# MATH 679: PERFECTOID SPACES

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Course Description. Perfectoid spaces are a class of spaces in arithmetic geometry introduced in 2012 by Peter Scholze in his PhD thesis [Sch12]. Despite their youth, these spaces have had stunning applications to many different areas of mathematics, including number theory, algebraic geometry, representation theory, and commutative algebra. The key to this success is that perfectoid spaces provide a functorial procedure to translate certain algebro-geometric problems from characteristic 0 (or mixed characteristic) to characteristic $p$; the latter can often be more accessible thanks to the magic of Frobenius.

A major portion of this class will be devoted to setting up the basic theory of perfectoid spaces. En route, we will encounter Huber's approach to nonarchimedean geometry via his language of adic spaces, Faltings' theory of 'almost mathematics' conceived in his proof of Fontaine's conjectures in $p$-adic Hodge theory, and the basic algebraic geometry of perfect schemes in characteristic $p$. The highlight of this part of the course will be the 'almost purity theorem', a cruder version of which forms the cornerstone of Faltings' aforementioned work. The rest of the course will focus on a single application of perfectoid spaces. There are several choices here: we could go either in the arithmetic direction (such as Scholze's work on the weight-monodromy conjecture or some recent progress in $p$-adic Hodge theory) or in a more algebraic direction (which is the lecturer's inclination). A final choice will be made during the semester depending on audience makeup and interest.

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# 1. January 4th

1.1. **Overview.** Today's lecture will be an overview of the class. The end goals are:

(1) *Direct summand conjecture* (conjectured by Hochster in the late 60's, resolved last year by Y. André): if $R$ is a regular ring and $f\colon R \hookrightarrow S$ a finite extension, then $f$ admits an $R$-linear splitting.

(2) *Hodge–Tate decomposition* (a $p$-adic analogue of Hodge decomposition; due to Tate, Faltings, Scholze): let $p$ be a prime, $K/\mathbf{Q}_p$ a finite extension, $\mathbf{C}_p = \widehat{\overline{K}}$ the completed algebraic closure, $X/K$ a smooth proper variety. Then, there exists a canonical $\mathrm{Gal}(\overline{K}/K)$-equivariant isomorphism

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes \mathbf{C}_p \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbf{C}_p(-j),$$

where $\mathbf{C}_p(n) \coloneqq \mathbf{C}_p(1)^{\otimes n}$ and $\mathbf{C}_p(1)$ is the cyclotomic character.

As a corollary, the action of $\mathrm{Gal}(\overline{K}/K)$ on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ determines the Hodge numbers $h^{i,j}(X) \coloneqq \dim_K H^j(X, \Omega^i_X)$ (by twisting the right-hand side of the decomposition by $j$ and taking $\mathrm{Gal}(\overline{K}/K)$-invariants).

The strategy, at a coarse level, to prove either of these statements is:

(1) Prove the result after a huge extension: $R \to R_\infty$ in (1), or $X_\infty \to X$ in (2).
(2) Descend down.

The objects $R_\infty$ and $X_\infty$ are instances of perfectoid algebras and spaces, respectively.

1.2. **Perfectoid fields.** The point objects are perfectoid fields, some examples of which are:

(1) (characteristic 0) $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$ with valuation ring $K^\circ = \mathbf{Z}_p[p^{1/p^\infty}]^\wedge$, and value group $\mathbf{Z}[1/p]$.
(2) (characteristic p) $K^\flat = (\mathbf{F}_p((t))_{\text{perf}})^\wedge$, with valuation ring $K^{\flat\circ} = (\mathbf{F}_p[[t]]_{\text{perf}})^\wedge$. Note that adjoining all $p$ power roots of $t$ is the same as taking the perfection in characteristic $p$.

Both $K$ and $K^\flat$ are both highly ramified (there is no nice choice of a uniformizer), but both non-archimedean fields. Today, $K$ and $K^\flat$ always denote these examples.

The baby example of the relation between the characteristic 0 and characteristic $p$ fields stems from this observation: $K^\circ = \big(\mathbf{Z}_p[t^{1/p^\infty}]/(t-p)\big)^\wedge$ and reducing this formula mod $p$ gives

$$K^\circ/p = \mathbf{F}_p[t^{1/p^\infty}]/t = K^{\flat\circ}/t.$$

This is a sort of coarse relation between a ring in characteristic 0 and a ring in characteristic $p$.

Observe the following:

$$K^{\flat\circ} = \varprojlim_m K^{\flat\circ}/t^m \simeq \varprojlim_n K^{\flat\circ}/t^{p^n} \simeq \varprojlim_{x \mapsto x^p} K^{\flat\circ}/t = \varprojlim_{\mathrm{Frob}} K^\circ/p.$$

The first equality follows because $K^{\flat\circ}$ is $t$-adically complete, and the second isomorphism follows because everything is perfect.

**Definition 1.1.** For any ring $R$, set $R^\flat := \varprojlim_{\mathrm{Frob}} R/p$.

The ring $R^\flat$ has characteristic $p$, because each $R/p$ does.

**Example 1.2.** $(K^\circ)^\flat = K^{\flat\circ}$. This follows from the previous calculation.

There is a diagram "connecting" $K$ and $K^\flat$:

$$\begin{tikzcd} & & K^\circ \arrow[r, hook] \arrow[d] & K \\ K^{\flat\circ} \arrow[r] \arrow[d, hook] & K^{\flat\circ}/t \arrow[r, "\simeq"] & K^\circ/p & \\ K^\flat & & & \end{tikzcd}$$

The fields $K$ and $K^\flat$ are obtained from $K^\circ$ and $K^{\flat\circ}$ by inverting $p$ and $t$, respectively.

**Example 1.3.** If $R = \mathbf{Z}_p$, then $R^\flat = \mathbf{F}_p$. If $R = \mathbf{Z}_p[t]$, then $R^\flat = \mathbf{F}_p$ (this is because $R/p = \mathbf{F}_p[t]$, but only the constant polynomials admit all $p$-power roots).

**Exercise 1.4.** What is $K^\flat$ for $K = \mathbf{Q}_p(\mu_{p^\infty})^\wedge$? Here, $\mu_{p^\infty}$ denotes all $p$-power roots of unity.

**Theorem 1.5.** [Fontaine-Wintenberger] *There exists a canonical isomorphism $G_K \simeq G_{K^\flat}$ of absolute Galois groups*[1]*. Better, there is an equivalence of categories between*

$$K_{\text{fét}} := \{\text{finite étale } K\text{-algebras}\} \simeq \{\text{finite étale } K^\flat\text{-algebras}\} =: (K^\flat)_{\text{fét}}.$$

**Corollary 1.6.** *The $\mathbf{F}_p$-cohomological dimension of $\mathrm{Gal}(\overline{K}/K)$ is $\leq 1$, i.e. $H^i(G_K, M) = 0$ for all $i > 1$ and for all $G_K$-representations $M$ on an $\mathbf{F}_p$-vector space.*

In characteristic $p$, this statement is true in greater generality: for any ring $R$, it follows from the Artin–Schreier sequence, which relates the étale and the coherent cohomology; in general, for an arbitrary scheme $X$ in characteristic $p$, the cohomological dimension is bounded above by the Krull dimension, in contrast to in characteristic 0 where it is bounded above by two times the Krull dimension.

**Example 1.7.** Assume $p \neq 2$. If $L = K(\sqrt{p})$, then the theorem of Fontaine-Wintenberger tells us that it corresponds to a quadratic extension of $K^\flat$, namely $L^\flat = K^\flat(\sqrt{t})$. That is, in this instance we can "replace $p$ with $t$" (though this does not work in general). Of course, if $p = 2$, $L = K$, so the statement is still true, but vacuous.

The easiest case of the Fontaine-Wintenberger theorem is the unramified one: consider the categories $K^{unr}_{\text{fét}} := K^\circ_{\text{fét}}$ and $K^{\flat,unr}_{\text{fét}} := K^{\flat\circ}_{\text{fét}}$. We claim that this equivalence carries $K^{unr}_{\text{fét}}$ to $K^{\flat,unr}_{\text{fét}}$. To see this, reduce mod $p$ and $t$ respectively:

[1] To speak of an absolute Galois group involves the choice of an algebraic closure, but the isomorphism is canonical in the sense that a choice of algebraic closure of $K$ determines a choice for $K^\flat$, and vice-versa.

$$\begin{tikzcd} K^{\mathrm{unr}}_{\mathrm{f\acute{e}t}} \arrow[r, "\simeq"] & (K^\circ/p)_{\mathrm{f\acute{e}t}} \arrow[d, "="] \\ K^{\flat,\mathrm{unr}}_{\mathrm{f\acute{e}t}} \arrow[r, "\simeq"] & \left(K^{\flat\circ}/t\right)_{\mathrm{f\acute{e}t}} \end{tikzcd}$$

The miracles are the equivalences $K^{\mathrm{unr}}_{\mathrm{f\acute{e}t}} \simeq (K^\circ/p)_{\mathrm{f\acute{e}t}}$ and $K^{\flat,\mathrm{unr}}_{\mathrm{f\acute{e}t}} \simeq \left(K^{\flat\circ}/t\right)_{\mathrm{f\acute{e}t}}$, which come from deformation theory. The insight is that the same argument works in general, using almost mathematics.

The rough idea (of almost mathematics): if $\mathfrak{m} \subset K^\circ$ is the maximal ideal (so $\mathfrak{m} = (p^{1/p^\infty})$ in our example), then $\mathfrak{m} = \mathfrak{m}^2$ (in the noetherian world, this would happen iff $\mathfrak{m} = 0$). So,

$$\{\mathfrak{m}\text{-torsion modules}\} \subseteq \{K^\circ\text{-modules}\}$$

is a Serre subcategory, i.e. it is closed under taking subobjects, quotients, and extensions. Therefore, you can define the category of almost modules over $K^\circ$ to be $\mathbf{Mod}^a_{K^\circ} \coloneqq \{K^\circ\text{-modules}\}/\{\mathfrak{m}\text{-torsion modules}\}$.

**Fact 1.8.** $\mathbf{Mod}^a_{K^\circ}$ is an abelian $\otimes$-category.

The upshot: you can develop all/most/almost all of commutative algebra in $\mathbf{Mod}^a_{K^\circ}$ (this is done systematically in the book by Gabber–Ramero [GR03]). To give a flavor, let's discuss flatness.

**Example 1.9.** A map $A \to B$ of commutative $K^\circ$-algebras is almost flat if $\mathrm{Tor}_i^A(B, M)$ is almost zero, i.e. $\mathfrak{m}$-torsion, for any $A$-module $M$ and $i > 0$.

We will mostly need this theory for almost étale property. This buys us the following:

**Theorem 1.10.** [Tate, Faltings] *If $L/K$ is a finite extension, then $L^\circ/K^\circ$ is almost étale, meaning it is almost flat, it is almost unramified (the module of Kähler differentials is almost zero), and almost of finite presentation). Similarly for finite extensions of $K^\flat$.*

Therefore, get the Fontaine-Wintenberger theorem by taking the previous diagram and removing the word unramified:

$$\begin{tikzcd} K_{\mathrm{f\acute{e}t}} & K^\circ_{a\,\mathrm{f\acute{e}t}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & (K^\circ/p)_{a\,\mathrm{f\acute{e}t}} \arrow[d, "="] \\ K^\flat_{\mathrm{f\acute{e}t}} & \left(K^{\flat\circ}\right)_{a\,\mathrm{f\acute{e}t}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \left(K^{\flat\circ}/t\right)_{a\,\mathrm{f\acute{e}t}} \end{tikzcd}$$

Here, $K^\circ_{a\,\mathrm{f\acute{e}t}}$ denotes the category of almost finite étale algebras over $K^\circ$, and similarly for $\left(K^{\flat\circ}\right)_{a\,\mathrm{f\acute{e}t}}$, $\left(K^{\flat\circ}/t\right)_{a\,\mathrm{f\acute{e}t}}$, and $(K^\circ/p)_{a\,\mathrm{f\acute{e}t}}$.

**Example 1.11.** Let $L = K(\sqrt{p})$ and $p \neq 2$. The main question is: what is $L^\circ$? A stupid observation: $p^{1/2} \in L^\circ$, $p^{1/2p} \in L^\circ$, and $p^{1/2p^n} \in L^\circ$ for all $n \geq 1$. Set $L_n^+ = K^\circ[p^{1/2p^n}] \subseteq L^\circ$ and one can check that $L^\circ = \bigcup_n L_n^+$.

The upside for us is that $L_n^+ = K^\circ[x_n]/f_n(x_n)$, where $f_n(x_n) = x_n^2 - p^{1/p^n}$, so we can compute the discriminant

$$\mathrm{disc}(L_n^+/K^\circ) \coloneqq \mathrm{val}_p\left(f'(x_n)\right) = \mathrm{val}_p\left(2p^{1/2p^n}\right) = \frac{1}{2p^n},$$

assuming we have normalized $\mathrm{val}_p(p) = 1$. Hence, $\mathrm{disc}(L_n^+/K^\circ) \to 0$ as $n \to \infty$. This is what it means for $L^\circ/K^\circ$ to be almost unramified.

1.3. **Perfectoid spaces.** We saw earlier that $K^{\flat\circ} = \varprojlim_{x \mapsto x^p} K^\circ/p$, i.e. $K^{\flat\circ}$ can be obtained from $K^\circ$.

**Claim 1.12.** The map of multiplicative monoids $\varprojlim_{x\mapsto x^p} K^\circ \to \varprojlim_{x\mapsto x^p} K^\circ/p$ is a bijection.

At every finite level this is not true, but in the limit the error disappears. The basic idea is to use the following fact many times: if $x \equiv y \bmod p$, then $x^p \equiv y^p \bmod p^2$.

Similarly, can show $K^\flat \xrightarrow{\simeq} \varprojlim_{x\mapsto x^p} K$. Therefore, we can rewrite this as

$$(\mathbf{A}^1_{K^\flat})(K^\flat) \xrightarrow{\simeq} \left(\varprojlim_{x\mapsto x^p} \mathbf{A}^1_K\right)(K).$$

The same holds true for points in finite extensions of $K$ and $K^\flat$.

To generalize, use the notion of adic spaces:

$$\{\text{Huber's adic spaces}\} \subseteq \{\text{locally ringed spaces equipped with valuations}^{2}\}.$$

There is an analytification functor:

$$\{\text{schemes of finite type over } K\} \to \{\text{adic spaces over } K\},$$

denoted $X \mapsto X^{\text{ad}}$. The previous observation can be lifted to:

**Theorem 1.13.** $\left(\mathbf{A}^1_{K^\flat}\right)^{\text{ad}} \simeq \varprojlim_{x\mapsto x^p} \left(\mathbf{A}^1_K\right)^{\text{ad}}$ *as topological spaces.*

## 2. January 9th

2.1. **Overview (continued).** The examples we were working with is $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$ and $K^\flat = (\mathbf{F}_p((t))_{\text{perf}})^\wedge$. This is the case of a point, and the theory of perfectoid spaces generalizes this to algebras. The two main things we want to generalize are the following:

(1) $K^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} K$ as multiplicative monoids;
(2) if $L/K$ is a finite separable extension, then $L^\flat/K^\flat$ is a finite separable extension. This bootstraps to an equivalence of categories $K_{\text{fét}} \simeq K^\flat_{\text{fét}}$.

**Definition 2.1.** A *perfectoid $K$-algebra* $R$ is a Banach $K$-algebra $R$ such that

(1) the subring $R^\circ = \{\text{power-bounded elements}\} \subseteq R$ is bounded;
(2) the Frobenius $\text{Frob}\colon R^\circ/p \to R^\circ/p$ is surjective.

The condition (2) is most important and it says that there are "lots of $p$-power roots mod $p$".

**Example 2.2.**

(1) The perfectoid analogue of a point is $R = K$.
(2) The perfectoid analogue of the affine line is the following: if $R^\circ = K^\circ[x^{1/p^\infty}]^\wedge$, then

$$R = K\langle x^{1/p^\infty}\rangle := R^\circ[1/p].$$

The Banach structure on $R$ is defined by declaring $R^\circ$ to be the unit ball.

The most basic theorem, which came as the biggest surprise in the story, is the tilting correspondence.

**Theorem 2.3.** [Tilting Correspondence] *There exists an equivalence of categories*

$$\{\text{perfectoid } K\text{-algebras}\} \xrightarrow{\sim} \{\text{perfectoid } K^\flat\text{-algebras}\},$$

*given by the functor* $R \mapsto R^\flat := (R^\circ)^\flat\,[1/t]$.

This should be thought of as a version of the Fontaine-Wintenberger theorem from last time.

**Example 2.4.** If $R = K\langle x^{1/p^\infty}\rangle$, then $R^\flat = K^\flat\langle x^{1/p^\infty}\rangle$. This is sort of an "accident"; one cannot usually write down the tilt this easily.

To each such $R$, we will attach a space $\mathrm{Spa}(R)$ (an adic space in the sense of Huber), defined as

$$\mathrm{Spa}(R) := \left\{ |\cdot|\colon R \to \Gamma\cup\{0\} \text{ semivaluation}\colon \begin{array}{c} \Gamma \text{ is a totally ordered abelian group,} \\ |\cdot| \text{ is continuous,} \\ |f|<1 \text{ for } f\in R^\circ \end{array} \right\} /\text{equivalence.}$$

The identity of the abelian group $\Gamma$ is always denoted by 1. Note that the value group $\Gamma$ varies point by point.

We can define subsets of $\mathrm{Spa}(R)$ via inequalities, e.g. subsets of the form

$$\{|\cdot| \in \mathrm{Spa}(R)\colon |f| \le |g|\}$$

for $f, g \in R$, are the basic building blocks of the theory (contrast to affine schemes, where the basic building blocks are defined by elements of the ring being non-zero); they will be the quasi-compact open subsets of $\mathrm{Spa}(R)$.

This is an example of a perfectoid space, and general perfectoid spaces will be defined by gluing these. The main theorems that we will prove about perfectoid spaces are the following.

**Theorem 2.5.** *Let $R$ be a perfectoid algebra.*

*(1)* [Huber] *Using such subsets (as above) as a basis, get a topology on* $\mathrm{Spa}(R)$ *that makes it a spectral space*[3].
*(2)* [Tilting the analytic topology] *There exists a natural homeomorphism* $\mathrm{Spa}(R) \simeq \mathrm{Spa}(R^\flat)$.
*(3)* [Tilting the étale topology] *There exists an equivalence of sites* $\mathrm{Spa}(R)_{\text{ét}} \xrightarrow{\simeq} \mathrm{Spa}(R^\flat)_{\text{ét}}$.
*(4)* [Structure sheaf] *There exists a sheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(R)$ such that $H^0(X, \mathcal{O}_X) = R$ and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*
*(5)* [Integral structure sheaf] *Set $\mathcal{O}_X^+ \subseteq \mathcal{O}_X$ via $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U)\colon |f(x)| \le 1 \text{ for all } x \in U\}$. Then, $\mathcal{O}_X^+$ is a sheaf and $H^0(X, \mathcal{O}_X^+) = R^\circ$ and $H^i(X, \mathcal{O}_X^+) \overset{a}{=} 0$ for all $i > 0$ (i.e. the higher cohomology is almost zero).*

Now let's start the course!

2.2. **Conventions.** By convention, a (complete) non-Archimedean (NA) field is a field $K$ equipped with a non-Archimedean valuation $|\cdot|\colon K^* \to \mathbf{R}_{>0}$ (we use the multiplicative notation) such that $K$ is complete for the valuation topology.

**Remark 2.6.**

(1) Often, we extend this valuation $|\cdot|$ to $|\cdot|\colon K \to \mathbf{R}_{\ge 0}$ via $|0| = 0$.
(2) Set the value group to be the subgroup $|K^*| \subseteq \mathbf{R}_{>0}$; the valuation ring is $K^\circ = \{f \in K\colon |f| \le 1\}$; the maximal ideal is $K^{\circ\circ} = \{f \in K\colon |f| < 1\} \subset K^\circ$; the residue field is $k := K^\circ/K^{\circ\circ}$ (sometimes write $\tilde{K}$); a pseudouniformizer is any non-zero $\pi \in K^{\circ\circ}$.

**Example 2.7.** Some basic examples are in the table below.

| $K^\circ$ | $K$ | pseudouniformizer | value group |
|---|---|---|---|
| $\mathbf{Z}_p$ | $\mathbf{Q}_p$ | $p$ | $\lvert p\rvert^{\mathbf{Z}} \subseteq \mathbf{R}_{>0}$ |
| $\mathcal{O}_K$ the ring of integers | $K/\mathbf{Q}_p$ finite extension | $p$ | $\lvert p\rvert^{\frac{1}{e}\mathbf{Z}}$, $e$ is the ramification index |
| $\mathbf{Z}_p[p^{1/p^\infty}]^\wedge$ | $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ | $p$ | $\lvert p\rvert^{\mathbf{Z}[1/p]}$ |
| $\mathcal{O}_{\mathbf{C}_p} = \widehat{\overline{\mathbf{Z}_p}}$ | $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}_p}}$ | $p$ | $\lvert p\rvert^{\mathbf{Q}}$ |
| $\mathbf{F}_p[[t]]^\wedge_{\text{perf}}$ | $\mathbf{F}_p((t))^\wedge_{\text{perf}}$ | $t$ | $\lvert t\rvert^{\mathbf{Z}[1/p]}$ |

Among these, the perfectoid examples are the final three.

**Exercise 2.8.**

[3]In the sense of Mel Hochster's thesis [Hoc69], a topological space is *spectral* if it is homeomorphic to the spectrum of a ring.

(1) A subset $S \subseteq K$ is *bounded* if there exists a pseudouniformizer $t$ such that $t \cdot S \subseteq K^\circ$ (equivalently, if you put the metric topology on $K$, then $S$ is a bounded set). An element $f \in K$ is *power-bounded* if $f^{\mathbf{N}} = \{f, f^2, f^3, \ldots\}$ is bounded. Check that $K^\circ$ is the set of power-bounded elements (in particular, you can recover $K^\circ$ intrinsically from the structure of a topological field – you don't need to know the valuation).
(2) Check that $K^{\circ\circ}$ coincides with the set of topologically nilpotent elements, i.e. those $f \in K$ such that $f^n \to 0$ as $n \to \infty$.
(3) Given a valuation ring $K^\circ \subseteq K$, check that you can reconstruct the valuation on $K$ (up to equivalence).

This exercise is the baby example of the fact that Banach algebras can be described completely algebraically in this non-Archimedean setting.

2.3. **Perfections and tilting.** There is a fixed prime $p$ throughout this course. Recall that a characteristic $p$ ring $R$ is *perfect* if the Frobenius $\phi\colon R \xrightarrow{\simeq} R$ is an isomorphism (in this course, any random $\phi$ that shows up is the Frobenius); the ring is *semiperfect* if $\phi$ is surjective (that is, if $p$-th roots exist but are not necessarily unique).

**Definition 2.9.** Let $R$ be a ring.

(1) If $R$ has characteristic $p$, then $R_{\mathrm{perf}} := \varinjlim_\phi R = R^{1/p^\infty}$ and $R^{\mathrm{perf}} := \varprojlim_\phi R$.
(2) [Fontaine] Set $R^\flat := (R/p)^{\mathrm{perf}} = \varprojlim_\phi R/p$, which we view as a topological ring via inverse limit topology (with $R/p$ having the discrete topology).

The elementary things to check are that $R_{\mathrm{perf}}$ and $R^{\mathrm{perf}}$ are perfect, $R_{\mathrm{perf}}$ is universal for maps to perfect rings of characteristic $p$, and $R^{\mathrm{perf}}$ is universal for maps from perfect rings of characteristic $p$.

**Example 2.10.**

(1) $\mathbf{F}_p[t]_{\mathrm{perf}} = \mathbf{F}_p[t^{1/p^\infty}]$ and $\mathbf{F}_p[t]^{\mathrm{perf}} = \mathbf{F}_p$.
(2) If $R/\overline{\mathbf{F}_p}$ is a finite type algebra, then $R^{\mathrm{perf}} = \left(\overline{\mathbf{F}}_p\right)^{\pi_0(\mathrm{Spec}(R))}$, whereas $R_{\mathrm{perf}}$ is quite large and not so easy to write down.
(3) $\left(\mathbf{F}_p[t^{1/p^\infty}]/(t)\right)^{\mathrm{perf}} = \mathbf{F}_p[t^{1/p^\infty}]^\wedge$, where $\wedge$ denotes the $t$-adic completion. More generally, if $R$ is a perfect and $f \in R$ is a non-zero divisor (nzd), then $(R/f)^{\mathrm{perf}}$ is the $f$-adic completion of $R$.

The way to see this is as follows: consider the towers

$$\begin{array}{ccccccc} \ldots \longrightarrow & R/f^{p^2} & \longrightarrow & R/f^p & \longrightarrow & R/f \\ & \downarrow{\scriptstyle \phi^{-2}} & & \downarrow{\scriptstyle \phi^{-1}} & & \downarrow{\scriptstyle =} \\ \ldots \longrightarrow & R/f & \xrightarrow{\phi} & R/f & \xrightarrow{\phi} & R/f \end{array}$$

where the horizontal maps in the top row are the standard ones. The inverse limit of the top row is $\widehat{R}$, and the inverse limit of the bottom row is $(R/f)^{\mathrm{perf}}$, hence they must coincide. In fact, we do not require that $f$ be an nzd, and this works more generally for any finitely-generated ideal of $R$.
(4) $\left(\mathbf{Z}_p[p^{1/p^\infty}]^\wedge\right)^\flat = \left(\left(\mathbf{Z}_p[t^{1/p^\infty}]/(t-p)\right)^\wedge\right)^\flat = \left(\mathbf{F}_p[t^{1/p^\infty}]/t\right)^{\mathrm{perf}} = \mathbf{F}_p[t^{1/p^\infty}]^\wedge$.

**Exercise 2.11.** If $R \to S$ is a surjective map of characteristic $p$ rings with nilpotent kernel (in the sense that some power of the kernel is the zero ideal), then $R^{\mathrm{perf}} \xrightarrow{\simeq} S^{\mathrm{perf}}$ and $R_{\mathrm{perf}} \xrightarrow{\simeq} S_{\mathrm{perf}}$.

**Proposition 2.12.** *If $R$ is a $p$-adically complete ring, the projection $R \to R/p$ induces a (multiplicative) bijection*

$$\varprojlim_{x \mapsto x^p} R \xrightarrow{\simeq} \varprojlim_{x \mapsto x^p} R/p = R^\flat.$$

If $R$ had characteristic $p$, this is obvious because both sides are literally the same.

**Lemma 2.13.** *If $R$ is a ring and $a, b \in R$ such that $a \equiv b \bmod p$, then $a^{p^k} \equiv b^{p^k} \bmod p^{k+1}$ for all $k \geq 1$.*

*Proof of Proposition 2.12.* Let us first prove injectivity. Given sequences $(a_n)$ and $(b_n)$ in $\varprojlim_{x \mapsto x^p} R$ such that $a_n \equiv b_n \bmod p$ for all $n$, we want to prove that $a_n = b_n$ for all $n$. Since they are elements of this inverse limit, we know that $a_{n+k}^{p^k} = a_n$ and similarly for $b_n$. Thus, if $a_{n+k} \equiv b_{n+k} \bmod p$, then by Lemma 2.13, $a_n \equiv b_n \bmod p^{k+1}$. Therefore, by $p$-adic separatedness, we get that $a_n = b_n$ for all $n$.

To prove surjectivity, we will explicitly construct an inverse. Fix a sequence $(\overline{b_n}) \in \varprojlim R/p$. Choose arbitrary lifts $a_n \in R$ of $\overline{b_n}$, then they satisfy $a_{n+k}^{p^k} \equiv a_n \bmod p$. Fix $n$. Consider the sequence $\{a_{n+k}^{p^k}\}_k$. Lemma 2.13 implies that this is a Cauchy sequence for the $p$-adic topology. Therefore, set $b_n = \lim_k a_{n+k}^{p^k}$ and check that $b_n \in \varprojlim_{x \mapsto x^p} R/p$ and it lifts $\overline{b_n}$. $\square$

The upshot is that we get a canonical multiplicative map

$$\begin{tikzcd} \varprojlim_{x \mapsto x^p} R \arrow[r, "\simeq"] \arrow[d] & \varprojlim_{x \mapsto x^p} R/p = R^\flat \arrow[dl, dashed, "\sharp"] \\ R & \end{tikzcd}$$

where $\varprojlim_{x \mapsto x^p} R \to R$ is the first projection.

The image of the map $\sharp$ consists of exactly those $f \in R$ that admit a compatible system $f^{1/p^n}$ of $p$-power roots; we call these elements *perfect* (this is not standard terminology).

**Exercise 2.14.** If $R$ is $p$-adically complete, then $\varprojlim_{x \mapsto x^p} R \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} R/p = R^\flat$ is a homeomorphism (via the $p$-adic topology on $R$).

## 3. January 11th

Two announcements: office hours are Monday & Wednesday from 2:30-4pm, and there is a learning seminar on adic spaces on Thursdays from 4-5:30pm (the plan will be 3-4 talks on Huber's paper [Hub93]).

Last time, we discussed the tilting functor: given a ring $R$, we define $R^\flat = (R/p)^{\mathrm{perf}}$ and proved that if $R$ is $p$-adically complete, there is an alternate description

$$\varprojlim_{x \mapsto x^p} R \xrightarrow{\simeq} \varprojlim_{\phi} R/p = R^\flat,$$

at least as multiplicative monoids. The upshot is that we get $\sharp \colon R^\flat \to R$. Here, $p$ a fixed prime throughout.

### 3.1. Perfectoid fields.

**Definition 3.1.** A *perfectoid field* $K$ is a non-Archimedean field with residue characteristic $p$ such that

(1) $|K^*| \subseteq \mathbf{R}_{>0}$ is not discrete;
(2) the Frobenius $K^\circ/p \to K^\circ/p$ is surjective, i.e. $K^\circ$ is semiperfect.

