \frac{1}{Z} \sum_n S_{n\,i}{}^z \exp(-E_n / k_B T) \tag{8.2}$$

where we sum over all possible spin configurations (labeled by the integer n). For a specific spin configuration, we assume its energy is E_n and the spin at site i is $S_{n\,i}{}^z$.

Q1: Everything on the right hand side are an analytic functions. However, $\langle S_i{}^z\rangle$ as a function of temperature T is obviously an non-analytic function with singularity at T_c. How can this be possible?

Q2: For the right hand side, E_n is even under $S_z \to -S_z$ but $S_{n\,i}{}^z$ is odd. Therefore, $S_{n\,i}{}^z \exp(-E_n / k_B T)$ is an odd function under $S_z \to -S_z$. For an odd function, by summing over all possible states n, the total value must be zero. How can we get nonzero $\langle S^z\rangle$?

In fact, for spontaneous symmetry breaking, there are two important ingredients,

(1) we need a small symmetry breaking field.

(2) the system size needs to be infinitely large.

The first ingredient is automatically satisfied in most real systems. For example, for the ferromagnetic transition, we always have some small but finite magnetic field in any experiments (lets assume it is along the +z direction), this field will favor the state with $\langle S_z\rangle > 0$. Therefore, in reality, what we obtained in an experiment is

$$\langle S_z\rangle_+ = \text{Lim}_{h\to 0^+}\langle S_z(h)\rangle \tag{8.3}$$

or

$$\langle S_z\rangle_- = \text{Lim}_{h\to 0^-}\langle S_z(h)\rangle \tag{8.4}$$

where h is a small external magnetic field and we try our best to reduce the strength of this field. When we approaches zero field, we have two options, from the + or - side. These two options give us $\langle S_z\rangle_+$ and $\langle S_z\rangle_-$ respectively. Due to the Ising symmetry, we know that $\langle S_z\rangle_+ = -\langle S_z\rangle_-$.

If $\langle S_z(h)\rangle$ is an analytic function (not singular $h = 0$), it is easy to prove that

$$\text{Lim}_{h\to 0^+}\langle S_z(h)\rangle = \text{Lim}_{h\to 0^-}\langle S_z(h)\rangle \tag{8.5}$$

So

$$\langle S_z\rangle_+ = \langle S_z\rangle_- \tag{8.6}$$

Because we know that $\langle S_z\rangle_+ = -\langle S_z\rangle_-$, we find immediately that

$$\langle S_z\rangle_+ = \langle S_z\rangle_- = 0 \tag{8.7}$$

However, if $\langle S_z(h)\rangle$ is NOT an analytic function at $h = 0$,

$$\text{Lim}_{h\to 0^+}\langle S_z(h)\rangle \neq \text{Lim}_{h\to 0^-}\langle S_z(h)\rangle \tag{8.8}$$

If we approach the zero field limit from the + side, we get $\langle S_z\rangle = \langle S_z\rangle_+$. If we approaches $h = 0$ from the - side, we get $\langle S_z\rangle = \langle S_z\rangle_- = -\langle S_z\rangle_+$. This is the reason, why we have two possible values for the magnetization in the low-temperature. Here, I used the fact $\langle S_z\rangle_+ = -\langle S_z\rangle_-$.

But we know that everything formula in statistic mechanics is analytic, e.g.

$$\langle S_i{}^z(h)\rangle = \frac{1}{Z}\sum_n S_{n\,i}{}^z(h)\exp(-E_n(h)/k_B T) \tag{8.9}$$

Q: How can we get an non-analytic $\langle S_z(h)\rangle$ using analytic functions?

A: This is because we have one singular object in our system, i.e. the system size $L \to +\infty$.

We know that if we have a series of functions $f_i(x)$ with $i = 1, 2, 3, \ldots$. If every function is analytic, the infinite i limit, $\text{Lim}_{i\to+\infty} f_i(x)$, may not necessarily be an analytic function. For example,

$$f_i(x) = \frac{x}{x^2 + 1/i^2} \tag{8.10}$$

For any value of i, $f_i(x)$ is analytic, but

$$\text{Lim}_{i\to+\infty} f_i(x) = \frac{1}{x} \tag{8.11}$$

This is an singular function at $x = 0$. If we approaches $x = 0$ from the + side, we find

$$\text{Lim}_{x\to 0^+}\text{Lim}_{i\to+\infty} f_i(x) = \text{Lim}_{x\to 0^+}\frac{1}{x} = +\infty \tag{8.12}$$

