SMALL CATEGORIES WITHOUT LOOPS AND COMPLEXES OF GROUPS

MATTHEW STEVENSON

Abstract. This is the final project for Higher Algebra 1, submitted to Prof. Mikaël Pichot in the fall of 2013. We present an introduction to small categories without loops (or scwols) and complexes of groups following the treatment of [1], which generalizes Bass-Serre theory to higher dimensions.

1. Small Categories Without Loops

1.1. Definitions and Examples.

Definition 1. A *small category without loops* (scwol) $\mathcal{X}$ consists of two sets, the vertices $V(\mathcal{X})$ and the edges $E(\mathcal{X})$, along with the initial and terminal vertex maps $i, t\colon E(\mathcal{X}) \to V(\mathcal{X})$. Further, let $E^{(2)}(\mathcal{X})$ be the composable edges i.e. pairs $(a, b) \in E(\mathcal{X}) \times E(\mathcal{X})$ such that $i(a) = t(b)$; then, we also have the composition map $E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$ given by $(a, b) \mapsto ab$. We require that it satisfies the following conditions:

(1) For $(a, b) \in E^{(2)}(\mathcal{X})$, $i(ab) = i(b)$ and $t(ab) = t(a)$.
(2) For $a, b, c \in E(\mathcal{X})$ such that $i(a) = t(b)$ and $i(b) = t(c)$, $(ab)c = a(bc)$.
(3) There does not exist $a \in E(\mathcal{X})$ such that $i(a) = t(a)$.

Definition 2. Given a scwol $\mathcal{X}$, a *subscwol* $\mathcal{X}'$ consists of subsets $V(\mathcal{X}') \subseteq V(\mathcal{X})$ and $E(\mathcal{X}') \subseteq E(\mathcal{X})$, along with the following closure conditions on the maps:

(1) If $a \in E(\mathcal{X}')$, then $i(a), t(a) \in V(\mathcal{X}')$.
(2) If $a, b \in E(\mathcal{X}')$ are such that $i(a) = t(b)$, then $ab \in E(\mathcal{X}')$.

Given $x, y \in V(\mathcal{X})$, we declare $x \sim y$ if there exists an edge $e \in E(\mathcal{X})$ such that $i(e) = x$ and $t(e) = y$. The equivalence classes of $V(\mathcal{X})$ are called the *connected components* of $\mathcal{X}$, and $\mathcal{X}$ is said to be *connected* if there is only one equivalence class. It follows that any scwol can be written as a disjoint union of connected subscwols.

Example 3. A poset $(X, <)$ can be realized as a scwol $\mathcal{X}$ as follows: let $V(\mathcal{X}) = X$ and $E(\mathcal{X})$ consist of pairs $(x, y) \in X^2$ such that $x < y$. In this case, the initial and terminal vertex maps are given by $i(x, y) = y$ and $t(x, y) = x$; the composition is defined by $(x, y)(y, z) := (x, z)$.

One can realize a scwol $\mathcal{X}$ as the set of arrows of a small category: $V(\mathcal{X})$ consists of the identity arrows and $E(\mathcal{X})$ consists of the non-identity arrows. In this manner, subscwols correspond to subcategories.

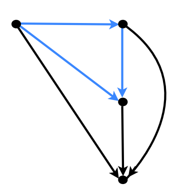

Figure 1. A scwol represented pictorially as a directed graph; the blue edges determine a subscwol.

Date: December 23, 2013.

1.2. **Morphisms and Group Actions.**

Definition 4. Given scwols $\mathcal{X}$ and $\mathcal{Y}$, a *morphism* $f\colon \mathcal{X} \to \mathcal{Y}$ is a map such that $f(V(\mathcal{X})) \subseteq V(\mathcal{Y})$ and $f(E(\mathcal{X})) \subseteq E(\mathcal{Y})$ that satisfies the following conditions:

1. For $a \in E(\mathcal{X})$, $i(f(a)) = f(i(a))$ and $t(f(a)) = f(t(a))$.
2. For $(a,b) \in E^{(2)}(\mathcal{X})$, $f(ab) = f(a)f(b)$.
3. For $x \in V(\mathcal{X})$, the restriction of f is a bijection

$$\{a \in E(\mathcal{X})\colon i(a) = x\} \longrightarrow \{b \in E(\mathcal{Y})\colon i(b) = f(x)\}. \tag{1.1}$$

Remark that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of scwols corresponds to a functor $\mathcal{X} \to \mathcal{Y}$ of categories, so we will say that the morphism f is an *isomorphism* if the corresponding functor is an isomorphism.

