

## 2

## Fermi liquid

### 2.1. Can we apply the free fermion approximation to a real metal?

Can we ignore interactions in a real metal? Experiments says yes (free fermion models work very well), but the theoretical answer is much more involving.

#### 2.1.1. Strong interaction vs. weak interaction

**Q:** Does interaction get stronger when electron density is higher?

**A naive answer:** Interaction is proportional to  $1/r$ , where  $r$  is the separation between two electrons. Higher density means that the average separation between electrons is smaller, so higher density means stronger interaction.

In an electron gas, the average distance between electrons  $r_s$  can be estimated as

$$\frac{V}{N} = \frac{4\pi}{3} r_s^3 \quad (2.1)$$

where  $V/N$  is the average volume occupied by one electron, and the density  $n$  is  $n = N/V$ . Thus

$$r_s = \left( \frac{3V}{4\pi N} \right)^{1/3} = \left( \frac{3}{4\pi n} \right)^{1/3} \quad (2.2)$$

The average Coulomb interaction one electron feels can be estimated as

$$\frac{E_{\text{int}}}{N} \propto \frac{e^2}{r_s} = e^2 \left( \frac{4\pi}{3} n \right)^{1/3} \propto n^{1/3} \quad (2.3)$$

Small  $r_s$  (or high density  $n$ ) means stronger interaction.

**However, the truth is opposite, because**

$$H = H_{\text{KE}} + H_{\text{Int}} \quad (2.4)$$

We need to compare interaction with the kinetic energy to determine whether we have a strongly correlated system or a weakly correlated system. Strong correlation means  $H_{\text{Int}} \gg H_{\text{KE}}$  and weak correlation means  $H_{\text{KE}} \gg H_{\text{Int}}$ . The total kinetic energy in a Fermi gas is also a function of  $r_s$  (or say density)

$$K.E. = 2V \int_{\epsilon < \epsilon_F} \frac{d^3 k}{(2\pi)^3} \frac{k^2}{2m} = 2V \int_0^{k_F} \frac{k^4 dk}{4\pi^2 m} = V \frac{k_F^5}{10\pi^2 m} \quad (2.5)$$

The particle number is

$$N = 2V \int_{\epsilon < \epsilon_F} \frac{d^3 k}{(2\pi)^3} = 2V \int_0^{k_F} \frac{k^2 dk}{2\pi^2 m} = V \frac{k_F^3}{3\pi^2} \quad (2.6)$$

So,  $K.E.$  per particle

$$\frac{K.E.}{N} = \frac{V \frac{k_F^5}{10\pi^2 m}}{V \frac{k_F^3}{3\pi^2 m}} = \frac{3 k_F^2}{10 m} \propto \frac{1}{m r_S^2} \quad (2.7)$$

More precisely,

$$k_F = \left( 3\pi^2 \frac{N}{V} \right)^{1/3} \quad (2.8)$$

Comparing with  $r_s = \left( \frac{3V}{4\pi N} \right)^{1/3} = \left( \frac{3}{4\pi n} \right)^{1/3}$

$$k_F = \frac{1}{r_s} \left( \frac{9\pi}{4} \right)^{1/3} \quad (2.9)$$

So kinetic energy per particle is

$$\frac{K.E.}{N} = \frac{3 k_F^2}{10 m} = \frac{3}{10} \left( \frac{9\pi}{4} \right)^{2/3} \frac{1}{m r_S^2} \quad (2.10)$$

The ratio between K.E. and interaction energy,

$$\frac{\text{Interaction}}{K.E.} \propto \frac{e^2}{r_S} \bigg/ \frac{1}{m r_S^2} = r_S \bigg/ \frac{1}{m e^2} = \frac{r_S}{a_0} \quad (2.11)$$

Here  $a_0 = 1/m e^2 = 0.529 \text{ \AA}$ . When  $r_S \gg a_0$  (low density), we have a strongly correlated system with the energy of interactions  $\gg$  K.E. When  $r_S \ll a_0$  (high density), we have a weakly correlated systems with K.E.  $\gg$  interaction.

**Lower density means stronger interaction.**

### 2.1.2. Perturbation theory

In the previous chapter, we showed that the diagrammatic expansion of Feynman is in fact a power-law expansion of interaction strength  $V$ .

Because  $\frac{\text{Interaction}}{K.E.} \sim \frac{r_S}{a_0}$ , the small parameter in this expansion is in fact  $r_S/a_0$

$$G = G_0 + \text{ii} \frac{r_S}{a_0} + \text{ii} \left( \frac{r_S}{a_0} \right)^2 + \text{ii} \left( \frac{r_S}{a_0} \right)^3 + \dots \quad (2.12)$$

If  $r_S \ll a_0$  (the high density limit), this expansion converge, i.e. higher order terms are small and can be dropped. In this regime, the free fermion approximation (ignoring interactions) is obviously a good approximation and the error bar is about  $r_S/a_0$ .

If  $r_S \gg a_0$  (the low density regime), one should use interactions the non-perturbed Hamiltonian and treat the kinetic energy as perturbation. There, to the leading order, we can set the kinetic energy to zero. The ground state is easy to find. Because electrons want to stay as far away from each other as possible to reduce the interaction energy, in 2D, they just form a close pack structure (a triangular lattice). This state is known as a Wigner crystal. Please notice that it is NOT a crystal of atoms. It is a crystal made by electrons, and the lattice back ground is irrelevant here.

**Reality: In typical metals,  $2 < r_S/a_0 < 6$ . Neither too small, nor too large. We can not ignore interactions, neither can be ignore the kinetic energy.**

With such a strong interaction, we cannot ignore interactions. So let's go to the first order and see what we get.

## 2.2. Hartree-Fock approximation

The first order perturbation theory here is known as the Hartree-Fock approximation. In this section, we compute the first order perturbation theory using quantum field theory we learned in the previous chapter. The final conclusion of this section is, after we include the first order corrections, the theory is even worse. In fact, when we compare theoretical results with experimental data, the free fermion theory works much better, and the Hartree-Fock approximation generates wrong results. This is due to the special property of  $1/r$  interactions.