If we approaches $x = 0$ from the - side, we get

$$\text{Lim}_{x\to 0^-}\text{Lim}_{i\to+\infty} f_i(x) = \text{Lim}_{x\to 0^-}\frac{1}{x} = -\infty \tag{8.13}$$

$$\text{Lim}_{x\to 0^+}\text{Lim}_{i\to+\infty} f_i(x) \neq \text{Lim}_{x\to 0^-}\text{Lim}_{i\to+\infty} f_i(x) \tag{8.14}$$

Similarly, for $\langle S_z(h)\rangle$ discussed above, if we take into account the system size L, $\langle S_z(h, L)\rangle$ is an analytic function for any finite L, but the $L \to \infty$ may be singular. If that is the case, the following two quantities may not be the same in general

$$\text{Lim}_{h\to+0}\text{Lim}_{L\to+\infty}\langle S_z(h, L)\rangle \neq \text{Lim}_{h\to-0}\text{Lim}_{L\to+\infty}\langle S_z(h, L)\rangle \tag{8.15}$$

This is what happens in an phase which spontaneously break some symmetry of the Hamiltonian.

8.3. The Ginzburg–Landau approach (phenomenological)

8.3.1. phenomenological construction

If we zoom out and coarse grain the lattice, we can define an order parameter in continuum.

$$m\left(\vec{r}\right) = \frac{1}{\delta \mathrm{V}} \sum_{i \in \delta \mathrm{V}} \langle S_i^z \rangle \tag{8.16}$$

Here, we draw a small box around the position $\vec{r}$, which we called δV. Inside δV, we get the total spin by summing over all sites inside this box. Then, we divided the number by the volume of this box δV. We choose δV to be much larger than the size of a unit cell, but much smaller than the system size. Under the Ising transformation, $S_z \to -S_z$, it is easy to check that $m\left(\vec{r}\right) \to -m\left(\vec{r}\right)$. With this quantity, we can write down the Ginzburg–Landau free-energy

$$F = \int d\vec{r} \left(\partial_i m \, \partial_i m + \alpha \, m^2 + \, g \, m^4 + \text{higher order terms}\right) \tag{8.17}$$

and the minimum of F give us the expectation value of $m\left(\vec{r}\right)$. Here, we refines the unit of length so that the $\partial_i m \, \partial_i m$ term has coefficient unity.

The Ginzburg–Landau theory is an phenomenological theory, which means that we cannot determine the value of the control parameters, which depends on the microscopic details of the system. The rule to write down the GL free energy is to write down every single term allowed by symmetry ($m \to -m$). For example, a term proportional to m or m^3 is not allowed. If we have those terms in the theory, it will violate the $m \to -m$ symmetry.

Q: why the spatial derivative starts from second order?

A: because $m \, \partial_i m = \partial_i \frac{m^2}{2}$, which is a total derivative term. The integral of a total derivative is zero (up to a surface term, which typically can be ignored, unless the system is a topological state).

8.3.2. Microscopic approach (fix the numbers)

The control parameters (e.g. α and g) can be determined using diagrammatic techniques we discussed early on. They are known as the vertex functions. Here, I briefly describe the construction and details can be found in D.J. Amit, *field theory, the renormalization group, and critical phenomena,* World Scientific.

First, we add some external fields to the Hamiltonian (notice that the $h_i = 0$ limit recovers the original Ising model)

$$H(\{h_i\}) = -J \sum_{\langle i,j \rangle} S_i^z \, S_j^z + \sum_i S_i^z \, h_i \tag{8.18}$$

Then, using the new Hamiltonian we compute the partition function Z, which now relies on $\{h_i\}$

$$Z(\{h_i\}) = \sum_n \exp(-E_n(\{h_i\})/k_B \, T) = \mathrm{tr}[\exp(-H(\{h_i\})/k_B \, T)] \tag{8.19}$$

Then, we compute take a log and get the Gibbs free energy

$$G(\{h_i\}) = -k_B \, T \ln Z(\{h_i\}) \tag{8.20}$$

Then, we define

$$\overline{\phi}_i = \frac{\delta \mathrm{G}(\{h\})}{\delta \mathrm{h_i}} = \frac{1}{Z(\{h\})} \mathrm{tr}[S_i^z \exp(-H(\{h_i\})/k_B \, T)] \tag{8.21}$$

which is the expectation value of S_i^z for site i.