Definition 5. Given a group G and a scwol $\mathcal{X}$, group action $G \curvearrowright \mathcal{X}$ is a morphism $G \to \mathrm{Aut}(\mathcal{X})$ such that for $a \in E(\mathcal{X})$ and $g \in G$,

1. $g \cdot i(a) \neq t(a)$,
2. if $g \cdot i(a) = i(a)$, then $g \cdot a = a$.

The first condition stipulates that the action of the group cannot construct loops, and the second implies that if $g \in G$ stabilizes the initial vertex $i(a)$ of the edge $a \in E(\mathcal{X})$, then it must stabilize the edge a as well. Consequently, the quotient by the group action is a scwol in the obvious way: let $V(\mathcal{X}/G) = V(\mathcal{X})/G$ and $E(\mathcal{X}/G) = E(\mathcal{X})/G$. The initial vertex of $\overline{a} \in V(\mathcal{X}/G)$ is $\overline{i(a)}$, and similarly the terminal vertex is $\overline{t(a)}$; remark that $\overline{i(a)} \neq \overline{t(a)}$ by the first condition. Furthermore, composition is defined by $(\overline{a}, \overline{b}) \mapsto \overline{ab}$ for $(a,b) \in E^{(2)}(\mathcal{X})$.

Proposition 6. *Given a group action $G \curvearrowright \mathcal{X}$, the natural projection $p : \mathcal{X} \to \mathcal{X}/G$ is a morphism of scwols.*

Proof. The p-equivariance of the initial and terminal vertex maps follows from the definition of $\mathcal{X}/G$ as a scwol; similarly, the second condition follows from the composition rule in $\mathcal{X}/G$. Finally, for any $x \in V(\mathcal{X})$, the restriction of p to $\{a \in E(\mathcal{X})\colon i(a) = x\}$ surjects onto $\{b \in E(\mathcal{X}/G)\colon i(b) = p(x)\}$, by construction of the set $E(\mathcal{X}/G)$. It remains to show injectivity of the restriction: if $p(a) = p(b)$, then there exists $g \in G$ such that $g \cdot a = b$. In particular, $g \cdot i(a) = i(b) = i(a) = x \Rightarrow g \cdot a = a$, i.e. $a = b$. □

1.3. **The Geometric Realization.** Let $E^{(k)}(\mathcal{X})$ denote the set of composable sequences of length k, i.e. $(a_1, \ldots, a_k) \in E(\mathcal{X})^k$ such that $(a_i, a_{i+1}) \in E^{(2)}(\mathcal{X})$ for all $i = 1, \ldots, k-1$. Also, let Δ^k denote the standard k-simplex in $\mathbb{R}^{k+1}$.

To realize $\mathcal{X}$ as a geometric object, we will glue together simplices in a manner prescribed by the scwol; to identify simplices, we define the following 'boundary' maps. If $k > 1$, let $\partial_i\colon E^{(k)}(\mathcal{X}) \to E^{(k-1)}(\mathcal{X})$ be given by

$$\partial_i(a_1, \ldots, a_k) := \begin{cases} (a_2, \ldots, a_k), & \text{if } i = 0, \\ (a_1, \ldots, a_i a_{i+1}, \ldots, a_k), & \text{if } i \in [1, k), \\ (a_1, \ldots, a_{k-1}), & \text{if } i = k. \end{cases} \tag{1.2}$$

If $k = 1$, $\partial_0 = i$ and $\partial_1 = t$. A simple computation yields the relation $\partial_i \partial_j = \partial_{j-1} \partial_i$ when $i < j$. Further, let $d_i\colon \Delta^{k-1} \to \Delta^k$ be given by

$$d_i(t_0, \ldots, t_{k-1}) := (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}). \tag{1.3}$$

Similarly, we have the relation $d_i d_j = d_{j+1} d_i$ if $i \leq j$. This map d_i can be thought of as the inclusion of Δ^{k-1} into Δ^k along the hyperplane $t_i = 0$.

Now, consider the space

$$X = \bigsqcup_{\substack{k \in \mathbb{Z}_{\geq 0} \\ A \in E^{(k)}(\mathcal{X})}} \Delta^k \times \{A\}. \tag{1.4}$$

Given $x \in \Delta^{k-1}$ and $A \in E^{(k)}(\mathcal{X})$, we declare $(d_i(x), A) \sim (x, \partial_i(A))$ as points in X. The quotient of X by this equivalence relation is called the geometric realization of the scwol $\mathcal{X}$, and is denoted $|\mathcal{X}|$. The geometric realization can be metrized in a natural way, by considering it as a piecewise Euclidean complex. By applying induction on k and using the relations on d_i and ∂_j, one can show that

Lemma 7. *For $A \in E^{(k)}(\mathcal{X})$, the quotient map $q\colon X \to |\mathcal{X}|$ restricted to $\Delta^k \times \{A\} \subset X$ is injective.*