### 2.2.1. Coulomb interaction in the momentum space

What is the Coulomb interaction in the  $k$ - $\omega$  space,  $V(q, i\Omega_n)$ ?

$$V(r) = \frac{e^2}{r} \quad (2.13)$$

Define  $U(r) = \frac{e}{r}$ , which is the potential induced by a point charge  $e$ . Maxwell told us that

$$\nabla^2 U(r) = -4\pi e \delta(r) \quad (2.14)$$

In  $q$ -space

$$-q^2 U(q) = -4\pi e \quad (2.15)$$

Therefore,

$$U(q) = 4\pi \frac{e}{q^2} \quad (2.16)$$

Notice that  $V(r) = e U(r)$ , so  $V(q) = e U(q)$

$$V(q) = \frac{4\pi e^2}{q^2} \quad (2.17)$$

This is our  $V(q, i\Omega_n)$ .

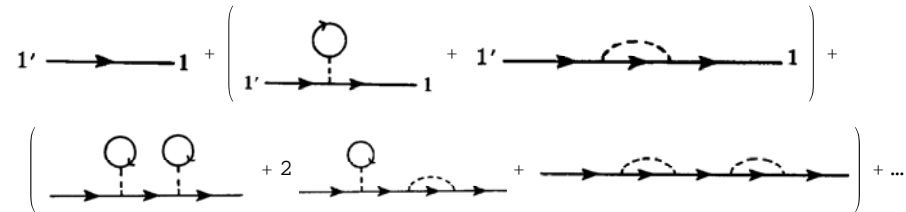
$$V(q, i\Omega_n) = V(q) = \frac{4\pi e^2}{q^2} \quad (2.18)$$

**Note 1:**  $V(q, i\Omega_n)$  is independent of  $i\Omega_n$ . This is because that Coulomb interaction is instantaneous (a good approximation in non-relativistic systems).

**Note 2:**  $V(q)$  is a singular function at  $q \rightarrow 0$ .

### 2.2.2. Hartree-Fock approximation

Consider these diagrams



Using Dyson's equation, we find that

$$G^{\text{HF}}(k, i\omega_n) = \frac{1}{\omega - \epsilon(k) - \Sigma_{\text{HF}}(k, i\omega_n)} \quad (2.19)$$

where

$$\Sigma_{\text{HF}}(k, i\omega_n) = \text{circle with dot} + \text{dashed line} = \mp \int \frac{dq}{(2\pi)^3} \frac{1}{\beta} \sum_{i\Omega_n} V(0, 0) G^{(0)}(q, i\Omega_n) - \int \frac{dq}{(2\pi)^3} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k+q, i\omega_n + i\Omega_n) V(q, i\Omega_n) \quad (2.20)$$

The first term (the Hartree term) is a constant, independent of  $k$  and  $\omega_n$ . This term just shift the single particle energy  $\epsilon(k)$  by a constant, which has no physical impact. The second term (the Fock term) is more important.

$$\Sigma_F(k, i\omega_n) =$$

$$\text{dashed line} = - \int \frac{dq}{(2\pi)^3} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k+q, i\omega_n + i\Omega_n) V(q, i\Omega_n) = - \int \frac{dq}{(2\pi)^3} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k+q, i\Omega_n) V(q, i\Omega_n - i\omega_n) \quad (2.21)$$

Here, in the last step, we redefine  $i\Omega_n$  as  $i\Omega_n \rightarrow i\Omega_n - i\omega$ .

## 2.3. Frequency sum

$$\Sigma_F(k, i\omega_n) = - \int \frac{dq}{(2\pi)^3} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k+q, i\Omega_n) V(q, i\Omega_n - i\omega_n) \quad (2.22)$$

The sum of frequency here  $\sum_{i\Omega_n}$  sums over all fermionic Matsubara frequencies (because  $i\Omega_n$  is the frequency of a fermionic Green's function  $G^{(0)}$ ). In general, if we consider an arbitrary function  $f(i\Omega_n)$  and sums over all fermionic Matsubara frequencies, we get

$$\sum_{i\Omega_n} f(i\Omega_n) = \sum_n f\left(\frac{2n+1}{\beta} \pi - i\mu\right) \quad (2.23)$$

**Q: How should we compute the sum?**

### 2.3.1. Cauchy's residue theorem

For a complex function  $f(z)$ , there is a very famous and very important integral formula, known as Cauchy's residue theorem. We can use this formula to compute the frequency sum. Let's consider a complex function  $f(z)$  and a contour on the complex plan  $C$ . We assign a direction for the contour  $C$  and compute the integral  $\oint_C f(z)$ . Here, the integral is computed along the direction of  $C$ .

**Inside and outside:** a contour separate the complex plane into two regions: inside and outside. If we follow the direction of the contour, the region on the right hand side of the contour is called "inside" and the other side is "outside". Notice that if we flip the direction of the contour  $C$ , we flip inside and outside.

**Here, we emphasize that inside and outside are relative concepts. The inside of  $C$  is the same as the outside of  $-C$ .**

Assume that inside/outside the contour  $C$ ,  $f(z)$  is an analytic function except for some discrete singularity points  $z_m$ , at which  $f(z_m) \rightarrow \infty$ . At each singularity point, we write  $f(z)$  as a power series.

$$f(z) = \dots + \frac{a_{-n}}{(z-z_m)^n} + \dots + \frac{a_{-1}}{z-z_m} + a_0 + a_1(z-z_m) + \dots + a_n(z-z_m)^n + \dots \quad (2.24)$$

The value of  $a_{-1}$  is known as the residue at  $z_m$ ,  $\text{Res}(f, z_m)$ . Another way to define the residue is

$$\text{Res}(f, z_m) = \lim_{z \rightarrow z_m} (z - z_m) f(z) \quad (2.25)$$

One can prove that

$$\oint_C f(z) = 2\pi i \sum_{m \text{ inside } C} \text{Res}(f, z_m) \quad (2.26)$$

and this is known as the Cauchy's residue theorem. We can do the same thing for contour  $-C$  (flip the direction of  $C$ )

$$\oint_{-C} f(z) = 2\pi i \sum_{m \text{ inside } -C} \text{Res}(f, z_m) \quad (2.27)$$

