

Economics 431
Homework 4
Due Wed July 25

Part I A variant of Stackelberg game

There are two firms. Demand is linear, $p = A - BQ$, and marginal cost is constant and equal to c . Suppose that firm 1 moves first and chooses either monopoly output (M), $q_M = \frac{A-c}{2B}$, or Cournot output (C), $q_c = \frac{A-c}{3B}$. Firm 2 *observes* the choice made by firm 1 and chooses one of two output levels: either Stackelberg output (S), $\frac{1}{2}q_M = \frac{A-c}{4B}$ or Cournot output (C), $q_c = \frac{A-c}{3B}$. Let $A = 20$, $B = 1$ and $c = 8$.

- a) How many strategies does each player have and what are they?
- b) Draw the tree of this game and write down its payoff matrix
- c) Find all the Nash equilibria of this game.
- d) Find all the subgame perfect equilibria of this game.

Part II. Indivisible plants and entry deterrence

Suppose that firms can only build an integer number of plants 0, 1, 2,... and each plant produces one unit of output. In other words, output can only be adjusted in increments of one unit. Building n plants costs $3.5n$. Once the plant is installed, variable cost of production is zero. Market price is

$$p = 6 - K$$

where K is industry capacity (total number of plants).

- a) Show that a monopolist will choose to install just one plant
- b) Consider a game between two firms who simultaneously choose their number of plants K_1 and K_2 . Show that the Cournot equilibrium is for each firm to build just one plant. (Hint: represent this game by a payoff matrix)
- c) Suppose that firm 1 builds before firm 2. Show that in a subgame perfect equilibrium firm 1 builds two plants and firm 2 stays out. (Hint: represent this game by a tree).

Part III Practice exams

Winter 2002 Second Midterm, Questions 4, 5
Fall 2003 Second Midterm, Question 5

Part IV End of Chapter and practice problems

End of chapter 11.3
Practice problem 12.3

Economics 431
Homework 4
Answer key

Part I A variant of Stackelberg game

There are two firms. Demand is linear, $p = A - BQ$, and marginal cost is constant and equal to c . Suppose that firm 1 moves first and chooses either monopoly output (M), $q_M = \frac{A-c}{2B}$, or Cournot output (C), $q_c = \frac{A-c}{3B}$. Firm 2 *observes* the choice made by firm 1 and chooses one of two output levels: either Stackelberg output (S), $\frac{1}{2}q_M = \frac{A-c}{4B}$ or Cournot output (C), $q_c = \frac{A-c}{3B}$.

a) How many strategies does each player have and what are they?

Player 1 has 2 strategies: $S_1 = \{M, C\}$

Player 2 has 4 strategies

Verbal description	What to do when see M	What to do when see C
Always S	S	S
C only if sees C	S	C
C only if sees M	C	S
Always C	C	C

b) Draw the tree of this game and write down its payoff matrix

First of all, we must compute the payoffs from each possible combination of outputs (possible outcomes of the game)

Suppose

$$q_1 = \frac{1}{n} \frac{A-c}{B}$$

$$q_2 = \frac{1}{m} \frac{A-c}{B}$$

Then

$$p - c = A - c - Bq_1 - Bq_2 = \left(1 - \frac{1}{n} - \frac{1}{m}\right) (A - c)$$

$$\pi_1(q_1, q_2) = (p - c) q_1 = \left(1 - \frac{1}{n} - \frac{1}{m}\right) \frac{1}{n} \frac{(A - c)^2}{B}$$

$$\pi_2(q_1, q_2) = (p - c) q_2 = \left(1 - \frac{1}{n} - \frac{1}{m}\right) \frac{1}{m} \frac{(A - c)^2}{B}$$