Proposition 8. *A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ induces a map $|f|\colon |\mathcal{X}| \to |\mathcal{Y}|$.*

Proof. Identifying 0-cells of $|\mathcal{X}|$ and $|\mathcal{Y}|$ with vertices of $\mathcal{X}$ and $\mathcal{Y}$ respectively, let $|f|$ send $x \in V(\mathcal{X})$ to $f(x) \in V(\mathcal{Y})$. As the geometric realizations are built out of simplices, for $|f|$ to be well-defined, we require that it be an affine transformation (i.e. it sends n-cells to n-cells). □

The injectivity of the quotient map restricted to each simplex implies that if $f\colon \mathcal{X} \to \mathcal{Y}$ is an isomorphism, then $|f|$ is an isometry. As a consequence, a group action $G \curvearrowright \mathcal{X}$ induces an action $G \curvearrowright |\mathcal{X}|$ by isometries, because the automorphism corresponding to $g \in G$ induces an isometry $|g|$ on $|\mathcal{X}|$ by the above Proposition.

Example 9. Given a connected scwol $\mathcal{X}$ such that $E^{(2)}(\mathcal{X}) = \emptyset$, the geometric realization $|\mathcal{X}|$ is a connected bipartite graph. The bipartition is given by vertices that are *sources* and *sink*: the sources are only ever the initial vertex of an edge and the sinks are only ever the terminal vertex of an edge.

Example 10. The geometric realization of the scwol given in Figure 1 is a standard 3-simplex; the geometric realization of the subscwol (which was determined by the blue edges) is a standard 2-simplex.

1.4. **Covering Theory of Scwols.**

Definition 11. Given scwols $\mathcal{X}, \mathcal{Y}$ with $\mathcal{X}$ connected[1] a morphism $\rho\colon \mathcal{Y} \to \mathcal{X}$ is a *covering* if for any $y \in V(\mathcal{Y})$,

(1) $\rho\colon \{b \in E(\mathcal{Y})\colon t(b) = y\} \longrightarrow \{a \in E(\mathcal{X})\colon t(a) = \rho(y)\}$ is a bijection,
(2) $\rho\colon \{b \in E(\mathcal{Y})\colon i(b) = y\} \longrightarrow \{a \in E(\mathcal{X})\colon i(a) = \rho(y)\}$ is a bijection.

Remark 12. If $\rho\colon \mathcal{Y} \to \mathcal{X}$ is a covering, it follows that the map $|\rho|\colon |\mathcal{Y}| \to |\mathcal{X}|$ which is induced on the geometric realization is a covering map of topological spaces.

Lemma 13. *Given a free action $G \curvearrowright \mathcal{X}$ such that $\mathcal{X}/G$ is connected, then the natural projection $p\colon \mathcal{X} \to \mathcal{X}/G$ is a covering.*

Proof. For any $x \in V(\mathcal{X})$, the restriction of p to $\{a \in E(\mathcal{X})\colon t(a) = x\}$ surjects onto $\{b \in E(\mathcal{X}/G)\colon t(b) = p(x)\}$, by construction of the set $E(\mathcal{X}/G)$. It remains to show injectivity of the restriction: if $p(a) = p(b)$, then there exists $g \in G$ such that $g \cdot a = b$. In particular, $g \cdot t(a) = t(b) = t(a) = x \iff g \cdot x = x \Rightarrow g = e$, as the action is free; thus, $a = b$ i.e. the restriction is injective. The second condition follows because $p\colon \mathcal{X} \to \mathcal{X}/G$ is a morphism of scwols. □

Let $E^{\pm}(\mathcal{X}) = E(\mathcal{X}) \sqcup E(\mathcal{X})^{op}$, where $E(\mathcal{X})^{op}$ consists of all the edges in $E(\mathcal{X})$ with the initial and terminal vertices swapped. An *edge path* in $\mathcal{X}$ joining $x, y \in V(\mathcal{X})$ is a sequence $c = (e_1, \ldots, e_k)$ with $e_i \in E^{\pm}(\mathcal{X})$ such that $i(e_1) = x$, $t(e_i) = e_{i+1}$, and $t(e_k) = y$. If $c' = (e'_1, \ldots, e'_\ell)$ is an edge path joining $y, z \in V(\mathcal{X})$, then denote the concatenation of c and c' by

$$c * c' = (e_1, \ldots, e_k, e'_1, \ldots, e'_\ell). \tag{1.5}$$

Given $a \in E(\mathcal{X})$, denote the opposite edge by $a^{op} \in E(\mathcal{X})^{op}$. One can then reduce an edge path c by eliminating subsequences in the following two ways:

$$(\ldots, e_{i-1}, a, a^{op}, e_{i+2}, \ldots) \mapsto (\ldots, e_{i-1}, e_{i+2}, \ldots), \tag{1.6}$$

$$(\ldots, e_{i-1}, a, b, e_{i+2}, \ldots) \mapsto (\ldots, e_{i-1}, ab, e_{i+2}, \ldots), \text{ if } (a, b) \in E^{(2)}(\mathcal{X}) \text{ or } (a^{op}, b^{op}) \in E^{(2)}(\mathcal{X}). \tag{1.7}$$