Notice that  $\oint_C f(z) = -\oint_{-C} f(z)$  and inside of  $-C$  is just outside of  $C$ ,

$$\oint_C f(z) = -2\pi i \sum_{m \text{ outside } C} \text{Res}(f, z_m) \quad (2.28)$$

This proves one conclusion, which we will use later

$$\sum_{m \text{ inside } C} \text{Res}(f, z_m) = - \sum_{m \text{ outside } C} \text{Res}(f, z_m) \quad (2.29)$$

**This singularity points are known as the poles, and thus Cauchy told us that  $\oint_C f(z)$  are purely determined by the residue at the poles.**

### 2.3.2. Residue of the Fermi distribution function

Now consider the Fermi distribution function,

$$f(z) = \frac{1}{\exp[\beta(z - \mu)] + 1} \quad (2.30)$$

**Q: What are the singularity points of  $f(z)$ ?**

**A: fermionic Matsubara frequencies**

This is because for  $f(z)$ , the singular points must satisfy  $\exp[\beta(z - \mu)] + 1 = 0$

$$\exp[\beta(z - \mu)] = -1 = \exp[(2n + 1)\pi i] \quad (2.31)$$

where  $n$  is an integer, so

$$\beta(z - \mu) = (2n + 1)\pi i \quad (2.32)$$

thus

$$z = \frac{(2n + 1)\pi i}{\beta} + \mu = i\omega_n \quad (2.33)$$

**Q:** What is the residue?

**A:**  $\text{Res}(f, i\omega_n) = -1/\beta$

This is because

$$\begin{aligned} f(z) &= \frac{1}{\exp[\beta(z - \mu)] + 1} = \frac{1}{\exp[\beta(z - i\omega_n) + (i\omega_n - \mu)] + 1} = \frac{1}{\exp[\beta(z - i\omega_n)] \exp[(i\omega_n - \mu)] + 1} = \\ &= \frac{1}{1 - \exp[\beta(z - i\omega_n)]} = \frac{1}{1 - \{1 + \beta(z - i\omega_n) + O(z - i\omega_n)^2\}} = -\frac{1}{\beta(z - i\omega_n) + O(z - i\omega_n)^2} = -\frac{1}{\beta(z - i\omega_n)} + O(z - i\omega_n)^0 \end{aligned} \quad (2.34)$$

### 2.3.3. Integrals that involve $f(z)$

Now, let's consider an arbitrary function  $g(z)$  and we assume that  $g(z)$  is an analytic function near  $i\omega_n$ . We draw a small circle around each  $i\omega_n$  on the complex plane and define our contour  $C$  as the combination of all these small circles. We choose the direction such that all of  $i\omega_n$  are inside our contour  $C$ . Now we consider the integral  $\oint_C f(z) g(z)$ , where  $f(z)$  is the Fermi distribution function.

$$\oint_C f(z) g(z) = 2\pi i \sum_{m \text{ inside } C} \text{Res}(f g, z_m) \quad (2.35)$$

If the circles are small enough,  $g(z)$  would not give any singularity inside  $C$ . Therefore, all the singularity points come from  $f(z)$ , which is singular at  $i\omega_n$ .

$$\oint_C f(z) g(z) = 2\pi i \sum_{i\omega_n} \text{Res}(f g, i\omega_n) \quad (2.36)$$

Because  $g(z)$  is analytic around  $i\omega_n$ , all the residue comes from the singularity of  $f(z)$

$$\oint_C f(z) g(z) = 2\pi i \sum_{i\omega_n} \text{Res}(f g, i\omega_n) = 2\pi i \sum_{i\omega_n} g(i\omega_n) \text{Res}(f, i\omega_n) = -\frac{2\pi i}{\beta} \sum_{i\omega_n} g(i\omega_n) \quad (2.37)$$

This can be proved using

$$\text{Res}(f g, i\omega_n) = \lim_{z \rightarrow i\omega_n} (z - i\omega_n) f(z) g(z) = g(i\omega_n) \lim_{z \rightarrow i\omega_n} (z - i\omega_n) f(z) = g(i\omega_n) \text{Res}(f, i\omega_n) = -g(i\omega_n)/\beta \quad (2.38)$$

Therefore,

$$\frac{1}{\beta} \sum_{i\omega_n} g(i\omega_n) = -\frac{1}{2\pi i} \oint_C f(z) g(z) \quad (2.39)$$

This means that if we want to compute the frequency sum for a function  $g(z)$ , as long as  $g(z)$  is analytic around  $i\omega_n$ , we can turn the sum into an integral. In fact, we can do better than that, notice that  $\oint_C = -\oint_{-C}$

$$\frac{1}{\beta} \sum_{i\omega_n} g(i\omega_n) = -\frac{1}{2\pi i} \oint_C f(z) g(z) = \frac{1}{2\pi i} \oint_{-C} f(z) g(z) \quad (2.40)$$

Here, we can use Cauchy's residue theorem again, but for  $-C$ .

$$\frac{1}{\beta} \sum_{i\omega_n} g(i\omega_n) = \frac{1}{2\pi i} \oint_{-C} f(z) g(z) = \sum_{m \text{ inside } -C} \text{Res}(f g, z_m) = \sum_{m \text{ outside } C} \text{Res}(f g, z_m) \quad (2.41)$$

Here, because  $f(z)$  is analytic outside  $C$ , all the residue should come from the singular points of function  $g(z)$ .

$$\frac{1}{\beta} \sum_{i \omega_n} g(i \omega_n) = \sum_{m \text{ outside } C} \text{Res}(f g, z_m) = \sum_{m \text{ outside } C} f(z_m) \text{Res}(g, z_m) \quad (2.42)$$

This is the way that we use to compute frequency sums.

**To compute**  $\frac{1}{\beta} \sum_{i \omega_n} g(i \omega_n)$

**Step 1:** analytic continuation:  $g(z)$

**Step 2:** find the pole of  $g(z)$  and compute  $\sum_{m \text{ outside } C} f(z_m) \text{Res}(g, z_m)$ , where  $f(z)$  is the Fermi distribution function.