Then, we make a Legendre transformation to get the free energy F

$$F\left(\{\overline{\phi}_i\}\right) = \sum_i \overline{\phi}_i \, h_i - G(\{h_i\}) \tag{8.22}$$

Now, the free energy is a function of $\overline{\phi}_i$, and one can easily prove that

$$\frac{\delta \, F\left(\{\overline{\phi}_i\}\right)}{\delta \, \overline{\phi}_i} = \sum_i h_i \tag{8.23}$$

Therefore, the extremum condition of $F(\{\bar{\phi}_i\})$ (i.e. $\frac{\delta F(\{\bar{\phi}_i\})}{\delta \bar{\phi}_i}$=0) coincides with the zero field condition. And the value of $\bar{\phi}_i$ that minimizes $F(\{\bar{\phi}_i\})$ are precisely the expectation value of spins at each site at zero field. i.e. $F(\{\bar{\phi}_i\})$ is our GL free energy. Then, we coarse grain the lattice and get a continues theory as describe above.

Comment 1, In reality, the computation is much easier, thanks to some tricks people found. It turns out that if we want the coefficient α, we just need to compute two-point correlation functions. If we want g we just need four-point correlation functions.

Comment 2, Because the coefficients in the GL free energy are obtained by computing n-point correlation functions, there the loop integral may have singular values and thus require UV cut off: $\alpha(\Lambda)$ and $g(\Lambda)$. This is called running couplings in renormalization group, i.e. the control parameter relies on cut offs.

8.3.3. Disordered and ordered phases

If we assume that $g > 0$ and higher order terms don't have any important contributions, we have two phases depending on the sign of α

$\alpha > 0$ is the disordered phase (the paramagnetic phase). Here, we have $\langle m\rangle = 0$ and the state has the same symmetry as the Hamiltonian (or say the GL free energy).

α<0 is the ordered phase (the ferromagnetic state). Here, $|\langle m\rangle| \neq 0$, but because the system has the $m \to -m$ symmetry, if we have a state with $\langle m\rangle = a$, there must be another possible state with $\langle m\rangle = -a$.

- In the ordered phase, we have (at least) two different possible states, and they have opposite $\langle m\rangle$.
- The energy barrier that separates these two states is infinitely high (if the system size is finite). Therefore, once the system fall into one of the two states, it will take infinite long time and infinite energy to flip into the other state.

Of course, in reality, the system size can never be infinite, but they are typically large $\sim 10^{23}$. This means that it will not take infinite long time for use to flip the ground states, but it will take a very long time, too long to observe.

8.3.4. Fluctuations around the expectation value

In the ordered phase (α<0), the GL free energy is minimized by setting $m = \pm\sqrt{-\alpha/2\,g}$. Now, let's studying fluctuations around this expectation value by consider m slightly away from $+\sqrt{-\alpha/2\,g}$ (we can repeat the same discussion about the other ordered state $m = -\sqrt{-\alpha/2\,g}$ and we will get the same result)

$$m(\vec{r}) = \sqrt{-\alpha/2\,g} + \delta\mathrm{m}(\vec{r}) \tag{8.24}$$

$$\begin{aligned} F = \int d\vec{r}\,(\partial_i m\,\partial_i m + \alpha\,m^2 + g\,m^4) = \int d\vec{r}\,\left\{\partial_i \delta\mathrm{m}(\vec{r})\,\partial_i \delta\mathrm{m}(\vec{r}) + \alpha\left[\sqrt{-\alpha/2\,g} + \delta\mathrm{m}(\vec{r})\right]^2 + g\left[\sqrt{-\alpha/2\,g} + \delta\mathrm{m}(\vec{r})\right]^4\right\} = \\ \int d\vec{r}\left[\partial_i \delta\mathrm{m}(\vec{r})\,\partial_i \delta\mathrm{m}(\vec{r}) + \alpha\,\delta\mathrm{m}(\vec{r})^2 - 3\,\alpha\,\delta\mathrm{m}(\vec{r})^2 + \text{higher order terms}\right] + \text{constant} = \\ \int d\vec{r}\left[\partial_i \delta\mathrm{m}(\vec{r})\,\partial_i \delta\mathrm{m}(\vec{r}) + 2\,|\alpha|\,\delta\mathrm{m}(\vec{r})^2\right] + \text{constant} + \ldots \end{aligned} \tag{8.25}$$