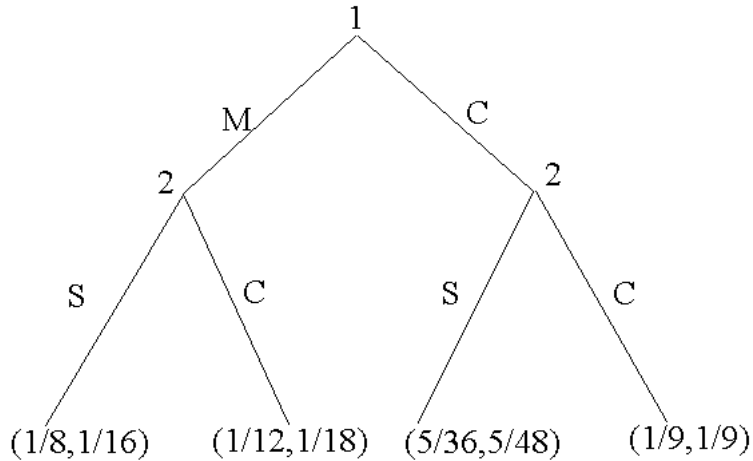


Figure 1:

Outcome	$p - c$	π_1	π_2
(M, S)	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
(M, C)	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{18}$
(C, S)	$\frac{5}{12}$	$\frac{5}{36}$	$\frac{5}{48}$
(C, C)	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$

Payoff matrix

	SS	SC	CS	CC
M	$\frac{1}{8}, \frac{1}{16}$	$\frac{1}{8}, \frac{1}{16}$	$\frac{1}{12}, \frac{1}{18}$	$\frac{1}{12}, \frac{1}{18}$
C	$\frac{5}{36}, \frac{5}{48}$	$\frac{1}{9}, \frac{1}{9}$	$\frac{5}{36}, \frac{5}{48}$	$\frac{1}{9}, \frac{1}{9}$

c) Find all the Nash equilibria of this game.

	SS	SC	CS	CC
M	$\frac{1}{8}, \frac{1}{16}$	$\frac{1}{8}, \frac{1}{16}$	$\frac{1}{12}, \frac{1}{18}$	$\frac{1}{12}, \frac{1}{18}$
C	$\frac{5}{36}, \frac{5}{48}$	$\frac{1}{9}, \frac{1}{9}$	$\frac{5}{36}, \frac{5}{48}$	$\frac{1}{9}, \frac{1}{9}$

d) Find all the subgame perfect equilibria of this game.

There are two Nash equilibria: (M, SC) and (C, CC) .

The first one, (M, SC) , is subgame perfect. If we are in the left subgame, this implies that firm 1 plays M . We have to check whether SC is a best response to M . This strategy tells firm 2 to play S after M , which is the best response.

If we are in the right subgame, this implies that firm 1 plays C . We have to check whether SC is a best response to C . This strategy tells firm 2 to play C after C , which is the best response.

If we are in the initial node (the subgame which is the game itself) and firm 2 plays SC , M is the best response.

We have established that every player plays the best response to the other player's strategy in every subgame. Therefore, (M, SC) is subgame perfect by definition.

In general, any equilibrium that you find by backwards induction will be a subgame perfect Nash equilibrium.

The second equilibrium, (C, CC) , is not subgame perfect. When we are in the left subgame, firm 1 plays M but the strategy CC tells firm 2 to play C which is not a best response.

Part II. Indivisible plants and entry deterrence

Suppose that firms can only build an integer number of plants 0, 1, 2,... and each plant produces one unit of output. In other words, output can only be adjusted in increments of one unit. Building n plants costs $3.5n$. Once the plant is installed, variable cost of production is zero. Market demand is

$$p = 6 - K$$

where K is industry capacity (total number of plants).

a) Show that a monopolist will choose to install just one plant

Note that if the number of plants is 3 price is already below the marginal cost of installing the plant.

Number	Price	Total cost	Profit
0	6	0	0
1	5	3.5	1.5
2	4	7	1

b) Consider a Cournot game between two firms who simultaneously choose their number of plants K_1 and K_2 . Show that the Nash equilibrium is for each firm to build just one plant.