**Q: How about bosons?**

**A: One should use the Bose distribution function.**

$$f(z) = \frac{1}{\exp[\beta(z - \mu)] - 1} \quad (2.43)$$

**Note: if  $g(z)$  has branch cut, one needs to make sure that we avoid the branch cut, which leads to additional contributions. This effect is important for some calculations but will not be discussed in this lecture.**

## 2.4. Hartree-Fock approximation: part II

Now let's compute the frequency sum of the Fock term

$$\Sigma_F(k, i\omega_n) = - \int \frac{dq}{(2\pi)^3} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k+q, i\Omega_n) V(q, i\Omega_n - i\omega_n) \quad (2.44)$$

$$\frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k+q, i\Omega_n) V(q) = \sum_{m \text{ outside } C} f(z_m) V(q) \text{Res}[G^{(0)}(k+q, z), z_m] \quad (2.45)$$

Here,  $V$  is independent of  $\omega$  and thus cannot give us any pole on the complex  $\omega$  plane. The residue can only come from  $G^{(0)}(k+q, z) = \frac{1}{z - \epsilon(k+q)}$ . This function has only one singular point at  $z = \epsilon(k+q)$  and the residue here is obviously 1.

$$\frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k+q, i\Omega_n) V(q) = f[\epsilon(k+q)] V(q) = f[\epsilon(k+q)] V(q) \quad (2.46)$$

Therefore,

$$\Sigma_F(k, i\omega_n) = - \int \frac{dq}{(2\pi)^3} V(q) f[\epsilon(k+q)] = - \int \frac{dq}{(2\pi)^3} V(k-q) f[\epsilon(q)] = - \int \frac{dq}{(2\pi)^3} V(k-q) \langle n(q) \rangle \quad (2.47)$$

In the last step, I used the fact that  $f[\epsilon(q)]$  is the average occupation number for the quantum state with momentum  $q$ . So the Green's function is

$$G_{\text{HF}}(k, i\omega_n) = \frac{1}{i\omega_n - \epsilon(k) - \Sigma_{\text{HF}}(k)} \quad (2.48)$$

This means that within the Hartree-Fock approximation (first order perturbation theory), we can still think of our interacting particles as "free particles", but the dispersion relation is shifted from  $\epsilon(k)$  into  $\epsilon(k) - \int dq v(k-q) \langle n(q) \rangle$ .

Consider a system with charge distribution  $n(r)$ , the potential energy felt by an electron at  $r_0$  is

$$\delta E = \int dr v(r - r_0) n(r) \quad (2.49)$$

So interactions with other electrons shift the energy of the electron we study by  $\delta E = \int dq v(r - r_0) n(r)$ . The HF approximation is the momentum space version of this formula. In a real system, one electron feels the electric field induced by other electrons. This field varies with time  $t$ , because other electrons are moving. The H-F approximation assumes that we can substitute this varying electric by its average value, which is time-independent. Therefore, in the H-F approximation, we treat all other electrons as a static (not moving) and uniform background. This charge background generates a static field, which shifts the energy of our electron. Because we only consider the average field, this approximation is also known as the **mean-field approximation**.

$$\Sigma_F(k, i\omega_n) = - \int \frac{d\mathbf{q}}{(2\pi)^3} V(k-q) \langle n(\mathbf{q}) \rangle = -4\pi \int d\mathbf{q} \frac{e^2}{\left( \left| \vec{k} - \vec{q} \right| \right)^2} \langle n(\mathbf{q}) \rangle \quad (2.50)$$

Here, the minus sign comes from the anti-commutator.

At  $T = 0$ ,  $\langle n(\mathbf{q}) \rangle = 1$  for  $q < k_F$  and  $\langle n(\mathbf{q}) \rangle = 0$  for  $q > k_F$ , so

$$\Sigma_F(k, i\omega_n) = -4\pi e^2 \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\left( \left| \vec{k} - \vec{q} \right| \right)^2} \langle n(\mathbf{q}) \rangle = -4\pi e^2 \int_{q < k_F} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\left( \left| \vec{k} - \vec{q} \right| \right)^2} \quad (2.51)$$

Let's assign the direction of  $\vec{k}$  to be the z direction

$$\begin{aligned} \Sigma_F(k, i\omega_n) &= -4\pi e^2 \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\left( \left| \vec{k} - \vec{q} \right| \right)^2} \langle n(\mathbf{q}) \rangle = \\ &= -4\pi e^2 \int_0^{k_F} \frac{q^2 d\mathbf{q}}{(2\pi)^3} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \frac{1}{(k-q\cos\theta)^2 + q^2\sin^2\theta} = -8\pi^2 e^2 \int_0^{k_F} \frac{dq}{(2\pi)^3} \int_{-1}^1 d\cos\theta \frac{q^2}{k^2 + q^2 - 2kq\cos\theta} = \\ &= -\frac{e^2}{\pi} \int_0^{k_F} dq \left( -\frac{q}{2k} \right) \ln \left| \frac{k^2 + q^2 - 2kq}{k^2 + q^2 - 2kq} \right| = -\frac{e^2}{\pi} \int_0^{k_F} dq \frac{q}{k} \ln \left| \frac{k+q}{k-q} \right| = \frac{e^2}{\pi} k_F \left( 1 + \frac{k_F^2 - k^2}{2kk_F} \ln \left| \frac{k_F+k}{k_F-k} \right| \right) \end{aligned} \quad (2.52)$$

Define  $x = k/k_F$

$$\Sigma_F(k, i\omega_n) = -\frac{e^2}{\pi} k_F \left( 1 + \frac{k_F^2 - k^2}{2kk_F} \ln \left| \frac{k_F+k}{k_F-k} \right| \right) = -\frac{e^2}{\pi} k_F \left( 1 + \frac{1-x^2}{2x} \ln \left| \frac{1+x}{1-x} \right| \right) \quad (2.53)$$

So our new particles have dispersion

$$\epsilon_{\text{eff}}(k) = \epsilon(k) + \Sigma(k) = \epsilon(k) - \frac{e^2}{\pi} k_F \left( 1 + \frac{k_F^2 - k^2}{2kk_F} \ln \left| \frac{k_F+k}{k_F-k} \right| \right) \quad (2.54)$$