**Example 3.2.**

(1) $\mathbf{Q}_p$ is not perfectoid because the value group is $\mathbf{Z}$ (though the Frobenius is surjective on the residue field $\mathbf{F}_p$). The condition that $|K^*|$ is not discrete is in the definition to rule out this example.
(2) $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$ is perfectoid. The value group is $|p|^{\mathbf{Z}[1/p]}$, which is not discrete, and we claim that the valuation ring is $\mathbf{Z}_p[p^{1/p^\infty}]$. This is indeed a valuation ring (because valuation rings are closed under direct limit and completions with respect to finitely-generated (or equivalently, principal) ideals); thus, $K^\circ$ and $\mathbf{Z}_p[p^{1/p^\infty}]$ are valuation rings with same fraction field and with one contained in the other, hence they must coincide. Moreover, it is easy to see that $K^\circ/p$ is semiperfect, because there is a presentation

$$\mathbf{Z}_p[p^{1/p^\infty}] = \Big(\mathbf{Z}_p[t^{1/p^\infty}]/(t-p)\Big)^\wedge$$

and when you reduce this presentation mod $p$, it simplifies to $\mathbf{F}_p[t^{1/p^\infty}]/(t)$.

(3) Let $K = \widehat{\overline{\mathbf{Q}_p}} =: \mathbf{C}_p$. The value group is $|p|^{\mathbf{Q}}$ (which is not discrete) and $K^\circ = \widehat{\overline{\mathbf{Z}_p}}$ is the valuation ring (by the same argument as above), where $\overline{\mathbf{Z}_p}$ is the absolute integral closure of $\mathbf{Z}_p$ (i.e. the integral closure of $\mathbf{Z}_p$ in an algebraic closure of $\mathbf{Q}_p$, also denoted $\mathbf{Z}_p^+$ in commutative algebra). Therefore, $K^\circ/p = \overline{\mathbf{Z}_p}/p$ is semiperfect and so the field $K$ is perfectoid.

(4) Let $K$ be a non-Archimedean field of characteristic $p$ and assume $K$ is nontrivially valued. Then, $K$ is perfectoid if and only if it is perfect.

Fix a perfectoid field $K$.

**Lemma 3.3.**

*(1) The value group $|K^*|$ is $p$-divisible;*
*(2) $(K^{\circ\circ})^2 = K^{\circ\circ}$;*
*(3) $K^\circ$ is not noetherian.*

The assertion (3) is there just for psychological purposes, but (2) is the key takeaway (it allows us to use almost mathematics).

*Proof.* Call $x \in K^\circ$ *small*[4] if $|p| < |x| < 1$, so $x$ is non-zero in $K^\circ/p$. As $K^\circ/p$ is semiperfect, we can choose $y \in K^\circ$ such that $y^p = x + pz$ for some $z \in K^\circ$. Taking norms, $|y|^p = |x|$ by the non-Archimedean property. Therefore, $|x|$ is $p$-divisible in $|K^*|$.

It suffices to show that $|x|$, for $x \in K^\circ$ small, generates. The value group $|K^*|$ is not discrete, so $|K^*| \supsetneq |p|^{\mathbf{Z}}$. Thus, there exists $x \in K^*$ such that $|x| \notin |p|^{\mathbf{Z}}$, i.e. $|p|^n < |x| < |p|^{n+1}$ for some $n \in \mathbf{Z}$. After scaling by $|p|^n$, we may assume that $|p| < |x| < 1$, so small elements exist. As $p$ does not divide $x$, $p = xy$ by the property of being a valuation ring. Then, $x$ and $y$ are both small, so $|p|^{\mathbf{Z}} \subseteq \langle |x| \colon x \text{ small}\rangle \subseteq |K^*|$. A similar argument shows that $|y| \in \langle |x| \colon x \text{ small}\rangle$ for all $y \in K^*$. Therefore, $|K^*|$ is $p$-divisible.

For (2), pick $f \in K^{\circ\circ}$. The perfectoid property tells us that $f = g^p + ph$ for some $g \in K^{\circ\circ}$ and $h \in K^\circ$. But, $g^p \in (K^{\circ\circ})^2$ because $g \in K^{\circ\circ}$ and $p = xy$ for $x, y \in K^{\circ\circ}$ by the proof of (1). Therefore, $f \in (K^{\circ\circ})^2$. This proves (2), and then (3) follows from (2) and Nakayama's lemma. $\square$

**Remark 3.4.** We showed in the proof of Lemma 3.3(1) that $|K^*|$ is generated by those $|x|$ with $|p| < |x| < 1$.

**Exercise 3.5.** If $K$ is a perfectoid field, then the module of Kähler differentials $\Omega^1_{K^\circ/\mathbf{Z}_p}$ is $p$-divisible. This says that we cannot use differential geometric techniques, but it will help us use deformation theoretic ones.

The $\sharp$-map is defined by the diagram

$$\begin{tikzcd} \varprojlim_{x\mapsto x^p} K^\circ \arrow[r, "\simeq"] \arrow[d] & \varprojlim_{x\mapsto x^p} K^\circ/p = K^{\circ,\flat} \arrow[dl, dashed, "\sharp"] \\ K^\circ & \end{tikzcd}$$

and we will prove that $\sharp$ has large image.

Fix a pseudouniformizer $\pi \in K^\circ$, i.e. a non-zero element of the maximal ideal, such that $|p| \leq |\pi| < 1$. The usual examples are $\pi = p$ in characteristic 0, and $\pi$ is anything in characteristic $p$.

**Lemma 3.6.** *There exists $t \in K^{\circ,\flat}$ such that $|t^\sharp| = |\pi|$.*

The proof of Lemma 3.6 will be discussed next class.

**Example 3.7.** If $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$ and $\pi = p$, then $K^{\circ,\flat} = \mathbf{F}_p[t^{1/p^\infty}]^\wedge$, where $t = (p, p^{1/p}, p^{1/p^2}, \ldots)$, and we have $t^\sharp = p$.

[4] This terminology is local to this proof.

From now on, assume that $\pi = t^\sharp$.

**Lemma 3.8.** *The ring $K^{\circ,\flat}$ is $t$-adically complete and the $t$-adic topology is the inverse limit topology.*

*Proof.* Consider the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & K^{\circ,\flat}/t^{p^2} \arrow[r] \arrow[d, "f\circ\phi^{-2}"] & K^{\circ,\flat}/t^p \arrow[r] \arrow[d, "f\circ\phi^{-1}"] & K^{\circ,\flat}/t \arrow[d, "f"] \\
\dots \arrow[r] & K^\circ/\pi \arrow[r, "\phi"] & K^\circ/\pi \arrow[r, "\phi"] & K^\circ/\pi
\end{tikzcd}$$

where the horizontal maps in the top row are the standard ones and $f\colon K^{\circ,\flat}/t \to K^{\circ,\pi}$ is the isomorphism (of rings) induced by the canonical map $K^{\circ,\flat} \to K^\circ/\pi$. The inverse limit of the bottom row is $K^{\circ,\flat}$ with the inverse limit topology, and the inverse limit of the top row is the $t$-adic completion of $K^{\circ,\flat}$, so the $t$-adic completion of $K^{\circ,\flat}$ coincides with $K^{\circ,\flat}$. $\square$

**Remark 3.9.** Recall that if $K = \mathbf{Q}_p(\mu_{p^\infty})^\wedge$ and $L = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$, then $K^\flat \simeq L^\flat \simeq (\mathbf{F}_p((t))_{\mathrm{perf}})^\wedge$ (though the isomorphism between $K^\flat$ and $L^\flat$ is very non-canonical).

The functor $K \mapsto K^\flat$ is *not* fully faithful on the category of perfectoid fields over $\mathbf{Q}_p$, as the above example demonstrates (or similarly at the level of valuation rings, which is all that we've defined so far). But, we'll see, it is so on perfectoid fields over a given perfectoid field, i.e. if you work over a fixed perfectoid base, then going from characteristic zero to characteristic $p$ is completely loss-less.

Choose $t \in K^{\circ\flat}$ such that $\pi = t^\sharp$. Set $K^\flat := K^{\circ,\flat}[1/t]$. Endow both $K^{\circ,\flat}$ and $K^\flat$ with the $t$-adic topology.

**Lemma 3.10.** *The multiplicative isomorphism $K^{\circ,\flat} \xrightarrow{\simeq} \varprojlim_{x\mapsto x^p} K^\circ$ extends to a multiplicative isomorphism*

$$K^\flat \xrightarrow{\simeq} \varprojlim_{x\mapsto x^p} K.$$

## 4. January 18th

4.1. **Perfectoid fields (continued).** Let's prove Lemma 3.6 from last time, which is restated below. Recall the setup: $K$ is a perfectoid field (assume $\mathrm{char}(K) = 0$ for simplicity; everything is true in characteristic $p$, but for degenerate reasons). Fix a pseudouniformizer $\pi \in K^\circ$ with $|p| \le |\pi| < 1$; in particular, $\pi$ divides $p$.

The relevant commutative diagram is the following:

$$\begin{tikzcd}
\varprojlim_{x\mapsto x^p} K^\circ \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\simeq"] & K^\circ \arrow[d] \\
K^{\circ,\flat} = \varprojlim_\phi K^\circ/p \arrow[r] \arrow[ur, dashed, "\sharp"] \arrow[d, "\simeq"] & K^\circ/p \arrow[d] \\
\varprojlim_\phi K^\circ/\pi \arrow[r] & K^\circ/\pi
\end{tikzcd}$$

The top square (or two triangles) consists of multiplicative maps of monoids, whereas the bottom square consists of ring maps. Recall that we always topologize $K^{\circ,\flat}$ via the inverse limit topology, where $K^\circ/p$ is equipped with the discrete topology.

**Lemma 4.1.** *There exists $t \in K^{\circ\flat}$ such that $|t^\sharp| = |\pi|$, and $\mathrm{pr}_0$ induces an isomorphism $K^{\circ\flat}/t \xrightarrow{\simeq} K^\circ/\pi$.*

The lemma is saying that, though we may not be able to hit $\pi$ with the $\sharp$ map, we can hit $\pi$ times a unit.

*Proof.* As the value group of any perfectoid field is $p$-divisible, we may choose $f \in K^\circ$ such that $|f|^p = |\pi|$, so $|f| > |\pi|$ and hence $f$ is non-zero in $K^\circ/\pi$. Lift $f$ to some $g \in K^{\circ,\flat}$ and set $t = g^p \in K^{\circ,\flat}$. We know $g^\sharp \equiv f \bmod \pi$ from the diagram; thus, $|g^\sharp| = |f|$, because $|f| > |\pi|$. Therefore, because $\sharp$ is a multiplicative map, $|(g^p)^\sharp| = |g^\sharp|^p = |f|^p = |\pi|$. In particular, $|t^\sharp| = |\pi|$. This proves the first part.

Remark that this shows that the image of $|\cdot|\colon K^\circ \to \mathbf{R}_{\geq 0}$ coincides with the image of $K^{\circ,\flat} \xrightarrow{\sharp} K^\circ \to \mathbf{R}_{\geq 0}$ (because the small elements from last time generate the value group).

We know that $t$ maps to zero in $K^\circ/\pi$, because it maps to $g^p$ and $g$ is a $p$-th root of $\pi$. Therefore,we get $K^{\circ,\flat}/t \twoheadrightarrow K^\circ/\pi$, which is surjective because the original map $K^{\circ,\flat} \twoheadrightarrow K^\circ/\pi$ was so. Say $h \in K^{\circ,\flat}$ is such that $h$ maps to zero in $K^\circ/\pi$, then the diagram tells you that $h^\sharp \in (\pi) = (t^\sharp)$, where the ideals are equal by the first part. Therefore, $h^\sharp = at^\sharp$ for $a \in K^\circ$. Then, $a_n := (h^{1/p^n})^\sharp/(t^{1/p^n})^\sharp \in K$ satisfies $a_n^{p^n} = a \in K^\circ$. As the valuation ring $K^\circ$ is integrally closed, $a_n \in K^\circ$. Therefore, we get that $(h^{1/p^n})^\sharp = (a_n(t^{1/p^n})^\sharp) \in \varprojlim_{x\mapsto x^p} K^\circ$, and so $h = \tilde{a}t$ in $K^{\circ,\flat}$. □

The corollary, which we proved last time, was:

**Corollary 4.2.** *The ring $K^{\circ,\flat}$ is $t$-adically complete and the natural topology is the $t$-adic topology.*

*Proof.* We showed there was an isomorphism of complexes

$$\begin{array}{ccccccc} \ldots \longrightarrow & K^{\circ,\flat}/t^{p^2} & \longrightarrow & K^{\circ,\flat}/t^p & \longrightarrow & K^{\circ,\flat}/t \\ & \downarrow & & \downarrow & & \downarrow \\ \ldots \longrightarrow & K^\circ/\pi & \xrightarrow{\phi} & K^\circ/\pi & \xrightarrow{\phi} & K^\circ/\pi \end{array}$$

and taking the inverse limit yields the result. □

**Proposition 4.3.** *If $R$ is a $p$-adically complete valuation ring (possibly of higher rank), then $R^\flat$ is also a valuation ring. In fact, if $R \xrightarrow{|\cdot|} \Gamma \cup \{0\}$ is the valuation on $R$, then $R^\flat \xrightarrow{\sharp} R \xrightarrow{|\cdot|} \Gamma \cup \{0\}$ gives the corresponding valuation on $R^\flat$.*

In this generality, the value groups of $R$ and of $R^\flat$ could be different (e.g. if $R = \mathbf{Z}_p$ and $R^\flat = \mathbf{F}_p$). However, this tilting procedure will not change the value group in the perfectoid case.

*Proof.* First, we prove that $R^\flat$ is a domain. Here, we think of $R^\flat$ as $\varprojlim_{x\mapsto x^p} R$. Fix two sequences $(a_n), (b_n) \in R^\flat$ such that $a_n b_n = 0$ for all $n$. Thus, $a_0 = 0$ or $b_0 = 0$ (as $R$ is a domain). As $a_n^{p^n} = a_0$ and similarly for $b_n$, either $a_n = 0$ for all $n$ or $b_n = 0$ for all $n$, so $R^\flat$ is a domain.

Second, we prove that $R^\flat$ is a valuation ring. There are many different characterizations of valuation rings, but the one we want to use is the following: a domain is a valuation ring iff for any two elements, one divides the other. Take $(a_n), (b_n) \in R^\flat$, then either $a_0$ divides $b_0$ or $b_0$ divides $a_0$ (as $R$ is a valuation ring). In $R$, $x|y$ iff $x^p|y^p$. As the transition maps are just raising to the $p$-th power, we conclude that either $a_n|b_n$ for all $n$, or vice versa. □

**Remark 4.4.** The proof shows that if $(a_n), (b_n) \in R^\flat$, then $(a_n)|(b_n)$ iff $a_0|b_0$ or equivalently, $(a_n)^\sharp|(b_n)^\sharp$ (because the sharp is the zeroth index). However, this says exactly that $|(a_n)^\sharp| \geq |(b_n)^\sharp|$. The upshot: denote the composition by $|\cdot|^\flat\colon R^\flat \xrightarrow{\sharp} R \xrightarrow{|\cdot|} \Gamma \cup \{0\}$, then given $a, b \in R^\flat$, $a|b$ iff $|a|^\flat \geq |b|^\flat$. Therefore, $|\cdot|^\flat$ is the valuation on $R^\flat$.

One should check that $|\cdot|^\flat$ satisfies the non-Archimedean inequality, which involves unraveling the definition (and this would give an alternate proof that it is the valuation on $R^\flat$).

**Proposition 4.5.** *Fix $t \in K^{\circ,\flat}$ as in Lemma 4.1.*

*(1) $K^{\circ,\flat}$ is a valuation ring and $K^\flat := K^{\circ,\flat}[1/t]$ is a field.*[5]
*(2) The ideal $(t^{1/p^\infty})$ is maximal, and $\dim(K^{\circ,\flat}) = 1$.*
*(3) The valuation topology on $K^{\circ,\flat}$ coincides with the $t$-adic topology. In particular, $K^\flat$ is a perfectoid field.*
*(4) The value groups $|K^*|$ and $|(K^\flat)^*|$ are identical.*

*Proof.* For (1), we know that the rank of $K^{\circ,\flat}$ is $\leq 1$; but, $0 < |t| < 1$, so the rank of $K^{\circ,\flat}$ is not zero (because the only way a valuation ring has rank zero is if it's a field), so the rank is 1. Therefore, inverting any non-zero element in the maximal ideal of $K^{\circ,\flat}$ gives a field.

For (2), as the rank is also the Krull dimension of the valuation ring, we've already shown that $\dim(K^{\circ,\flat}) = 1$. It remains to show $(t^{1/p^\infty})$ is maximal. We know $K^{\circ,\flat}/t \simeq K^\circ/\pi$, so the maximal ideal of $K^{\circ,\flat}/t$ is nilpotent (because it is true for $K^\circ/\pi$), but that means that it is actually equal to $(t^{1/p^\infty})$. (Here, we are using that if $R$ is perfect and $I = (f^{1/p^\infty})$, then $R/I$ is perfect, hence $I$ is radical)

For (3), observe that $|f|^\flat$ is small iff $t$ to some large power divides $f$. Thus, open neighborhoods of 0 in both topologis coincide. Therefore, $K^{\circ,\flat}$ is complete for the valuation topology, and hence $K^\flat$ is a perfectoid field (it's clear that $K^\flat$ is perfect, we just needed completeness).

For (4), we know that for any $\pi \in K^\circ$ such that $|p| \leq |\pi| < 1$, there exists a $t \in K^{\circ,\flat}$ such that $|t|^\flat = |\pi|$. There is the obvious containment $|K^{\circ,\flat}|^\flat \subseteq |K^\circ|$. But, generators of $|K^\circ|$ are given by $|\pi|$, with $|\pi|$ as above. Therefore, the value groups are identical. ☐

**Remark 4.6.** A similar proof shows that there is a bijection

$$\Big\{\text{continuous}^{6} \text{ valuation } K \to \Gamma \cup \{0\}\Big\} \to \Big\{\text{continuous valuation on } K^\flat\Big\},$$

where the map is $|\cdot| \mapsto |\cdot|^\flat$. (Here, $K$ is still perfectoid.) To prove this, you show that the valuation rings of the continuous valuations correspond bijectively to valuation rings lying between $K^{\circ\circ}$ and $K^\circ$, or said differently, valuation subrings of the residue field (which is the same for both $K$ and $K^\flat$).

As an example of such a continuous valuation, take $k = \mathbf{F}_p(t)_{\mathrm{perf}}$ and $K = \left(W(k)[p^{1/p^\infty}]^\wedge\right)[1/p]$, where $W(k)$ is the ring of Witt vectors of $k$. This is a "large" perfectoid field, where the residue field is not finite. Choose a nontrivial valuation ring $\overline{R} \subseteq k$ (this is a rank-1 valuation ring), and set $R$ to be the preimage of $\overline{R}$ under the map $K^\circ \twoheadrightarrow k$ (this is rank-2 valuation ring). Therefore, the valuation associated to $R$ gives a continuous rank-2 valuation on $K$.

**Theorem 4.7.** [Almost purity in dimension zero] *Let $K$ be perfectoid, $L/K$ a finite extension (necessarily separable), and give $L$ its natural topology as a finite-dimensional $K$-vector space*[7] *Then:*
*(1) $L$ is perfectoid;*
*(2) $L^\flat/K^\flat$ is a finite extension of the same degree as $L/K$;*
*(3) $L \mapsto L^\flat$ gives an equivalence $K_{\text{fét}} \xrightarrow{\simeq} K^\flat_{\text{fét}}$;*
*Here, $L^\flat := (L^\circ)^\flat[1/t]$ for $t \in K^\flat$ a pseudouniformizer.*

The proof of almost purity will have to wait until we have set up the machinery of almost mathematics, but once we have done that, the key ingredient will be the following.

**Proposition 4.8.** [Kedlaya] *If $K^\flat$ is algebraically closed, then so is $K$.*

Of course, if we knew (3) from the almost purity theorem, this would be obvious. The following proof appears in detail in [KL15, Lemma 3.5.5], as well as in [Sch12, Proposition 3.8].

*Proof.* Assume $\mathrm{char}(K) = 0$ (else there is nothing to prove). Choose a monic polynomial $P(T) \in K^\circ[T]$ of degree $d$. The strategy is to construct a sequence $x_n \in K^\circ$ such that

[5]This is an abuse of notation: $K^\flat$ is not the tilt of $K$.

[7]As the valuation on $K$ is complete, there is a unique extension of the valuation on $K$ to one on $L$, and the corresponding valuation topology coincides with the topology $L$ acquires by viewing it as $K^n$ for some $n$.

(1) $|P(x_n)| \leq |p|^n$;
(2) $|x_{n+1} - x_n| \leq |p|^{n/d}$.

Then, (2) implies that the sequence $(x_n)$ converges to some $x \in K^\circ$, and (1) implies that $P(x) = 0$.

Set $x_0 = 0$. Recursively, define $x_0, x_1, \ldots, x_n$ satisfying (1) and (2). Consider $P(T + x_n) = \sum_{i=0}^d b_i T^i$. We may assume that $b_0 \neq 0$ (otherwise, take $x_{n+1} = x_n$). Consider the quantity

$$C = \min \left\{ \left| \frac{b_0}{b_j} \right|^{1/j} : j > 0 \ , \ b_j \neq 0 \right\}.$$

Considering the case $j = d$ (so $b_d = 1$ because $P$ is monic), $C \leq |b_0|^{1/d} \leq 1$. As $K^\flat$ is algebraically closed, $|K^*| = |(K^\flat)^*|$ is uniquely divisible (i.e. as an abelian group, it is $n$-divisible and has no $n$-torsion for all $n$). We can choose $u \in K^\circ$ such that $C = |u|$. Also, $\frac{b_i}{b_0} \cdot u^i \in K^\circ$. Moreover, there exists an $i$ such that $\frac{b_i}{b_0} \cdot u^i$ is a unit (namely, the index $i$ achieving the minimum in the definition of $C$).

**Claim 4.9.** Consider a lift $Q(T) \in K^{\circ,\flat}[T]$ such that $Q(T) \mapsto \sum_{i=0}^d \frac{b_i}{b_0} u^i T^i$ in $(K^\circ/p)[T]$. Then, there exists a unit $v \in \left(K^{\circ,\flat}\right)^*$ such that $Q(v) = 0$.

The proof of the claim uses Newton polygons, and it will not be discussed further. Now, set $x_{n+1} = x_n + u \cdot v^\sharp$. Then,

$$P(x_{n+1}) = P(x_n + uv^\sharp) = \sum_{i=0}^d b_i u^i (v^\sharp)^i = b_0 \underbrace{\left( \sum_{i=0}^d \frac{b_i}{b_0} u^i (v^\sharp)^i \right)}_{\substack{= Q(v) \bmod p \\ = 0 \bmod p}}$$

Therefore, $|P(x_{n+1})| \leq |b_0| \cdot |p| = |P(x_n)| \cdot |p| \leq |p|^n \cdot |p| = |p|^{n+1}$. This shows (1), and (2) is proven similarly. $\square$

## 5. January 23rd

Today, we will begin discussing the basics of almost mathematics. The notes distributed by Bhargav have been integrated into these notes (verbatim in many sections), though here the material is presented in the same order as during the lecture.

5.1. **Almost mathematics.** Let $R$ be a commutative ring, and $I \subset R$ an ideal. There is a forgetful functor $i_* \colon \mathbf{Mod}_{R/I} \to \mathbf{Mod}_R$ given by restriction of scalars, and it has adjoints on both sides:

(1) a left adjoint $i^* \colon \mathbf{Mod}_R \to \mathbf{Mod}_{R/I}$ given by $i^*(M) = M \otimes_R R/I$;
(2) a right adjoint $i^! \colon \mathbf{Mod}_R \to \mathbf{Mod}_{R/I}$ given by $i^!(M) = \mathrm{Hom}_R(R/I, M) = M[I] = \{m \in M : I \cdot m = 0\}$.

Let us recall the context in which these types of functors first arose: for a topological space $X$, the inclusion $j \colon U \hookrightarrow X$ of an open subset, and the inclusion $i \colon Z \hookrightarrow X$ where $Z = X \backslash U$, we have similar functors. More precisely, we get the picture

$$\begin{tikzcd} \mathrm{Shv}(Z) \arrow[r, "i_*"] & \mathrm{Shv}(X) \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \arrow[r, "j^*"] & \mathrm{Shv}(U). \arrow[l, bend right, "j_*"'] \arrow[l, bend left, "j_!"] \end{tikzcd}$$

Here, $\mathrm{Shv}(\cdot)$ denotes the category of sheaves of sets (or possibly abelian groups), and, by convention, a left adjoint appears directly above its right adjoint. More precisely, the adjoint pairs are $(i_*, i^!)$, $(i^*, i_*)$, $(j_!, j^*)$, and $(j^*, j_*)$. There are some relations between them, which we won't write out.

The goal is to build an analog using $\mathbf{Mod}_{R/I}$ instead of $\mathrm{Shv}(Z)$, and $\mathbf{Mod}_R$ instead of $\mathrm{Shv}(X)$. As $\mathbf{Mod}_R$ consists of quasicoherent sheaves on $\mathrm{Spec}(R)$ and $\mathbf{Mod}_{R/I}$ consists of quasicoherent sheaves on $\mathrm{Spec}(R/I)$, the obvious guess would be to look at quasicoherent sheaves on the complement of $\mathrm{Spec}(R/I)$ in $\mathrm{Spec}(R)$, but this does not work. Rather, one should think of the quotient (which will be the category $\mathbf{Mod}_R^a$ of almost modules) as quasicoherent sheaves on some non-existent open set $\overline{U} \subseteq X$ that contains $U$.

The following are our standing assumptions for today: $I \subset R$ is flat and $I^2 = I$ (in the noetherian case, this happens only if $I$ is generated by idempotents). This implies that $I \otimes_R I \simeq I^2 \simeq I$, so $I$ is an "idempotent in the ring of $R$-modules". Almost mathematics attempts to "project along this idempotent".

**Example 5.1.** The main examples are the following:

(1) Let $R = K^\circ$ for a perfectoid field $K$ and $I = K^{\circ\circ}$. As torsion-free modules over valuation rings are flat, it is easy to see that $I$ is flat. In fact, we can be more explicit: if $t \in K^{\flat,\circ}$ is a pseudo-uniformizer, then $a = t^\sharp$ has a compatible system of $p$-power roots, and $I = (a^{1/p^\infty})$. In particular, $I = \operatorname{colim}_n(a^{1/p^n})$ is a countable union of free $K^\circ$-modules, so it is flat of projective dimension $\leq 1$.

(2) Let $R$ be a perfect ring of characteristic $p$, and $I = (f^{1/p^\infty})$ for some $f \in R$. It is clear that $I^2 = I$. To show that $I$ is flat, there is an easy case: if $f \in R$ is a non-zero divisor, then $I = \bigcup_n (f^{1/p^n}) \simeq \bigcup_n R \cdot f^{1/p^n}$ is free, because a filtered limit of flat modules is flat. In general (i.e. if we allow $f$ to be a zero divisor), consider the inductive system defined by $M_i = R$ for $i > 0$, where the transition map $M_i \to M_{i+1}$ is given by multiplication by $f^{\frac{1}{p^i} - \frac{1}{p^{i+1}}}$. There is a map of inductive systems, described below:

$$\begin{tikzcd}
M \coloneqq \operatorname{colim}_n \Big( & M_0 \arrow[r, "f^{1-\frac{1}{p}}"] \arrow[d, "1 \mapsto f"] & M_1 \arrow[r, "f^{\frac{1}{p} - \frac{1}{p^2}}"] \arrow[d, "1 \mapsto f^{\frac{1}{p}}"] & M_2 \arrow[r] \arrow[d] & \dots \arrow[r] & M_n \arrow[r, "f^{\frac{1}{p^n} - \frac{1}{p^{n+1}}}"] \arrow[d, "1 \mapsto f^{\frac{1}{p^n}}"] & M_{n+1} \arrow[r] \arrow[d, "1 \mapsto f^{\frac{1}{p^{n+1}}}"] & \dots \Big) \\
I = \operatorname{colim}_n \Big( & (f) \arrow[r, hook] & (f^{\frac{1}{p}}) \arrow[r, hook] & (f^{\frac{1}{p^2}}) \arrow[r, hook] & \dots \arrow[r, hook] & (f^{\frac{1}{p^n}}) \arrow[r, hook] & (f^{\frac{1}{p^{n+1}}}) \arrow[r, hook] & \dots \Big)
\end{tikzcd}$$

As each $M_i$ is free, $M$ is flat (again using that a filtered colimit of flat modules is flat). Therefore, it suffices to show that the induced map $M \to I$ is an isomorphism. It is clearly surjective (the generator of each ideal lies in the image). To prove the injectivity of $M \to I$, take $\alpha \in M_n$ such that $\alpha$ maps to $0 \in I$. Then, we know $\alpha f^{1/p^n} = 0 \in I$, so $\alpha^{p^n} f = 0$, and hence $\alpha^{p^m} f = 0$ for all $m \geq n$. If $R$ is perfect, then $\alpha f^{1/p^m} = 0$ for all $m \geq n$, and hence the transition map $M_n \to M_{n+1}$ kills $\alpha$. Thus, $\alpha = 0$ in $M$ and $M \to I$ is injective.