[1]If $\mathcal{X}$ is not connected, a covering of $\mathcal{X}$ is an epimorphism $\rho\colon \mathcal{Y} \to \mathcal{X}$ such that for any $\mathcal{X}'$ is a connected component of $\mathcal{X}$, then the restriction $\rho\colon \rho^{-1}(\mathcal{X}') \to \mathcal{X}'$ is a covering in the sense above. However, we will only consider connected scwols in this section.

If the edge path c' can be obtained from c by applying the above reductions or their inverses, then they are said to be *homotopic*. Assume in addition that the scwol $\mathcal{X}$ is connected, then the homotopy classes of edge paths joining $x \in V(\mathcal{X})$ (i.e. loops) form a group under the concatenation operation, where the inverse of c is given by $c^{-1} = (e_k^{op}, \dots, e_1^{op})$. This group is called the *fundamental group* of $\mathcal{X}$ at x, denoted $\pi_1(\mathcal{X}, x)$.[2]

Theorem 14. *Given $y \in V(\mathcal{Y})$, a covering $\rho\colon \mathcal{Y} \to \mathcal{X}$ induces a monomorphism $\overline{\rho}\colon \pi_1(\mathcal{Y}, y) \to \pi_1(\mathcal{X}, \rho(y))$.*

Proof. If $c' = (e'_1, \dots, e'_k)$ is a loop in $\mathcal{Y}$ based at y, then it is said to be a *lifting* of a loop $c = (e_1, \dots, e_k)$ in $\mathcal{X}$ based at $\rho(y)$ if $e_i = \rho(e'_i)$ for all i. As a covering is a 'local bijection,' given a loop c in $\mathcal{X}$ there is a unique lifting in $\mathcal{Y}$. Furthermore, homotopic edge paths in $\mathcal{Y}$ are mapped to homotopic edge paths in $\mathcal{X}$, as the covering is a morphism. Consequently, for each homotopy class of loops in $\mathcal{Y}$ based at y, there is a unique homotopy class of loops in $\mathcal{X}$ based at $\rho(y)$ to which they are mapped by ρ; this induces an injective morphism $\pi_1(\mathcal{Y}, y) \hookrightarrow \pi_1(\mathcal{X}, \rho(y))$ of groups. □

Definition 15. Given a connected scwol $\mathcal{X}$ and group G, a *Galois covering* of $\mathcal{X}$ with *Galois group* G is a covering $\rho\colon \mathcal{Y} \to \mathcal{X}$ and a free action $G \curvearrowright \mathcal{Y}$ such that $\mathcal{Y}/G \simeq \mathcal{X}$.

Example 16. Given a connected scwol $\mathcal{X}$, one can construct a simply connected cover $\tilde{\mathcal{X}}$ in a canonical way (see page 529 of [1]). Given a base vertex $x \in V(\mathcal{X})$, the associated covering $\rho\colon \tilde{\mathcal{X}} \to \mathcal{X}$ is a Galois covering with Galois group $\pi_1(\mathcal{X}, x)$. In fact, any action $G \curvearrowright \mathcal{X}$ can be lifted to an action on $\tilde{\mathcal{X}}$.

2. Complexes of Groups

2.1. Definitions and Examples.

Definition 17. Given a scwol $\mathcal{X}$, a *complex of groups* $G(\mathcal{X}) = (G_x, \psi_a, g_{a,b})$ over $\mathcal{X}$ is the triple determined by

(1) For $x \in V(\mathcal{X})$, there is a group G_x, which is called the *local group* at x.
(2) For $a \in E(\mathcal{X})$, there is a monomorphism $\psi_a\colon G_{i(a)} \to G_{t(a)}$.
(3) For $(a, b) \in E^{(2)}(\mathcal{X})$, there is a *twisting element* $g_{a,b} \in G_{t(a)}$ that satisfies the following two properties: if we denote conjugation by $g_{a,b}$ as $\mathrm{Ad}(g_{a,b}) \in \mathrm{Aut}(G_{t(a)})$, then
$$\mathrm{Ad}(g_{a,b})\psi_{ab} = \psi_a\psi_b, \tag{2.1}$$
and for any $(a, b, c) \in E^{(3)}(\mathcal{X})$, the twisting elements satisfy the *cocycle condition*
$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c} \in G_{t(a)}. \tag{2.2}$$

If all of the twisting elements are trivial, we say that the complex of groups $G(\mathcal{X})$ over $\mathcal{X}$ is *simple*.