So the effective velocity of our particles are

$$v(k) = \frac{d\epsilon_{\text{eff}}(k)}{dk} = \frac{d\epsilon(k)}{dk} - \frac{e^2}{\pi} k_F \frac{d}{dk} \left( 1 + \frac{k_F^2 - k^2}{2kk_F} \ln \left| \frac{k_F+k}{k_F-k} \right| \right) = v_{\text{non-interacting}}(k) - \frac{e^2 k_F}{\pi} \frac{2kk_F - (k^2 + k_F^2) \ln \left| \frac{k_F+k}{k_F-k} \right|}{2k^2 k_F} \quad (2.55)$$

The Fermi velocity is

$$v_F = \lim_{k \rightarrow k_F} v(k) = v_{F, \text{non-interacting}} - \frac{e^2}{\pi} \left( 1 - \ln \left| \frac{2k_F}{k_F - k} \right| \right)_{k \rightarrow k_F} \approx \lim_{k \rightarrow k_F} \ln \left| k_F - k \right| = \infty \quad (2.56)$$

Within the H-F approximation, the Fermi velocity is infinite, which is in direct contradiction to experimental results.

**Q: What's wrong?**

**A: The  $1/r$  interaction is a long-range interaction (non-local). It results in  $v(q=0) = \infty$ , which makes the diagrammatic expansion diverge.**

**Naively speaking, this means that particles with  $1/r$  interaction is always strongly-correlated (perturbation theory fails, no matter how weak the interaction is). However, it turns out that there is one important physics here, which actually makes the interaction short-range and weak, and thus the power-law expansion becomes sound again.**

## 2.5. Screening and the Random Phase approximation

To understand why the free theory (the 0th order approximation) works better than the Hartree-Fock approximation (the 1st order approximation), one needs to take into account a very important phenomenon: screening. This effect is covered by higher order diagrams, which we ignored in the Hartree-Fock approximation. (This is why the HF approximation fails).

### 2.5.1. Interactions between electrons in a solid

For two electrons in vacuum, the Coulomb's law tells us that the interaction energy between these two electrons is

$$V(r) = e^2 / r \tag{2.57}$$

**Q: What's the Coulomb's law inside a solid material?**

**A: It depends on whether the material is a metal or an insulator.**

For an insulator, we know that the Coulomb's law is

$$V(r) = \frac{1}{\epsilon} \frac{e^2}{r} \tag{2.58}$$

where  $\epsilon$  is the dielectric constant (permittivity) of the solid. Here, inside an insulator, the solid medium modifies the interaction strength by a factor  $1/\epsilon$ , but the potential still decays as  $1/r$ .

### 2.5.2. Screening

For a metal, the screening effect changes the  $1/r$  interaction into a short range interaction which decays exponentially.

**Q: What is the  $E$  field in a metal? (consider only the static  $E$  field)**

**A:  $E = 0$**

This is because we have Ohm's law

$$j = \sigma E \tag{2.59}$$

In equilibrium,  $j = 0$ , so  $E = 0$ . This means that the electric field is reduced dramatically in a solid, and therefore we expect the interaction strength between two charge particles to become much weaker in a metal. To see this, we consider two test charge with charge  $e$  inside a metal. Because electrons can move in a metal, electrons will redistribute after we introduces these two test charge. For a positive test charge, electrons will be attracted to the test charge and thus the total electric charge near the test charge (test charge+electrons) is almost zero. (Same will happen for negative charge). In other words, conducting electrons "screen" the test charge, which reduces the interaction dramatically. This effect can be described as

$$V(q, \omega) = \frac{V^{(0)}(q)}{\epsilon(q, \omega)} \tag{2.60}$$

Here,  $V(q, \omega)$  is the effective interactions that electrons inside a metal feel in the momentum-frequency space.  $V^{(0)}(q) = 4\pi e^2 / q^2$  is the Coulomb potential in vacuum. The denominator  $\epsilon(q, \omega)$  is the dielectric function. For insulators, the dielectric constant is the  $\omega \rightarrow 0$  and  $q \rightarrow 0$  limit of  $\epsilon(q, \omega)$ . For a metal, we will show below that  $\epsilon(q = 0, \omega = 0) = \infty$ , which makes the effective potential  $V(q, \omega)$  finite at  $q = 0$  and  $\omega = 0$ . If  $V(q, \omega)$  is finite at  $q = 0$  and  $\omega = 0$ , the effective interaction  $V(r, \omega = 0)$  decays exponentially as a function of  $r$  (instead of  $1/r$ ). In other words, in a metal the e-e interactions are short-range interactions (not  $1/r$ ). In H-F, we use the  $1/r$  interaction, instead of the correct effective interaction. And this is the reason why the theory fails.

### 2.5.3. $V(q, i\omega_n)$ and Dyson's equation

Here, we consider the following diagrams

$$\text{Double line} = \text{Single line} + \text{Single line with loop} + \text{Single line with two loops} + \text{Single line with three loops} + \dots \tag{2.61}$$

If we define effective interaction  $V(q, \omega)$  as

$$\text{Double dashed line} = \text{Single dashed line} + \text{Single dashed line with loop} + \text{Single dashed line with two loops} + \dots \tag{2.62}$$

and then all the diagrams above can be represented as a simple first order diagram.

$$\text{Double line} = \text{Single line} + \text{Single line with dashed loop} \tag{2.63}$$



This formula only contains the first order perturbation in terms of the effective interaction strength  $V$ , but it includes infinite diagrams in terms of  $V^{(0)}$ . The physical meaning here is, if you substitute the interaction  $V^{(0)}$  by the effective interaction  $V$ , then the Fock diagram contains a lot more higher order contributions. We can further improve the accuracy by including more and more terms in effective interaction  $V$ . Here, the strategy is, we treat interaction (dashed line) as a Green's function (for some bosonic particles), and then we plot all higher order corrections to the interaction strength. The sum gives us an effective interaction. When we compute any physical observables, e.g. the Fermion Green's function, we will not use the bare interaction  $V^{(0)}$ , but the effective interaction  $V$ . The physical meaning is that inside a metal, electrons will screen the Coulomb interaction, and thus effectively the interaction between electrons is  $V$ , instead of  $V^{(0)}$ . And please notice that here we didn't include any more diagram. It is just a way to organize the diagrams to make the calculation more efficient.