Because we know that the total number of plants in the industry will be at most two, each firm effectively has two strategies: 0 plants or 1 plant. The game can then be represented by a payoff matrix:

		Firm 2	
		0 plants	1 plant
Firm 1	0 plants	0, 0	0, 1.5
	1 plant	1.5, 0	0.5, 0.5

Installing 1 plant is a dominant strategy for both firms in this 2 x 2 game. The unique Nash equilibrium is (1 plant, 1 plant).

c) Suppose that firm 1 builds before firm 2. Show that in a Stackelberg equilibrium firm 1 builds two plants and firm 2 stays out. The game tree is represented on the figure. Firm 1 has three strategies that give it non-negative profit: 0, 1 or 2 plants. If it builds 0, firm 2 will enjoy monopoly power. Firm 2 will then choose to build 1 plant and will have the profit of 1.5. If firm 1 builds two plants, firm 2 cannot build any more plants, because it results in negative profit, so firm 2 builds 0 plants. Finally, if firm 1 builds just one plant, firm 2 will build another one, which will earn each firm a profit of 0.5. In a subgame perfect equilibrium, firm 1 will choose to build 2 plants, because it gives the highest profit of 1.

Winter 2002 Second Midterm

4) (30 points) Consider a senate race game. Player 1 (the incumbent senator) decides whether to launch an ad campaign or not (A, N). Player 2 (challenger) *simultaneously and independently* decides whether to enter the race or stay out (I, O). The payoffs from each action combination are given by (Player 1's payoff is the first entry in each cell).

		Player 2	
		I	O
Player 1	A	1, 1	3, 3
	N	2, 4	4, 2

a) **(8 points)** Find all the Nash equilibria of this game. Which player(s), if any, have a dominant strategy?

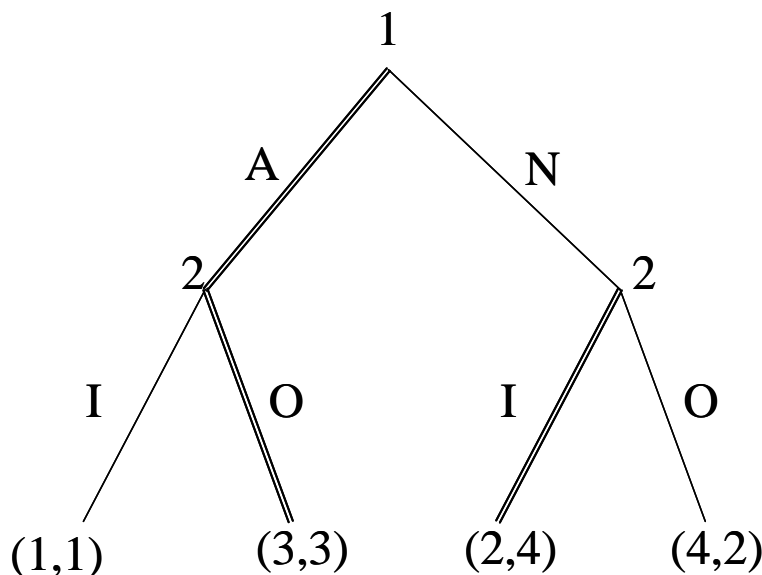
		Player 2	
		I	O
Player 1	A	1, 1	3, <u>3</u>
	N	<u>2, 4</u>	<u>4, 2</u>

The only Nash equilibrium is (N, I) . Player 1 has a dominant strategy, player 2 does not.

b) **(10 points)** Now suppose that Player 1 moves first by choosing either A or N . Player 2 *observes* player 1's action and then chooses I or O . For every action combination, the players' payoffs are the same as in the above payoff matrix. Draw a tree of this new game. How many strategies does player 1 have and what are they? How many strategies does player 2 have and what are they? Find all the subgame perfect equilibria of this game.

$$S_1 = \{A, N\}$$

$$S_2 = \{II, IO, OI, OO\}$$



Subgame perfect equilibrium is (A, OI) .

c) **(12 points)** Now find *all* the Nash equilibria of the game in part b). Are there any Nash equilibria that are not subgame perfect? Which Nash equilibrium is more believable? Explain. Does Player 1 receive a higher payoff in a static game of part a) or in a dynamic game of part b)? Why do you think this difference arises?