**Proposition 5.2.**

*(1) The image of $i_*$ is closed under subquotients and extensions. In particular, $i_*$ realizes $\mathbf{Mod}_{R/I}$ as an abelian Serre subcategory of $\mathbf{Mod}_R$, so the quotient $\mathbf{Mod}_R^a \coloneqq \mathbf{Mod}_R / \mathbf{Mod}_{R/I}$ exists by general nonsense (see* [Sta17, Tag 02MS]*).*

*(2) The quotient functor $q\colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ admits fully faithful left and right adjoints. Thus, $q$ commutes with all limits and colimits.*

*(3) The image of $i_*$ is a "tensor ideal" of $\mathbf{Mod}_R$, so the quotient $\mathbf{Mod}_R^a$ inherits a symmetric monoidal $\otimes$-product from $\mathbf{Mod}_R$.*

*(4) The $\otimes$-structure on $\mathbf{Mod}_R^a$ is closed, i.e. to $X, Y \in \mathbf{Mod}_R^a$, one can functorially associate an object $\operatorname{alHom}(X, Y) \in \mathbf{Mod}_R^a$ equipped with a functorial isomorphism*

$$\operatorname{Hom}(Z \otimes X, Y) \simeq \operatorname{Hom}(Z, \operatorname{alHom}(X, Y)).$$

To understand the quotient $\mathbf{Mod}_R^a$, we will construct it explicity. Consider the full subcategory

$$\mathcal{A} = \{M \in \mathbf{Mod}_R \colon I \otimes_R M \xrightarrow{\simeq} M\},$$

or equivalently, $\mathcal{A}$ is the essential image of the idempotent functor $M \mapsto I \otimes_R M$ on $\mathbf{Mod}_R$. As $I$ is flat, one can check that $\mathcal{A}$ is an *abelian* subcategory of $\mathbf{Mod}_R$, closed under taking kernels, cokernels, images, and extensions.

The category $\mathcal{A}$ will be the category of almost modules, but we first need to construct 3 auxiliary functors in order to realize $\mathcal{A}$ as a quotient of $\mathbf{Mod}_R$.

(1) Let $j_!\colon \mathcal{A} \to \mathbf{Mod}_R$ be the inclusion, which is exact.

(2) The inclusion $j_!$ has an exact right adjoint $j^*\colon \mathbf{Mod}_R \to \mathcal{A}$ given by $j^*(M) = I \otimes_R M$. In addition, the unit map $N \to j^* j_! N$ is an isomorphism for any $N \in \mathcal{A}$, hence $j^*$ is fully faithful.

*Proof.* We first note that $I \otimes_R M \in \mathcal{A}$, as $I \otimes_R I \simeq I$, so we have a well-defined functor. The exactness is clear from the flatness of $I$. For adjointness, fix some $N \in \mathcal{A}$ and $M \in \mathbf{Mod}_R$. We must show that $\mathrm{Hom}_{\mathcal{A}}(N, I \otimes_R M) \simeq \mathrm{Hom}_{\mathbf{Mod}_R}(N, M)$. Using the exact triangle

$$I \otimes_R M \to M \to M \otimes^L_R R/I,$$

it is enough to show that $\mathrm{RHom}_R(N, M \otimes^L_R R/I) \simeq 0$. By adjointness, we have

$$\mathrm{RHom}_R(N, M \otimes^L_R R/I) \simeq \mathrm{RHom}_{R/I}(N \otimes^L_R R/I, M \otimes^L_R R/I).$$

But, our hypothesis on $N$ tells us that $N \otimes_R I \simeq N$, and that this tensor product is derived (by flatness of $I$). By associativity of (derived) tensor products, it is enough to show that $I \otimes^L_R R/I \simeq 0$. This follows from the flatness of $I$ and the hypothesis $I = I^2$.

Granting adjointness, the assertion about the unit map results from the isomorphism $I \simeq I \otimes_R I$. The full faithfulness of $j_!$ is immediate from the definition. $\square$

(3) The functor $j^*$ has a right adjoint $j_*\colon \mathcal{A} \to \mathbf{Mod}_R$ given by $j_*(M) = \mathrm{Hom}_R(I, M)$. In addition, the counit map $j^* j_* M \to M$ is an isomorphism for any $M \in \mathcal{A}$, hence $j_*$ is fully faithful.

*Proof.* Fix $N \in \mathbf{Mod}_R$. Then,

$$\mathrm{Hom}_{\mathcal{A}}(j^*N, M) = \mathrm{Hom}_R(I \otimes_R N, M) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_R(I, M)) = \mathrm{Hom}_R(N, j_*(M)),$$

which proves the adjointness; here we use Hom-$\otimes$ adjunction to get the second isomorphism.

For the rest, fix $M \in \mathcal{A}$. We must show that $I \otimes_R \mathrm{Hom}_R(I, M) \simeq M$ via the natural "evaluation" map. As tensoring with $I$ is exact, it suffices to show the stronger statement that $I \otimes^L_R \mathrm{RHom}_R(I, M) \simeq M$. As $M \in \mathcal{A}$, we have $I \otimes^L_R M \simeq M$, so it is enough to check that the natural map $M \to \mathrm{RHom}_R(I, M)$ induces an isomorphism after tensoring with $I$. Using the exact triangle

$$\mathrm{RHom}_R(R/I, M) \to M \to \mathrm{RHom}_R(I, M),$$

this reduces to showing that tensoring with $I$ kills the term on the left. However, the term on the left admits the structure of an $R/I$-complex, so we can write

$$I \otimes^L_R \mathrm{RHom}_R(R/I, M) \simeq I \otimes^L_R R/I \otimes^L_R \mathrm{RHom}_R(R/I, M).$$

We now conclude using that $I \otimes^L_R R/I \simeq 0$, as before.

The full faithfulness of $j_*$ is a formal consequence of the counit being an isomorphism. $\square$

All of the relevant functors are collected in the following diagram:

$$\begin{tikzcd} \mathbf{Mod}_{R/I} \arrow[r, "i_*"] & \mathbf{Mod}_R \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \arrow[r, "j^*"] & \mathcal{A}. \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

By convention, a left adjoint appears directly above its right adjoint, so we have adjoint pairs $(i^*, i_*)$, $(i_*, i^!)$, $(j_!, j^*)$, and $(j^*, j_*)$.

**Lemma 5.3.** *(1) The compositions $i^* j_!$, $i^! j_*$, and $j^* i_*$ are all naturally isomorphic to the zero functor.*

*(2) The kernel of $j^*$ is exactly $\mathbf{Mod}_{R/I}$.*

*Proof.* To see that the composition $i^* j_!$ is zero, we must show that if $M \simeq I \otimes_R M$, then $M \otimes_R R/I \simeq 0$. This follows by observing that $I \otimes_R R/I \simeq 0$.

To see that the composition $i^! j_*$ is zero, we must show that if $M \simeq I \otimes_R M$, then $\mathrm{Hom}_R(R/I, \mathrm{Hom}_R(I, M)) \simeq 0$. But, the Hom-$\otimes$ adjunction identifies this with $\mathrm{Hom}_R(R/I \otimes_R I, M)$, so we conclude using $R/I \otimes_R I \simeq 0$.

To see that the composition $j^*i_*$ is zero, we must show that $M \otimes_R I \simeq 0$ if $M$ is $I$-torsion, but this follows from $M \otimes_R I \simeq M \otimes_R R/I \otimes_{R/I} I \simeq 0$, where the last equality uses that $R/I \otimes_R I \simeq 0$.

For (2), given $M \in \mathbf{Mod}_R$ with $j^*(M) = I \otimes_R M \simeq 0$, we must check that $M$ is $I$-torsion. Tensoring the standard exact sequence $0 \to I \to R \to R/I \to 0$ with $M$ shows that $M \simeq M/IM$, so $M$ is $I$-torsion. ☐

The following remark explains why a more naive definition of the category $\mathcal{A}$ runs into trouble.

**Remark 5.4.** For any $M \in \mathbf{Mod}_R$, the action map $I \otimes_R M \to M$ has image inside $IM \subseteq M$, and thus if $M \in \mathcal{A}$, then $IM = M$. However, the converse need not be true.

Consider $R = k[t]_{\mathrm{perf}}$ for a perfect field $k$ of characteristic $p$. Let $I = (t^{1/p^\infty})$, so $R/I \simeq k$ is the residue field at the origin. Let $M \subset R/t$ be the maximal ideal inside the local ring $R/t$; this is the image of $I/t$ inside $R/t$, but it does not coincide with $I/t$. As $I = I^2$, it is easy to see that $M = IM$. However, the action map $I \otimes_R M \to M$ need not be injective.

The kernel of the action map is $\mathrm{Tor}_1^R(R/I, M)$. To calculate this, we use the defining exact sequence

$$0 \to M \to R/t \to R/I \to 0.$$

Tensoring with $R/I$ and using that[8] $\mathrm{Tor}_i^R(R/I, R/I) = 0$ for $i > 0$, we learn that $\mathrm{Tor}_1^R(M, R/I) \simeq \mathrm{Tor}_1^R(R/t, R/I)$. The latter group is computed to be non-zero using the standard resolution $\left(R \xrightarrow{t} R\right)$ of $R/t$, so the claim follows.

*Proof of Proposition 5.2.* For (1), it is clear that a subquotient of an $I$-torsion $R$-module is also $I$-torsion. For extensions: if an $R$-module $M$ can be realized as an extension of two $R$-modules killed by $I$, then $M$ is itself killed by $I^2$, and thus also by $I$, as $I = I^2$. The rest is by category theory, but we shall construct an explicit candidate for $\mathbf{Mod}_R^a$ in the proof of (2).

For (2), we claim that the functor $j^* \colon \mathbf{Mod}_R \to \mathcal{A}$ introduced above provides an explicit realization of the quotient functor $q \colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$. To check this, we must show that:

- $j^*(\mathbf{Mod}_{R/I}) = 0$: this amounts to showing that $I \otimes_R M = 0$ for an $R$-module $M$ killed by $I$. But, for such $M$, we have $I \otimes_R M = I \otimes_R R/I \otimes_{R/I} M = I/I^2 \otimes_R M = 0$, as $I = I^2$.
- $j^*$ is exact: this follows from the description $j^*(M) = I \otimes_R M$ and the flatness of $I$.
- $j^*$ is universal with the previous 2 properties: let $q' \colon \mathbf{Mod}_R \to \mathcal{B}$ be an exact functor of abelian categories such that $q'(\mathbf{Mod}_{R/I}) = 0$. Fix $M \in \mathbf{Mod}_R$. We then have the canonical action map $I \otimes_R M \to M$. The kernel and cokernel of this map are identified with $\mathrm{Tor}_1^R(R/I, I)$ and $R/I \otimes_R I$, respectively. In particular, the kernel and cokernel are killed by $I$, and thus also by $q'$. As $q'$ is exact, we learn that $q'(I \otimes_R M) \simeq q'(M)$; however, $I \otimes_R M = j_! j^*(M)$, so we have shown that $q' \simeq q' j_! j^*$, so $q$ factors through $j^*$, as desired.

Granted this, the existence of fully faithful left and right adjoint was verified above, via the construction of $j_!$ and $j_*$.

For (3), if $M$ is killed by $I$, so too is $M \otimes_R N$ for any $R$-module $N$. By category theory, this implies that the symmetric monoidal structure on $\mathbf{Mod}_R$ passes to the quotient $\mathcal{A}$. In particular, $j^*$ is symmetric monoidal.

For (4), given $M, N \in \mathbf{Mod}_R$, we simply set $\mathrm{alHom}\,(j^*(M), j^*(N)) := j^* \mathrm{Hom}_R(M, N)$; this is well-defined by (3), as $\mathrm{Hom}_R(M, N)$ is $I$-torsion if either $M$ or $N$ is so. The rest of the proof is left as an exercise (or one can read [GR03, 2.2.11]). ☐

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[8]This is a general fact about perfect rings. In our case, we may prove this as follows: writing $I = \bigcup_n I_n$ with $I_n := (t^{1/p^n})$, we get $\mathrm{Tor}_i^R(R/I, R/I) \simeq \mathrm{colim}_n \mathrm{Tor}_i^R(R/I_n, R/I_n)$; here, we use that Tor commutes with direct limits in either variable. Using the standard resolution $\left(R \xrightarrow{t^{1/p^n}} R\right)$ of $R/I_n$, we see that the Tor's vanish for $i > 1$. For $i = 1$, we have a canonical identification $\mathrm{Tor}_1^R(R/I_n, R/I_n) \simeq I_n/I_n^2$. We now observe that the transition maps $R/I_n \to R/I_{n+1}$ induce the zero map $I_n/I_n^2 \to I_{n+1}/I_{n+1}^2$, as $I_n \subset I_{n+1}^2$. It follows that $\mathrm{colim}_n \mathrm{Tor}_1^R(R/I_n, R/I_n) = 0$, as desired.

**Exercise 5.5.** Let $k$ be a perfect field of characteristic $p$, let $R = k[t^{1/p^\infty}]$, and let $I = (t^{1/p^\infty})$. Show that the extension of scalars functor $\mathbf{Mod}_R \to \mathbf{Mod}_{R[t^{-1}]}$ given by $M \mapsto M \otimes_R R[t^{-1}]$ factors through $\mathbf{Mod}_R^a$, and that the resulting functor $\mathbf{Mod}_R^a \to \mathbf{Mod}_{R[t^{-1}]}$ is not an equivalence.

## 6. January 25th

6.1. **Almost mathematics (continued).** Last time, $R$ was a ring, $I \subset R$ was an ideal, and our standing assumption for almost mathematics was the following: $I$ is a flat ideal and $I^2 = I$. The example most relevant to us is $R = K^\circ$ for a perfectoid field $K$, and $I = K^{\circ\circ}$ (or even any rank-1 valuation ring whose value group is not discrete).

If you have the crucial condition $I^2 = I$, $\mathbf{Mod}_{R/I} \subseteq \mathbf{Mod}_R$ is an abelian Serre subcategory. Because of this, we can talk about the quotient: by general categorical nonsense, there is a quotient functor $q\colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a \coloneqq \mathbf{Mod}_R/\mathbf{Mod}_{R/I}$. The goal from last time was to construct the quotient $q$ explicitly, without the categorical nonsense.

The strategy is to consider the full subcategory $\mathcal{A} = \{M \in \mathbf{Mod}_R \colon I \otimes M \xrightarrow{\simeq} M\}$ of $\mathbf{Mod}_R$. Remark that the condition that $I \otimes M \xrightarrow{\simeq} M$ is stronger than asking that $IM = M$. Moreover, we have 3 important functors:

1. The inclusion $j_! \colon \mathcal{A} \to \mathbf{Mod}_R$, which is obviously exact.
2. The right adjoint $j^* \colon \mathbf{Mod}_R \to \mathcal{A}$ to $j_!$ (this will be our quotient functor $q$). Explicitly, $j^*$ is given by $M \mapsto I \otimes M$, and this is exact because $I$ is flat.
3. The right adjoint $j_* \colon \mathcal{A} \to \mathbf{Mod}_R$ to $j^*$, which is left exact (because it is a right adjoint) but it is not necessarily exact. Explicitly, $j_*$ is given by $M \mapsto \mathrm{Hom}(I, M)$.

**Theorem 6.1.** *The quotient functor $q\colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ is equivalent to $j^*\colon \mathbf{Mod}_R \to \mathcal{A}$.*

*Proof.* The defining property of the Serre quotient is that it is exact, it kills $\mathbf{Mod}_{R/I}$, and it is universal with these properties. Thus, we need to check the following:

1. $j^*$ is exact, which is ok;
2. $j^*(\mathbf{Mod}_{R/I}) = 0$: if $M$ is $I$-torsion, then $I \otimes_R M = I \otimes_R R/I \otimes_{R/I} M = I/I^2 \otimes_{R/I} M = 0$;
3. $j^*$ is universal with these properties: if $q' \colon \mathbf{Mod}_R \to \mathcal{B}$ satisfies (1) and (2), fix $M \in \mathbf{Mod}_R$. Both the kernel and cokernel of the action map $I \otimes M \to M$ are $R/I$-modules, so they are killed by $q'$. Exactness implies that $q'(I \otimes M) \xrightarrow{\simeq} q'(M)$, i.e.
$$q'(j_!j^*M) = q'(I \otimes M) \xrightarrow{\simeq} q'(M),$$
so we get a functorial factorization $q' = q'j_!j^*$ of $q'$ through $j^*$.

□

We will often forget the $j$ in the notation, which is inherited from topology, to be consistent with [GR03].

**Definition 6.2.** Fix $M \in \mathbf{Mod}_R$. Define
$$M^a \coloneqq j^*(M) \in \mathbf{Mod}_R^a,$$
$$M_* \coloneqq j_*(j^*M) = j_*(M^a) = \mathrm{Hom}_R(I, M) \in \mathbf{Mod}_R,$$
$$M_! \coloneqq j_!(M^a) = I \otimes_R (M^a)_* \in \mathbf{Mod}_R.$$
The almost module $M^a$ is often called the *almostification* of $M$, and the module $M_*$ is often called the *module of almost elements* of $M$.

**Remark 6.3.**

1. $M_!$ and $M_*$ make sense for any object $M \in \mathbf{Mod}_R^a$.
2. We have maps $M_! \to M \to M_*$ in $\mathbf{Mod}_R$, which are all almost isomorphisms (i.e. they are isomorphisms after applying the almostification functor $(-)^a \coloneqq q$).

**Example 6.4.** For this example, $R = K^\circ$ for a perfectoid field $K$ and $I = K^{\circ\circ}$.

(1) If $M$ is $I$-torsion, then $M_! = M_* = 0$.
(2) If $M$ is a torsion-free $R$-module, then $M_* = \{m \in M \otimes_I K : \epsilon m \in M \text{ for all } \epsilon \in I\}$. Indeed, there is an inclusion $M_* = \operatorname{Hom}(I, M) \supset M = \operatorname{Hom}(R, M)$, which is an isomorphism after applying $- \otimes_{K^\circ} K$, and hence $M_* \subseteq M \otimes_{K^\circ} K$. One should think of the ideal $I$ as being very close to the unit ideal, and elements $\epsilon \in I$ should be thought of as very close to 1.
(3) $I_* = K^\circ = R$.
(4) If $J \subset K^\circ$ is any ideal, then $J_*$ is principal exactly when $c = \sup\{|x| : x \in J\} \in |K^*|$. The proof is an easy exercise in valuation theory. Note that $J_* = J$; in particular $R_* = R$.
(5) If $t \in K^\circ$ is a pseudouniformizer, then $(K^\circ/t)_* \supsetneq K^\circ/tK^\circ = (K^\circ)_*/(tK^\circ)_*$. That is, there is a short exact sequence
$$0 \to K^\circ \xrightarrow{t} K^\circ \to K^\circ/t \to 0$$
and applying $(-)_*$ gives
$$0 \to (K^\circ)_* \xrightarrow{t} (K^\circ)_* \to (K^\circ/t)_*,$$
so one gets the inclusion $K^\circ/tK^\circ \subseteq (K^\circ/tK^\circ)_*$, but it may not be surjective. For example, the surjectivity fails for any field $K$ that is not spherically complete (e.g. $K = \mathbf{F}_p((t))^\wedge_{\text{perf}}$).

**Remark 6.5.** There is a reasonable notion of $\operatorname{Ext}^i_{R^a}(M^a, N^a)$ for $M^a, N^a \in \mathbf{Mod}^a_R$. One could define it e.g. as $\operatorname{Ext}^i_R(M_!, N) \simeq \operatorname{Ext}^i_R(M, \underline{\operatorname{RHom}}^i(I, N))$. In particular, if you have a short exact sequence
$$0 \to M' \to M \to M'' \to 0$$
in $\mathbf{Mod}^a_R$, then we obtain a long-exact sequence
$$0 \to M'_* \to M_* \to M''_* \to \operatorname{Ext}^1_R(I, M') \to \operatorname{Ext}^1_R(I, M) \to \dots.$$

**Definition 6.6.** Let $M \in \mathbf{Mod}_R$, so $M^a \in \mathbf{Mod}^a_R$.

(1) $M$ (or $M^a$) is *almost flat* if $\operatorname{Tor}^R_i(M-) \overset{a}{=} 0$ for all $i > 0$ (i.e. the Tor group is almost zero); equivalently, the functor $M^a \otimes -$ is exact in $\mathbf{Mod}^a_R$.
(2) $M$ (or $M^a$) is *almost projective* if $\operatorname{Ext}^i_R(M, -) \overset{a}{=} 0$ for all $i > 0$.
(3) $M$ (or $M^a$) is *almost finitely generated* if for every $\epsilon \in I$, there exists a finitely generated $R$-module $N_\epsilon$ and a map $f_\epsilon : N_\epsilon \to M$ whose kernel and cokernel are killed by $\epsilon$. Similarly, one defines *almost finitely presented* modules. If the number of generators of $N_\epsilon$ is bounded independent of $\epsilon$, then we say $M$ is *uniformly almost finitely generated*.

**Remark 6.7.** The definition of an almost projective module is not the same as saying that $M^a$ is a projective object in $\mathbf{Mod}^a_R$ (if so, then the Ext groups would actually be zero, but we only want them to almost zero). In fact, $R^a$ is almost projective but not projective: indeed, $\operatorname{Hom}_{\mathbf{Mod}_{R^a}}(R^a, M^a) = (M^a)_*$ and $(-)_*$ is not exact.

**Example 6.8.** Let $p \neq 2$, $R = K^\circ = \mathbf{Z}_p[p^{1/p^\infty}]^\wedge$, $I = K^{\circ\circ}$, and $L = K(\sqrt{p})$. We claim that $L^\circ$ is uniformly almost finitely generated and almost projective, but it is not finitely generated.

It suffices to show the following: for any $n \geq 0$, there exists a finite free $K^\circ$-module $R_n$ (of rank 2) and an injective map $R_n \to L^\circ$ with cokernel killed by $p^{1/p^n}$. Set $R_n = K^\circ \oplus K^\circ p^{1/2p^n} \subseteq L^\circ$. We know that $L^\circ = (\bigcup_n R_n)^\wedge = \left(\operatorname{colim}_n \mathbf{Z}_p[p^{1/2p^n}]\right)^\wedge$, because the filtered colimit of the discrete valuation rings $\mathbf{Z}_p[p^{1/2p^n}]$ is again a valuation ring, it sits inside $L^\circ$, and the two valuation rings have the same fraction field, so they must coincide.

We claim that the cokernel of $R_n \to R_{n+1}$ is killed by $p^{1/p^n}$. The proof comes from the following equality:
$$p^{1/p^n} \cdot p^{1/2p^{n+1}} = p^{\frac{(p+1)/2}{p^{n+1}}} \cdot p^{1/2p^n} \in R_n.$$
Therefore, $p^{1/p^n} \cdot R_{n+1} \subseteq R_n$, which gives the claim. The claim implies that the cokernel of $R_n \to \operatorname{colim}_m R_m$ is killed by $\prod_{m \geq n} p^{1/p^m} = p^{\frac{p}{p^n(p-1)}}$, and hence by $p$ (this is an abuse of notation, as this infinite product does not

actually lie in the ring). Therefore, as $R_0$ is $p$-adically complete, this implies that $\mathrm{colim}_m R_m$ is also $p$-adically complete[9]. Hence, $L^\circ = \mathrm{colim}_n R_n$.

Similarly, as the cokernel of $R_n \to L^\circ$ is killed by $p^{\frac{p}{p^n(p-1)}}$, one gets the uniform almost finite generation of $L^\circ$. Moreover, it implies that $\mathrm{Ext}^{>0}_{K^\circ}(L^\circ, -)$ is killed by $p^{\frac{p}{p^n(p-1)}}$. As this holds for all $n$, one gets that $\mathrm{Ext}^{>0}_{K^\circ}(L^\circ, -) \overset{a}{=} 0$, i.e. $L^\circ$ is almost projective.

### 6.2. Almost étale extensions.

**Definition 6.9.** A map $A \to B$ of $R^a$-algebras is *almost finite étale* if

(1) [Finiteness] $B$ is almost finitely presented over $A$.
(2) [Unramified[10] ] There exists a diagonal idempotent in $(B \otimes_A B)_*$, i.e. an element $e \in (B \otimes_A B)_*$ such that $e^2 = e$ and, if $\mu \colon B \otimes_A B \to B$ denotes the multiplication map, $\ker(\mu_*) \cdot e = 0$ and $\mu_*(e) = 1$.
(3) [Flatness] $B$ is almost flat over $A$.

Write $A_{\text{fét}}$ for the category of almost finite étale maps.

The above definition is the one that appears in [Sch12], but [GR03] uses a different definition (so that they may discuss étale maps, as opposed to just finite étale ones).

**Remark 6.10.** The functor $(-)_*$ takes algebras to algebras; in particular, any algebra is the almostification of an almost algebra because $A = (A_*)^a$ for any almost $R^a$-algebra $A$.

## 7. January 30th

Let $(R, I)$ be our almost setup (i.e. $R$ is a ring and $I \subset R$ is a flat ideal such that $I = I^2$), and we have functors as in the diagram below:

$$\begin{tikzcd} \mathbf{Mod}_R \arrow[r, "M \mapsto M^a"] & \mathbf{Mod}_R^a \arrow[l, bend right, "N^a_! := I \otimes N \leftarrow N^a"'] \arrow[l, bend left, "N^a_* := \mathrm{Hom}(I,N) \leftarrow N^a"] \end{tikzcd}$$

### 7.1. Almost étale extensions (continued).

**Example 7.1.** If $R = K^\circ$ for a perfectoid field $K$ and $M \in \mathbf{Mod}_R$ is torsion-free, then recall that

$$M_* = \{x \in M \otimes_I K \colon \epsilon x \in M \text{ for all } \epsilon \in K^{\circ\circ}\}.$$

Consider the case when $K = \mathbf{Q}_p[p^{1/p^\infty}]^\wedge$; $K$ has an extension $L = W(k)[p^{1/p^\infty}]^\wedge[1/p]$, where $k = \mathbf{F}_p((t))_{\text{perf}}$ and $W(k)$ is the ring of Witt vectors of $k$. There was an interesting valuation ring in $L$ given by

$$R = \{f \in L^\circ \colon \overline{f} \in \mathbf{F}_p[[t]]_{\text{perf}}\},$$

where $\overline{f} = f \bmod (p^{1/p^\infty})$. We saw that $R$ is a rank-2 valuation ring such that $L^{\circ\circ} \subset R \subset L^\circ$. Note that both $R$ and $L^\circ$ are $K^\circ$-algebras. Also, $R_* = L^\circ$, because $L^\circ/R$ is killed by $K^{\circ\circ} = (p^{1/p^\infty})$, and so it is almost zero.

This is is typical of what will occur when we define perfectoid spaces, i.e. the difference between a higher-rank point and the rank-1 points close to it will be almost zero.

**Remark 7.2.** There exist enough injectives in $\mathbf{Mod}_R^a$, and they are all of the form $M^a$ for $M \in \mathbf{Mod}_R$ injective. Proving this is an exercise in using the adjoint functors in the diagram above. The upshot is that you can do homological algebra in this category.

**Definition 7.3.** A map $A \to B$ of $R^a$-algebras is *almost finite étale* if

[9] Given a $p$-adically complete ring, any extension by something killed by $p$ remains $p$-adically complete.

[10] There is an equivalent definition of almost unramified in terms of an infinitesimal lifting criterion, but since a square-zero extension of a perfectoid algebra is not necessarily a perfectoid algebra, the definition above seems more appropriate.

(1) $B$ is almost finitely presented over $A$;
(2) $B$ is almost flat over $A$;
(3) there exists a "diagonal idempotent" $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\ker(\mu_*) \cdot e = 0$, and[11] $\mu_*(e) = 1$, where $\mu\colon B \otimes_A B \to B$ is the multiplication map.

**Example 7.4.** If $p \neq 2$, $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$, and $L = K(\sqrt{p})$, then we claim that $L^\circ/K^\circ$ is almost finite étale. We already saw that $L^\circ/K^\circ$ is almost finite projective (almost finite free, even) and $L^\circ = \mathbf{Z}_p[p^{1/2p^\infty}]^\wedge$. This is not unramified over $K^\circ$ in the usual sense, because it is not of finite type.

Note that $L/K$ is a quadratic Galois extension, so there is an isomorphism $L \otimes_K L \simeq L \times L$ as rings. Explicitly, $a \otimes b \mapsto (ab, a\sigma(b))$, where $\sigma\colon L \to L$ is the Galois involution. Then, we claim that

$$e = \frac{1}{2\sqrt{p} \otimes 1}\left(1 \otimes \sqrt{p} + \sqrt{p} \otimes 1\right) \in L \otimes_K L$$

is the diagonal idempotent, because it maps to $(1, 0) \in L \times L$ (as composing with projection onto the first factor gives the multiplication map).

We need to check that $e \in (L^\circ \otimes_{K^\circ} L^\circ)_*$. Note that $pe \in L^\circ \otimes_{K^\circ} L^\circ$, but we need that the product of $e$ with any small power of $p$ lies in $L^\circ \otimes_{K^\circ} L^\circ$. Now, observe that

$$e = \frac{1}{2\sqrt{p^{1/p^n}} \otimes 1}\left(1 \otimes \sqrt{p^{1/p^n}} + \sqrt{p^{1/p^n}} \otimes 1\right)$$

for all $n \geq 1$. This formula tells you that $p^{1/p^n}e \in L^\circ \otimes_{K^\circ} L^\circ$. As this holds for all $n$, get $e \in (L^\circ \otimes_{K^\circ} L^\circ)_*$. Finally, one can check the remaining conditions in $L \otimes_K L$, which we can do because $p$ is a non-zero divisor in $L^\circ \otimes_{K^\circ} L^\circ$. Thus, $L^\circ/K^\circ$ is almost finite étale.

Hopefully this example convinces you that this notion of almost finite étale is capturing something interesting. The goal now is to prove the almost purity theorem, which is a more general statement about almost finite étale maps. The first step is to prove it in characteristic $p$.