Here we compute  $V$  by considering diagrams shown above. This approximation (ignoring all other diagrams) are called the Random Phase Approximation (RPA).

$$V^{\text{RPA}}(q, i\omega_n) = V^{(0)}(q, i\omega_n) + V^{(0)}(q, i\omega_n) \Pi^{(0)}(q, i\omega_n) V^{(0)}(q, i\omega_n) + V^{(0)}(q, i\omega_n) \Pi^{(0)}(q, i\omega_n) V^{(0)}(q, i\omega_n) \Pi^{(0)}(q, i\omega_n) V^{(0)}(q, i\omega_n) \dots \quad (2.64)$$

Here, the leading order correction is  $\Pi^{(0)}$  is called a bubble. This name comes from the fact that the Feynman diagram for  $\Pi^{(0)}$  looks like a bubble.

$$\Pi^{(0)}(q, i\omega_n) = \text{bubble} = \int \frac{dk}{(2\pi)^d} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k, i\Omega_n) G^{(0)}(k+q, i\Omega_n + i\omega_n) \quad (2.65)$$

If we include all other diagram, we can get more accurate  $\Pi(q, i\omega_n)$ ,

$$V(q, i\omega_n) = V^{(0)}(q, i\omega_n) + V^{(0)}(q, i\omega_n) \Pi(q, i\omega_n) V^{(0)}(q, i\omega_n) + V^{(0)}(q, i\omega_n) \Pi^{(0)}(q, i\omega_n) V^{(0)}(q, i\omega_n) \Pi^{(0)}(q, i\omega_n) V^{(0)}(q, i\omega_n) \dots \quad (2.66)$$

where

$$\Pi(q, i\omega_n) = \text{bubble} + \text{bubble with vertical dashed line} + \text{bubble with horizontal dashed line} + \dots \quad (2.67)$$

Using Dyson's equation, we find

$$V(q, i\omega_n) = \frac{V^{(0)}(q, i\omega_n)}{1 - \Pi(q, i\omega_n) V^{(0)}(q, i\omega_n)} = \frac{1}{V^{(0)}(q, i\omega_n)^{-1} - \Pi(q, i\omega_n)} \quad (2.68)$$

Notice that

$$V(q, i\omega_n) = \frac{V^{(0)}(q, i\omega_n)}{1 - \Pi(q, i\omega_n) V^{(0)}(q, i\omega_n)} = \frac{V^{(0)}(q, i\omega_n)}{\epsilon(q, i\omega_n)} \quad (2.69)$$

This tells us that the dielectric function for a metal is

$$\epsilon(q, i\omega_n) = 1 - \Pi(q, i\omega_n) V^{(0)}(q, i\omega_n) = 1 - \frac{4\pi e^2}{q^2} \Pi(q, i\omega_n). \quad (2.70)$$

As will be shown later, in the limit of  $\omega \rightarrow 0$  and  $q \rightarrow 0$ ,  $\Pi(q, \omega)$  of a metal goes to a finite constant value

$$\lim_{q \rightarrow 0} \epsilon(q, \omega = 0) = O\left(\frac{1}{q^2}\right) = \infty \quad (2.71)$$

i.e. the dielectric constant in a metal is infinite. Since both  $V^{(0)}(q)$  and  $\epsilon(q, \omega = 0)$  diverge as  $1/q^2$  in the small  $q$ , the effective interaction is NOT singular at  $\omega = 0$  and  $q = 0$

$$V(q, \omega = 0) = \frac{V^{(0)}(q)}{\epsilon(q, \omega = 0)} = \frac{O(1/q^2)}{O(1/q^2)} = \text{finite at } q \rightarrow 0 \quad (2.72)$$

This tell us that screening effect helps us remove the singularity in the interaction strength at small  $q$ . This will cure all the problem we had in the HF approximation.

#### 2.5.4. The random phase approximation

Here we use the RPA, where we ignore all higher order term in  $\Pi(q, i\omega_n)$  beyond the bubble diagram.

$$\Pi(q, i\omega_n) = \Pi^{(0)}(q, i\omega_n) + O(V^1) \quad (2.73)$$

$$\Pi^{(0)}(q, i\omega_n) = \bigcirc = \int \frac{dk}{(2\pi)^d} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k, i\Omega_n) G^{(0)}(k+q, i\Omega_n + i\omega_n) \quad (2.74)$$

Once again, we compute the frequency sum using the Fermi distribution function

$$\frac{1}{\beta} \sum_{i\omega_n} g(i\omega_n) = \sum_{m \text{ outside } C} f(z_m) \text{Res}(g, z_m) \quad (2.75)$$

Here,  $g(i\Omega_n) = G^{(0)}(k, i\Omega_n) G^{(0)}(k+q, i\Omega_n + i\omega_n)$ .

$$g(z) = G^{(0)}(k, z) G^{(0)}(k+q, z+i\omega_n) = \frac{1}{z-\epsilon(k)} \frac{1}{z+i\omega_n-\epsilon(k+q)} \quad (2.76)$$

Obviously,  $g(z)$  has two singularity points:  $z_1 = \epsilon(k)$  and  $z_2 = \epsilon(k+q) - i\omega_n$ . Residue at  $z_1$  can be found using

$$\text{Res}[g(z), z_1] = \lim_{z \rightarrow z_1} g(z) (z - z_1) = \lim_{z \rightarrow z_1} \frac{1}{z - z_1} \frac{1}{z - z_2} (z - z_1) = \lim_{z \rightarrow z_1} \frac{1}{z - z_2} = \frac{1}{z_1 - z_2} \quad (2.77)$$