	<i>II</i>	<i>IO</i>	<i>OI</i>	<i>OO</i>
<i>A</i>	1, 1	1, 1	<u>3, 3</u>	3, 3
<i>N</i>	<u>2, 4</u>	<u>4, 2</u>	2, <u>4</u>	<u>4, 2</u>

We can see that there are two Nash equilibria: (A, OI) and (N, II) . In the first one, player 2's strategy prescribes him to always do what's best for him, no matter which node of the game he is in. The second strategy involves bluff, or non-credible threat. Player 2 says, "I will always enter, even if it hurts me". Knowing this, it is in player 1's best interest not to advertise. Player 2 expects player 1 not to advertise, and so he knows the game will never reach the point where his bluff is called, so *II* is a best response to *N*.

Which Nash equilibrium is more believable? In the second equilibrium, player 2 makes a threat that he has no incentive to carry out. So why should player 1 believe him? "I don't care what you say you'll do, I will just advertise, and we'll see what happens". Every time a player plays a strategy he himself has no incentive to stick to, we have this credibility problem.

Player 1 gets a higher payoff of 3 in the subgame perfect equilibrium of the dynamic game. By moving first, he is able to take advantage of the different responses of player 2 to his different actions, and therefore can better manipulate the outcome of the game.

5) (30 points) Market demand is given by $p = 12 - Q$. There are two firms: the incumbent firm (I) and the entrant firm (E). Incumbent moves first by choosing quantity q_I from the interval $[0, 4]$. The entrant observes q_I and decides whether or not to enter and how much to produce if he enters (q_E). There is *no fixed cost of entry*. If the entrant decides to stay out, his profit is zero and the incumbent enjoys a monopoly position. Suppose that both incumbent and entrant have identical marginal costs equal to $c = 8$.

a) **(10 points)** What is the subgame perfect equilibrium of this game? What are the quantities produced by the incumbent and entrant? What are their profits?

The entrant always enters, because he can earn at least zero profit no matter what quantity the incumbent produces. Then this is a Stackelberg game.

$$q_I = q_M = \frac{A - c}{2B} = 2$$

$$q_E = \frac{A - c}{4B} = 1$$

$$\pi_I = \frac{\pi_M}{2} = \frac{(A - c)^2}{8B} = 2$$

$$\pi_E = \frac{\pi_M}{4} = \frac{(A - c)^2}{16B} = 1$$

b) **(6 points)** What is the minimum quantity that must be produced by the incumbent to deter entry (to make entry unprofitable)? Will the incumbent ever try to deter entry by increasing quantity?

The dominant strategy for the entrant is

$$q_E^*(q_I) = \frac{A - c}{2B} - \frac{q_I}{2}$$

Whenever this strategy tells the entrant to produce positive quantity, the entrant is able to earn positive profit and therefore chooses to enter. Then the only quantity that can deter entry is

$$q_I = \frac{A - c}{B} = 4.$$

The incumbent will never choose this quantity because it yields zero profit, he can do better with lower quantities.

Now suppose that before the game begins the incumbent can purchase new equipment: he can either stay with the old equipment that leaves his marginal cost at $c = 8$ or spend an additional amount $K = 5$ on new equipment which cuts his marginal

cost to $c_L = 6$. If the incumbent does not purchase new equipment, his marginal cost stays at $c = 8$, and the game proceeds as in part *a*).

c) **(14 points)** If the incumbent had purchased new equipment and anticipates entry, what quantity does he produce? Will he deter entry? What is the incumbent's payoff? At the beginning of the game, will the incumbent choose to purchase new equipment? (Hint: does the entrant's dominant strategy change in any way when the incumbent's marginal cost changes?)