If we ignore higher order terms and go to the k-space we find that

$$F = \sum_k \left[\left(k^2 + 2\,|\alpha|\right)\delta\mathrm{m}^2\right] \tag{8.26}$$

In analogy to what we learned in the field theory, the coefficient of a quadratic term is our single-particle dispersion. If we compare this with the dispersion of a relativistic particle, we find that

$$\epsilon^2 = k^2 + 2\,|\alpha| \tag{8.27}$$

$$\epsilon = \sqrt{k^2 + 2\,|\alpha|} \tag{8.28}$$

Thus, we got a particle with mass 2|α|. In the order phases, α<0, this particle has a finite mass (i.e. a massive particle).

8.4. Continuous symmetry breaking (the X-Y model)

Assume that we have a spin lattice and the coupling between the spins are along the X and Y directions.

$$H = -J\sum_{\langle i,j\rangle} S_i^x\,S_j^x - J\sum_{\langle i,j\rangle} S_i^y\,S_j^y \tag{8.29}$$

Here, the high temperature state has zero magnetization (same as the Ising model discussed above), but the low temperature phase is very different from the Ising model. Here, the magnetization would align in the XY plane, instead of the z-direction. Therefore, the order parameter is a 2D vector

$$\vec{m} = (m_x,\, m_y) \tag{8.30}$$

Therefore, the GL free energy should take the form

$$F = \int d\vec{r}\left[\partial_i \vec{m}\, \partial_i \vec{m} + \alpha\, \vec{m}\cdot\vec{m} + g\left(\vec{m}\cdot\vec{m}\right)^2 + \text{higher order terms}\right] \tag{8.31}$$

Again, we have two phases: (1) the paramagnetic phase with $\alpha > 0$ and (2) the ferromagnetic phase at $\alpha < 0$.

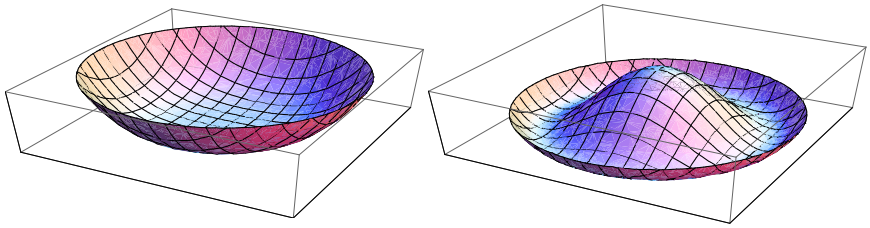

Fig. 1. F as a function of $\vec{m} = (m_x, m_y)$

8.4.1. Fluctuations around the expectation value

In the ordered phase (α<0), the GL free energy is minimized by setting $\left|\vec{m}\right| = \sqrt{-\alpha/2\,g}$. Now, let's studying fluctuations around this expectation value by consider $\vec{m}$ slightly away from one of the ordered state

$$m\left(\vec{r}\right) = \sqrt{-\alpha/2\,g}\ \vec{e}_x + \delta\mathrm{m}\left(\vec{r}\right) \tag{8.32}$$

In other words

$$m_x\left(\vec{r}\right) = \sqrt{-\alpha/2\,g}\ + \delta\mathrm{m}_x\left(\vec{r}\right) \tag{8.33}$$

$$m_y\left(\vec{r}\right) = \delta\mathrm{m}_y\left(\vec{r}\right) \tag{8.34}$$

$$\begin{aligned} F = &\int d\vec{r}\left(\partial_i \vec{m}\, \partial_i \vec{m} + \alpha\, \vec{m}\cdot\vec{m} + g\left(\vec{m}\cdot\vec{m}\right)^2\right) = \\ &\int d\vec{r}\left\{(\nabla\delta\mathrm{m}_x)^2 + (\nabla\delta\mathrm{m}_y)^2 + \alpha\left[\left(\sqrt{-\alpha/2\,g}\ + \delta\mathrm{m}_x\right)^2 + \delta\mathrm{m}_y{}^2\right]^2 + g\left[\left(\sqrt{-\alpha/2\,g}\ + \delta\mathrm{m}_x\right)^2 + \delta\mathrm{m}_y{}^2\right]^2\right\} = \\ &\int d\vec{r}\left[(\nabla\delta\mathrm{m}_x)^2 + (\nabla\delta\mathrm{m}_y)^2 + 2\left|\alpha\right|\ \delta\mathrm{m}_x{}^2 + \alpha\,\delta\mathrm{m}_y{}^2 - \alpha\,\delta\mathrm{m}_y{}^2 + \text{higher order terms}\right] + \text{constant} = \\ &\int d\vec{r}\left[(\nabla\delta\mathrm{m}_x)^2 + 2\left|\alpha\right|\ \delta\mathrm{m}_x{}^2\right] + \int d\vec{r}\left[(\nabla\delta\mathrm{m}_y)^2\right] + \text{constant} + \ldots \end{aligned} \tag{8.35}$$