Similarly, we find that  $\text{Res}[g(z), z_2] = \frac{1}{z_2 - z_1}$ , therefore,

$$\frac{1}{\beta} \sum_{i\Omega_n} g(i\Omega_n) = \sum_{m \text{ outside } C} f(z_m) \text{Res}(g, z_m) = \frac{f(z_1)}{z_1 - z_2} + \frac{f(z_2)}{z_2 - z_1} = \frac{f(z_1) - f(z_2)}{z_1 - z_2} = \frac{f[\epsilon(k)] - f[\epsilon(k+q)]}{i\omega_n + \epsilon(k) - \epsilon(k+q)} \quad (2.78)$$

So

$$\Pi^{(0)}(q, i\omega_n) = \bigcirc = \int \frac{dk}{(2\pi)^d} \frac{1}{\beta} \sum_{i\Omega_n} G^{(0)}(k, i\Omega_n) G^{(0)}(k+q, i\Omega_n + i\omega_n) = \int \frac{dk}{(2\pi)^d} \frac{f[\epsilon(k)] - f[\epsilon(k+q)]}{i\omega_n + \epsilon(k) - \epsilon(k+q)} \quad (2.79)$$

This integral can be computed analytically at  $T = 0$  (see Mahan, page 437). Here, we will only look at the limit of  $\omega \rightarrow 0$  and  $q \rightarrow 0$ . At  $\omega = 0$  and  $q \rightarrow 0$

$$\Pi^{(0)}(q, \omega = 0) = \int \frac{dk}{(2\pi)^d} \frac{f[\epsilon(k)] - f[\epsilon(k+q)]}{\epsilon(k) - \epsilon(k+q)} = \int \frac{dk}{(2\pi)^d} \frac{df(\epsilon)}{d\epsilon} = - \int \frac{dk}{(2\pi)^d} \frac{df(\epsilon)}{d\mu} \quad (2.80)$$

Here, we used the fact that

$$\frac{d}{d\epsilon} \frac{1}{\exp[\beta(\epsilon - \mu)] + 1} = - \frac{d}{d\mu} \frac{1}{\exp[\beta(\epsilon - \mu)] + 1} \quad (2.81)$$

Therefore,

$$\Pi^{(0)}(q, \omega = 0) = - \int \frac{dk}{(2\pi)^d} \frac{df(\epsilon)}{d\mu} = - \frac{d}{d\mu} \int \frac{dk}{(2\pi)^d} f(\epsilon) = - \frac{dN}{d\mu} = -N(0) \quad (2.82)$$

Here,  $N = \int \frac{dk}{(2\pi)^d} f(\epsilon)$  is the total number of particles.  $\frac{dN}{d\mu}$  is the density of states  $N(0)$  at the Fermi level. At  $T = 0$ , for a Fermi liquid,

$$\text{Number of states} = 4\pi g \int \frac{k^2 dk}{(2\pi)^3} = \frac{g}{2\pi^2} \int k^2 dk = \frac{g}{2\pi^2} \int 2m\epsilon d\sqrt{2m\epsilon} = \frac{g}{\sqrt{2}\pi^2} m^{3/2} \int \sqrt{\epsilon} d\epsilon \quad (2.83)$$

So

$$N(0) = \frac{g}{\sqrt{2}\pi^2} m^{3/2} \sqrt{\epsilon_F} \quad (2.84)$$

And thus the dielectric function

$$\epsilon^{\text{RPA}}(q, i\omega_n) = 1 - \frac{4\pi e^2}{q^2} \Pi^{\text{RPA}}(q, i\omega_n) \quad (2.85)$$

In the  $\omega = 0$  and  $q \rightarrow 0$  limit

$$\lim_{q \rightarrow 0} \epsilon^{\text{RPA}}(q, \omega = 0) = 1 + \frac{4\pi e^2}{q^2} N(0) \quad (2.86)$$

One can define the Thomas-Fermi screening wave vector  $q_{\text{TF}}$

$$q_{\text{TF}}^2 = \frac{6\pi e^2 n}{\epsilon_F} \quad (2.87)$$

here  $n$  is the density of electron and  $n = \frac{2}{3} \epsilon_F N(0)$

$$n = \int_0^{\epsilon_F} d\epsilon N(\epsilon) = \frac{g}{\sqrt{2}\pi^2} m^{3/2} \int d\epsilon \sqrt{\epsilon} = \frac{g}{\sqrt{2}\pi^2} \frac{2}{3} (m\epsilon_F)^{3/2} = \frac{2}{3} \epsilon_F \frac{g}{\sqrt{2}\pi^2} m^{3/2} \sqrt{\epsilon_F} = \frac{2}{3} \epsilon_F N(0) \quad (2.88)$$

Therefore,

$$q_{\text{TF}}^2 = \frac{6\pi e^2 n}{\epsilon_F} = \frac{6\pi e^2}{\epsilon_F} \frac{2}{3} \epsilon_F N(0) = 4\pi e^2 N(0) \quad (2.89)$$

So

$$\lim_{q \rightarrow 0} \epsilon^{\text{RPA}}(q, \omega = 0) = 1 + \frac{q_{\text{TF}}^2}{q^2} \quad (2.90)$$

In other words, the dielectric constant for a metal is  $\infty$ , which diverges as  $1/q^2$  in the  $q \rightarrow 0$  limit. As for the effective interaction,

$$V(q, \omega) = \frac{V^{(0)}(q, \omega)}{\epsilon(q, \omega)} = \frac{\frac{4\pi e^2}{q^2}}{\epsilon(q, \omega)} \quad (2.91)$$

At  $\omega = 0$ , because both denominator and numerator diverge as  $1/q^2$  in the small  $q$  limit,  $V(q = 0, \omega = 0)$  is finite. This is very different from the bare interaction  $V^{(0)}(q)$ , which is singular at  $q = 0$ .

$$\lim_{q \rightarrow 0} V(q, \omega = 0) = \frac{\frac{4\pi e^2}{q^2}}{1 + \frac{q_{\text{TF}}^2}{q^2}} = \frac{4\pi e^2}{q^2 + q_{\text{TF}}^2} \quad (2.92)$$

If we convert this formula into the real space, the small  $q$  limit of  $V(q)$  give us the large  $r$  limit of  $V(r)$ .