7.2. **Construction (finite étale algebras and idempotents).** Say $R$ is a ring and $R \to S$ is a finite étale extension, then the unramified-ness implies that there exists $e \in S \otimes_R S$ such that $e$ is idempotent and $e$ cuts out $\operatorname{Spec}(S) \hookrightarrow \operatorname{Spec}(S \otimes_R S)$, i.e. the diagonal is a connected component and $e$ is the corresponding idempotent. Equivalently, we have a product decomposition $S \otimes_R S \xrightarrow{\simeq} S \times S'$, where $e \mapsto (1, 0)$. Write

$$e = \sum_{i=1}^n a_i \otimes b_i$$

for $a_i, b_i \in S$. Consider the map $\alpha\colon S \to R^n$ given by $\alpha(f) \coloneqq \left(\operatorname{Tr}_{S/R}(f \cdot a_i)\right)_{i=1}^n$, where $n$ is the same as the $n$ appearing in the decomposition of $e$. Note that the reason the trace $\operatorname{Tr}_{S/R}$ makes sense is because the map $R \to S$ is finite flat. Consider also the map $\beta\colon R^n \to S$ given by $\beta\left((g_i)_{i=1}^n\right) \coloneqq \sum_{i=1}^n g_i b_i$.

**Claim 7.5.** The composition $S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$ is the identity,

*Proof.* We will just show that $\beta \circ \alpha(1) = 1$ (the rest of the proof is the similar, but with more notation). Explicitly, we'd like to show that $\sum_{i=1}^n \operatorname{Tr}_{S/R}(a_i) \cdot b_i = 1$. We have $j\colon S \otimes_R S \xrightarrow{\simeq} S \times S'$, where $e \mapsto (1, 0)$. If $i_2\colon S \to S \otimes_R S$ denotes the second inclusion $s \mapsto 1 \otimes s$, then $\operatorname{Tr}_{i_2}(e) = 1$. Indeed, using additivity of traces under products of finite étale maps, get $\operatorname{Tr}_{i_2}(e) = \operatorname{Tr}_{\mathrm{pr}_1 \circ j \circ i_2}(1) = 1$. But,

$$1 = \operatorname{Tr}_{i_2}(e) = \operatorname{Tr}_{i_2}\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n \operatorname{Tr}_{S/R}(a_i) \cdot b_i.$$

$\square$

[11] Given $M \in \mathbf{Mod}_R$, the element $1 \in M_* \coloneqq \operatorname{Hom}_R(I, M)$ denotes the composition $I \hookrightarrow R \to M$, where $I \hookrightarrow R$ is the inclusion and $R \to M$ is the structure map.

7.3. **Almost purity in characteristic $p$.** The upshot of Claim 7.5 is that we get an explicit presentation of $S$ as a summand of a free $R$-module from a formula for $e$. From this, we get a "non-categorical" version of the almost purity theorem in positive characteristic.

**Proposition 7.6.** [Almost purity in characteristic $p$, I] *If $\eta\colon R \to S$ is an integral map of perfect rings over $\mathbf{F}_p$ and $\eta[1/t]$ is finite étale for some $t \in R$, then $\eta$ is almost finite étale with respect to the ideal $I = (t^{1/p^\infty})$.*

Here, $\eta[1/t]$ denotes the map $R[1/t] \to S[1/t]$ induced by $\eta$.

*Proof.* The first step is to reduce to the case where both $R$ and $S$ are $t$-torsion free.

**Claim 7.7.** The subring $R[t^\infty] := \{r \in R\colon t^n r = 0 \text{ for all } n \gg 0\} \subseteq R$ is almost zero.

*Proof of Claim 7.7.* If $f \in R[t^\infty]$, then $t^n f = 0$ for some $n > 0$, so $t^n f^{p^m} = 0$ for all $m \gg 0$. By perfectness, can take $p$-th roots on both sides to get $t^{n/p^m} f = 0$ for all $m \gg 0$, i.e. $f$ is almost zero. □

We may replace $R$ with $R/R[t^\infty]$ and $S$ with $S/S[t^\infty]$ in order to assume that $R \subseteq R[1/t]$ and $S \subseteq S[1/t]$, i.e. we now assume that $t$ is a non-zero divisor.

The second step is to reduce to the case where $R$ and $S$ are both integrally closed in $R[1/t]$ and $S[1/t]$, respectively. Let $R_{\mathrm{int}}$ be the integral closure of $R$ in $R[1/t]$.

**Claim 7.8.** The inclusion $R \subseteq R_{\mathrm{int}}$ is an almost isomorphism.

*Proof of Claim 7.8.* If $f \in R_{\mathrm{int}}$, then we know that $f^{\mathbf{N}} := \{f^n : n \geq 0\} \subseteq t^{-c}R$ for some $c > 0$. Thus, $t^c f^{\mathbf{N}} \in R$, so $t^c f^{p^n} \in R$ for all $n \geq 0$. By perfectness, $t^{c/p^n} f \in R$ for all $n \gg 0$, hence $f \in R_*$. Therefore, $R \to R_{\mathrm{int}}$ is an almost isomorphism. □

Therefore, we may assume that $R \subseteq R[1/t]$ and $S \subseteq S[1/t]$ are integrally closed (using that the integral closure of a perfect ring is perfect).

The third step is to check that $R \to S$ is almost unramified. Let $e \in (S \otimes_R S)[1/t]$ be the diagonal idempotent. We know that $t^c e \in S \otimes_R S$ for some $c > 0$, so $t^c e^{p^m} = t^c e \in S \otimes_R S$, hence $e \in (S \otimes_R S)_*$. Therefore, $R \to S$ is almost unramified.

In the fourth and final step, it remains to show that $S$ is almost finite projective over $R$. Fix some $m \geq 0$, then the third step tells you that $t^{1/p^m} e \in S \otimes_R S$. We can expand

$$t^{1/p^m} e = \sum_{i=1}^{n} a_i \otimes b_i,$$

where $a_i, b_i \in S$. Consider the composition $S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$, where recall that $\alpha(f) = \left(\mathrm{Tr}_{S/R}(f a_i)\right)_{i=1}^n$ and $\beta\left((g_i)_{i=1}^n\right) = \sum_{i=1}^n g_i b_i$. The trace $\mathrm{Tr}_{S/R}$ makes sense because $R \subseteq R[1/t]$ is integrally closed.

**Claim 7.9.** We have that $\beta \circ \alpha = t^{1/p^m}$.

*Proof of Claim 7.9.* We know $\beta \circ \alpha = t^{1/p^m}$ is true after inverting $t$, and thus true as $t$ is an nzd on $S$. □

It follows immediately from Claim 7.9 that $S$ is almost finite projective (but not necessarily uniformly so), which concludes the proof of the proposition. □

The "categorical version" of the almost purity in characteristic $p$ is the following:

**Theorem 7.10.** [Almost purity in characteristic $p$, II] *If $R$ is a perfect ring of characteristic $p$, $t \in R$ is any element, and $I = (t^{1/p^\infty})$ is the ideal of almost mathematics, then inverting $t$ produces an equivalence*

$$R_{a\,\text{fét}} \xrightarrow{\simeq} R[1/t]_{\text{fét}},$$

*given by $(R \to S) \mapsto (R[1/t] \to S[1/t])$.*

Recall that $R[1/t]_{\text{fét}}$ is the category of finite étale $R[1/t]$-algebras, and $R_{a\,\text{fét}}$ is the category of almost finite étale $R^a$-algebras. To prove this version of the almost purity theorem, we need the following definition (due to Gabber and Ramero) and a lemma from commutative algebra.

**Definition 7.11.** A ring map $A \to B$ is *weakly étale* if both $A \to B$ and $B \otimes_A B \to B$ are flat.

**Fact 7.12.** If $A \to B$ is finitely-presented, then it is weakly étale iff it is étale.

**Lemma 7.13.** *If $A \to B$ is a weakly étale map of $\mathbf{F}_p$-algebras, then*

$$\begin{tikzcd} A \arrow[r, "Frob_A"] \arrow[d] & A \arrow[d] \\ B \arrow[r, "Frob_B"] & B \end{tikzcd}$$

*is a pushout diagram, where $Frob_A$ and $Frob_B$ denote the Frobenius morphisms on $A$ and $B$, respectively. In particular, if $A$ is perfect, then $B$ is perfect.*

We will prove the lemma and the theorem next class.

## 8. February 1st

8.1. **Almost purity in characteristic $p$ (continued).** Our goal from last time was to prove the following version of the almost purity theorem in characteristic $p$.

**Theorem 8.1.** *If $R$ is a perfect ring of characteristic $p$, $t \in R$, and $I = (t^{1/p^\infty})$ is the ideal of almost mathematics, then there is an equivalence $R_{a\,\text{fét}} \xrightarrow{\simeq} R[1/t]_{\text{fét}}$.*

**Remark 8.2.** In the "almost setup" $(R, I)$, if $I = (1)$ is the unit ideal, then "almost mathematics" is "usual mathematics"; that is, the notion of almost finite étale coincides with finite étale In Theorem 8.1, the ideal $I = (t^{1/p^\infty})$ becomes the unit ideal after inverting $t$, so $R[1/t]_{\text{fét}} = R[1/t]_{a\,\text{fét}}$ In particular, the morphism $R \to R[1/t]$ induces a functor $R_{a\,\text{fét}} \to R[1/t]_{a\,\text{fét}} = R[1/t]_{\text{fét}}$, which is the functor asserted to be an equivalence in the statement of Theorem 8.1.

The primitive version of the almost purity theorem from last time is the key ingredient used to show that the functor $R_{a\,\text{fét}} \to R[1/t]_{\text{fét}}$ is essentially surjective, so it remains to check that this functor is fully faithful. To do so, we need the following lemma:

**Lemma 8.3.** [GR03, 3.5.13] *If $A \to B$ is a weakly étale map of $\mathbf{F}_p$-algebras (i.e. both $A \to B$ and $B \otimes_A B \to B$ are flat), then*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{Frob}_A"'] & B \arrow[d, "\mathrm{Frob}_B"] \\ A \arrow[r] & B \end{tikzcd}$$

*is a pushout square of commutative rings. In particular, if $A$ is perfect, then $B$ is perfect.*

Lemma 8.3 (along with the same proof) adopts to the almost world, but there are no new ideas from the usual proof we give below.

We will not develop the full theory of the weakly étale maps here, but one should think of them as étale maps without the finite type assumptions. They satisfy some basic stability properties, listed below.

**Fact 8.4.**

(1) Weakly étale maps are stable under base change and composition.
(2) If $A \to B_1$ and $A \to B_2$ are weakly étale, then any $A$-algebra map $B_1 \to B_2$ is also weakly étale.

The harder fact, which we will not prove and we do not need, is that weakly étale maps are inductive limits of etale maps; see [BS15, Theorem 1.3].

*Proof of Lemma 8.3.* Set $B^{(1)} := B \otimes_{A,\mathrm{Frob}_A} A$ to be the pushout of $A \to B$ along $\mathrm{Frob}_A$ (also known as the Frobenius twist), so you get a natural map $F_{B/A} \colon B^{(1)} \to B$ known as the relative Frobenius. We would like to show $F_{B/A}$ is an isomorphism. The relevant morphisms are summarized in the following diagram:

$$\begin{tikzcd}
 & & B \\
B \arrow[r, "1\otimes\mathrm{Frob}_A"] \arrow[urr, bend left, "\mathrm{Frob}_B"] & B^{(1)} \arrow[ur, "F_{B/A}"] & \\
A \arrow[u] \arrow[r, "\mathrm{Frob}_A"] & A \arrow[u] \arrow[uur, bend right] &
\end{tikzcd}$$

As $A \to B^{(1)}$ and $A \to B$ are weakly étale, so too is $F_{B/A}$. On the other hand, a small diagram chase implies that $F_{B/A}$ factors Frob on both $B^{(1)}$ and $B$; that is, $\mathrm{Frob}_B$ and $\mathrm{Frob}_{B^{(1)}}$ both factor over $F_{B/A}$, e.g. there is a map $B \to B^{(1)}$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & B \arrow[dr] & \\
B^{(1)} \arrow[ur, "F_{B/A}"] \arrow[rr, "\mathrm{Frob}_{B^{(1)}}"] & & B^{(1)}
\end{tikzcd}$$

It suffices to show that if $\alpha \colon R \to S$ is a weakly étale map that factors Frobenius on both $R$ and $S$, then $\alpha$ must be an isomorphism. The first claim is that such a map $\alpha$ is faithfully flat. Were it not faithfully flat, it would carry some module to zero, and we can WLOG assume it carries a cyclic module to zero. Say $I \subset R$ is an ideal such that $R/I \otimes_I S = 0$, then we want to show that $R/I = 0$. If one first checks that the assumptions are compatible with base change (the condition about weakly étale maps is fine, but one must check the factoring of Frobenius is preserved), then we may replace $\alpha$ with $\alpha \otimes_R R/I$. This reduces to the case where $R \to 0$ is weakly étale and factors Frobenius. The map to the zero ring is always étale. If $R \neq 0$, then Frobenius on $R$ is also non-zero, so this is not possible unless $R = 0$. Therefore, $\alpha$ is faithfully flat.

Now, observe that for $\alpha \colon R \to S$ as above, the multiplication map $\mu \colon S \otimes_R S \to S$ is also weakly étale and it factors a power of Frobenius on both sides. While it is more or less clear that $\mu$ is weakly étale, it is not clear that it factors a power of Frobenius; this requires drawing many more diagrams, which we will not do here. Therefore, $\mu$ is faithfully flat by the above paragraph; however, $\mu$ is also surjective, hence an isomorphism.

Consider the diagram

$$\begin{tikzcd}
R \arrow[r, "\alpha"] \arrow[d, "\alpha"] & S \arrow[d, "i_2"] \arrow[ddr, bend left, "\mathrm{id}"] & \\
S \arrow[r] \arrow[drr, bend right, "\mathrm{id}"] & S \otimes_R S \arrow[dr, "\mu"] & \\
 & & S
\end{tikzcd}$$

where $i_1, i_2 \colon S \to S \otimes_R S$ denote the first and second inclusions. The diagram implies that $i_1$ is an isomorphism (because $\mu$ and id are isomorphism), but (the vertical) $\alpha$ is faithfully flat, which implies that (the horizontal) $\alpha$ must also be an isomorphism. This concludes the proof. □

Lemma 8.3 can now be used to prove the "categorical version" of the almost purity theorem.

*Proof of Theorem 8.1.* Last time, we showed essential surjectivity, the proof of which we recall here: if $S \in R[1/t]_{\text{fét}}$, then Lemma 8.3 implies that $S$ is perfect. Consider $S_{\text{int}}$, the integral closure of $R$ in $S$. Then, $R \to S_{\text{int}}$ is an integral extension of perfect rings (the ring $S_{\text{int}}$ is perfect because the integral closure inside a perfect ring is perfect) and such that inverting $t$ gives us back the extension that we started with, i.e. $R[1/t] \to S_{\text{int}}[1/t] = S$. This was exactly the setting of Proposition 7.6 from last time, which proves that the map $R \to S_{\text{int}}$ is almost finite étale. Therefore, we have essential surjectivity.

For the fully faithfulness of functor, fix $S \in R_{a\,\text{fét}}$.

**Claim 8.5.** If $T$ is the integral closure of $R$ in $S[1/t] = S_*[1/t]$, then $S \simeq T^a$.

It is not too hard to see that Claim 8.5 implies the fully faithfulness, because formation of integral closures is a functorial operation.

*Proof of Claim 8.5.* Note that neither inclusion to show is obvious. Observe that $S$ is perfect[12] by using (the almost version of) Lemma 8.3. Also, $T$ is perfect (because both $R$ and $S[1/t]$ are perfect). Furthermore, by using the trick in Claim 7.7 from last time, we may assume that $t$ is a non-zero divisor on $R$, $T$, and $S_*$. Integrally, we have the two maps

$$\begin{tikzcd} & T \\ R \arrow[ur] \arrow[dr] & \\ & S_* \end{tikzcd}$$

and rationally the two become the same map $R[1/t] \longrightarrow T[1/t] = S[1/t]$. What we will actually prove is that $T_* = S_*$, as subrings of $T[1/t] = S_*[1/t]$. Granted this, then we may pass to the almost world to get that $T^a = S$.

For the inclusion $T_* \subseteq S_*$, it suffices to show that $T \subseteq S_*$. Fix $f \in T$. By integrality, we know that $f^{\mathbf{N}}$ is contained in a finitely generated $R$-submodule of $T[1/t] = S_*[1/t]$, which is also a finitely generated $S_*$-submodule of $S_*[1/t]$. Thus, $f^{\mathbf{N}} \in t^{-c} S_*$ for some $c \in \mathbf{Z}_{\geq 0}$ (by clearing denominators in the finite generating set). This says that $t^c f^{p^n} \in S_*$ for all $n \geq 0$. As everything is perfect, we can take $p$-th roots: $t^{c/p^n} f \in S_*$ for all $n \geq 0$, so $f \in (S_*)_* = S_*$.

For the reverse inclusion $S_* \subseteq T_*$, we can repeat the same argument: fix $f \in S_*$, then we know that $S_*$ is almost finitely generated over $R$, so $tf^{\mathbf{N}}$ lies in a finitely-generated $R$ submodule of $S_*$. This implies that $tf^{\mathbf{N}}$ lies in a finitely generated $R$-submodule of $S_*[1/t] = T_*[1/t]$. Thus, $f^{\mathbf{N}} \in t^{-c} T$ for some $c \in \mathbf{Z}_{\geq 0}$, and we can repeat the previous argument to see that $f \in T_*$. □

This completes the proof of Theorem 8.1. □

Later, we will need the following lemma:

**Lemma 8.6.** *Let $K$ be a perfectoid field, $R = K^\circ$, $I = K^{\circ\circ}$, $t \in K$ a pseudouniformizer, and $M \in \mathbf{Mod}_R^a$, where $(R, I)$ is our almost setup.*

*(1) $M$ is almost flat iff $M_*$ is flat iff $M_!$ is flat.*

*(2) Assume $M$ is almost flat, then $M$ is $t$-adically complete iff $M_*$ is so iff $M_!$ is so.*

[12] As $S$ is an almost module, we must specify what it means for it to be perfect. One can say $S$ is perfect if $S_*$ is, or equivalently, take (an honest module) $S_0$ such that $S = S_0^a$, then the Frobenius morphism on $S_0$ induces an endomorphism on $S$, and we demand that this endomorphism be an isomorphism.

Here, we say that the almost module $M$ is *$t$-adically complete* if the natural map $M \xrightarrow{\simeq} \varprojlim_n M/t^n M$ is an isomorphism (where all operations are done in the almost category).

8.2. **Non-Archimedean Banach algebras (algebraically).** We will soon start discussing perfectoid algebras. There are 3 equivalent definitions of perfectoid algebras, two involving almost mathematics and one using non-Archimedean Banach algebras, and we will need all three notions. For this reason, we will discuss some of the general theory of non-Archimedean Banach algebras.

Fix a non-Archimedean field $k$ and a non-trivial valuation $|\cdot|: K \to \mathbf{R}_{\geq 0}$.

**Definition 8.7.** A *$K$-Banach algebra* is a $K$-algebra $R$ equipped with a function $|\cdot|: R \to \mathbf{R}_{\geq 0}$ such that

(1) $|\cdot|$ extends the valuation $|\cdot|$ on $K$.
(2) [Norm] If $|f| = 0$ for $f \in R$, then $f = 0$.
(3) [Submultiplicative] $|f + g| \leq |f||g|$ for $f, g \in R$.
(4) [non-Archimedean property] $|f + g| \leq \max\{|f|, |g|\}$ for $f, g \in R$.
(5) [Completeness] $R$ is complete for the metric $d(f, g) \coloneqq |f - g|$.

These form the category of $K$-Banach algebras, where maps are just continuous $K$-algebra maps (i.e. one does not require them to preserve the norm on the nose). One can have isomorphic $K$-Banach algebras with different norms, but the same topology on the ring.

**Definition 8.8.** For a $K$-Banach algebra $R$, the set $R^\circ$ of *power-bounded elements* of $R$ is given by

$$R^\circ \coloneqq \{f \in R \colon f^{\mathbf{N}} \text{ is a bounded subset}\}.$$

In terms of the norm on $R$, the set $R^\circ$ can be described as

$$R^\circ = \{f \in R \colon |f^n| \text{ is bounded as } n \text{ varies}\}.$$

We list some of the basic properties of the set $R^\circ$ of power-bounded elements:

(1) $R^\circ$ is open in $R$ (this is because it contains $R_{\leq 1} = \{f \in R \colon |f| \leq 1\}$, which is open).
(2) $R^\circ$ is a subring of $R$.
(3) $R^\circ$ only depends on $R$ as a topological ring.
(4) $R^\circ$ is a $K^\circ$-algebra.

For proofs of these facts, see Rankeya Datta's notes [Dat17] from the adic spaces learning seminar.

We will (try to) follow the following convention: the integral rings will be denoted by $A, B, \ldots$ and the rational ones will be $R, S, \ldots$ and so on.

**Example 8.9.** Assume $K$ is discretely valued and $t \in K$ is a uniformizer. Fix a $t$-adically complete, $t$-torsion free $K^\circ$-algebra $A$ and set $R = A[1/t]$. The $K$-algebra $R$ is a $K$-Banach algebra with the norm

$$|f| = \min_{n \in \mathbf{Z}} \{|t^n| \colon f \in t^n A\}.$$

One can check that this makes $R$ into a $K$-Banach algebra, and the topology on $R$ has a neighborhood basis of 0 given by $\{t^n A \colon n \in \mathbf{Z}\}$. In particular, the collections $\{t^n A\}$ and $\{R_{\leq |t|^n}\}$ (the set of closed balls of varying radii) are cofinal amongst each other.

**Remark 8.10.** In Example 8.9, there are containments of rings $A \subseteq R_{\leq 1} \subset R^\circ$ and $R_{\leq 1} \subseteq R^\circ$. We will generally be interested in situations where these are all (essentially) the same, but in general they could be different from one another! $R^\circ$ is integrally closed in $R$. Therefore, in practice, $A$ can be much smaller than $R^\circ$ (e.g. if $A$ was not integrally closed inside $A[1/t]$). In fact, even $R_{\leq 1}$ can be bigger than $A$. An explicit example is given below.

**Example 8.11.** If $K$ is discretely-valued and $A \coloneqq K^\circ[x]/(x^2)$, then one can form the $K$-Banach algebra $R$ as in Example 8.9 and the ring of power-bounded elements is $R^\circ = K^\circ \oplus K \cdot x$, which is much larger than $A$.

Next time, we will put hypotheses on our Banach algebras so as to rule out examples such as Example 8.9 and Example 8.11.

## 9. February 6th

9.1. **Non-Archimedean Banach algebras (continued).** Let $K$ be a non-Archimedean field, and let $R$ be a $K$-Banach algebra (i.e. a $K$-algebra with a norm that extends the norm on $K$, and is complete and non-Archimedean). Associated to such an algebra, we had the subring of power-bounded elements $R^\circ$, which was a $K^\circ$-algebra.

Given a suitable $K^\circ$-algebra, we saw last time in Example 8.9 how to construct an associated $K$-Banach algebra, when $K$ was discretely-valued. This is generalized (to arbitrary $K$) in the following example.

**Example 9.1.** Fix a pseudouniformizer $t \in K$. For each $\gamma \in |K|$, choose $t_\gamma \in |K|$ such that $|t_\gamma| = \gamma$. If $A$ is a $K^\circ$-algebra which is $t$-adically complete and $t$-torsion free, then we want to make $R = A[1/t] = A \otimes_{K^\circ} K$ into a $K$-Banach algebra. Define norm on $R$ as follows: if $f \in R$, set

$$|f| \coloneqq \inf_\gamma \{\gamma \in |K| \colon f \in t_\gamma \cdot A\}.$$

Note that this norm does not depend on the choice of $t_\gamma$'s, because $K_{\le\gamma}A = t_\gamma A$ as sets. One can check that this norm makes $R$ into a $K$-Banach algebra, and in fact the collection of "open balls" $\{R_{\le|t|^n}\}$ and the collection $\{t^n A\}$ are cofinal amongst each other. Said differently, the $t$-adic topology on $A$ is equal to the subspace topology on $A$ inherited from $R$.

**Example 9.2.** We have $A \subseteq R_{\le 1} \subset R^\circ$, but $A \ne R^\circ$ in general. Indeed, if $A$ has nilpotents, then equality may not occur, as the following example demonstrates: if $A = K^\circ[\epsilon]/(\epsilon^2)$, then $R = K[\epsilon]/(\epsilon^2)$, but $R^\circ = K^\circ \oplus K\epsilon \supsetneq A$.

The problem in the above example is that $R^\circ$ is not bounded in $R$, which motivates the following definition.

**Definition 9.3.** A $K$-Banach algebra $R$ is *uniform* if $R^\circ$ is bounded (i.e. $R^\circ \subseteq R_{\le r}$ for some $r > 0$).

**Exercise 9.4.** A uniform Banach ring is reduced (but the converse is not true!).

To describe uniform Banach algebras algebraically, we require the following notion from commutative algebra.

**Definition 9.5.** Given an inclusion $A \subseteq B$ of rings, an element $b \in B$ is *totally integral*[13] if $b^{\mathbf{N}}$ lies in a finitely-generated $A$-submodule of $B$. We say $A$ is *totally integrally closed* in $B$ if for any totally integral element $b \in B$ over $A$, we have $b \in A$.

**Remark 9.6.**

(1) An integral element over $A$ is a totally integral element over $A$; in particular, totally integrally closed implies integrally closed.
(2) If $A$ is noetherian, then totally integral elements and integral elements are the same; in particular, totally integrally closed and integrally closed coincide.
(3) If $A'$ denotes the set of all totally integral elements of $B$ over $A$, then $A'$ is a subring of $B$; however, $A'$ may not be totally integrally closed. Nevertheless, there does exist a total integral closure of $A$ in $B$ (attained by repeatedly taking all totally integrally closed elements, possibly trans-finitely many times).

Now we can state the comparison theorem between these two algebraic and metric properties.

**Theorem 9.7.** *If $K$ is a non-Archimedean field and $t \in K$ is any pseudouniformizer, then there is an equivalence of categories*

$$\mathcal{C} \coloneqq \{\text{uniform } K\text{-Banach algebras}\} \xrightarrow{\simeq} \left\{\begin{matrix} t\text{-adically complete, } t\text{-torsion free } K^\circ\text{-algebras } A \\ \text{such that } A \text{ is totally integrally closed in } A[1/t] \end{matrix}\right\} =: \mathcal{D}_{\mathrm{tic}},$$

*where the functor $\mathcal{C} \to \mathcal{D}_{\mathrm{tic}}$ is given by $R \mapsto R^\circ$ and the inverse functor $\mathcal{D}_{\mathrm{tic}} \to \mathcal{C}$ is given by $A \mapsto A[1/t]$, equipped with the norm of example Example 9.1.*

[13] In [Sta17, Tag 00GW], this notion is called an *almost integral* element (which is a bad choice of terminology, as we will also be using almost mathematics). In [Mat89, Exercise 9.5], the term *completely integrally closed* is used instead of totally integrally closed.

**Remark 9.8.** Everything in Theorem 9.7 (that is, both categories $\mathcal{C}$ and $\mathcal{D}_{\mathrm{tic}}$, and the two functors between them) is independent of the choice of pseudouniformizer $t \in K^{\circ\circ}\backslash\{0\}$, because for two different pseudouniformizers $t$ and $t'$, the collections $\{t^n A\}$ and $\{(t')^n A\}$ are cofinal in one another.

*Proof.* Define $F\colon \mathcal{C} \to \mathcal{D}_{\mathrm{tic}}$ by setting $F(R) = R^\circ$, then we need to show that $R^\circ$ is $t$-adically complete, $t$-torsion free, and totally integrally closed in $R^\circ[1/t] = R$. Torsion freeness is clear (because it is a subring of a torsion free ring). For $t$-adic completeness, we need to use uniformity: as $R$ is uniform, $R^\circ$ is bounded, so $R^\circ \subseteq R_{\leq r}$ for some $r > 0$. But, $\bigcap_n t^n R_{\leq r} = 0$, because any element of $t^n R_{\leq r}$ has norm $\leq r|t|^n$, so an element of the intersection must have norm 0. Thus, $R^\circ$ is $t$-adically separated. Moreover, $R_{\leq 1} \subseteq R^\circ$ is an open subgroup (because it's open in $R$), so $R^\circ$ is $t$-adically complete iff $R_{\leq 1}$ is so (because $R^\circ/R_{\leq 1} \subseteq R_{\leq r}/R_{\leq 1}$, and the latter is $t$-power torsion), but this is obvious. Therefore, $R^\circ$ is $t$-adically complete.

To see that $R^\circ$ is totally integrally closed in $R^\circ[1/t] = R$, take $f \in R$ totally integral over $R^\circ$, then $f^{\mathbf{N}}$ is contained in a finitely generated $R^\circ$-submodule of $R$. By clearing denominators, we may assume $f^{\mathbf{N}} \in t^{-c}R^\circ$ for some $c > 0$. As $R$ is uniform, $R^\circ \subseteq R_{\leq r}$ for some $r > 0$. In particular, $f^{\mathbf{N}} \in t^{-c}R_{\leq r}$, which is bounded; hence $f$ is power bounded, i.e. $f \in R^\circ$.

Therefore, $F\colon \mathcal{C} \to \mathcal{D}_{\mathrm{tic}}$ is a functor. Conversely, define $G\colon \mathcal{D}_{\mathrm{tic}} \to \mathcal{C}$ by sending $A \mapsto R \coloneqq A[1/t]$, where the norm is as in Example 9.1. We must show that $R$ is uniform, or equivalently that $R^\circ$ is bounded. In fact, we prove the following stronger statement.