Remark 18. If $\mathcal{X}$ is a 1-dimensional scwol[3], then (2.1) is a vacuous condition, and a complex of groups $G(\mathcal{X})$ over $\mathcal{X}$ is the same as a graph of groups, as in [4]. Moreover, (2.2) is trivially satisfied when the dimension of $\mathcal{X}$ is ≤ 2, as in that case $E^{(3)}(\mathcal{X}) = \emptyset$.

Definition 19. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of scwols, and let $G(\mathcal{X}) = (G_x, \psi_a, g_{a,b}), G(\mathcal{Y}) = (G_y, \psi_{a'}, g_{a',b'})$ be complexes of groups over $\mathcal{X}, \mathcal{Y}$ respectively. A *morphism over* f is a pair $\phi = (\phi_x, \phi(a))$ such that

(1) For any $x \in V(\mathcal{X})$, there is a group morphism $\phi_x\colon G_x \to G_{f(x)}$.
(2) For any $a \in E(\mathcal{X})$, there is an element $\phi(a) \in G_{t(f(a))}$ such that
$$\mathrm{Ad}(\phi(a))\psi_{f(a)}\phi_{i(a)} = \phi_{t(a)}\psi_a, \tag{2.3}$$
and for any $(a, b) \in E^{(2)}(\mathcal{X})$,
$$\phi_{t(a)}(g_{a,b}) \cdot \phi(ab) = \phi(a) \cdot \psi_{f(a)}(\phi(b)) \cdot g_{f(a),f(b)}. \tag{2.4}$$

[2]In fact, one can show that the fundamental group $\pi_1(\mathcal{X}, x)$ of a connected scwol $\mathcal{X}$ is isomorphic to the fundamental group $\pi_1(|\mathcal{X}|, x)$ of its geometric realization, where the vertex $x \in V(\mathcal{X})$ is identified with the corresponding 0-cell in $|\mathcal{X}|$.

[3]The dimension of a scwol $\mathcal{X}$ is defined to be the maximal $k \in \mathbb{Z}_{\geq 0}$ for which $E^{(k)}(\mathcal{X})$ is non-empty. Equivalently, the dimension is the maximal dimension of a simplex in the geometric realization $|\mathcal{X}|$.

In the case where f is an isomorphism of scwols and each ϕ_x is an isomorphism of groups, we say that ϕ is an *isomorphism* of complexes of groups. If each morphism ϕ_x is injective, we say that ϕ is *injective on local groups.*

Example 20. A group can be viewed as a complex of groups over the trivial scwol (i.e. a single vertex).

The above definitions appear more natural if we consider the complex of groups instead as a category. Indeed, given a complex of groups $G(\mathcal{X})$ over a scwol $\mathcal{X}$, let $CG(\mathcal{X})$ be the small category whose objects are the vertices $V(\mathcal{X})$ and whose arrows are the pairs (g,α) for $\alpha \in \mathcal{X}$ and $g \in G_{t(\alpha)}$. The source and range of an arrow (g,α) are given by the initial and terminal vertex of α, respectively. Further, given arrows (g,α) and (h,β), they are composable if α and β are composable; in this case,

$$(g,\alpha)\circ(h,\beta) = (g\cdot\psi_\alpha(h)\cdot g_{\alpha,\beta}, \alpha\beta). \tag{2.5}$$

In this context, the conditions (2.1) and (2.2) are exactly equivalent to this composition rule being associative. Moreover, a morphism $\phi\colon G(\mathcal{X}) \to G(\mathcal{Y})$ over $f\colon \mathcal{X} \to \mathcal{Y}$ induces a functor $CG(\mathcal{X}) \to CG(\mathcal{Y})$: on objects, this functor is just the restriction of f to vertices and on arrows, it is given by

$$(g,\alpha) \mapsto (\phi_{t(\alpha)}(g)\cdot\phi(\alpha), f(\alpha)). \tag{2.6}$$

The conditions (2.3) and (2.4) are equivalent to this functor preserving composition of arrows.

2.2. **Developability.** Let $G \curvearrowright \mathcal{X}$ with $\mathcal{Y} = \mathcal{X}/G$, and let $p\colon \mathcal{X} \to \mathcal{Y}$ be the natural projection. Given $y \in V(\mathcal{Y})$, let $\overline{y} \in V(\mathcal{X})$ be such that $p(\overline{y}) = y$. As p is a 'local bijection,' given $a \in E(\mathcal{Y})$ with $i(a) = y$, there exists a unique lift $\overline{a} \in E(\mathcal{X})$ with $i(\overline{a}) = \overline{y}$. It is not true in general that $t(\overline{a})$ is $\overline{t(a)}$; however, they are in the same G-orbit, so there exists $h_a \in G$ such that $h_a \cdot t(\overline{a}) = \overline{t(a)}$.