The entrant's dominant strategy depends on the *entrant's* marginal cost, not the incumbent's. It is still

$$q_E^*(q_I) = \frac{A - c}{2B} - \frac{q_I}{2} = 2 - \frac{q_I}{2}$$

The incumbent chooses q_I so that it maximizes

$$\begin{aligned} \max_{q_I} (A - c_L - Bq_I - Bq_E^*(q_I)) \cdot q_I &= \\ &= \max_{q_I} \left(A - c_L - Bq_I - B \left(\frac{A - c}{2B} - \frac{q_I}{2} \right) \right) \cdot q_I = \\ &= \max_{q_I} \left(12 - 6 - q_I - \left(\frac{12 - 8}{2} - \frac{q_I}{2} \right) \right) \cdot q_I = \\ &= \max_{q_I} \left(4 - \frac{q_I}{2} \right) \cdot q_I \\ &q_I^* = 4, q_E^* = 0 \end{aligned}$$

Entry is deterred. The incumbent has a payoff of

$$\pi_I = (p - c_L) \cdot q_I^* = (12 - q_I^* - c_L) \cdot q_I^* = 2 \cdot 4 = 8.$$

At the beginning of the game, if the incumbent purchases new equipment, it involves no up front spending and leaves him with a payoff of 2. If he buys new equipment, he spends 5 but gets a profit of 8, leaving him with a net payoff of 3. The incumbent chooses to buy new equipment.

Fall 2003 Second Midterm

5) (20 points) Consider the following sequential game of entry deterrence on a market with linear demand given by

$$p = 9 - Q$$

The incumbent (firm 1) first chooses whether or not to invest in new technology. The investment costs K . If the incumbent invests, his marginal cost is $c_L = 1$, and if he does not invest, it is $c_H = 6$. Then the incumbent and the entrant (firm 2), whose marginal cost is $c_H = 6$ play a Cournot quantity game. The profit functions are

$$\pi_1 = \begin{cases} (A - c_L - B(q_1 + q_2))q_1 - K & \text{if invests} \\ (A - c_H - B(q_1 + q_2))q_1 & \text{if does not invest} \end{cases}$$

$$\pi_2 = (A - c - B(q_1 + q_2))q_2$$

a) **(10 points)** Show that if the incumbent invests, the Nash equilibrium of the subsequent Cournot game has the entrant produce zero. How much does the incumbent produce if the entrant produces zero? (Hint: will the entrant be better off producing zero if making $q_2 > 0$ drops the price below his marginal cost?)

If the entrant produces zero, the incumbent's best response is to produce monopoly output that corresponds to his marginal cost c_L

$$q_M = \frac{A - c_L}{2B} = 4$$

If at this output level the market price is below c_H ,

$$p = \frac{A + c_L}{2} = 5 < 6 = c_H,$$

then the entrant will have a negative profit from any $q_2 > 0$. Therefore, entrant chooses to produce zero as a best response. The strategies that are best responses to each other constitute a Nash equilibrium.

b) **(10 points)** What are the equilibrium profits of the two firms if the incumbent does not invest? Find all values of K for which the incumbent will choose to invest in new technology.

If the incumbent does not invest, both firms have equal marginal costs c_H , and they each get