**Claim 9.9.** $A = R^\circ$.

As $A \subseteq R_{\leq 1}$, it follows immediately from Claim 9.9 that $R^\circ$ is bounded, and hence that $G\colon \mathcal{D}_{\mathrm{tic}} \to \mathcal{C}$ is a functor.

*Proof of Claim 9.9.* It is clear that $A \subseteq R^\circ$. Conversely, pick $f \in R^\circ$, so $f^{\mathbf{N}}$ is bounded. Then, $f^{\mathbf{N}} \in t^{-c}A$ for some $c > 0$, because the topology is the $t$-adic one. Thus, $f$ is totally integral over $A$, because $t^{-c}A$ is a finitely generated $A$-submodule of $A[1/t]$, and hence $f \in A$ as $A$ is totally integrally closed in $R = A[1/t]$. $\square$

It follows from Claim 9.9 that $F \circ G \simeq \mathrm{id}$, at least at the level of objects. It remains to show that $G \circ F \simeq \mathrm{id}$. Fix $R \in \mathcal{C}$, then $G(F(R)) = R^\circ[1/t]$, but possibly with a different norm. Let $|\cdot|$ denote the norm on $R$, and let $|\cdot|_{\mathrm{new}}$ denote the norm on $G(F(R)) = R$. The unit ball for $|\cdot|_{\mathrm{new}}$ is $R^\circ$ (by total integrality), and $R^\circ$ is contained in the unit ball of $|\cdot|$. Conversely, uniformity implies that the unit ball of $|\cdot|_{\mathrm{new}}$, which is equal to $R^\circ$, is contained in $R_{\leq r}$ for some $r > 0$. Therefore, this formally implies that the identity map $(R, |\cdot|) \xrightarrow{\simeq} (R, |\cdot|_{\mathrm{new}})$ is an isomorphism of $K$-Banach algebras. $\square$

One should think of Theorem 9.7 as a way of recognizing the subring $R^\circ$ of a uniform $K$-Banach algebra $R$ algebraically.

**Corollary 9.10.** *If $K$ is a perfectoid field*[14] *and $t \in K$ is a pseudouniformizer, then the category $\mathcal{D}_{\mathrm{tic}}$ is equivalent to the following categories:*

$$\mathcal{D}_{\mathrm{ic}} \coloneqq \left\{ \begin{matrix} t\text{-adically complete, } t\text{-torsion free } K^\circ\text{-algebras } A \\ \text{such that } A \text{ is integrally closed in } A[1/t]\text{, and } A \simeq A_*. \end{matrix} \right\},$$

*and*

$$\mathcal{D}_{\mathrm{prc}} \coloneqq \left\{ \begin{matrix} t\textit{-adically complete, } t\textit{-torsion free } K^\circ\textit{-algebras } A \\ \textit{such that if } f \in A[1/t] \textit{ and } f^p \in A\textit{, then } f \in A\textit{, and } A \simeq A_*. \end{matrix} \right\}$$

Here, the subscript "ic" stands for integrally closed, and "prc" stands for $p$-root closed. This terminology is not standard.

[14] Corollary 9.10 only requires that $K$ be a non-Archimedean field with divisible value group.

*Proof.* Choose $t \in K$ to be perfect. If $A \in \mathcal{D}_{\mathrm{tic}}$, then we want to show that $A = A_*$. If $f \in A_*$, then $f^n \in A_*$ for all $n \geq 0$ (because $A_*$ is a ring), and so $tf^n \in A$ for all $n \geq 0$. It follows that $f^{\mathbf{N}} \in t^{-1}A$ and hence $f \in A$, because $A$ is totally integrally closed in $A[1/t]$. Thus, $A_* = A$.

Therefore, we get fully faithful functors $\mathcal{D}_{\mathrm{tic}} \subseteq \mathcal{D}_{\mathrm{ic}} \subseteq \mathcal{D}_{\mathrm{prc}}$. To show that all three categories are the same, take $A \in \mathcal{D}_{\mathrm{prc}}$, then it suffices to show that $A$ is totally integrally closed in $A[1/t]$. If $f \in A[1/t]$ is totally integral over $A$, then $f^{\mathbf{N}} \in t^{-c}A$ for some $c > 0$, by definition; it follows that $t^c f^{\mathbf{N}} \in A$. In particular, $t^c f^{p^n} \in A$ for all $n \geq 0$. As $A$ is $p$-roots closed and $t$ is perfect, we can extract $p$-th roots from both sides of this equation to get $t^{c/p^n} f \in A$ for all $n \geq 0$. Thus, $f \in A_* = A$. □

**Corollary 9.11.** *The category* $\mathcal{C} = \{\text{uniform } K\text{-Banach algebras}\}$ *has all limits and colimits.*

The proof of Corollary 9.11 is left as an exercise (briefly, limits will be limits as $K$-algebras; on the other hand, the colimit will be computed by taking the colimit as $K$-algebras, completing, killing torsion, and taking the integral closure).

9.2. **Perfectoid algebras.** Fix a perfectoid field $K$. Fix a pseudouniformizer $t \in K^\flat$ such that $|p| \leq |t|^\flat < 1$, where $|\cdot|^\flat$ is the norm on $K^\flat$ (see Proposition 4.5). Set $\pi := t^\sharp \in K^\circ$, so $|\pi| = |t|^\flat$.

Recall the rough picture connecting $K$ and $K^\flat$:

$$\begin{tikzcd}
\varprojlim_{x \mapsto x^p} K^\circ \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\simeq"'] & K^\circ \arrow[r, hook] \arrow[d] & K \\
K^{\flat,\circ} \simeq (K^\circ/\pi)^{\mathrm{perf}} \arrow[r] \arrow[ur, "\sharp"] \arrow[d, hook] & K^{\flat,\circ}/t \simeq K^\circ/\pi & \\
K^\flat & &
\end{tikzcd}$$

We want to define a perfectoid algebra over a perfectoid field, and later to relate perfectoid algebras over $K$ to perfectoid algebras over $K^\flat$ via the above diagram. For this reason, we will need certain intermediate notions, i.e. perfectoid algebras over certain objects in the diagram.

**Definition 9.12.**

(1) A Banach $K$-algebra $R$ is *perfectoid* if it is uniform and $\mathrm{Frob}\colon R^\circ/\pi^{1/p} \to R^\circ/\pi$ is surjective. Write $\mathrm{Perf}_K$ for the category of such algebras, where the morphisms are continuous $K$-algebra maps.

(2) A $K^{\circ a}$-algebra $A$ is *perfectoid* if
  - (a) $A$ is $\pi$-adically complete and $\pi$-torsion free,
  - (b) $\mathrm{Frob}\colon A/\pi^{1/p} \to A/\pi$ is an isomorphism.

  Write $\mathrm{Perf}_{K^{\circ a}}$ for the category of such algebras, where the morphisms are $K^{\circ a}$-algebra maps.

(3) A $K^{\circ a}/\pi$-algebra $A$ is *perfectoid* if
  - (a) $A$ is flat over $K^{\circ a}/\pi$,
  - (b) $\mathrm{Frob}\colon A/\pi^{1/p} \to A = A$ is an isomorphism. Note that $A = A/\pi$.

  Write $\mathrm{Perf}_{K^{\circ a}/\pi}$ for the category of such algebras, where the morphisms are $K^{\circ a}/\pi$-algebra maps.

We will later see that the definition of a perfectoid algebra is independent of the choice of $\pi$.

## 10. February 8th

A classical reference on Banach algebras is the book [BGR84]. For the modern modern results from last time, certain sections of the papers [KL15, KL16, Ked17, Mih16, And16] provide useful references.

10.1. **Perfectoid algebras (continued).** Let $K$ be a perfectoid field (of characteristic 0). The diagram relating $K$ and its tilt $K^\flat$ was the following:

$$\begin{tikzcd}
\varprojlim\limits_{x\mapsto x^p} K^\circ \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\simeq"'] & K^\circ \arrow[r, hook] \arrow[d] & K \\
K^{\flat,\circ} := \varprojlim\limits_{\phi} K^\circ/p \arrow[r] \arrow[ur, "\sharp"] \arrow[d, hook] & K^\circ/p & \\
K^\flat & &
\end{tikzcd}$$

Recall that $\sharp$ is a multiplicative map, but it is not a ring homomorphism. We showed that, though $\sharp$ may not be surjective, it is surjective in terms of the valuation; that is, there exists $t \in K^{\flat,\circ}$ such that $t^\sharp = p \cdot (\text{unit}) =: \pi$. In particular, the expression $\pi^{1/p^n}$ makes sense.

The goal of today is to define perfectoid algebras, and to describe how one can "move" perfectoid algebras from $K$ to $K^\flat$ via the above diagram.

**Definition 10.1.**

(1) The category $\mathrm{Perf}_K$ of perfectoid $K$-algebras consists of uniform $K$-Banach algebras $R$ such that $\mathrm{Frob}\colon R^\circ/p \to R^\circ/p$ is surjective.
(2) The category $\mathrm{Perf}_{K^{\circ a}}$ of perfectoid $K^{\circ a}$-algebras consists $\pi$-adically complete, $\pi$-torsion free $K^{\circ a}$-algebras $A$ such that $\mathrm{Frob}\colon A/\pi^{1/p} \xrightarrow{\simeq} A/\pi$.
(3) The category $\mathrm{Perf}_{K^{\circ a}/\pi}$ of perfectoid $K^{\circ a}/\pi$-algebras consists of flat $K^{\circ a}/\pi$-algebras $A$ such that $\mathrm{Frob}\colon A/\pi^{1/p} \xrightarrow{\simeq} A$.

The notion of $\pi$-adic completeness does not depend on the choice of $\pi$. Moreover, to say a $K^\circ$-algebra is $\pi$-torsion free is equivalent to it being flat over $K^\circ$, and thus it too does not depend on $\pi$. Furthermore, the Frobenius map $A/\pi^{1/p} \to A/\pi$ makes sense because $\pi = p \cdot (\text{unit})$, so $A/\pi$ has characteristic $p$.

**Remark 10.2.**

(1) The category $\mathrm{Perf}_{K^{\circ a}}$ can be realized as consisting of those flat, $\pi$-adically complete $K^{\circ a}$-algebras $A$ such that $K^{\circ a}/p \to A/p$ is relatively perfect[15]. In particular, the category $\mathrm{Perf}_{K^{\circ a}}$ does not depend on the choice of $\pi$.
(2) Using the functor $(-)_*$, the category $\mathrm{Perf}_{K^{\circ a}}$ can also be described as those flat, $\pi$-adically complete $K^\circ$-algebras $A$ such that $\mathrm{Frob}\colon A/\pi^{1/p} \xrightarrow{\simeq} A/\pi$ and $A \xrightarrow{\simeq} A_*$. This description may seem more appealing as there is no almost mathematics involved; however, the functor $M \mapsto M_*$ does not commute with reduction mod $\pi$, so we will not use this characterization.
(3) Any non-zero $A \in \mathrm{Perf}_{K^{\circ a}}$ is faithfully flat over $K^{\circ a}$, and similarly for $\mathrm{Perf}_{K^{\circ a}/\pi}$. The proof is left as an exercise.

**Example 10.3.**

(1) If $R$ is a uniform Banach $K^\flat$-algebra, then $R$ is perfectoid iff $R$ is perfect.

[15] A map $A \to B$ of characteristic $p$ rings is *relatively perfect* if the map $B^{(1)} := B \otimes_{A,\mathrm{Frob}_A} A \xrightarrow{F_{B/A}} B$ is an isomorphism. If $A = \mathbf{F}_p$, then $A \to B$ is relatively perfect iff $B$ is perfect. In general, to say $A \to B$ is relatively perfect means that $B$ fails to be perfect to the same extent that $A$ fails to be perfect.

(2) If $A = K^\circ[x^{1/p^\infty}]^\wedge$, then $A^a \in \mathrm{Perf}_{K^{\circ a}}$ and $R = A[1/\pi] \in \mathrm{Perf}_K$. In fact, $R$ can be written more explicitly as follows:

$$R = \left\{ \sum_{i \in \mathbf{N}[1/p]} a_i x^i : \text{for any } U \subseteq K \text{ open, all but finitely-many } a_i \text{ are in } U \right\}.$$

This is the perfectoid analogue of a Tate algebra, and it is one of the most important examples of a perfectoid algebra to keep in mind.

To see that $R \in \mathrm{Perf}_K$, one must check that $A \subset A[1/\pi] = R$ is totally integrally closed, and hence by Theorem 9.7 from last time, $A = R^\circ$. It then follows that $R^\circ/p \to R^\circ/p$ is surjective by construction, because $A/\pi = (K^\circ/\pi)\,[x^{1/p^\infty}]$.

**Theorem 10.4.** *There are equivalences*

$$\mathrm{Perf}_K \simeq \mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_{K^{\circ a}/\pi}$$

*The functor* $\mathrm{Perf}_K \to \mathrm{Perf}_{K^{\circ a}}$ *is given by* $R \mapsto R^{\circ a}$*, with inverse given by* $A \mapsto A_*[1/\pi]$*; the functor* $\mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_{K^{\circ a}/\pi}$ *is given by* $A \mapsto A/\pi$*. Furthermore, there is a similar statement over* $K^\flat$*.*

As $K^\circ/\pi \simeq K^{\flat,\circ}/t$, Theorem 10.4 immediately yields the promised tilting corrrespondence.

**Corollary 10.5.** [Tilting Correspondence] *There is a diagram of equivalences*

$$\begin{array}{ccccc} \mathrm{Perf}_K & \xrightarrow{\simeq} & \mathrm{Perf}_{K^{\circ a}} & \xrightarrow{\simeq} & \mathrm{Perf}_{K^{\circ a}/\pi} \\ & & & & \downarrow\simeq \\ \mathrm{Perf}_{K^\flat} & \xrightarrow{\simeq} & \mathrm{Perf}_{K^{\flat,\circ a}} & \xrightarrow{\simeq} & \mathrm{Perf}_{K^{\flat,\circ a}/\pi} \end{array}$$

*Proof of Theorem 10.4.* We will first show the equivalence $\mathrm{Perf}_K \simeq \mathrm{Perf}_{K^{\circ a}}$. Given $R \in \mathrm{Perf}_K$, set $A = (R^\circ)^a$. It is immediate from the definitions that $A$ is $\pi$-adically complete, $\pi$-torsion free, and the Frobenius $A/\pi \to A/\pi$ is surjective; indeed, the Frobenius is surjective prior to almostifying, and almostification is exact. Thus, in order to show that the assignment $R \mapsto R^{\circ a}$ is a functor, we need to show that $\mathrm{Frob}\colon R^\circ/\pi \to R^\circ/\pi$ has kernel equal to $\pi^{1/p}R^\circ/\pi R^\circ$.

If $f \in R^\circ$ is such that $f^p \in \pi R^\circ$, consider $g \coloneqq \pi^{-1/p} f \in R$; this is well-defined because $\pi$ is invertible in $R$. This element $g$ satisfies $g^p = \pi^{-1} f^p \in R^\circ$. As $R^\circ$ is integrally closed in $R$, we must have $g \in R^\circ$, hence $f \in \pi^{1/p}R^\circ$, as required.

Now, to show that $A \mapsto A_*[1/\pi]$ is functor, fix $A \in \mathrm{Perf}_{K^{\circ a}}$. By Corollary 9.10 from last time, uniform $K$-Banach algebras are the same as $\pi$-adically complete, $\pi$-torsion free $K^\circ$-algebras $B$ such that $B$ is $p$-root closed in $B[1/\pi]$, and $B \simeq B_*$. The category of such objects was denoted by $\mathcal{D}_{\mathrm{prc}}$.

**Claim 10.6.** $A_* \in \mathcal{D}_{\mathrm{prc}}$.

Claim 10.6 implies that $R \coloneqq A_*[1/\pi]$ is uniform, and $R^\circ = A_*$ (this is not yet enough to show that $R$ is perfectoid – it remains to show the surjectivity of the Frobenius).

*Proof of Claim 10.6.* It is known that $A_*$ is $\pi$-adically complete (by Lemma 8.6(2)) and $\pi$-torsion free (because the multiplication-by-$\pi$ map $A \to A$ has no kernel, and $(-)_*$ is left exact). Next, we will show that $A_*$ is $p$-root closed. Consider the commutative diagram

$$\begin{array}{ccc} A_*/\pi^{1/p} & \xrightarrow{\mathrm{Frob}} & A_*/\pi \\ \big\downarrow & & \big\downarrow \\ \left(A/\pi^{1/p}\right)_* & \xrightarrow{\mathrm{Frob}} & (A/\pi)_* \end{array}$$

The vertical maps are injective because $(-)_*$ is left exact, and $\mathrm{Frob}\colon \left(A/\pi^{1/p}\right)_* \to (A/\pi)_*$ is an isomorphism because $A$ is perfectoid and, of course, $(-)_*$ maps isomorphisms to isomorphisms. It follows that $\mathrm{Frob}\colon A_*/\pi^{1/p} \to A_*/\pi$ is injective; equivalently, if $f \in A_*$ satisfies $f^p \in \pi A_*$, then $f \in \pi^{1/p} A_*$. Call this observation $(*)$.

Fix $x \in A_*[1/\pi]$ such that $x^p \in A_*$. We want to show that $x \in A_*$. Write $y = \pi^{k/p} x \in A_*$ for some $k \geq 1$, then we will show that $k$ can be decreased by 1. By an inductive argument, this will imply that $x \in A_*$. As $x^p \in A_*$, we have $y^p = \pi^k x^p \in \pi^k A_*$ because $x^p \in A_*$. If $k \geq 1$, then observation $(*)$ implies that $y \in \pi^{1/p} A_*$; in particular, $\pi^{-1/p} y \in A_*$, so $\pi^{\frac{k-1}{p}} x \in A_*$, as required.

Finally, as $A_{**} = A_*$, we conclude that $A_* \in \mathcal{D}_{\mathrm{prc}}$. $\square$

To show $R \in \mathrm{Perf}_K$, it suffices to show that the Frobenius map $R^\circ/\pi \to R^\circ/\pi$ is surjective. Observe that it is almost surjective, because $A$ is perfectoid.

Pick $x \in A_*$ and $c \in (0,1) \cap \mathbf{Q}$ (e.g. $c = 1/p$). The almost surjectivity of $\mathrm{Frob}\colon A_*/\pi \to A_*/\pi$ tells us that there exists $y \in A_*$ such that $y^p = \pi^c x \bmod \pi$. Set $z = \frac{y}{\pi^{c/p}} \in A_*[1/\pi]$, then $z^p = \frac{y^p}{\pi^c} \in A_*$, because have $y^p = \pi^c x + \pi w$ for some $w \in A_*$. As $A_*$ is $p$-root closed, $z \in A_*$. As $z^p = x + \pi^{1-c} w \in A_*$, $z$ is a $p$-th root of $x$ modulo $K^{\circ\circ} A$; that is, $\mathrm{Frob}\colon A_*/K^{\circ\circ} A_* \to A_*/K^{\circ\circ} A_*$ is surjective. Now, we may use the following general fact:

**Lemma 10.7.** *If $M \to N$ is an almost surjective map of $K^\circ$-modules and $M/K^{\circ\circ} \to N/K^{\circ\circ} N$ is surjective, then $M \to N$ is surjective.*

*Proof.* If $M \to N \to Q \to 0$ is the cokernel, then the fact that $M/K^{\circ\circ} \to N/K^{\circ\circ}$ is surjective implies that $Q/K^{\circ\circ} = 0$. As $M \to N$ is almost surjective, it follows that $Q$ is almost zero, so $Q \cdot K^{\circ\circ} = 0$. Thus, $Q = 0$. $\square$

Therefore, we have constructed a functor $F\colon \mathrm{Perf}_K \to \mathrm{Perf}_{K^{\circ a}}$ given by $R \mapsto R^{\circ a}$, and $G\colon \mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_K$ given by $A \mapsto A_*[1/\pi]$. We know that $G \circ F \simeq \mathrm{id}$ by Corollary 9.10 from last time. Moreover, $(FG)(A) = F(A_*[1/\pi]) = (A_*[1/\pi])^\circ = A_*$, where the last equality follows because $A \in \mathcal{D}_{\mathrm{prc}}$. Therefore, we have the equivalence $\mathrm{Perf}_K \simeq \mathrm{Perf}_{K^{\circ a}}$. $\square$

Next time, we will review the deformation theory required to show the other equivalence $\mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_{K^{\circ a}/\pi}$. We will need to show that perfectoid algebras have no interesting deformations; the perfect analogue of the required statement is the following exercise.

**Exercise 10.8.** If $R$ is a perfect $\mathbf{F}_p$-algebra and $A, B$ are two flat $\mathbf{Z}/p^2$-algebras lifting $R$, then there is a canonical isomorphism $A \simeq B$.

## 11. February 13th

Let $K$ be a perfectoid field (of characteristic 0), and let $\pi \in K$ be a perfect element (i.e. $\pi$ has all $p$-power roots) such that $|\pi| = |p|$ (hence $\pi = p \cdot (\text{unit})$). We had functors

$$\mathrm{Perf}_K \longleftarrow \mathrm{Perf}_{K^{\circ a}} \longrightarrow \mathrm{Perf}_{K^{\circ a}/\pi},$$

where $\mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_K$ is given by $A \mapsto A_*[1/\pi]$; and we showed last time that this is an equivalence. The functor $\mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_{K^{\circ a}/\pi}$ is given by $A \mapsto A/\pi$, and today we discuss the necessary deformation theory to show that it too is an equivalence.

The equivalence $\mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_K$ is less surprising if one has seen some non-Archimedean geometry, and it works for any reasonable finite-type algebra; however, the equivalence $\mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_{K^{\circ a}/\pi}$ is more surprising.

Recall the following result due to Witt:

**Theorem 11.1.** [Witt] *There is an equivalence of categories*

$$\Big\{p\text{-adically complete, } p\text{-torsion free rings } A \text{ such that } A/p \text{ is perfect}\Big\} \xrightarrow{\simeq} \Big\{\text{perfect rings of characteristic } p\Big\},$$

*where the functor is given by $A \mapsto A/p$.*

Witt proved Theorem 11.1 by constructing an inverse functor, but we will instead prove it using deformation (more precisely, by using the cotangent complex). A very similar proof will then apply in the perfectoid setting.

**Remark 11.2.** In Theorem 11.1, the inverse functor is given $B \mapsto W(B)$, where $W(B)$ denotes the ring of Witt vectors of $B$. While the ring of Witt vectors makes sense for any ring of characteristic $p$, it behaves best for when certain conditions are imposed (e.g. on perfect rings, or if there is no $p$-torsion).

11.1. **Cotangent complexes.** References for today's material are the following: Quillen's lecture notes [Qui], as well as his paper [Qui70], are concise. For a full development, see [Ill71, Ill72].

The idea is to "derive" the Kähler differentials functor, but this does not exactly make sense, because the category of rings is not abelian.

**Construction**. Fix a ring $A$ and a set $S$. Consider $A[S]$, the polynomial algebra over $A$ with variables indexed by $S$. The functor $S \mapsto A[S]$ is the left-adjoint to the forgetful functor $A$-algebras to sets (i.e. to give a map $A[S] \to B$ is the same as giving a set-theoretic map $S \to B$).

Consider an $A$-algebra $B$. Thinking of $B$ as a set, we get a map $\eta_B \colon A[B] \to B$, the counit map of the adjunction. Furthermore, there are two maps

$$A[A[B]] \rightrightarrows A[B],$$

one of which is $A[\eta_B]$ and the other is $\eta_{A[B]}$, and these are actually different! Continue this way get a "resolution"

$$\underbrace{\left(\ldots \overset{\to}{\underset{\to}{\rightrightarrows}} A[A[A[B]]] \overset{\to}{\underset{\to}{\rightrightarrows}} A[A[B]] \rightrightarrows A[B]\right)}_{=:\mathcal{P}^\bullet_{B/A}} \xrightarrow{\eta_B} B$$

The resolution $\mathcal{P}^\bullet_{B/A}$ is a simplicial $A$-algebra. As all procedures were functorial in the input $B$, $\mathcal{P}^\bullet_{B/A}$ is called the *canonical $A$-algebra resolution* of $B$. This construction is analogous to how one constructs a resolution of any group $G$ by free abelian groups by first taking the free abelian group $G_1$ on $G$, then taking the free abelian group $G_2$ on $G_1$, and so on.

Quillen's philosophy is to take a functor that behaves well for polynomial rings and apply this functor to a resolution; this leads to the definition of the cotangent complex.

**Definition 11.3.** [Quillen] Let $L_{B/A} \coloneqq \Omega^1_{\mathcal{P}^\bullet_{B/A}/A} \otimes_{\mathcal{P}^\bullet_{B/A}} B$; that is, $L_{B/A}$ is the simplicial $B$-module given by

$$\ldots \overset{\to}{\underset{\to}{\rightrightarrows}} \Omega^1_{\mathcal{P}^1_{B/A}} \otimes_{\mathcal{P}^1_{B/A}} B \rightrightarrows \Omega^1_{\mathcal{P}^0_{B/A}} \otimes_{\mathcal{P}^0_{B/A}} B$$

Taking alternating sums of the differentials yields a chain complex called the *cotangent complex* of $B/A$., which we also denote by $L_{B/A}$.

**Remark 11.4.** We always view $L_{B/A}$ as an object in the derived category $\mathbf{D}(B)$ of $B$-modules. In particular, any isomorphisms between complexes are isomorphisms in $\mathbf{D}(B)$ (or equivalently, quasi-isomorphisms of the complexes).

**Fact 11.5.** We may use any polynomial $A$-algebra resolution[16] of $B$ to calculate $L_{B/A} \in \mathbf{D}(B)$.

Fact 11.5 is needed to show that $L_{B/A}$ has certain good behavior. For example, the tensor product of two canonical resolutions is just a polynomial resolution, not the canonical resolution of the tensor product; nonetheless, the tensor product of the canonical resolutions can be used to compute the cotangent complex by the above fact.

[16] A simplicial $A$-algebra is a resolution of $B$ if the map to $B$ is a quasi-isomorphism of chain complexes, where the simplicial $A$-algebra is viewed as a complex by taking alternating sums of the differentials, and $B$ is viewed as a complex by placing it in degree 0.

**Remark 11.6.** For any ring map $A \to B$, there is a canonical isomorphism $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$; in particular, the cotangent complex is behaves sort of like a derived functor. This is proved by staring directly at the canonical free resolution $\mathcal{P}^\bullet_{B/A}$.

**Proposition 11.7.** [Properties of the cotangent complex]

*(1) If $B$ is a polynomial $A$-algebra, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$.*
*(2)* [Künneth formula] *If $B, C$ are flat $A$-algebras, then $L_{B \otimes_A C/A} \simeq \big(L_{B/A} \otimes_A C\big) \bigoplus \big(B \otimes_A L_{C/A}\big)$.*
*(3) Given maps $A \to B \to C$ of rings, there is an exact triangle*

$$L_{B/A} \otimes_B C \to L_{C/A} \to L_{C/B} \xrightarrow{+1}$$

*in $\mathbf{D}(C)$. This is called the* transitivity triangle*.*
*(4) Given a flat map $A \to B$ and any map $A \to C$, there is an isomorphism*

$$L_{B/A} \otimes_A C \xrightarrow{\simeq} L_{B \otimes_A C/C}.$$

*(5) If $A \to B$ is étale, then $L_{B/A} \simeq 0$.*
*(6) If $B \to C$ is an étale map of $A$-algebras, then $L_{B/A} \otimes_A C \xrightarrow{\simeq} L_{C/A}$.*
*(7) If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$.*

*Proof.* For (1), compute $L_{B/A}$ using the constant resolution $B \to B$.

For (2), use that $\mathcal{P}^\bullet_{B/A} \otimes_A \mathcal{P}^\bullet_{C/A}$ (tensored together termwise) is a polynomial $A$-algebra resolution of $B \otimes_A C$, because $B, C$ are flat. In fact, one only needs that $B$ and $C$ are Tor-independent, i.e. $\mathrm{Tor}_i^A(B, C) = 0$ for all $i > 0$. Here, we are using that the Dold-Kan functor

$$\{\text{simplicial } A\text{-modules}\} \to \{\text{chain complexes of } A\text{-modules}\},$$

given by sending a simplicial $A$-module $\mathcal{P}^\bullet$ to the same underlying complex of groups with the differential being the alternating sum of differentials of $\mathcal{P}^\bullet$, is compatible with tensor products (at least after passing to the derived category $\mathbf{D}(A)$).

For (3), consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \arrow[r] & C \\ A \arrow[r] & \mathcal{P}^\bullet_{B/A} \arrow[r] \arrow[u] & \mathcal{P}^\bullet_{C/\mathcal{P}^\bullet_{B/A}} \arrow[u] \end{tikzcd}$$

As $A \to \mathcal{P}^\bullet_{B/A}$ and $\mathcal{P}^\bullet_{B/A} \to \mathcal{P}^\bullet_{C/\mathcal{P}^\bullet_{B/A}}$ are termwise polynomial algebras, this reduces to showing the corresponding statement for $\Omega^1$ for polynomial rings; see e.g. [Har77, Proposition 8.3A].

For (4), argue as in (3) (and again, it suffices to assume Tor-independence rather than flatness). Derived algebraic geometry would allow one to drop any flatness conditions, if one is willing to work over simplicial rings.

For (5), assume first that $B$ is a localization of $A$ (i.e. this is an open immersion on Spec), then $B$ is flat over $A$, and the multiplication map $\mu \colon B \otimes_A B \xrightarrow{\simeq} B$ is an isomorphism. By (2),

$$\big(L_{B/A} \otimes_A B\big) \oplus \big(B \otimes_A L_{B/A}\big) \simeq L_{B \otimes_A B/A} \simeq L_{B/A}.$$

This can be rewritten as an isomorphism

$$L_{B/A} \oplus L_{B/A} \xrightarrow{\simeq} L_{B/A}.$$

This map turns out to be the sum map; thus, $L_{B/A} \simeq 0$.