If we ignore higher order terms and go to the k-space we find that

$$F = \sum_k \left[\left(k^2 + 2\left|\alpha\right|\right)\delta\mathrm{m}_x{}^2\right] + \sum_k k^2\,\delta\mathrm{m}_y{}^2 \tag{8.36}$$

Here, we find two modes (which is not surprising, because we have m_x and m_y in our original GL free energy). For the mode $\delta\mathrm{m}_x$, it is a massive particle with mass $2|\alpha|$. For the $\delta\mathrm{m}_y$ mode, it a massless particle with mass zero.

Because $\delta\mathrm{m}_x$ changes the amplitude of m_x (which is obviously by looking at $m\left(\vec{r}\right) = \sqrt{-\alpha/2\,g}\ \vec{e}_x + \delta\mathrm{m}\left(\vec{r}\right)$), it is called the amplitude mode. The mode $\delta\mathrm{m}_y$ titles the direction of the order parameter, because direction is described by an angle and an angle is no different from a phase (mathematically), this is called a phase model.

Generically, if we spontaneously breaks a continuous symmetry (e.g. rotation along the z axis here), a phase mode results in a massless particle while the amplitude mode is a massive particle. The massless mode of this type is known as a *Goldstone mode.*

Comment: In physics, we believe that there shall be a reason for everything. If we find a particle have zero mass, there should also be a reason, and the Goldstone mode is (almost) the only reason we know. In principle, some particles may accidentally pick zero as the value of their mass, but the chance for such a case to arise in our universe is tiny ($P = 0$). Therefore, if we find a massless particle, it must be a Goldstone particle.

This conclusion is so far correct. In the early days (which means a few years ago), some people think that there may be one exception, neutrinos, which was believed to be massless and they are NOT Goldstone modes. But we now know that neutrinos are massive. Although their mass are very small (in the order of eVs), it is not zero.

8.5. Continuous symmetry breaking (the Heisenberg model)

If we repeat the same discussion for a Heisenberg model,

$$H = -J \sum_{\langle i,j\rangle} S_i^x S_j^x - J \sum_{\langle i,j\rangle} S_i^y S_j^y - J \sum_{\langle i,j\rangle} S_i^z S_j^z \tag{8.37}$$

we find one amplitude mode (massive) and two massless Goldstone modes. For example, for

$$m(\vec{r}) = \sqrt{-\alpha/2g}\ \vec{e}_x + \delta\mathrm{m}(\vec{r}) \tag{8.38}$$

we get

$$F = \sum_k \left[\left(k^2 + 2|\alpha|\right)\delta\mathrm{m}_x{}^2\right] + \sum_k k^2\, \delta\mathrm{m}_y{}^2 + \sum_k k^2\, \delta\mathrm{m}_z{}^2 \tag{8.39}$$

8.6. 3D nematic transition

8.6.1. Order parameter

If we consider a liquid made by rod-like molecules, when we cool down the systems, there is a isotropic liquid-nematic liquid transition. At high T, the rods are pointing in random directions, but at low T, their directions align up, which is known as the nematic phase. For many cases, this rod is invariant under space inversion $r \to -r$, which means that it makes no difference no matter we align the rod along $\vec{r}$ and $-\vec{r}$.