$$V(r, \omega = 0) = \int \frac{d^d q}{(2\pi)^d} \exp(i\vec{q} \cdot \vec{r}) V(q, \omega = 0) \quad (2.93)$$

When  $r$  is large enough, the small  $q$  part of the integral dominates the whole integral (large  $q$  and large  $r$  means very rapid oscillation in  $\exp(i\vec{q} \cdot \vec{r})$ , which is zero after integral). So at long distance,

$$\begin{aligned} V(r, \omega = 0) &\approx \int \frac{d^3 q}{(2\pi)^3} \exp(i\vec{q} \cdot \vec{r}) \frac{4\pi e^2}{q^2 + q_{\text{TF}}^2} = \frac{1}{4\pi^2} \int_0^\infty q^2 dq \int_0^\pi d\theta \sin\theta \exp(iqr \cos\theta) \frac{4\pi e^2}{q^2 + q_{\text{TF}}^2} = \\ &\frac{1}{4\pi^2} \int_0^\infty q^2 dq \frac{2 \sin(qr)}{qr} \frac{4\pi e^2}{q^2 + q_{\text{TF}}^2} = \frac{e^2}{i\pi r} \left[ \int_0^\infty \frac{q \exp(iqr)}{q^2 + q_{\text{TF}}^2} dq - \int_0^\infty \frac{q \exp(-iqr)}{q^2 + q_{\text{TF}}^2} dq \right] = \\ &\frac{e^2}{i\pi r} \left[ \int_{-\infty}^\infty \frac{q \exp(iqr)}{q^2 + q_{\text{TF}}^2} dq \right] = \frac{e^2}{i\pi r} 2\pi i \text{Res} \left[ \frac{q \exp(iqr)}{q^2 + q_{\text{TF}}^2}, q = iq_{\text{TF}} \right] = \frac{e^2}{r} \exp(-q_{\text{TF}} r) \end{aligned} \quad (2.94)$$

The interaction decays exponential at long distance with decay rate  $q_{\text{TF}}$ .

### 2.5.5. Fermion self-energy within the random phase approximation

For a Fermi liquid, the RPA shows pretty good agreement with experiments and is the minimum approximation to describe an interacting Fermi liquid.

$$G(k, i\omega_n) = \frac{1}{i\omega_n - \epsilon(k) - \Sigma^{\text{RPA}}(k, i\omega_n)} \quad (2.95)$$

where

$$\Sigma^{\text{RPA}}(k, i\omega_n) = \text{---} \text{---} \text{---} \quad (2.96)$$

and the double dashed line is the effective interaction computed using the RPA approximation. We will not go into the detailed calculation. Instead, we just present the result here. After analytic continuation, we get

$$G(k, \omega) = \frac{1}{\omega - \epsilon(k) - \Sigma^{\text{RPA}}(k, \omega)} \quad (2.97)$$

For a Fermi liquid at T=0 K, we focus on  $k \sim k_F$  and  $z \sim 0$ .

Near  $k_F$ ,  $\epsilon(k) = \mu + v_F(k - k_F) + O(k - k_F)^2$ .

$$G^{(0)}(k, \omega) = \frac{1}{\omega - v_F(k - k_F) - \mu + O(k - k_F)^2} \quad (2.98)$$

Using this  $G^{(0)}$ , we can compute  $\Sigma^{\text{RPA}}$ . For  $\Sigma^{\text{RPA}}(k, z)$ , we expand it around  $k \sim k_F$  and  $z \sim 0$ .

$$G(k_F, \omega) = \frac{1}{\omega - v(k - k_F) - \mu - \Sigma^{\text{RPA}}(k_F, 0) - a(k - k_F) + b\omega + O(k - k_F)^2 + O(z)^2} \quad (2.99)$$

It is important to notice that  $b$  is real and positive (the  $\omega^2$  term has a complex coefficient). The imaginary part of  $\Sigma^{\text{RPA}}$  will only appear at order  $\omega^2$  and above. Therefore, in the limit  $k \sim k_F$  and  $\omega \sim 0$ , the imaginary part of the self-energy is much smaller than the real part, and thus we can ignore the imaginary part in the low-energy limit  $\omega \rightarrow 0$ . Therefore,

$$G^{\text{RPA}}(k_F, \omega) = \frac{Z}{\omega - \tilde{v}(k - k_F) - \tilde{\mu}} + O(k - k_F)^2 + O(z)^2 \quad (2.100)$$

here  $Z = 1/(1 + b) < 1$ .  $\tilde{v} = \frac{v+a}{1+b}$  and  $\tilde{\mu} = \frac{\mu + \Sigma^{\text{RPA}}(k_F, 0)}{1+b}$ . Notice that in the limit of  $k \rightarrow k_F$  and  $\omega \rightarrow 0$ ,  $G^{\text{RPA}}(k_F, \omega)$  looks just like the Green's function for a free fermion, with a renormalized Fermi velocity and a renormalized chemical potential. This is why we can treat electrons as "free" particles in a Fermi liquid, although the interaction is pretty strong. In addition, we have an extra factor  $Z$ , which is known as the Fermi residue

$$G^{\text{RPA}}(k_F, z) = Z G^{\text{Free}}(k_F, z) \quad (2.101)$$

This extra factor is  $0 < Z < 1$ . The physical meaning is that now at  $k_F$  the jump in occupation number  $n(k_F - \delta) - n(k_F + \delta) = Z$  not 1.

The imaginary part of the self-energy (inverse life-time of our particle) is proportional to  $\omega^2$ , which means that at zero frequency and at  $k = k_F$ , our particles has infinite life time.

## 2.5.6. Summary

In Landau's Fermi liquid theory, we can consider the low-energy excitations (near the Fermi surface) in an interacting Fermi liquid as quasi-particles. In the limit  $\omega \rightarrow 0$ , the life time of our quasi-particles diverge and they behave just like free particles.

These quasi-particles carry the same charge and spin as an electron. However, the dispersion (Fermi velocity) and the chemical potential of these quasi-particles are renormalized by interactions. In addition, interaction reduces the Fermi residue from 1 to some finite value  $0 < Z < 1$ .

This is why we can treat metals as if the electrons are free particles, although the interaction is in fact pretty strong.