In general, if $A \to B$ is étale, then the multiplication map $\mu \colon B \otimes_A B \to B$ is an open immersion. Thus, $L_{B/B \otimes_A B} = 0$ by the previous case. It is an exercise to use the transitivity triangle for $B \xrightarrow{i_1} B \otimes_A B \xrightarrow{\mu} B$ to conclude.

For (6), use the transitivity triangle plus the fact that $L_{C/B} \simeq 0$ by (5).

For (7), if $A \to B$ factors as $A \to P \to B$, where $A \to P$ is a polynomial algebra and $P \to B$ is étale, then the assertion holds for $A \to P$ by (1), for $P \to B$ by (5), and these can be put together by using the transitivity triangle. Here, we need that for any $A \to B$, $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$, as in Remark 11.6. In general, we can reduce to this case via Zariski localization (that is, after inverting some elements of $B$, we are in this setting). □

The moral of the story is that the familiar properties of Kähler differentials extend to the cotangent complex, possibly with extra flatness hypotheses.

The key result (which we will not prove) is the following:

**Theorem 11.8.** *For any commutative ring $A$, consider the category*

$$\mathcal{C}_A := \Big\{\text{flat } A\text{-algebras } B \text{ such that } L_{B/A} \simeq 0\Big\}.$$

*For any surjective map $\tilde{A} \to A$ with nilpotent kernel (i.e. some power is zero), the functor*[17] *$\mathcal{C}_{\tilde{A}} \to \mathcal{C}_A$, given by $B \mapsto B \otimes_{\tilde{A}} A$, is an equivalence.*

The crucial statement part of Theorem 11.8 is that every object in $\mathcal{C}_A$ has a unique lift to $\mathcal{C}_{\tilde{A}}$.

**Proposition 11.9.** *If $A$ is a ring of characteristic $p$ and $B$ is a flat $A$-algebra which is relatively perfect (i.e. the relative Frobenius map $F_{B/A} \colon B^{(1)} := B \otimes_{A,\mathrm{Frob}} A \to B$ is an isomorphism), then $L_{B/A} \simeq 0$.*

*Proof.* For any flat $A$-algebra $C$, the relative Frobenius map $F_{C/A} \colon L_{C^{(1)}/A} \to L_{C/A}$ is the zero map (this follows from the fact that $F_{P/A} \colon \Omega^1_{P^{(1)}/A} \to \Omega^1_{P/A}$ is the zero map for any polynomial $A$-algebra $P$, and using the canonical free resolution to compute the cotangent complexes).

Therefore, the relative Frobenius map $F_{B/A}$ induces the zero map on $L_{(-)/A}$; however, $F_{B/A}$ is an isomorphism, hence must go to an isomorphism by functoriality. Thus, $L_{B^{(1)}/A} \to L_{B/A}$ is both the zero map and an isomorphism, and this cannot occur unless $L_{B/A} \simeq 0$. □

**Corollary 11.10.** *Fix $n \geq 1$. Then, there is an equivalence of categories*

$$\mathcal{C}_n := \Big\{\text{flat } \mathbf{Z}/p^n\text{-algebras } A \text{ such that } A/p \text{ is perfect}\Big\} \simeq \Big\{\text{perfect rings of characteristic } p\Big\} =: \mathcal{C}_1,$$

*given by $B \mapsto B/p$.*

*Proof.* It suffices to show that $\mathcal{C}_n \subseteq \mathcal{C}_{\mathbf{Z}/p^n}$; more precisely, if $A \in \mathcal{C}_n$, then we must show that $L_{A/(\mathbf{Z}/p^n)} \simeq 0$. It follows from Proposition 11.9 (and flat base change) that $L_{A/(\mathbf{Z}/p^n)} \otimes_{\mathbf{Z}/p^n} \mathbf{Z}/p \simeq L_{(A/p)/(\mathbf{Z}/p)} \simeq 0$, because $A/p$ is perfect. Now, we can use the following general fact: if $K \in D(\mathbf{Z}/p^n)$ is a complex such that $K \otimes_{\mathbf{Z}/p^n} \mathbf{Z}/p = 0$, then $K = 0$. Therefore, $L_{A/(\mathbf{Z}/p^n)} = 0$, so $A \in \mathcal{C}_{\mathbf{Z}/p^n}$. □

Granted Corollary 11.10, we can pass to the limit over all $n$ to get the equivalence

$$\varprojlim_n \mathcal{C}_n = \Big\{\text{flat } p\text{-adically complete } \mathbf{Z}_p\text{-algebra } A \text{ such that } A/p \text{ is perfect } \Big\} \simeq \Big\{\text{perfect rings of char } p\Big\} = \mathcal{C}_1.$$

This is precisely the statement of Theorem 11.1.

Next time, we will show the perfectoid analogue of Theorem 11.1 to complete the proof of Theorem 10.4.

[17]Note that the assignment $\mathcal{C}_{\tilde{A}} \ni B \mapsto B \otimes_{\tilde{A}} A \in \mathcal{C}_A$ is a functor because of the flatness condition.

## 12. February 15th

12.1. **Cotangent complexes (continued).** Last time, we asserted the following:

**Theorem 12.1.** *If $A$ is a ring, let $\mathcal{C}_A := \{$flat $A$-algebras $B$ such that $L_{B/A} \simeq 0\}$. If $\tilde{A} \to A$ is a surjection of rings with nilpotent kernel (i.e., a "thickening"), then $\mathcal{C}_{\tilde{A}} \xrightarrow{\simeq} \mathcal{C}_A$.*

This was Theorem 11.8 from last time.

**Remark 12.2.** If $B \in \mathcal{C}_A$ and $B$ is finitely-presented over $A$, then $B$ is étale over $A$; that is, for finitely-presented algebras, the vanishing of the cotangent complex characterizes the property of being étale. In this setting, Theorem 12.1 is equivalent to the topological invariance of the étale site.

Furthermore, we proved the following:

**Proposition 12.3.** *If $A$ is a ring of characteristic $p$ and $B$ is a flat $A$-algebra such that $B$ is relatively perfect over $A$, then $B \in \mathcal{C}_A$, i.e. $L_{B/A} \simeq 0$.*

This was Proposition 11.9 from last time. Finally, we sketched the following corollary, which we now prove.

**Corollary 12.4.** *If $\mathcal{C}_n$ denotes the category of flat $\mathbf{Z}/p^n$-algebras $B$ such that $B/p$ is perfect, then $\mathcal{C}_{n+1} \xrightarrow{\simeq} \mathcal{C}_n$ via the functor $B \mapsto B/p^n$.*

*Proof.* If $B \in \mathcal{C}_n$, then there is an isomorphism $L_{B/(\mathbf{Z}/p^n)} \otimes^L_{\mathbf{Z}/p^n} \mathbf{Z}/p \xrightarrow{\simeq} L_{(B/p)/(\mathbf{Z}/p)}$ by Proposition 11.7(4). As $B/p$ is perfect, $L_{(B/p)/(\mathbf{Z}/p)} \simeq 0$.

We claim that this implies that $L_{B/(\mathbf{Z}/p^n)} \simeq 0$. Consider the special case $n = 2$. If $K \in \mathbf{D}(\mathbf{Z}/p^2)$ is any complex of $\mathbf{Z}/p^2$-modules, then there exists an exact triangle

$$K \otimes^L_{\mathbf{Z}/p^2} \mathbf{Z}/p \to K \to K \otimes^L_{\mathbf{Z}/p^2} \mathbf{Z}/p,$$

obtained by taking the usual short exact sequence $0 \to \mathbf{Z}/p \to \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0$ and tensoring by $K$. Thus, $K = 0$ iff $K \otimes^L_{\mathbf{Z}/p^2} \mathbf{Z}/p = 0$. Applying this to our case, it follows that $L_{B/(\mathbf{Z}/p^n)} \simeq 0$, and hence $B \in \mathcal{C}_n$.

The above argument gives an inclusion $\mathcal{C}_n \hookrightarrow \mathcal{C}_{\mathbf{Z}/p^n}$, hence we have a diagram

$$\begin{tikzcd} \mathcal{C}_{n+1} \arrow[r, hook] \arrow[d] & \mathcal{C}_{\mathbf{Z}/p^{n+1}} \arrow[d, "\simeq"] \\ \mathcal{C}_n \arrow[r, hook] & \mathcal{C}_{\mathbf{Z}/p^n} \end{tikzcd}$$

It follows immediately from the definitions that this is a fibre square of categories, since $\mathcal{C}_n \hookrightarrow \mathcal{C}_{\mathbf{Z}/p^n}$ and $\mathcal{C}_{n+1} \hookrightarrow \mathcal{C}_{\mathbf{Z}/p^{n+1}}$ are fully faithful; that is, to give an object of $\mathcal{C}_{n+1}$ is to give an object of $\mathcal{C}_{\mathbf{Z}/p^{n+1}}$, whose image in $\mathcal{C}_{\mathbf{Z}/p^n}$ lies in $\mathcal{C}_n$. As the functor $\mathcal{C}_{\mathbf{Z}/p^{n+1}} \xrightarrow{\simeq} \mathcal{C}_{\mathbf{Z}/p^n}$ is an equivalence by Theorem 12.1, and because this is a fibre square, it follows that $\mathcal{C}_{n+1} \to \mathcal{C}_n$ is also an equivalence. $\square$

**Remark 12.5.** The inverse to $\mathcal{C}_n \overset{\simeq}{\to} \mathcal{C}_1$ is given by $B \mapsto W_n(B)$, where $W_n(B)$ denotes the Witt vectors on $B$ of level $n$.

The reasoning used in Corollary 12.4 also shows the following:

**Theorem 12.6.** *If $A$ is a ring of characteristic $p$ and $I \subseteq A$ is a nilpotent ideal, then base change along $A \to A/I$ induces an equivalence*

$$\{\text{flat and relatively perfect } A\text{-algebras}\} \xrightarrow{\simeq} \{\text{flat and relatively perfect } A/I\text{-algebras}\}.$$

*Proof.* By the equivalence $\mathcal{C}_A \simeq \mathcal{C}_{A/I}$ and Proposition 12.3, we already have full faithfulness. Moreover, any flat and relatively perfect $A/I$-algebra $\overline{B}$ lifts to a unique flat $A$-algebra $B$ such that $L_{B/A} \simeq 0$. We must show that $B$ is relatively perfect over $A$, i.e. $F_{B/A}$ is an isomorphism. This follows by combining the following two facts:

(1) The formation of $F_{B/A}$ commutes with arbitrary base change, so $F_{B/A} \otimes A/I \simeq F_{\overline{B}/(A/I)}$.
(2) If $f\colon M \to N$ is a map of flat $A$-modules such that $f \otimes A/I$ is an isomorphism, so is $f$.

To see (2), set $Q = \operatorname{cone}(f) \in \mathbf{D}(A)$ and we want to show that $Q \simeq 0$. We know that $Q \otimes^L_A A/I \simeq 0$ (because $M$ and $N$ are flat, so the cone of $f \otimes_A A/I$ is precisely the cone $Q$ mod $I$). Now, for any complex $K \in \mathbf{D}(A)$, $K \simeq 0$ iff $K \otimes^L_A A/I \simeq 0$ (as in the proof of Corollary 12.4, filter $A$ by $A/I$-modules). Thus, $Q \simeq 0$, as required. $\square$

**Corollary 12.7.** *If $\mathcal{C}_\infty$ denotes the category of $p$-adically complete, $p$-torsion free (or flat) $\mathbf{Z}_p$-algebras $B$ such that $B/p$ is perfect, then $\mathcal{C}_\infty \xrightarrow{\simeq} \mathcal{C}_n$ via the functor $B \mapsto B/p^n$.*

*Proof.* We have functors

$$\begin{array}{ccccc} \mathcal{C}_\infty & \longrightarrow & \varprojlim_n \mathcal{C}_n & \xrightarrow{\simeq} & \mathcal{C}_n \\ B & \longmapsto & (B/p^n) & & \\ & & (R_n) & \longmapsto & R_n \end{array}$$

It suffices to show that there is an equivalence

$$\left\{\begin{matrix} p\text{-adically complete, } p\text{-torsion free} \\ \mathbf{Z}_p\text{-modules } M \end{matrix}\right\} \xrightarrow{\simeq} \varprojlim \{\text{flat } \mathbf{Z}/p^n\text{-modules}\} := \left\{ (M_n, \varphi_n)\colon \begin{matrix} M_n \text{ flat } \mathbf{Z}/p^n\text{-modules,} \\ \varphi_n\colon M_{n+1}/p^n \xrightarrow{\simeq} M_n \end{matrix} \right\},$$

where the functor is given by $M \mapsto (M/p^n, \operatorname{std}_n)$ and $\operatorname{std}_n\colon (M/p^n)/p^{n-1} \to M/p^{n-1}$ is the standard map; the inverse functor is given by $(M_n, \varphi_n) \mapsto \varprojlim_n M_n$, where the transition maps of the inverse limit are the maps $M_{n+1} \to M_n$ induced by $\varphi_n$. The proof of this fact[18] is left as an exercise. $\square$

**Remark 12.8.** More generally, a similar argument shows the following: if $K/\mathbf{Q}_p$ is a non-Archimedean field extension, then there is a functor

$$\left\{\begin{matrix} \text{flat, } p\text{-adically complete } K^\circ\text{-algebras } B \\ \text{such that } K^\circ/p \to B/p \text{ is relatively perfect} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{flat } K^\circ/p\text{-algebra } \overline{B} \\ \text{such that } K^\circ/p \to \overline{B} \text{ is relatively perfect} \end{matrix}\right\},$$

given by $B \mapsto \overline{B} := B/p$. The exact same reasoning shows that this is also an equivalence. If $K = \mathbf{Q}_p$, this is the equivalence $\mathcal{C}_\infty \xrightarrow{\simeq} \mathcal{C}_1$ of Corollary 12.7. If $K/\mathbf{Q}_p$ is finite, the inverse functor is again given by the Witt vectors; in general, a more complicated construction is required.

For $K/\mathbf{Q}_p$ perfectoid, an almost version of the previous argument shows the following equivalence:

$$\operatorname{Perf}_{K^{\circ a}} := \left\{\begin{matrix} p\text{-adically complete, } p\text{-torsion free} \\ K^{\circ a}\text{-algebras } B \text{ satisfying } (*) \end{matrix}\right\} \xrightarrow{\simeq} \left\{\begin{matrix} \text{flat } K^{\circ a}/p\text{-algebras } \overline{B} \\ \text{such that } K^{\circ a}/p \to \overline{B} \text{ is relatively perfect} \end{matrix}\right\} =: \operatorname{Perf}_{K^{\circ a}/p},$$

where $(*)$ is the condition that $K^{\circ a}/p \to B/p$ is relatively perfect. Condition $(*)$ is equivalent to demanding that $\operatorname{Frob}\colon B/p^{1/p} \xrightarrow{\simeq} B/p$, provided $p^{1/p}$ makes sense (and if it does not make sense, then replace $p$ with a different pseudouniformizer $\pi$). This shows the second half of the tilting correspondence Theorem 10.4.

**Remark 12.9.** Let $K/\mathbf{Q}_p$ be perfectoid and let $\pi \in K$ be a pseudouniformizer such that $\pi$ is perfect (i.e. it has all $p$-power roots) and $1 > |\pi| \geq |p|$ (this ensures that $K^\circ/\pi$ has characteristic $p$). There is a map $K^\circ/p \to K^\circ/\pi$, which induces $\operatorname{Perf}_{K^{\circ a}/p} \xrightarrow{\simeq} \operatorname{Perf}_{K^{\circ a}/\pi}$, given by $B \mapsto B/\pi$; this follows from the almost version of Theorem 12.6. That is, these categories are insensitive to e.g. killing $p$ versus killing $\sqrt{p}$. In particular, the category $\operatorname{Perf}_{K^{\circ a}/\pi}$ is independent of the choice of $\pi$.

Let us take stock of what we have shown. If $t^\sharp = \pi$, then

[18] The analogous fact is true over any noetherian ring, where $p$ is replaced with some ideal.

$$\begin{array}{ccccccccc} \mathrm{Perf}_K & \xleftarrow{\simeq} & \mathrm{Perf}_{K^{\circ a}} & \xrightarrow{\simeq} & \mathrm{Perf}_{K^{\circ a}/\pi} & = & \mathrm{Perf}_{K^{\flat\circ a}/t} & \xleftarrow{\simeq} & \mathrm{Perf}_{K^{\flat\circ a}} & \xrightarrow{\simeq} & \mathrm{Perf}_{K^\flat} \\ A_*[1/\pi] & \longleftarrow & A & & & & & & A & \longmapsto & A_*[1/t] \\ & & A & \longmapsto & A/\pi & & A/t & \longleftarrow & A & & \end{array}$$

There is a concrete way to check whether a perfectoid $K^\flat$-algebra corresponds to a perfectoid $K$-algebra under this chain of equivalences.

**Corollary 12.10.** *If $R \in \mathrm{Perf}_K$, then $R$ corresponds to $R^\flat \in \mathrm{Perf}_K$ iff there is an identification $R^{\circ a}/\pi \xrightarrow{\simeq} R^{\flat\circ a}/t$ over the identification $K^{\circ a}/\pi \xrightarrow{\simeq} K^{\flat\circ a}/t$.*

**Example 12.11.**

(1) If $R = K$, $R^\flat = K^\flat$.
(2) If $R = K\langle x^{1/p^\infty}\rangle$, then $R^\flat = K^\flat\langle x^{1/p^\infty}\rangle$.

*Proof.* By definition, $R = K^\circ[x^{1/p^\infty}]^\wedge[1/\pi]$ and $R^\flat = K^{\flat\circ}[x^{1/p^\infty}]^\wedge[1/t]$. As $K^\circ[x^{1/p^\infty}]$ and $K^{\flat\circ}[x^{1/p^\infty}]$ are totally integrally closed in $K^\circ[x^{1/p^\infty}][1/p]$ and $K^{\flat\circ}[x^{1/p^\infty}][1/t]$ respectively, and being totally integrally closed passes to completion, so we have $R^\circ = K^\circ[x^{1/p^\infty}]^\wedge$ and $R^{\flat\circ} = K^{\flat\circ}[x^{1/p^\infty}]^\wedge$. Thus, $R^\circ/\pi = (K^\circ/\pi)[x^{1/p^\infty}]$ and $R^{\flat\circ}/t = (K^{\flat\circ}/t)[x^{1/p^\infty}]$, hence they coincide under the isomorphism $K^\circ/\pi \simeq K^{\flat\circ}/t$ (even without almostifying!). □

For rings that are not "toric", the tilting functor can be badly behaved; e.g., it may mix up finite type and non-finite type rings.

## 12.2. Tilting by Fontaine's functors.

**Theorem 12.12.** *Given $R \in \mathrm{Perf}_K$, we have*

$$R^\flat = (R^\circ)^\flat\,[1/t] = \left(\varprojlim_\phi R^\circ/p\right)[1/t] \in \mathrm{Perf}_{K^\flat}\,.$$

Thus, the abstract tilting equivalence between $\mathrm{Perf}_{K^{\circ a}}$ and $\mathrm{Perf}_{K^{\flat\circ a}}$ is given by Fontaine's tilting functor from Definition 2.9.

*Proof.* It suffices to show that the perfect ring $(R^\circ)^\flat$ is a flat $t$-adically complete $K^{\flat\circ}$-algebra lifting $K^{\flat\circ}/t = K^\circ/\pi$ to $R^\circ/\pi$. Said differently, there are maps $K^{\flat\circ} \to R^{\circ\flat}$ and $K^\circ \to R^\circ$ and we require that

$$\begin{tikzcd} K^{\flat\circ}/t \arrow[r] \arrow[d, equal] & R^{\circ\flat}/t \arrow[d, dashed, "\exists\,\simeq"] \\ K^\circ/\pi \arrow[r] & R^\circ/\pi \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
a_1\colon & K^{\flat\circ}/t \simeq K^\circ/\pi \arrow[r, "\mathrm{std}"] & R^\circ/\pi \\
a_2\colon & K^{\flat\circ}/t^p \arrow[r, "\mathrm{std}\circ\phi^{-1}"] \arrow[u, "\mathrm{std}"] & R^\circ/\pi \arrow[u, "\phi"'] \\
\vdots & \vdots \arrow[u, "\mathrm{std}"] & \vdots \arrow[u, "\phi"'] \\
a_n\colon & K^{\flat\circ}/t^{p^n} \arrow[r, "\mathrm{std}\circ\phi^{-n}"] \arrow[u, "\mathrm{std}"] & R^\circ/\pi \arrow[u, "\phi"'] \\
\vdots & \vdots \arrow[u, "\mathrm{std}"] & \vdots \arrow[u, "\phi"']
\end{tikzcd}$$

As in the above diagram, $a_n \coloneqq \mathrm{std} \circ \phi^{-n}\colon K^{\flat\circ}/t^{p^n} \xrightarrow{\phi^{-n}} K^{\flat\circ}/t \simeq K^\circ/\pi \xrightarrow{\mathrm{std}} R^\circ/\pi$, and $a_n$ is flat and relatively perfect because $\mathrm{std}\colon K^\circ/\pi \to R^\circ/\pi$ is so. One can check that $a_n$ is a lift of $a_{n-1}$ along the standard map $K^{\flat\circ}/t^{p^n} \to K^{\flat\circ}/t^{p^{n-1}}$. By Theorem 12.6, $a_n$ is the (necessarily unique) deformation of $a_1$ along the standard map $K^{\flat\circ}/t^{p^n} \to K^{\flat\circ}/t$. Taking the inverse limit of the $a_n$'s (and arguing as in Corollary 12.7), it follows that $K^{\flat\circ} \to R^{\circ\flat}$ is the unique flat and relatively perfect map lifting $a_1$. As this property also characterized the image of $R^\circ$ in $\mathrm{Perf}_{K^{\flat\circ a}}$, it follows that $R^{\circ\flat}$ is the image of $R$ in $\mathrm{Perf}_{K^{\flat\circ a}}$. The theorem follows by inverting $t$. $\square$

If $R \in \mathrm{Perf}_K$, then there is a bijective multiplicative map $R^{\circ\flat} \coloneqq \varprojlim_\phi R^\circ/\pi \xleftarrow{\simeq} \varprojlim_{x\mapsto x^p} R^\circ$, hence we get a multiplicative map $\sharp\colon R^{\circ\flat} \to R^\circ$ that sits in the following diagram:

$$\begin{tikzcd}
\varprojlim_{x\mapsto x^p} R^\circ \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\simeq"] & R^\circ \arrow[d] \\
R^{\circ\flat} = \varprojlim_\phi R^\circ/\pi \arrow[r] \arrow[ur, dashed, "\sharp"] & R^\circ/\pi
\end{tikzcd}$$

The image of $\sharp\colon R^{\circ\flat} \to R^\circ$ consists of exactly those $f \in R^\circ$ that admit a compatible system of $p$-power roots; such elements are said to be *perfect*.

## 13. February 20th

Let $K$ be a perfectoid field, and let $t \in K^\flat$ pseudouniformizer such that $\pi = t^\sharp$ satisfies $|\pi| \geq |p|$. We have shown the following:

$$\begin{tikzcd}
 & & A \arrow[rr, mapsto] & & A/\pi \\
R \arrow[rr, mapsto] & & R^{\circ a} & & \\
\mathrm{Perf}_K & & \mathrm{Perf}_{K^{\circ a}} \arrow[ll, "\simeq"'] \arrow[rr, "\simeq"] & & \mathrm{Perf}_{K^{\circ a}/\pi} \arrow[dddd, equal] \\
 & & A \arrow[dd, mapsto] & & \\
 & \begin{array}{c}\text{Banach algebra}\\ \text{theory}\end{array} \arrow[uu, dashed] \arrow[dd, dashed] & & \begin{array}{c}\text{Deformation}\\ \text{theory}\end{array} \arrow[uu, dashed] \arrow[dd, dashed] & \\
 & & A^\flat & & \\
\mathrm{Perf}_{K^\flat} & & \mathrm{Perf}_{K^{\flat\circ a}} \arrow[ll, "\simeq"] \arrow[rr, "\simeq"'] & & \mathrm{Perf}_{K^{\flat\circ a}/t}
\end{tikzcd}$$

To $R \in \mathrm{Perf}_K$, we associate $R^\flat = (R^\circ)^\flat[1/t] \in \mathrm{Perf}_{K^\flat}$. Last time, we explained that if $R \in \mathrm{Perf}_K$, then $R^{\flat\circ} \overset{a}{\simeq} R^{\circ\flat} := \varprojlim_\phi (R^\circ/\pi)^{\mathrm{perf}}$; the discussion took place in the almost category, because it was only there that we had uniqueness of lifts (as in the proof of Theorem 12.12).

**Exercise 13.1.**

(1) Check that $R^{\flat\circ} = R^{\circ\flat}$. *Hint: verify that $R^{\circ\flat}$ is totally integrally closed in $R^{\circ\flat}[1/t]$.* In particular, one has an explicit formula for the tilt $R^\flat$ and $R^{\flat\circ}$.

(2) In order to explicitly reconstruct $R$ from $R^\flat$, one must use facts about the Witt vectors: if $A$ is a perfect $\mathbf{F}_p$-algebra and $B$ is a $p$-adically complete ring, then

$$\mathrm{Hom}_{\mathbf{F}_p\text{-alg}}(A, B/p) \simeq \mathrm{Hom}_{\mathbf{Z}_p\text{-alg}}(W(A), B),$$

where $W(A)$ denotes the ring of Witt vectors of $A$. Therefore, for any $R \in \mathrm{Perf}_K$, take $A = R^\circ$ to get a map $\theta\colon W(A^\flat) \to A$ adjoint to the canonical map $A^\flat \to A/p$. The map $\theta$ plays an important role in $p$-adic Hodge theory, and it is known as Fontaine's functor $\theta$. Furthermore, the diagram

$$\begin{array}{ccc} W(K^{\circ\flat}) & \xrightarrow{\theta} & K^\circ \\ \downarrow & & \downarrow \\ W(A^\flat) & \xrightarrow{\theta} & A \end{array}$$

is a pushout. In other words, one can recover $R^\circ$ as

$$R^\circ = K^\circ \otimes_{W(K^{\circ\flat})} W(R^{\circ\flat}).$$

Therefore, we have formula recovering $R^\circ$ (and thus $R$) from $R^\flat$. This is the approach used in [KL15, KL16] to prove the tilting correspondence, thus avoiding the deformation theory argument.

13.1. **Almost purity.** As motivation, recall the Zariski–Nagata purity theorem. If $X/C$ is a smooth variety and $D \subseteq X$ is a smooth divisor, then we would like to describe the fundamental group of the complement $X\backslash D$; there is one obvious loop (namely, the one around $D$) and the purity theorem asserts that this is basically all that can occur. More precisely, every finite étale cover $V \to U := X\backslash D$ extends to a finite étale cover of $X$, after extracting an $n$-th root of the equation defining $D$.

Locally, if $R$ is a complete regular local $\mathbf{C}$-algebra and $f \in R$ is a regular parameter, then the purity theorem asserts that

$$\pi_1^{\text{ét}}(\mathrm{Spec}(R)\backslash\{f=0\}) \simeq \widehat{\mathbf{Z}}.$$

The idea is that taking the quotient $\pi_1^{\text{ét}} \twoheadrightarrow \mathbf{Z}/n$ corresponds to extracting an $n$-th root of $f$.

The almost purity theorem is the perfectoid analogue of the above statement.

**Theorem 13.2.** [Almost purity theorem] *Let $K$ be a perfectoid field, let $R \in \mathrm{Perf}_K$, and let $S = R^\flat \in \mathrm{Perf}_{K^\flat}$.*

*(1)* [Almost purity in characteristic $p$]*: inverting $t$ gives an equivalence $S^{\circ a}_{a\,\text{fét}} \xrightarrow{\simeq} S_{\text{fét}}$.*

*(2)* [Almost purity in characteristic 0]*: inverting $\pi$ gives an equivalence $R^{\circ a}_{a\,\text{fét}} \xrightarrow{\simeq} R_{\text{fét}}$.*

*(3) The tilting and untilting functors give an equivalence $R^{\circ a}_{a\,\text{fét}} \simeq S^{\circ a}_{a\,\text{fét}}$.*

**Remark 13.3.** Originally, Faltings showed (2) for rings of the form $R^\circ = K^\circ[x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty}]^\wedge$ and étale algebras over $R^\circ$. This is the main case one needs for $p$-adic Hodge theory applications, but for direct summand applications we really need the more general statement.

The assertion (1) was Theorem 7.10, and it has already been proven. Today, we will construct fully faithful functors $a, b, c, d$ such that

$$S_{\text{fét}} \xleftarrow{a} S^{\circ a}_{a\,\text{fét}} \xrightarrow{b} (S^{\circ a}/t)_{a\,\text{fét}} \simeq (R^{\circ a}/\pi)_{a\,\text{fét}} \xleftarrow{c} R^{\circ a}_{a\,\text{fét}} \xrightarrow{d} R_{\text{fét}}$$

and moreover show that $a, b, c$ are equivalences. We will only show that $d$ is an equivalence when $R$ is a perfectoid field. Once we set up the machinery of adic spaces, we will deduce that $d$ is an equivalence for general $R$ from the corresponding assertion over a field.

*Proof.* The functor $a$ was shown to exist and be an equivalence in Theorem 7.10. The functors $b$ and $c$ are the obvious ones (reduce mod $t$, or mod $\pi$) and they are equivalences by the following result.

**Theorem 13.4.** [GR03, Theorem 5.3.27] *The map $S^{\circ a}/t^{n+1} \to S^{\circ a}/t^n$ gives an equivalence on almost finite étale covers.*

If one ignores the almost mathematics, this assertion is the topological invariance of the étale site.