$$\pi_1 = \pi_2 = \frac{1}{9} \frac{(A - c_H)^2}{B} = 1$$

If the incumbent invests, he gets

$$\frac{1}{4} \frac{(A - c_L)^2}{B} - K = 16 - K$$

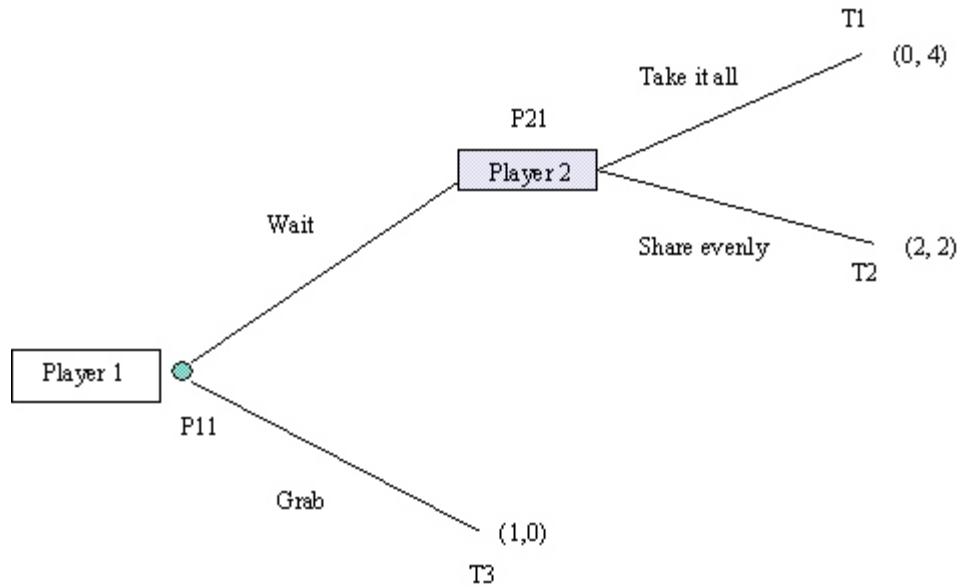
The incumbent invests if $16 - K > 1$, i.e.

$$K < 15.$$

Problem 11.3

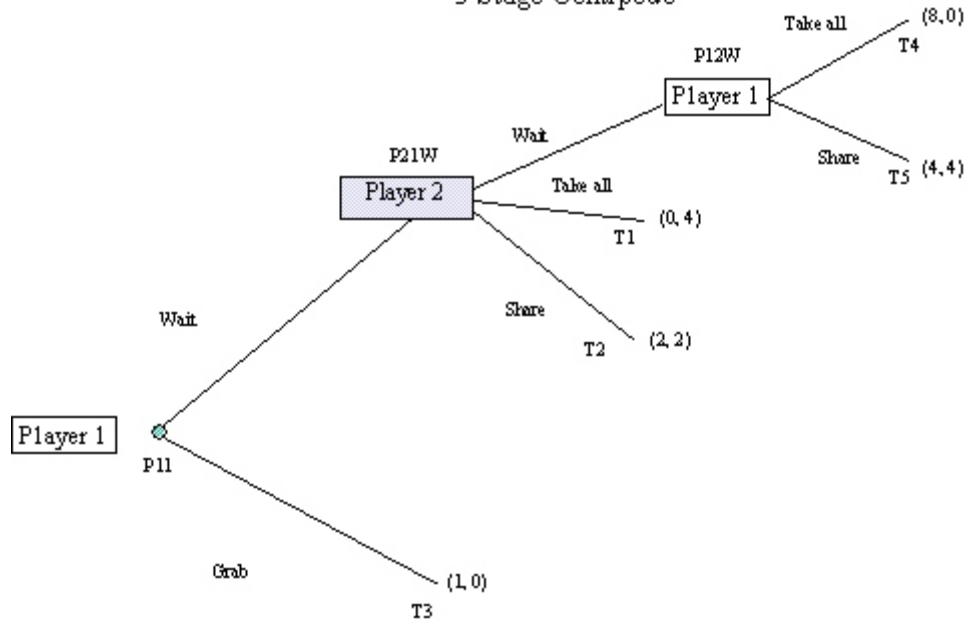
(a)

Centipede Game Tree



(b) The strategy of splitting the money is never an equilibrium since once the game reaches the point P21, the optimal strategy for the Player 2 is to take the entire \$4. Because Player 1 knows this will be the outcome at P2, Player 1 will always choose “Grab” and the outcome will be T3 with Player 1 getting \$1 and Player 2 getting nothing.