The nematic state is similar to the Heisenberg model, if we use a 3D unit vector $\vec{n}$ to mark the direction of each rod. However, we need to notice that $\vec{n}$ and $-\vec{n}$ describes the same state. In other words, the direction of a rod is a director, not a vector. To taken care of the ambiguity between $\vec{n}$ and $-\vec{n}$, the order parameter should be defined as a rank two tensor

$$Q_{\mathrm{ij}}(\vec{r}) = \frac{1}{\delta\mathrm{V}} \sum_{\vec{r}_k \in \delta\mathrm{V}} \left[n_i(\vec{r}_k)\, n_j(\vec{r}_k) - \frac{1}{3}\delta_{\mathrm{ij}}\right] \tag{8.40}$$

Here, again we cut a small box around $\vec{r}$. Then for all rods inside this box $(\vec{r}_k)$, we sum over the $n_i\, n_j$ for every rod. Here, the sub index i and j run over x, y and z. Then, we normalize everything using $1/\delta\mathrm{V}$. This tensor Q_{ij} is the correct order parameter and it respect all necessary symmetries, neither does it care about the ambiguity between $\vec{n}$ and $-\vec{n}$. Here, we subtract $-\frac{1}{3}\delta_{\mathrm{ij}}$ to make the matrix Q_{ij} traceless. This is because the trace part doesn't carry any useful information (beyond the particle density).

It is easy to check that Q is a tensor under rotation

$$Q \to Q' = U_R\, Q\, U_R{}^{\mathrm{T}} \tag{8.41}$$

where U_R is the rotational matrix, which is in general an orthogonal matrix.

The order parameter Q is a symmetric traceless tensor.

8.6.2. The GL free energy

Here we construct the GL free energy based on symmetry. (Here we assume $w > 0$ and $u > 0$)

$$F = \int d\vec{r}\left[\frac{1}{2}r\left(\frac{3}{2}\mathrm{Tr}\,Q^2\right) - w\left(\frac{9}{2}\mathrm{Tr}\,Q^3\right) + u\left(\frac{3}{2}\mathrm{Tr}\,Q^2\right)^2\right] \tag{8.42}$$

This free energy is very different from spin modes discussed above, because here we have a Q^3 term. This term is allowed here, due to the special symmetry of directors. It turns out that all possible nematic order parameters can be reduces into a diagonal matrix by rotating the axis. There, the most generic order parameter one can write down is:

$$Q = \begin{pmatrix} 2S/3 & 0 & 0 \\ 0 & -S/3+\eta & 0 \\ 0 & 0 & -S/3-\eta \end{pmatrix} \tag{8.43}$$

For most nematic liquids, $\eta = 0$, which is known as uni-axial nematic. The case with $\eta \neq 0$ is known as bi-axial nematic. Here, we assume η=0 for simplicity

$$Q = \begin{pmatrix} 2S/3 & 0 & 0 \\ 0 & -S/3 & 0 \\ 0 & 0 & -S/3 \end{pmatrix} \tag{8.44}$$

Using S and η, we find that

$$F = \int d\vec{r}\Big[\frac{1}{2}r\left(\frac{3}{2}\mathrm{Tr}\,Q^2\right) - w\left(\frac{9}{2}\mathrm{Tr}\,Q^3\right) + u\left(\frac{3}{2}\mathrm{Tr}\,Q^2\right)^2\Big] = \frac{r}{2}S^2 - wS^3 + uS^4 \tag{8.45}$$

If $w = 0$, we get a second order phase transition, same as discussed above for the spin models. However, if $w \neq 0$, the phase transition is first order. For $r > 9w^2/16u$, the GL free energy only has one minimum at $S = 0$ (the disordered phase, i.e. the isotropic liquid).

For $r < 9w^2/16u$, the system has two minimum at $S = 0$ and $\frac{-3w+\sqrt{-16ru+9w^2}}{8u}$. The former has GL free energy $F = 0$, the latter has $F = -\frac{\left(3w+\sqrt{-16ru+9w^2}\right)^2\left(-8ru+w\left(3w+\sqrt{-16ru+9w^2}\right)\right)}{2048u^3}$

For $w^2/2u < r < 9w^2/16u$, the trivial solution $S = 0$ has lower GL free energy, $F = 0$

For $r < w^2/2u$, the nontrivial solution $S = \frac{-3w+\sqrt{-16ru+9w^2}}{8u}$ has lower GL free energy $F = -\frac{\left(3w+\sqrt{-16ru+9w^2}\right)^2\left(-8ru+w\left(3w+\sqrt{-16ru+9w^2}\right)\right)}{2048u^3}$.

This implies that $r = w^2/2u$ is a first order transition point, at which S jump from 0 to $w/2u$.