To show that $d$ exists and is fully faithful, it suffices to show the following: if $A \in R^{\circ a}_{a\,\text{fét}}$, then

1. $A$ is perfectoid (i.e. $A \in \mathrm{Perf}_{K^{\circ a}}$);
2. $A_*[1/\pi] \in R_{\text{fét}}$;
3. $A$ is the almostification of the total integral closure of $R^\circ$ in $A_*[1/\pi]$.

The assertion (2) implies that there is a faithful functor $d\colon R^{\circ a}_{a\,\text{fét}} \to R_{\text{fét}}$, and (3) implies that we may recover $A$ from $d(A)$ (both at the level of objects and morphisms), so $d$ is fully faithful. For (1), if $A \in R^{\circ a}_{a\,\text{fét}}$, then:

(i) $A$ is almost finite projective over $R^{\circ a}$;
(ii) $R^{\circ a}/\pi \to A/\pi$ is weakly étale, and thus relatively perfect.

Condition (i) implies that $A$ is flat over $K^{\circ a}$, and (ii) implies that $K^{\circ a}/\pi \to A/\pi$ is also relatively perfect (because it factors as $K^{\circ a}/\pi \to R^{\circ a}/\pi \to A/\pi$, and the composition of relatively perfect maps is relatively perfect). Moreover, $A$ is $\pi$-adically complete, because it is a direct summand of a free module, which is complete. Therefore, $A$ is a perfectoid $K^{\circ a}$-algebra.

For (2), this is clear (given an algebra that is almost finite étale integrally, it becomes finite étale rationally). For (3), it is left as an exercise (the proof is similar to Corollary 9.10; it is in this step that one must use (1)).

It remains to show that if $R = K$, then $d$ is an equivalence. There is a fully faithful functor $K^\flat_{\text{fét}} \to K_{\text{fét}}$, denoted $L \mapsto L^\sharp$. We also know that it preserves degrees, it preserves automorphism groups, and thus it preserves Galois extensions. Using Galois theory, it suffices to show that each $N/K$ finite extension embeds into some $L^\sharp$. Let $M = \widehat{\overline{K^\flat}}$, the completed algebraic closure of $K^\flat$; this is a perfectoid field in characteristic $p$ that is algebraically closed. By Proposition 4.8, $M^\sharp$ is also algebraically closed. Every $N/K$ finite embeds into $M^\sharp$, so one must argue that it embeds into some finite version of $M^\sharp$. Set

$$M^\sharp_{\text{nc}} \coloneqq \varinjlim_{L \subseteq M} L^\sharp \subseteq M^\sharp.$$

As $M = \left(\varinjlim_{L\subseteq M} L\right)^\wedge$, it follows that $M^\sharp_{\text{nc}} \subseteq M^\sharp$ is dense. But, $M^\sharp$ is algebraically closed, so the dense subfield $M^\sharp_{\text{nc}}$ must also be algebraically closed (by Krasner's lemma plus separability). As $M^\sharp_{\text{nc}}/K$ is algebraically closed, each finite extension $N/K$ embeds into $M^\sharp_{nc} = \varinjlim L^\sharp$, so each $N/K$ embeds into some $L^\sharp$, as required. ☐

13.2. **Adic spaces.** The basic objects in our discussion are Tate rings (this is a slightly more restrictive setting than in other sources, but it is all that is need for our purposes).

**Definition 13.5.** A topological ring $A$ is *Tate* if there exists an open subring $A_0 \subseteq A$ such that the induced topology on $A_0$ is $t$-adic, where $t \in A_0$ is some element that is invertible in $A$.

Any such pair $(A_0, t)$ is called a *couple of definition* (cod), $A_0$ is called a *ring of definition* (rod), and $(t) \subseteq A_0$ is called an *ideal of definition* (iod).

**Example 13.6.**

1. If $K$ is a non-Archimedean field with its usual topology, then $K$ is Tate: $(K^\circ, \pi)$ is a couple of definition, for any pseudouniformizer $\pi \in K$, i.e. $\pi \in K^{\circ\circ}\backslash\{0\}$.
2. More generally, if $R$ is a $K$-Banach algebra, then $R$ is Tate: $(R_{\leq 1}, \pi)$ is a couple of definition, and $\pi \in K$ is a pseudouniformizer.
3. Algebraically, the way to construct these Tate rings is: given any ring $B_0$ and an element $t \in B_0$, get a Tate ring $B \coloneqq B_0[1/t]$ with couple of definition equal to $\left(\frac{B_0}{t^\infty\text{-torsion}}, t\right)$. Note that, in order to specify

the topology on $B$, it is enough to specify the couple of definition. As an example of this construction, take $B_0 = \mathbf{Z}_{(p)}$ and $t = p$, then $B = \mathbf{Q}$ with the $p$-adic topology.

We will often restrict to the complete case when dealing with these objects, but there are occasions (e.g. when discussing the structure sheaf on an adic space) when this restriction is burdensome, so it is best to not impose completeness in the definition.

**Exercise 13.7.** If $R$ is Tate and $(R_0, t)$ is a couple of definition, then $R = R_0[1/t]$.

## 14. February 22nd

Last time, we defined Tate rings; recall that $A$ is a *Tate ring* iff there is an open subring $A_0 \subseteq A$ such that the induced topology on $A_0$ is $t$-adic for some $t \in A_0$ that is invertible in $A$. The subring $A_0$ is called a *ring of definition*, $(t)$ is called an *ideal of definition*, and $(A_0, t)$ is called a *couple of definition*.

The main example, algebraically speaking, is the following: given a ring $B_0$ and an element $t \in B_0$, $B := B_0[1/t]$ is a Tate ring with couple of definition $(B_0/t^\infty\text{-torsion}, t)$.

14.1. **Basic properties of Tate rings.** Let $A$ be a Tate ring. Some of basic properties and concepts of Tate rings are listed below.

(1) For any couple of definition $(A_0, t)$ of $A$, we have $A = A_0[1/t]$.
(2) *Bounded sets.* If $S \subseteq A$ is a subset, then $S$ is *bounded* if $S \subseteq t^{-c}A_0$ for some $c \geq 0$ and some couple of definition $(A_0, t)$.

The notion of boundedness is independent of the choice of couple of definition $(A_0, t)$ (though changing the couple of definition may change the value $c$). Bounded subsets satisfy many stability properties; e.g. the union of two bounded sets is bounded, and the $A_0$-submodule of $A$ generated by a bounded set is bounded.

If $B_0 \subseteq A$ is an open and bounded subgroup, then $t^n A_0 \subseteq B_0 \subseteq t^{-n} A_0$ for some $n \geq 0$. Using this observation, one can show the following:

**Fact 14.1.** If $B_0$ is a subring of $A$, then $B_0$ is a ring of definition of $A$ iff $B_0$ is open and bounded.

One should think of Fact 14.1 as saying that we can recover the rings of definition from the topological ring $A$ (as the open and bounded subrings). From Fact 14.1 and basics about bounded sets, we get the following:

**Fact 14.2.** The collection of rings of definition of $A$ is filtered; that is, any two rings of definition of $A$ are contained in a third.

(3) *Powerboundedness.* An element $f \in A$ is *powerbounded* if $f^{\mathbf{N}}$ is a bounded set. More explicitly, this means that $f^{\mathbf{N}} \subseteq t^{-c}A_0$ for some couple of definition $(A_0, t)$ and $c \geq 0$ (or equivalently, $t^c f^{\mathbf{N}} \in A_0$ for some couple of definition $(A_0, t)$ and $c \geq 0$).

Observe that $A^\circ := \{f \in A \colon f \text{ is powerbounded}\}$ is a subring of $A$. Moreover, $A^\circ$ is integrally closed in $A$, and it contains any ring of definition (hence it is open). In fact,

$$A^\circ = \varinjlim_{A_0 \subseteq A} A_0 \subseteq A,$$

where the colimit ranges over rings of definition $A_0 \subseteq A$. Though any ring of definition of $A$ is bounded, this is not necessarily the case for $A^\circ$.

**Definition 14.3.** We say $A$ is *uniform* if $A^\circ$ is bounded (and thus $A^\circ$ is a ring of definition).

The advantage to discussing $A^\circ$, as opposed to a ring of definition of $A$, is that it is intrinsic to $A$ (as a topological ring).

**Example 14.4.** [Non-example] If $A_0$ is any ring and $t \in A$ is a non-zero-divisor and non-unit such that $A_0$ is $t$-adically separated, and if $\epsilon \in A_0$ is a nontrivial nilpotent, then $A \coloneqq A_0[1/t]$ is a Tate ring with couple of definition $(A_0, t)$, and we claim that $A$ is not uniform.

Indeed, since $A^\circ$ is integrally closed, it must contain every nilpotent element, so $t^{-n}\epsilon \in A^\circ$ for all $n \geq 0$; however, $\{t^{-n}\epsilon \colon n \geq 0\}$ is an unbounded set (if it was bounded, then $\epsilon \in \bigcap_n t^n A_0$, which is not possible by the $t$-adic separatedness of $A_0$).

(4) *Topologically nilpotent elements.*

**Definition 14.5.** An element $f \in A$ is *topologically nilpotent* if $f^n \to 0$ as $n \to +\infty$.

Colloquially, $f \in A$ is topologically nilpotent if every small neighbourhood of 0 contains "most" powers of $f$. Set $A^{\circ\circ}$ to be the collection of topologically nilpotent elements of $A$, then it is not so hard to see that $A^{\circ\circ} \subseteq A^\circ$ and moreover $A^{\circ\circ}$ is an ideal of $A^\circ$.

An element $f \in A$ is a *pseudouniformizer* (pu) if $f \in A^{\circ\circ}$ and $f$ is a unit in $A$. As an example, if $A$ is a Banach algebra over a non-Archimedean field $K$, then any non-zero element of $K^{\circ\circ}$ (the maximal ideal of the valuation ring $K^\circ$ of $K$) is a pseudouniformizer.

**Fact 14.6.** An element $f \in A$ is a pseudouniformizer iff there exists a couple of definition of $A$ of the form $(A_0, f)$.

One direction is clear: if $(A_0, f)$ is a couple of definition, then certainly $f$ is a unit, but the topology on $A_0$ is generated by powers of $f$, so $f^n$ must go to zero. Using Fact 14.6, one can also show that $A^{\circ\circ} = \sqrt{(f)}$ (as ideals of $A^\circ$) for any pseudouniformizer $f \in A$.

(5) *Completions.*

**Definition 14.7.** We say $A$ is *complete* if $A_0$ is $t$-adically complete for one (equivalently, for any) couple of definition $(A_0, t)$.

**Example 14.8.** If $K$ is a non-Archimedean field (by convention, $K$ is complete) and $\pi \in K$ is a pseudouniformizer, then $K$ is complete as a Tate ring, and $K\langle X\rangle = (K^\circ[X]^\wedge)[1/\pi]$ is a complete Tate ring (where $^\wedge$ denotes the $\pi$-adic completion) with couple of definition $(K^\circ[X]^\wedge, \pi)$. More generally, any $K$-Banach algebra is a complete Tate ring.

**Fact 14.9.** The obvious inclusion

$$\{\text{complete Tate rings}\} \hookrightarrow \{\text{Tate rings}\}$$

has a left adjoint $A \mapsto \widehat{A}$, where $\widehat{A}$ is called the *completion* of $A$. Explicitly, given a couple of definition $(A_0, t)$ of $A$, then $\widehat{A} = \widehat{A_0}[1/t]$, and a couple of definition of $\widehat{A}$ is $(\widehat{A_0}, t)$.

As topological groups, there is a more explicit description of the completion:

$$\widehat{A} \simeq \varprojlim_n A/t^n A_0.$$

From this description, it is not even obvious that this is a ring, because $t$ is invertible in $A$. See [Dat17] for two descriptions of the multiplication on $\widehat{A}$ using this description.

14.2. **Affinoid (pre-)adic spaces.**

**Definition 14.10.** Fix a Tate ring $A$.

(1) A *ring of integral elements* is an open, integrally closed subring $A^+ \subseteq A^\circ$.
(2) An *affinoid Tate ring* is a pair $(A, A^+)$, where $A$ is a Tate ring and $A^+$ is a ring of integral elements.
(3) A map $(A, A^+) \to (B, B^+)$ of affinoid Tate rings is a continuous map $f \colon A \to B$ such that $f(A^+) \subseteq B^+$.
(4) An affinoid Tate ring $(A, A^+)$ is *complete* if $A$ is so.

**Remark 14.11.**

(1) We often choose $A^+ = A^0$, though subsequent constructions may destroy this property.
(2) The difference between $(A, A^+)$ and $(A, A^\circ)$ is "small" in 2 ways: it is a higher-rank phenomenon (that is, if one restricts to rank-1 valuations, then there is no difference), and it is almost zero (more precisely, $A^\circ/A^+$ is an almost zero module).
(3) We have $A^{\circ\circ} \subseteq A^+$ as an ideal.

   *Proof.* If $f \in A^{\circ\circ}$, then $f^n \to 0$ as $n \to +\infty$. As $A^+$ is open, $f^n \in A^+$ for $n \gg 0$, hence $f \in A^+$ because $A^+$ is integrally closed. The rest is left as an exercise. $\square$

The relationships between the various rings are as follows: there exists a ring of definition $A_0$ of $A$ such that

$$\begin{tikzcd} A_0 \arrow[r, hook] & A^+ \arrow[r, hook] & A^\circ \arrow[r, hook] & A \\ & A^{\circ\circ} \arrow[u, hook] \arrow[ur, hook] & & \end{tikzcd}$$

However, unless $A^+ = A^\circ$, it is not true that all rings of definition lie in $A^+$.

**Exercise 14.12.** We have

$$A^+ = \bigcup_{A_0 \subseteq A^+} A_0,$$

where the union is taking over rings of definition $A_0$ of $A$ that lie in $A^+$.

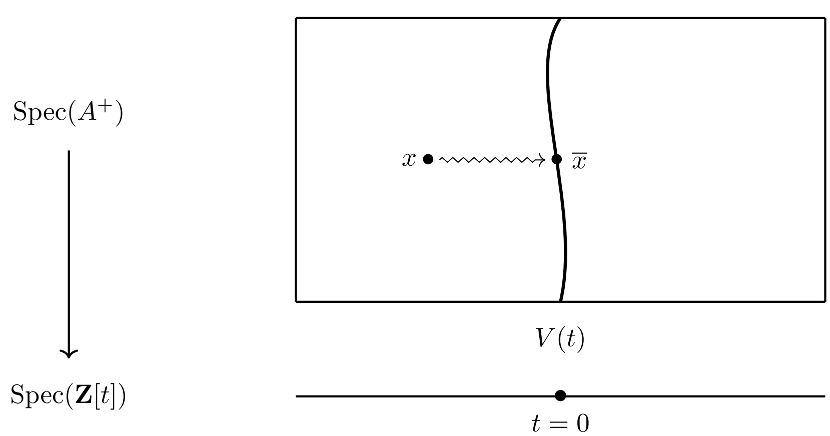
**Lemma 14.13.** *The inclusion*

$$\{\text{complete affinoid Tate rings } (A, A^+)\} \hookrightarrow \{\text{affinoid Tate rings}\}$$

*has a left adjoint* $(A, A^+) \mapsto (\widehat{A}, \widehat{A}^+)$*, where* $(\widehat{A}, \widehat{A}^+)$ *is called the* completion *of* $(A, A^+)$*.*

The idea of the proof of Lemma 14.13 is to set $\widehat{A}$ to be the completion of $A$, and $\widehat{A}^+$ to be the integral closure of the image of $A^+ \otimes_{A_0} \widehat{A_0} \to \widehat{A}$, for some ring of definition $A_0 \subseteq A^+$.

To such pairs $(A, A^+)$, we want to attach a space $\mathrm{Spa}(A, A^+)$. To describe the points of $\mathrm{Spa}(A, A^+)$, consider the following picture:




A point of $\mathrm{Spa}(A, A^+)$ will consist of a point $x$ of $\mathrm{Spec}(A)$ plus a specialization to a point $\overline{x}$ of $V(t)$. Here, we are using the facts that $\mathrm{Spec}(A) = \mathrm{Spec}(A^+) \backslash V(t)$ and (as a set) $V(t) = \mathrm{Spec}(A^+/t) = \mathrm{Spec}(A^+/A^{\circ\circ})$.

Recall our conventions for valuations: if $A$ is any ring, a valuation (or a seminorm) on $A$ is a map $x \colon A \to \Gamma \cup \{0\}$, where $\Gamma$ is a totally ordered abelian group with identity element 1, satisfying:

(1) $x$ is multiplicative;
(2) $x(0) = 0$, $x(1) = 1$;

(3) $x$ satisfies the non-Archimedean triangle inequality: $x(f+g) \leq \max\{x(f), x(g)\}$ for all $f, g \in A$.

Any valuation $x$ determines the following:

(1) $\mathfrak{p}_x \coloneqq \ker(x) = \{f \in A \colon x(f) = 0\}$ a prime ideal of $A$;
(2) a valuation ring
$$R_x \coloneqq \left\{\frac{a}{b} \in \kappa(\mathfrak{p}_x) \colon x(a) \leq x(b) \neq 0\right\} \subseteq \kappa(\mathfrak{p}_x),$$
where $\kappa(\mathfrak{p}_x) =$ is the residue field of $A$ at the prime ideal $\mathfrak{p}_x$.

Two such valuations $x, y$ are equivalent if $\mathfrak{p}_x = \mathfrak{p}_y$ and $R_x = R_y$. There is no topology here; we are simply parametrizing a point of $\operatorname{Spec}(A)$ along with a valuation ring on the residue field at that point.

**Definition 14.14.** If $(A, A^+)$ is an affinoid Tate ring, then
$$\operatorname{Spa}(A, A^+) \coloneqq \left\{\text{valuations } x \text{ on } A \colon \begin{array}{l} 1.\ x(f) \leq 1 \text{ for } f \in A^+ \\ 2.\ x \text{ is continuous} \end{array}\right\} / \sim$$
where $\sim$ denotes equivalence of valuations. To say that $x$ is continuous means with respect to the order topology on the value group; however, in the presence of the first condition, the continuity of $x$ is equivalent to the assertion that $x(t^n) \to 0$ as $n \to +\infty$ for any (or equivalently, for one) pseudouniformizer $t \in A$.

The $\operatorname{Spa}(A, A^+)$ is topologized using the following sets as a (sub)basis of open sets: for $f, g \in A$, set
$$\operatorname{Spa}(A, A^+)\left(\frac{f}{g}\right) \coloneqq \left\{x \in \operatorname{Spa}(A, A^+) \colon x(f) \leq x(g) \neq 0\right\}.$$
The topological space $\operatorname{Spa}(A, A+)$ is called the *adic spectrum* of $(A, A^+)$.

Fix an affinoid Tate ring $(A, A^+)$ and a point $x \in \operatorname{Spa}(A, A^+)$. Associated to $x$, we defined two invariants: $\mathfrak{p}_x$, called the *support* of $x$, and the valuation ring $R_x \subseteq \kappa(\mathfrak{p}_x)$. The conditions ensuring that $x$ lives in the adic spectrum can be reformulated in terms of $\mathfrak{p}_x$ and $R_x$, as hinted at by the facts below.

Fix a pseudouniformizer $t \in A$. Then,

(1) $x(t) \neq 0$ (because $t$ is a unit), and $x(t) < 1$ (because $x(t^n) \to 0$ as $n \to \infty$); these translate to the assertions that $t \notin \mathfrak{p}_x$ and $t$ does not belong to the maximal ideal of $R_x$.
(2) as $x(t^n) \to 0$, $t$ belongs to every non-zero prime ideal of $R_x$ (i.e. $x$ is *microbial*).

*Proof.* Given any such prime ideal $\mathfrak{q}$ of $R_x$, choose a non-zero element $g \in \mathfrak{q}$, then there exists $n \gg 0$ such that $x(t^n) \leq x(g)$, so $g$ divides $t^n$, hence $t^n \in \mathfrak{q}$. It follows that $t \in \mathfrak{q}$, because the ideal is prime. ☐

(3) $x(f) \leq 1$ for all $f \in A^+$ iff there is a commutative diagram
$$\begin{tikzcd} A^+ \arrow[r, dashed] \arrow[d, hook] & R_x \arrow[d, hook] \\ A \arrow[r, "\mathrm{can}"] & \kappa(\mathfrak{p}_x) \end{tikzcd}$$
That is, the image of $A^+$ under the canonical map $A \to \kappa(\mathfrak{p}_x)$ lands in the valuation ring $R_x$.

## 15. March 6th

Last time, we began talking about adic spaces: recall that the basic object was an affinoid Tate ring $(A, A^+)$, to which we associate the auxiliary rings
$$\begin{tikzcd} A_0 \arrow[r, hook] & A^+ \arrow[r, hook, "\text{integrally}", "\text{closed}"'] & A^\circ \arrow[r, hook] & A = A_0[1/t] \\ A_0 \cap A^{\circ\circ} \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & A^{\circ\circ} \arrow[u, hook] \arrow[r, equal] \arrow[d, equal] & A^{\circ\circ} \arrow[u, hook] \arrow[d, equal] & \\ \sqrt{tA_0} & \sqrt{tA^+} & \sqrt{tA^\circ} & \end{tikzcd}$$

where $A$ is a Tate ring, $A^\circ$ is the subring of powerbounded elements, $A^+$ is a ring of integral elements, $A^{\circ\circ}$ is the ideal of topologically nilpotent elements, $A_0$ is a ring of definition of $A$ contained in $A^+$, and $t \in A_0$ is a pseudouniformizer. While only the pair $(A, A^+)$ is of interest, it will prove convenient to have the other auxiliary rings.

### 15.1. **Affinoid adic spaces (continued).**

**Definition 15.1.** The *adic spectrum* is the set

$$\operatorname{Spa}(A, A^+) := \left\{ \text{(semi)valuations } x \text{ on } A \colon \begin{array}{l} 1.\ x(f) \leq 1 \text{ for } f \in A^+ \\ 2.\ x(t^n) \to 0 \text{ as } n \to +\infty \end{array} \right\} \Big/ \sim$$

where $\sim$ denotes equivalence of valuations; it is equipped with the topology generated by sets of the form

$$\operatorname{Spa}(A, A^+)\left(\frac{f}{g}\right) := \left\{ x \in \operatorname{Spa}(A, A^+) \colon |f(x)| \leq |g(x)| \neq 0 \right\}.$$

for $f, g \in A$. Here, $|f(x)| := x(f)$ for all $f \in A$. This is a standard abuse of notation that occurs in the subject, but it suggests the intuition that elements of $A$ are functions on $\operatorname{Spa}(A, A^+)$.

**Remark 15.2.** The open sets of the adic spectrum are defined by non-strict inequalities, as opposed to the strict inequalities that appear in the definition of Berkovich spaces (thus removing the need for a Grothendieck topology).

**Remark 15.3.**

(1) The kernel map $\ker\colon \operatorname{Spa}(A, A+) \to \operatorname{Spec}(A)$ is given by

$$x \mapsto \ker(x) = \{f \in A \colon |f(x)| = 0\} =: \mathfrak{p}_x.$$

One can easily check that $\mathfrak{p}_x$ is a prime ideal of $A$. Moreover, observe that

$$\ker^{-1}(D(f)) = \{x \in \operatorname{Spa}(A, A^+) \colon |f(x)| \neq 0\} = \operatorname{Spa}(A, A^+)\left(\frac{f}{f}\right),$$

where $D(f) := \{\mathfrak{p} \in \operatorname{Spec}(A) \colon f \notin \mathfrak{p}\}$ denotes the basic open set of $\operatorname{Spec}(A)$. In particular, the kernel map is continuous.

(2) The topological space $\operatorname{Spa}(A, A^+)$ only depends on the map $A^+ \to A$ of abstract rings (i.e. one does not need the topology on $A$). Indeed, the condition (1) (i.e. that $x(f) \leq 1$ for all $f \in A^+$) only depends on $A^+$. To see that (2) only depends on the map $A^+ \to A$, one can check that $t \in A^+$ is a pseudouniformizer iff $\sqrt{tA^+} = A^{\circ\circ}$, which is the radical ideal defining the complement of the open immersion $\operatorname{Spec}(A) \subseteq \operatorname{Spec}(A^+)$; thus, the notion of pseudouniformizer only depends on the map $A^+ \to A$. Therefore, both (1) and (2) are "algebraic". Finally, it is clear that the definition on the topology on $\operatorname{Spa}(A, A^+)$ only depends on $A^+ \to A$.

(3) Given $x \in \operatorname{Spa}(A, A^+)$, we get the following:
   (a) the kernel $\mathfrak{p}_x = \ker(x)$;
   (b) the valuation ring $R_x \subseteq \kappa(\mathfrak{p}_x)$.

The condition that $x \in \operatorname{Spa}(A, A^+)$ guarantees that there is a commutative diagram

$$\begin{tikzcd} A^+ \arrow[r, dashed, "\phi_x"] \arrow[d, hook] & R_x \arrow[d, hook] \\ A \arrow[r, "\text{can}"] & \kappa(\mathfrak{p}_x) \end{tikzcd}$$

such that, for any pseudouniformizer $t \in A^+$, $\phi_x(t) \neq 0$ and $\phi_x(t)^n \to 0$ in the valuation topology.

### 15.2. Microbial valuation rings.

**Definition 15.4.** Fix a valuation ring $V$ and $f \in V$. We say that $f$ is a *pseudouniformizer* if $f$ is non-zero and $f^n \to 0$ in the valuation topology on $V$. If such an $f$ exists, we say that $V$ is *microbial.*

**Remark 15.5.** If $V$ is a valuation ring and $f \in V$ is a pseudouniformizer, then it follows from the condition that $f^n \to 0$ in the valuation topology that $f$ is a non-unit. In particular, a field is not a microbial valuation ring.

**Example 15.6.**

(1) As in Remark 15.3(3), if $x \in \mathrm{Spa}(A, A^+)$, then the valuation ring $R_x$ is microbial with pseudouniformizer $\phi_x(t)$.
(2) Any valuation ring $V$ of finite rank (or equivalently, of finite Krull dimension by [Bou85, Chapter 6]) is microbial.

The condition of microbiality is perhaps better understood via one of the following equivalent characterizations.

**Exercise 15.7.** If $V$ is a valuation ring, then the following are equivalent:

(1) $V$ is microbial;
(2) $V$ has a height-1 prime ideal;
(3) there exists a non-zero element $f \in V$ such that $V$ is $f$-adically separated;
(4) $\mathrm{Frac}(V)$ is a Tate ring, in the valuation topology;
(5) there exists a non-unit $f \in V$ such that $\mathrm{Frac}(V) = V[1/f]$.

If $V$ is microbial, the height-1 prime ideal is the radical ideal generated by any pseudouniformizer. If (3) holds, then $f$ is a pseudouniformizer. For further details, see [Dat17, Proposition 3.2.6].

There is a standard construction that takes as input a valuation ring, and outputs a microbial valuation ring.

**Lemma 15.8.** *If $V$ is a valuation ring and $f \in V$ is a non-zero non-unit, then the maximal $f$-adically separated quotient*

$$\overline{V} \coloneqq V \Big/ \bigcap_n f^n V$$

*and the $f$-adic completion*

$$\widehat{\overline{V}} \coloneqq \varprojlim_n V/f^n V$$

*are microbial valuation rings with pseudouniformizer $f$. Moreover, the canonical map $\overline{V} \to \widehat{\overline{V}}$ is faithfully flat.*

*Proof.* It is a general fact that the maximal separated quotient of a valuation ring with respect to some element is again a valuation ring, and the completion of a valuation ring with respect to some element is again a valuation ring. It remains to show that $\overline{V}$ and $\widehat{\overline{V}}$ are microbial. It suffices to show the following more general claim (which is also one of the equivalent characterizations of microbiality in Exercise 15.7.

**Claim 15.9.** If $W$ is a valuation ring and $g \in W$ is a non-zero element such that $W$ is $g$-adically separated, then $W$ is microbial with pseudouniformizer $g$.

*Proof of Claim 15.9.* Fix a non-zero element $f \in W$, then we need to show that $|g^n| < |f|$ for $n \gg 0$. As $W$ is $g$-adically separated, $g$ is not a unit, hence $|g| < 1 \in \Gamma$. Therefore, it suffices to show that $|g^n| \leq |f|$ for $n \gg 0$ (because then $|g^{n+1}| < |f|$). This is equivalent to $f|g^n$ for all $n \gg 0$. By $g$-adic separatedness, $f \notin g^n W$ for $n \gg 0$ (otherwise, $f \in \bigcap_n g^n W = 0$, but $f$ is non-zero), so $g^n \in fW$; here, we use that $W$ is a valuation ring, so any two elements are comparable by divisibility. Thus, $f|g^n$, as required. $\square$

Therefore, $\overline{V}$ and $\widehat{\overline{V}}$ are both microbial valuation rings with pseudouniformizer $f$, by Claim 15.9.