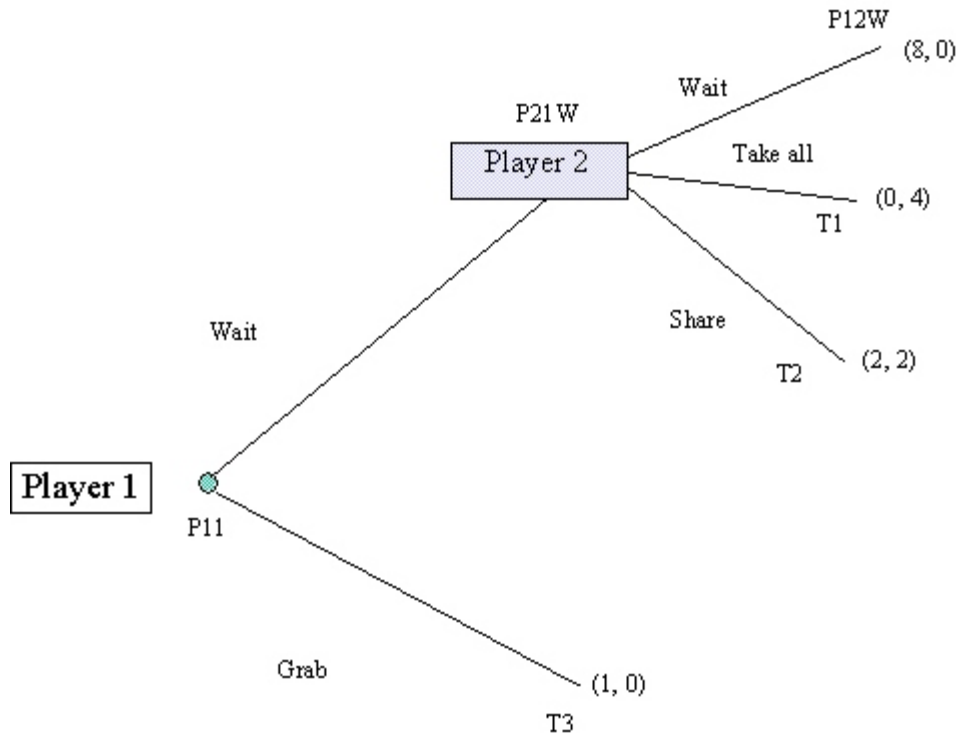
Game Tree
3 Stage Centipede



(c)

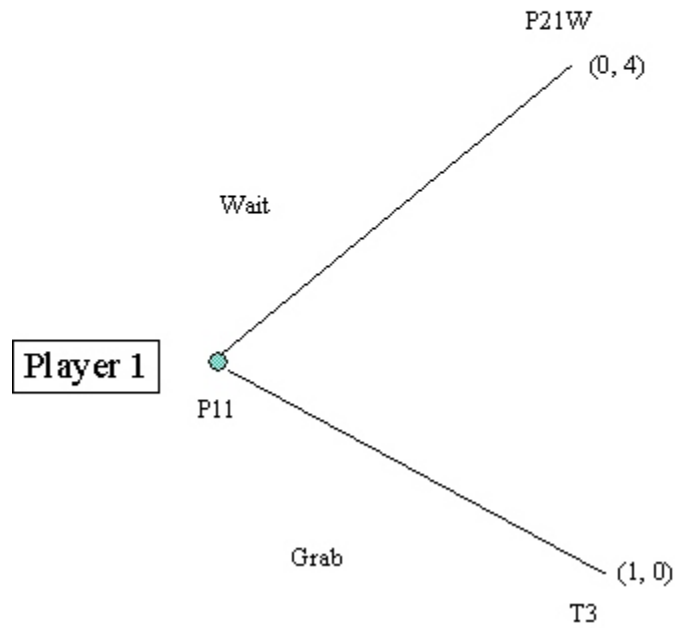
It is clear that in the third stage Player 1 will choose to keep all the money. Thus we can eliminate this choice from the tree and consider the new game with the final node removed (pruned). This will give

Game Tree
3 Stage Centipede



It is now obvious that Player 2 will choose to take it all at node P21W. Thus we can eliminate this node from the tree and replace it with the payoffs to both players when Player 2 chooses to take it all. This will give

Game Tree 3 Stage Centipede



It is now clear that Player 1 will grab the money at the initial node and the final payoff will be (1,0).