For $y \in V(\mathcal{Y})$, let $G_y := \mathrm{Stab}_G(\overline{y})$ and for any $a \in E(\mathcal{Y})$, consider the morphism $\psi_a\colon G_{i(a)} \to G$ given by

$$\psi_a(g) = h_a \cdot g \cdot h_a^{-1}. \tag{2.7}$$

In fact, $\mathrm{im}(\psi_a) \subseteq G_{t(a)}$, so it is considered as a morphism $\psi_a\colon G_{i(a)} \to G_{t(a)}$. Finally, for each $(a,b) \in E^{(2)}(\mathcal{Y})$, define the twisting element $g_{a,b} = h_a h_b h_{ab}^{-1} \in G_{t(a)}$.

Proposition 21. *The triple $G(\mathcal{Y}) = (G_y, \psi_a, g_{a,b})$ defines a complex of groups over $\mathcal{Y}$.*

Proof. Each ψ_a is an inner automorphism, and thus a monomorphism. Next if $(a,b) \in E^{(2)}(\mathcal{X})$, then for $g \in G_{t(a)}$

$$\mathrm{Ad}(g_{a,b})\circ\psi_{ab}(g) = h_a h_b h_{ab}^{-1}(h_{ab} g h_{ab}^{-1}) h_{ab} h_b^{-1} h_a^{-1} = h_a h_b g h_b^{-1} h_a^{-1} = h_a \psi_b(g) h_a^{-1} = \psi_a \circ \psi_b(g).$$

If $(a,b,c) \in E^{(3)}(\mathcal{X})$, then

$$\psi_a(g_{b,c}) g_{a,bc} = h_a h_b h_c h_{bc}^{-1} h_a^{-1} h_a h_{bc} h_{abc}^{-1} = h_a h_b h_c h_{abc}^{-1} = h_a h_b h_{ab}^{-1} h_{ab} h_c h_{abc}^{-1} = g_{a,b} g_{ab,c}.$$

□

Example 22. If 0 denotes the trivial scwol, then there exists a degenerate morphism $f\colon \mathcal{Y} \to 0$ for any scwol $\mathcal{Y}$. Now if $G(\mathcal{Y})$ is as above, then there exists a canonical morphism $\phi = (\phi_y, \phi(a))\colon G(\mathcal{Y}) \to G$ over f, where $\phi_y\colon G_y \hookrightarrow G$ is the natural inclusion map and $\phi(a) := h_a \in G_{t(a)}$. It is clear that ϕ is injective on local groups.

Definition 23. A complex of groups $G(\mathcal{Y})$ is said to be *developable* if it is isomorphic to some complex of groups which arises from a group action $G \curvearrowright \mathcal{X}$ with $\mathcal{Y} \simeq \mathcal{X}/G$.

Proposition 24. *Let $\mathcal{Y}$ be a connected scwol. If $G(\mathcal{Y})$ is a developable complex of groups over $\mathcal{Y}$, then it is isomorphic to some complex of groups which arises from a group action $G \curvearrowright \mathcal{X}$, where $\mathcal{X}$ is simply connected.*

Proof. There exists an action $G \curvearrowright \mathcal{X}$ such that $\mathcal{X}/G \simeq \mathcal{Y}$ and such that the complex of groups arising from the action is isomorphic to $G(\mathcal{Y})$. Take a connected component $\mathcal{X}_0 \subset \mathcal{X}$, and let $G_0 = \{g \in G\colon g|_{\mathcal{X}_0} \in \mathrm{Aut}(\mathcal{X}_0)\}$, then since $\mathcal{X}/G$ is connected, $\mathcal{X}_0/G_0 \simeq \mathcal{X}/G \simeq \mathcal{Y}$. The construction on page 541 of [1] then allows us to construct a universal covering space $\tilde{\mathcal{X}_0}$ of $\mathcal{X}_0$ and a group action $\tilde{G} \curvearrowright \tilde{\mathcal{X}_0}$ such that $\tilde{\mathcal{X}_0}/\tilde{G} \simeq \mathcal{X}_0/G \simeq \mathcal{Y}$, as required. □

The following result is known as the 'basic construction' in the literature.

Theorem 25. *Let $G(\mathcal{Y}) = (G_y, \psi_a, g_{a,b})$ be a complex of groups over a scwol $\mathcal{Y}$ and let G be a group. Given a morphism $\phi\colon G(\mathcal{Y}) \to G$, there exists a scwol $D(\mathcal{Y}, \phi)$ and an action $G \curvearrowright D(\mathcal{Y}, \phi)$ such that $D(\mathcal{Y}, \phi)/G \simeq \mathcal{Y}$. If in addition ϕ is injective on local groups, then the action $G \curvearrowright D(\mathcal{Y}, \phi)$ gives rise to $G(\mathcal{Y})$ and ϕ is the canonical morphism.*

Furthermore, if $G(\mathcal{Y})$ arises from the action $G \curvearrowright \mathcal{X}$ and $\phi\colon G(\mathcal{Y}) \to G$ is the canonical morphism, then there exist a G-equivariant isomorphism $\mathcal{X} \xrightarrow{\sim} D(\mathcal{Y}, \phi)$ such that the induced quotient map on $\mathcal{Y}$ is the identity.

We say that the scwol $D(\mathcal{Y}, \phi)$ is the *development* of $\mathcal{Y}$ with respect to ϕ.

Proof. The development $D(\mathcal{Y}, \phi)$ is defined in terms of pairs depending on the cokernels of the local groups under ϕ; indeed, let

$$V(D(\mathcal{Y}, \phi)) := \{(g \cdot \phi_y(G_y), y) \colon y \in V(\mathcal{Y}), g \cdot \phi_y(G_y) \in G/\phi_y(G_y)\}, \tag{2.8}$$

$$E(D(\mathcal{Y}, \phi)) := \{(g \cdot \phi_{i(a)}(G_{i(a)}), a) \colon a \in E(\mathcal{Y}), g \cdot \phi_{i(a)}(G_{i(a)}) \in G/\phi_{i(a)}(G_{i(a)})\}. \tag{2.9}$$

It is then natural to define the initial and terminal vertex maps as follows: if $(g \cdot \phi_{i(a)}(G_{i(a)}), a) \in E(D(\mathcal{Y}, \phi))$,

$$i(g \cdot \phi_{i(a)}(G_{i(a)}), a) = (g \cdot \phi_{i(a)}(G_{i(a)}), i(a)), \tag{2.10}$$

$$t(g \cdot \phi_{i(a)}(G_{i(a)}), a) = (g\phi(a)^{-1} \cdot \phi_{t(a)}(G_{t(a)}), t(a)). \tag{2.11}$$

Now, we declare two edges $(g \cdot \phi_{i(a)}(G_{i(a)}), a), (h \cdot \phi_{i(b)}(G_{i(b)}), b) \in E(D(\mathcal{Y}, \phi))$ to be composable if $(a, b) \in E^{(2)}(\mathcal{Y})$ and in addition $g \cdot \phi_{i(a)}(G_{i(a)}) = h\phi(b)^{-1} \cdot \phi_{i(a)}(G_{i(a)})$. In this case,

$$(g \cdot \phi_{i(a)}(G_{i(a)}), a) \circ (h \cdot \phi_{i(b)}(G_{i(b)}), b) := (h \cdot \phi_{i(b)}(G_{i(b)}), ab). \tag{2.12}$$

The latter condition is needed to ensure that $D(\mathcal{Y}, \phi)$ satisfies the axioms of a scwol. There is an obvious action $G \curvearrowright D(\mathcal{Y}, \phi)$, which is left multiplication i.e. for $g, h \in G$ and $\alpha \in \mathcal{Y}$,

$$h \cdot (g \cdot \phi_{i(\alpha)}(G_{i(\alpha)}), \alpha) := (hg \cdot \phi_{i(\alpha)}(G_{i(\alpha)}), \alpha). \tag{2.13}$$

Given an element of $D(\mathcal{Y}, \phi)$, its G-orbit is uniquely determined by the element of $\mathcal{Y}$ in the second coordinate; it follows that $D(\mathcal{Y}, \phi)/G \simeq \mathcal{Y}$. The action $G \curvearrowright D(\mathcal{Y}, \phi)$ induces a complex of groups $G'(\mathcal{Y}) = (G'_y, \psi'_a, g'_{a,b})$ over $D(\mathcal{Y}, \phi)/G \simeq \mathcal{Y}$. In the notation of Proposition 21, $\overline{y} = (y, G_y)$ and $h_a = \phi(a)$, which forces the data of $G'(\mathcal{Y})$ to be given by $G'_y = \phi_y(G_y)$, the restriction $\psi'_a = \mathrm{Ad}(\phi(a))\colon \phi_{i(a)}(G_{i(a)}) \to \phi_{t(a)}(G_{t(a)})$, and the twisting elements $g'_{a,b} = \phi(a)\phi(b)\phi(ab)^{-1} \in \phi_{t(a)}(G_{t(a)})$. If we assume in addition that the ϕ is injective on local groups, then it is obvious that the local groups of $G(\mathcal{Y})$ and $G'(\mathcal{Y})$ are all isomorphic. As a consequence, the monomorphisms and twisting elements of $G(\mathcal{Y})$ and of $G'(\mathcal{Y})$ are the same, up to this isomorphism; we conclude that $G(\mathcal{Y}) \simeq G'(\mathcal{Y})$ i.e. $G(\mathcal{Y})$ is developable.

Now, assume $G(\mathcal{Y})$ arises from the action $G \curvearrowright \mathcal{X}$ for some scwol $\mathcal{X}$, and let $p\colon \mathcal{X} \to \mathcal{X}/G \simeq \mathcal{Y}$ be the associated projection. As ϕ is injective on local groups, any element of $D(\mathcal{Y}, \phi)$ can be written as $(g \cdot G_{i(\alpha)}, \alpha)$ for some $\alpha \in \mathcal{Y}$. Recall from the covering theory of scwols that for each $\alpha \in \mathcal{Y}$, there must exist a unique $\overline{\alpha} \in \mathcal{X}$ such that $p(\overline{\alpha}) = \alpha$ and $i(\overline{\alpha}) = \overline{i(\alpha)}$. Consider the map $\theta\colon D(\mathcal{Y}, \phi) \to \mathcal{X}$ given by

$$\theta\colon (g \cdot G_{i(\alpha)}, \alpha) \mapsto g \cdot \overline{\alpha}. \tag{2.14}$$

By construction, this map is G-equivariant and is such that the induced quotient map on $\mathcal{Y}$ is the identity. One can show that θ in fact defines an isomorphism of scwols. □

It follows immediately from the basic construction that

Corollary 26. *The complex of groups $G(\mathcal{Y})$ is developable iff there exists a group G and a morphism $\phi\colon G(\mathcal{Y}) \to G$ that is injective on local groups.*

Lemma 27. *Let G be a group that admits two free actions on a set X such that the two quotients have the same cardinality, then there exists a permutation of X that conjugates the first action to the second.*

Proof. Let H be the permutation group of X, then consider the two free actions as two group morphisms, denoted $\varphi, \varphi' \colon G \to H$. Let $X_0, X_0' \subset X$ consists of representations from the G-orbits of X for the two actions. As the quotients have equal cardinality, there exists a bijection $f \colon X_0 \to X_0'$; since the action is free, for each $x \in X$, there exists a unique $x_0 \in X_0$ and $g_0 \in G$ such that $z = \varphi(g_0)(x_0)$. The analogous statement also holds for X_0'. We can define a bijection $g \colon X \to X$ given by

$$x \mapsto \varphi'(g_0)(f(x_0)). \tag{2.15}$$

However, we can write $x = \varphi(g_0)(x_0)$; rearranging, $\varphi(g_0)(x_0) = (f^{-1} \circ \varphi'(g_0) \circ f)(x_0)$, for $f \in H$. □

This study culminates in the following theorem of Bass & Serre on the developability of graphs of groups.

Corollary 28. *A complex of groups $G(\mathcal{Y})$ over a scwol $\mathcal{Y}$ of dimension 1 is developable.*

Proof. Write $G(\mathcal{Y}) = (G_y, \psi_a, g_{a,b})$, and define the set

$$X = \prod_{y \in V(\mathcal{Y})} G_y. \tag{2.16}$$

Let G denote the permutation group of X. Each local group G_y acts freely on X by left multiplication on the appropriate factor, and trivially otherwise. This defines monomorphisms $\phi_y \colon G_y \to G$ for each $y \in V(\mathcal{Y})$. Now, for any $a \in E(\mathcal{Y})$, the Lemma gives us that there exists $\phi(a) \in G_{t(a)}$ such that $\phi(a)$ conjugates $\phi_{t(a)}\psi_a$ with $\phi_{i(a)}$; that is,

$$\phi_{t(a)}\psi_a = \mathrm{Ad}(\phi(a))\phi_{i(a)}.$$

Finally, as $E^{(2)}(\mathcal{Y}) = \emptyset$, there are no twisting elements by our previous remarks. Therefore, $\phi = (\phi_y, \phi(a)) \colon G(\mathcal{Y}) \to G$ defines a morphism of complexes of groups and ϕ is injective on local groups by construction. In particular, $G(\mathcal{Y})$ is developable.

□

References

[1] M. Bridson, A. Häfliger. *Metric Spaces of Non-Positive Curvature.* Springer-Verlag (2009).
[2] J. Corson. *Complexes of groups.* Proc. London Math. Soc. (3) 65 (1992), no. 1, pp. 199-224.
[3] A. Haefliger. *Extension of complexes of groups.* Ann. Inst. Fourier (1992) 42(1-2), pp. 275-311.
[4] J-P. Serre. *Trees.* Springer-Verlag (1980).