For the faithful flatness of the canonical map $\alpha\colon \overline{V} \to \widehat{\overline{V}}$, we know that $\alpha$ is injective, hence $\alpha$ is flat (indeed, any injective map of valuation rings is flat, because flatness is equivalent to torsion-freeness over a valuation ring). Furthermore, $\mathrm{Spec}(\alpha)$ hits the closed point of $\mathrm{Spec}(\overline{V}) \subseteq \mathrm{Spec}(V)$, so $\alpha$ is faithfully flat by [Sta17, Tag 00HQ]. □

**Proposition 15.10.** *Fix an affinoid Tate ring $(A, A^+)$ and a pseudouniformizer $t \in A^+$. Then, there is a natural bijection between* $\mathrm{Spa}(A, A^+)$ *and the set*

$$S := \left\{ A^+ \xrightarrow{\phi} V : \begin{array}{l} 1.\ V \text{ is a microbial valuation ring} \\ 2.\ \phi(t) \in V \text{ is a pseudouniformizer} \end{array} \right\} \Big/ \sim,$$

*where the equivalence relation $\sim$ is defined as follows: if $A^+ \xrightarrow{\phi} V$ is as above and $\psi\colon V \to W$ is a faithfully flat map between microbial valuation rings such that $\psi(\phi(t))$ is a pseudouniformizer, then we declare $\phi \sim \psi \circ \phi$.*

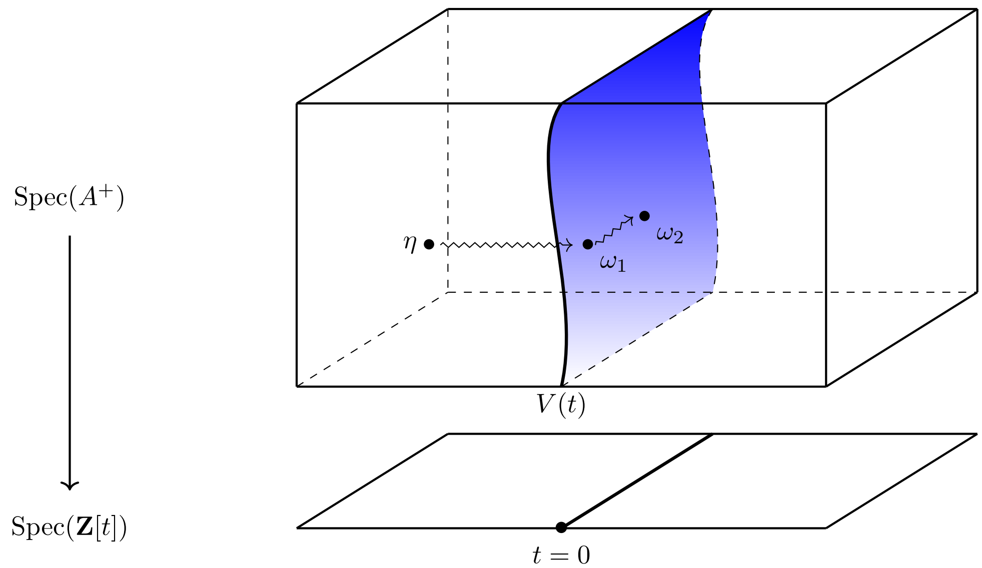
**Remark 15.11.**
(1) The set $S$ is independent of the choice of $t$ (i.e. if $t, t' \in A^+$ are pseudouniformizers and $\phi\colon A^+ \to V$ is a ring map to a microbial valuation ring $V$, then $\phi(t)$ is a pseudouniformizer in $A^+$ iff $\phi(t')$ is so).
(2) By Lemma 15.8, we may restrict to $\phi(t)$-adically complete microbial valuation rings in defining $S$.

*Proof.* Given $x \in \mathrm{Spa}(A, A^+)$, get a map $\phi_x\colon A^+ \to R_x$ as before. This gives an element of $S$. Conversely, given $\phi\colon A^+ \to V$ in $S$, get a diagram

$$\begin{tikzcd} A^+ \arrow[r, "\phi_x"] \arrow[d, hook] & V \arrow[d, hook] \\ A = A^+[1/t] \arrow[r, dashed, "\phi"] & V[1/\phi(t)] = \mathrm{Frac}(V) \end{tikzcd}$$

This map $\phi$ from $A$ into the valued field $\mathrm{Frac}(V)$ induces a valuation $x_\phi$ on $A$, and the conditions (1) and (2) (in the definition of the set $S$) guarantee that $x_\phi$ lies in $\mathrm{Spa}(A, A^+)$. It remains to show that $\phi$ and $\psi \circ \phi$ induce equivalent valuations on $A$, and moreover that these processes invert one another. This is left as an exercise. □

Given a map $\phi\colon A^+ \to V$, the picture to have in mind is described below: $\mathrm{Spec}(V)$ consists of a generic point $\eta$, a height-1 prime $\omega_1$, and other primes $\omega_2, \ldots$. The affine scheme $\mathrm{Spec}(A^+)$ can be written as the disjoint union of $V(t) = \mathrm{Spec}(A^+/t)$ (shaded below) and $\mathrm{Spec}(A)$ and, under the map $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(V) \to \mathrm{Spec}(A^+)$, $\eta$ is sent into $\mathrm{Spec}(A)$ and the $\omega_i$'s are sent into $V(t)$. If $V$ has rank 2, this is pictured below:




This sequence of specializations in $\operatorname{Spec}(A^+)$ is the point of $\operatorname{Spa}(A, A^+)$ determined by $\phi\colon A^+ \to V$. Note that the corresponding Berkovich point is obtained by only taking the "first non-trivial specialization" $\eta \rightsquigarrow \omega_1$, and forgetting about the specializations to $\omega_i$ for $i > 1$.

Next time, we will discuss certain properties of the affinoid Tate ring $(A, A^+)$ that can be recovered from the adic spectrum $\operatorname{Spa}(A, A^+)$.

## 16. March 8th

Today, let $(A, A^+)$ be an affinoid Tate ring and choose a couple of definition $(A_0, t)$ of $(A, A^+)$ such that $t \in A_0 \subseteq A^+ \subseteq A^0 \subseteq A$. Recall that $A$ is complete iff $A_0$ is $t$-adically complete.

### 16.1. **Properties of affinoid adic spaces.**

We will discuss how to translate between properties of $(A, A^+)$ and properties of $\operatorname{Spa}(A, A^+)$.

**Proposition 16.1.** *Let $(A, A^+)$ be as above.*

*(1) The canonical map $(A, A^+) \to (\widehat{A}, \widehat{A}^+)$ induces a homeomorphism $\operatorname{Spa}(\widehat{A}, \widehat{A}^+) \xrightarrow{\simeq} \operatorname{Spa}(A, A^+)$.*
*(2) $\operatorname{Spa}(A, A^+) = \emptyset$ iff $\widehat{A} = 0$.*
*(3) [Adic Nullstellensatz] $A^+ = \{f \in A\colon |f(x)| \le 1 \quad \forall x \in \operatorname{Spa}(A, A^+)\}$.*
*(4) If $(A, A^+)$ is complete, then $f \in A$ is a unit iff $|f(x)| \neq 0$ for all $x \in \operatorname{Spa}(A, A^+)$.*
*(5) If $x \rightsquigarrow y$ is a specialization in $\operatorname{Spa}(A, A^+)$ (i.e. $y$ is in the closure of $x$), then $\mathfrak{p}_x = \mathfrak{p}_y$ and there is a commutative diagram*

$$\begin{tikzcd} R_y \arrow[r, dashed] \arrow[d, hook] & R_x \arrow[d, hook] \\ \kappa(\mathfrak{p}_y) \arrow[r, equal] & \kappa(\mathfrak{p}_x) \end{tikzcd}$$

*where the induced map $R_y \to R_x$ is given by localization at a prime ideal. In fact, for a fixed $y \in \operatorname{Spa}(A, A^+)$, there is a homeomorphism*

$$\{\text{generalizations } x \text{ of } y\} \simeq \operatorname{Spec}(R_y) \setminus \{\text{generic point}\},$$

*where the set of generalizations of $y$ is equipped with the subspace topology inherited from $\operatorname{Spa}(A, A^+)$*

The generic point of $\operatorname{Spec}(R_y)$ cannot correspond to a generization of $y$ in $\operatorname{Spa}(A, A^+)$: indeed, such a point corresponds to a map $A^+ \to (R_y)_{(0)} = \kappa(\mathfrak{p}_y)$, but $\kappa(\mathfrak{p}_y)$ does not have a pseudouniformizer.

**Remark 16.2.** Proposition 16.1(4) is the adic version of the fact that $f \in A$ is a unit iff the image of $f$ in the residue field at every prime ideal in $\mathrm{Spec}(A)$ is a unit.

*Proof.* We will just prove (3) and (4). For (3), the inclusion $\subseteq$ is clear. For the opposite inclusion, pick $f \in A$ such that $|f(x)| \leq 1$ for all $x \in \mathrm{Spa}(A, A^+)$. Consider the subring $A^+[f^{-1}] \subseteq A[f^{-1}]$, i.e. $A^+[f^{-1}]$ is the subring of $A[f^{-1}]$ generated by $A^+$ and $f^{-1}$. We may assume that $f^{-1} \in A^+[f^{-1}]$ is not a unit: indeed, if $f^{-1}$ is a unit, then there is a monic equation in $A^+[x]$ satisfied by $f$ (after clearing denominators), and hence $f \in A^+$ because $A^+$ is integrally closed in $A$.

Assuming that $f^{-1} \in A^+[f^{-1}]$ is not a unit, we may find a maximal ideal $\mathfrak{m} \subseteq A^+[f^{-1}]$ containing $f^{-1}$. Choose a minimal prime ideal $\mathfrak{p} \subseteq A^+[f^{-1}]$ that is contained in $\mathfrak{m}$; therefore, we get a specialization $\mathfrak{p} \rightsquigarrow \mathfrak{m}$ in $\mathrm{Spec}(A^+[f^{-1}])$. By [Sta17, Tag 01J8], we can find a valuation ring $V$ and a map of schemes

$$\mathrm{Spec}(V) \to \mathrm{Spec}(A^+[f^{-1}])$$

such that the generic point of $\mathrm{Spec}(V)$ maps to $\mathfrak{p}$, and the closed point maps of $\mathrm{Spec}(V)$ maps to $\mathfrak{m}$.

**Claim 16.3.** Under the composition $A^+ \to A^+[f^{-1}] \to V$, the pseudouniformizer $t \in A^+$ maps to a non-zero non-unit of $V$.

From Claim 16.3, we can deduce (3): the maximal separated quotient $W = \overline{V} = V/\bigcap_n t^n V$ of $V$ with respect to $t$ is a microbial valuation ring with pseudouniformizer $t$ by Lemma 15.8, so $W$ defines a point $x \in \mathrm{Spa}(A, A^+)$ such that $|f^{-1}(x)| < 1$, or equivalently $|f(x)| > 1$, a contradiction.

*Proof of Claim 16.3.* We first show that the image of $t$ is non-zero in $V$; this is equivalent to the assertion that the generic point of $\mathrm{Spec}(V)$ is sent into $\mathrm{Spec}(A) = \mathrm{Spec}(A^+[1/t])$ under the map $\mathrm{Spec}(V) \to \mathrm{Spec}(A^+)$. We have the factorization

$$\mathrm{Spec}(V) \to \mathrm{Spec}(A^+[f^{-1}]) \to \mathrm{Spec}(A^+),$$

where the generic point of $\mathrm{Spec}(V)$ maps to a minimal prime ideal $\mathfrak{p}$ of $A^+[f^{-1}]$. Moreover, consider the commutative diagram

$$\begin{array}{ccc} A^+ & \hookrightarrow & A \\ \downarrow & & \downarrow \\ A^+[f^{-1}] & \hookrightarrow & A[f^{-1}] \end{array}$$

As $A^+[f^{-1}] \subseteq A[f^{-1}]$, all generic points of $\mathrm{Spec}(A^+[f^{-1}])$ are images of generic points of $\mathrm{Spec}(A[f^{-1}])$. Therefore, $t$ is invertible at all generic points of $\mathrm{Spec}(A^+[f^{-1}])$, and hence $\mathrm{Spec}(V) \to \mathrm{Spec}(A^+[f^{-1}])$ factors through $\mathrm{Spec}(A[f^{-1}])$, as required.

A similar argument (using $\mathfrak{m}$ instead of $\mathfrak{p}$) shows that $t$ is a non-unit of $V$. The proof is left as an exercise (use the fact that $f^{-1} \in \mathfrak{m}$ to see that $t$ lies in the maximal ideal of $V$, and conclude). $\square$

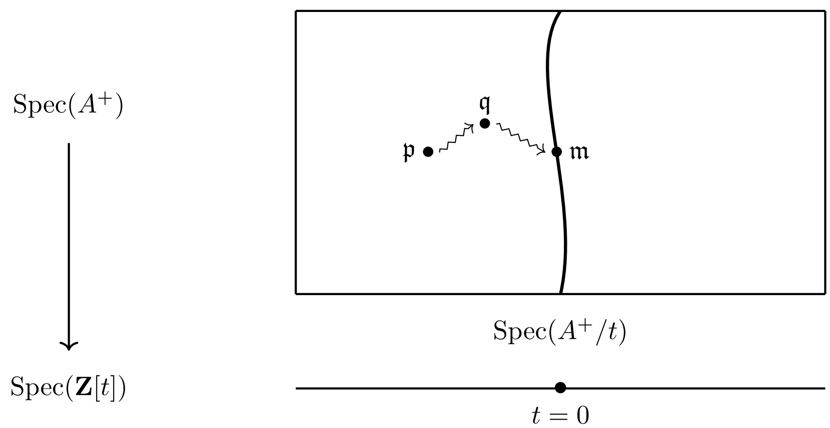
For (4), if $f \in A$ is a unit, it is clear that $|f(x)| \neq 0$ for any $x \in \mathrm{Spa}(A, A^+)$, because it cannot be sent to zero under a multiplicative map. Conversely, we will show that $f$ is invertible in the residue field $\kappa(\mathfrak{p})$ for each prime $\mathfrak{p} \in \mathrm{Spec}(A)$, from which it follows that $f$ must be a unit.

The fact that $A^+$ is complete is equivalent to the fact that $A_0$ is $t$-adically complete, so $t$ lies in the Jacobson radical of $A_0$ (i.e. the intersection of all maximal ideals of $A_0$). As this holds for each ring of definition $A_0$ contained in $A^+$, $t$ belongs to the Jacobson radical of $A^+$ (because $A^+$ is the filtered colimit of the rings of definition that it contains). The upshot is that each point $\mathfrak{q} \in \mathrm{Spec}(A^+)$ specializes to a point of the closed subset $\mathrm{Spec}(A^+/t)$ of $\mathrm{Spec}(A^+)$. Scale by $t^N$ for $N \gg 0$ to ensure that $f \in A^+$ (i.e. replace $f$ with $t^N f$).

We want to show that for all $\mathfrak{p} \in \mathrm{Spec}(A)$, $f$ is invertible in $\kappa(\mathfrak{p})$. Choose a map of schemes $\mathrm{Spec}(V) \to \mathrm{Spec}(A^+)$ such that the generic point of $\mathrm{Spec}(V)$ maps to $\mathfrak{p}$, and the closed point of $\mathrm{Spec}(V)$ maps into $\mathrm{Spec}(A^+/t)$. Set $W = V/\bigcap_n t^n V$, then $\mathrm{Spec}(W) \to \mathrm{Spec}(A^+)$ is such that the generic point maps to some prime $\mathfrak{q} \in \mathrm{Spec}(A)$, and the closed point maps into $\mathrm{Spec}(A^+/t)$.

Therefore, we get a point $x \in \operatorname{Spa}(A, A^+)$ corresponding $A^+ \to W$ such that $\ker(x) = \mathfrak{q}$. The hypothesis on $f$ ensures that $|f(x)| \neq 0$, so $f$ is non-zero in $\kappa(\mathfrak{q})$, and hence $f$ is a unit of the local ring $A_\mathfrak{q}$. Thus, $f$ is a unit of any further localization; in particular, $f$ is a unit of $A_\mathfrak{p}$, as $\mathfrak{p} \rightsquigarrow \mathfrak{q}$, and so $f$ is a unit of $\kappa(\mathfrak{p})$. As this holds for all $\mathfrak{p}$, $f$ is a unit of $A$.

The picture associated to this proof of (4) is the following:




The specialization $\mathfrak{p} \rightsquigarrow \mathfrak{q} \rightsquigarrow \mathfrak{m}$ is the image of $\operatorname{Spec}(V) \to \operatorname{Spec}(A^+)$, and the specialization $\mathfrak{q} \rightsquigarrow \mathfrak{m}$ is the image of $\operatorname{Spec}(W) \to \operatorname{Spec}(A^+)$. $\square$

16.2. **Global structure of affinoid adic spaces.**

**Definition 16.4.** [Hoc69] A quasi-compact topological space $X$ is *spectral* if

(1) $X$ is *sober*: every irreducible closed subset $Z \subseteq X$ is of the form $Z = \overline{\{z\}}$ for some $z \in Z$, i.e., $Z$ has a unique generic point;
(2) the collection $\mathcal{B} \coloneqq \{\text{quasi-compact (qc) opens } U \subseteq X\}$ is a basis for $X$.

While this definition may seem odd, Hochster showed that this is exactly the class of spaces obtained as prime spectra of commutative rings.

**Theorem 16.5.** [Hoc69, Theorem 6] *$X$ is spectral iff there is a homeomorphism $X \simeq \operatorname{Spec}(R)$ for some commutative ring $R$.*

There is another useful criterion for spectrality.

**Fact 16.6.** [Hoc69, Proposition 10] If $X$ is a topological space, then the following are equivalent:

(1) $X$ is spectral;
(2) there is a homeomorphism $X \simeq \varprojlim_{i \in I} X_i$, where each $X_i$ is a finite $T_0$-space[19] and the set $I$ is filtered.

**Warning 16.7.** Given a spectral space $X$, there is no natural ring $R$ such that $X \simeq \operatorname{Spec}(R)$; that is, one cannot construct $R$ functorially in $X$.

**Exercise 16.8.** Consider the 3-point space $X = \{\eta, s_1, s_2\}$, where the topology is defined by specializations as in the diagram below:

$$\begin{array}{ccc} \eta & \rightsquigarrow & s_1 \\ \wr\downarrow & & \\ s_2 & & \end{array}$$

[19]A topological space is $T_0$ if, given any 2 points, there is an open set containing one but not the other.

Given this space $X$, find a ring $R$ such that $X \simeq \mathrm{Spec}(R)$; that is, find ring $R$ with 3 prime ideals such that the poset of primes is as in the diagram.

Spectral spaces enjoy many nice properties; for example, the cohomology of sheaves vanishes above the Krull dimension (that is, the cohomological dimension is bounded above by the Krull dimension). This is an analogue of Grothendieck's theorem for schemes.

**Theorem 16.9.** [Hub93, Theorem 3.5] *If $(A, A^+)$ is an affinoid Tate ring, then:*

*(1)* $\mathrm{Spa}(A, A^+)$ *is spectral;*
*(2) a basis of quasi-compact opens is given by the* rational subsets *of* $\mathrm{Spa}(A, A^+)$*: subsets of the form*

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) := \left\{x \in \mathrm{Spa}(A, A^+) \colon |f_i(x)| \leq |g(x)| \text{ for } i = 1, \ldots, n\right\},$$

*where $f_1, \ldots, f_n, g \in A$ and $(f_1, \ldots, f_n) = (1)$ as ideals of $A$.*
*(3) For a map $f \colon (A, A^+) \to (B, B^+)$ of affinoid Tate rings, the pullback $f^\sharp \colon \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is* spectral*; that is, $f^\sharp$ is continuous and the preimage of any quasi-compact open is quasi-compact.*

The spectral maps defined in Theorem 16.9(3) are the correct notion of maps between spectral spaces, and they are (by definition) the morphisms of the category of spectral spaces.

**Remark 16.10.**

(1) In Theorem 16.9(2), we have: $|g(x)| \neq 0$ for all $x \in \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right)$.

*Proof.* If $|g(x)| = 0$ for some $x \in \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right)$, then $|f_i(x)| \leq |g(x)| = 0$, and so $|f_i(x)| = 0$ for all $i = 1, \ldots, n$. As $1 \in (f_1, \ldots, f_n)$, it follows that $|1| = 0$, a contradiction. $\square$

(2) For $f_1, \ldots, f_n, g$ as in Theorem 16.9(2), we have

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) = \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n, t^N}{g}\right)$$

for any pseudouniformizer $t \in A^+$ and any $N \gg 0$.

*Proof.* The inclusion $\supseteq$ is clear. For the opposite inclusion, we have

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) = \bigcup_{N > 0} \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n, t^N}{g}\right).$$

Indeed, observe that if $|g(x)| \neq 0$, then $|g(x)| \geq |t^N(x)|$ for $N \gg 0$. Now, the claim follows from the quasi-compactness of the rational subset. $\square$

(3) Similarly, we may always assume $f_n = g$, because $\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) = \mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n, g}{g}\right)$.

## 17. March 20th

The notation for today is as follows: $(A, A^+)$ is an affinoid Tate ring, and $(A_0, t)$ is a couple of definition, and typically we assume that $A_0 \subseteq A^+ \subseteq A^\circ \subseteq A$. The first 3 are "integral objects", where $A^+$ and $A^\circ$ contain the ideal $A^{\circ\circ}$ of topologically-nilpotent elements, and $\sqrt{tA_0} = A^{\circ\circ} \cap A_0$. Recall the adic spectrum is defined as

$$X = \mathrm{Spa}(A, A^+) := \left\{\text{valuations } x \colon A \to \Gamma \cup \{0\} \colon \begin{array}{l} 1.\ |f(x)| \leq 1 \text{ for } f \in A^+ \\ 2.\ |t^n(x)| \to 0 \text{ as } n \to +\infty \end{array} \right\} \Big/ \sim .$$

We saw in Proposition 15.10 that there is a bijection between $\mathrm{Spa}(A, A^+)$ and equivalence classes of maps $\phi \colon A^+ \to V$, where $V$ is a microbial valuation ring and $\phi(t)$ is a pseudouniformizer.

## 17.1. Global structure of affinoid adic spaces (continued).

**Theorem 17.1.** [Hub93, Theorem 3.5] *The adic spectrum $X = \mathrm{Spa}(A, A^+)$ is a spectral topological space (i.e. it is homeomorphic to the prime spectrum of a ring) with a basis of quasi-compact opens of the form*

$$X\left(\frac{f_1, \dots, f_n}{g}\right) := \{x \in X : |f_i(x)| \le |g(x)| \text{ for all } i = 1, \dots, n\}$$

*such that $f_1, \dots, f_n, g \in A$ and $(f_1, \dots, f_n) = (1)$. The open sets of $X$ of this form are called* rational subsets.

**Remark 17.2.** With notation as in Theorem 17.1:

(1) $|g(x)| \neq 0$ for all $x \in X\left(\frac{f_1,\dots,f_n}{g}\right)$

(2) We may assume (after adding a new $f_i$) that $f_{n+1} = g$; similarly, we can assume that one of the $f_i$'s is a pseudouniformizer.

(3) The collection of rational subsets is stable under finite intersection[20]: suppose $U = X\left(\frac{f_1,\dots,f_n}{g}\right)$ and $V = X\left(\frac{a_1,\dots,a_m}{b}\right)$, with $(a_1, \dots, a_m) = (1) = (f_1, \dots, f_n)$, are two rational open subsets. By (2), we may assume that $f_n = g$ and $a_m = b$. Then, we claim that

$$U \cap V = X\left(\frac{T}{gb}\right),$$

where $T = \{f_i a_j : 1 \le i \le n, 1 \le j \le m\}$. The inclusion $\subseteq$ is clear. Conversely, if $x \in X\left(\frac{T}{gb}\right)$, then

$$|f_i(x)||a_m(x)| = |f_i(x)||b(x)| \le |g(x)||b(x)|$$

and, since $|b(x)| \neq 0$, it follows that $|f_i(x)| \le |g(x)|$, so $x \in U$. By symmetry, we also get that $x \in V$.

(4) The defining opens of $X$ are typically not quasi-compact. Recall that the defining open subsets are of the form

$$\mathrm{Spa}(A, A^+)\left(\frac{f}{g}\right) = \{x \in X : |f(x)| \le |g(x)| \neq 0\},$$

for $f, g \in A$. Such an open subset is not necessarily a rational subset because $f$ is not assumed to be a unit in $A$. We can write

$$\mathrm{Spa}(A, A^+)\left(\frac{f}{g}\right) = \bigcup_{n \ge 0} \mathrm{Spa}(A, A^+)\left(\frac{f, t^n}{g}\right).$$

Indeed, the inclusion $\supseteq$ is clear, because $|t^n(x)| \neq 0$ as $t$ is a unit. For the inclusion $\subseteq$, if $|g(x)| \neq 0$, then $|g(x)| \ge |t^n(x)|$ for $n \gg 0$ (this is because $|t^n(x)| \to 0$, and so it must eventually be smaller than any other non-zero element). Moreover, the sets $\mathrm{Spa}(A, A^+)\left(\frac{f,t^n}{g}\right)$ are rational, hence quasi-compact. Thus, $\mathrm{Spa}(A, A^+)(\frac{f}{g})$ is an infinite union of quasi-compacts, but it need not be quasi-compact in general.

For example, take $f = g$: then, $\mathrm{Spa}(A, A^+)\left(\frac{g}{g}\right) = \{x : |g(x)| \neq 0\}$ is a Zariski open, and $\mathrm{Spa}(A, A^+)(\frac{g,t^n}{g})$ is the complement in $\mathrm{Spec}(A^+)$ of the tube of radius $|t^n|$ around $\{g = 0\}$.

(5) Adic spaces do not look like classical schemes: for all $x \in \mathrm{Spa}(A, A^+)$, the set $G(x) := \{\text{generalizations of } x\}$ forms a totally ordered set. However, the analogous statement fails for all noetherian schemes of dimension $\ge 2$ (this follows by observing that there is a bijection between the set of generalizations of a point $x$ of a scheme $X$ and $\mathrm{Spec}(\mathcal{O}_{X,x})$, as in [Sta17, Tag 01J7]). In particular, adic spaces are prime spectra of non-noetherian rings.

**Corollary 17.3.** *Fix $f \in A$. If $X = \mathrm{Spa}(A, A^+)$, then $f \in A^{\circ\circ}$ iff $|f(x)|^n \to 0$ as $n \to \infty$ for all $x \in X$.*

---

[20] The stability of rational subsets under finite intersections is analogous to the fact that $D(f) \cap D(g) = D(fg)$ in scheme theory.

Corollary 17.3 is analogous to the scheme-theoretic statement that an element $f \in A$ is nilpotent if and only if it is nilpotent at each stalk of $\mathrm{Spec}(A)$. Here, nilpotence is replaced by topological nilpotence.

*Proof.* If $f \in A^{\circ\circ}$, there exists $N > 0$ such that $f^N \in tA^+$, hence $f^N = tg$ for some $g \in A^+$. Then, $|t^n(x)| \to 0$ as $n \to \infty$ and $|g(x)| \leq 1$ for all $x \in X$, so it follows that $|f^n(x)| \to 0$ as $n \to \infty$. Conversely, assume that $|f(x)|^n \to 0$ for all $x \in X$. We can write

$$X = \bigcup_{n \geq 0} X\left(\frac{f^n}{t}\right),$$

where the inclusion $\subseteq$ follows from the hypothesis on $|f(x)|$. By quasi-compactness, we have $X = X(\frac{f^n}{t})$ for some $n > 0$. Therefore, the element $\frac{f^n}{t} \in A$ satisfies $|(\frac{f^n}{t})(x)| \leq 1$ for all $x \in X$, and hence $\frac{f^n}{t} \in A^+$ by Proposition 16.1(3). Thus, $f^N \in tA^+ \subseteq A^{\circ\circ}$. As $A^{\circ\circ}$ is a radical ideal, it follows that $f \in A^{\circ\circ}$. □

**Remark 17.4.** There is a *specialization map*[21] $\mathrm{sp}\colon \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A^+/A^{\circ\circ})$, defined as follows: given $x \in \mathrm{Spa}(A, A^+)$, we get a map $\mathrm{Spec}(R_x) \to \mathrm{Spec}(A^+)$ sending the closed point of $\mathrm{Spec}(R_x)$ (i.e. the *center* of $x$) to a point $\mathrm{sp}(x)$ of $\mathrm{Spec}(A^+/A^{\circ\circ}) \subseteq \mathrm{Spec}(A^+)$. It is easy to check that sp is a spectral map, i.e. it is continuous and the sp-preimage of a quasi-compact open set is again quasi-compact. (By contrast, the support map is not spectral.) Using the specialization map, one can describe $\mathrm{Spa}(A, A^+)$ as the inverse limit of all admissible modifications of $\mathrm{Spec}(A^+)$. Compare this to Raynaud's theory, where the generic fiber of a formal scheme (as a rigid-analytic space) is obtained as the inverse limit over all admissible blowups; see [Ray74].

17.2. **The structure presheaf.** Fix an affinoid Tate ring $(A, A^+)$ with adic spectrum $X = \mathrm{Spa}(A, A^+)$. If $U = X\left(\frac{f_1, \ldots, f_n}{g}\right)$ is a rational open subset of $X$, we want to assign to $U$ an affinoid Tate ring and, from this, build a presheaf on $X$. If this was a sheaf, then it would determine the value on all opens, because the rational subsets form a basis for the topology. It will be worthwhile to work with the presheaf, rather than its sheafification.

**Theorem 17.5.** [Hub94, Proposition 1.3] *For any rational open subset $U \subseteq X$, there exists a unique complete affinoid Tate $(A, A^+)$-algebra $\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big)$ satisfying the following universal property:*

*(1) $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ has image contained in $U$;*
*(2) the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is universal with property (1) with respect to complete affinoid Tate rings.*

*Moreover, the induced map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to U$ is a homeomorphism.*

*Proof.* By scaling with a high power of $t$, we may assume that $f_1, \ldots, f_n, g \in A_0$ and $t^N \in (f_1, \ldots, f_n) \subseteq A_0$. Set $B_0 = A_0[\frac{f_i}{g}\colon i = 1, \ldots, n] \subseteq A_0[\frac{1}{g}]$; this will be the new ring of definition. Set $B = A[1/g]$, then we claim that $B = B_0[1/t]$; this follows from the fact that $(f_1, \ldots, f_n) \ni t^N$ in $A_0$. Therefore, $B$ is a Tate $A$-algebra with couple of definition $(B_0, t)$. Set $B^+$ to be the integral closure of the subring of $B$ generated by the image of $A^+$ and $\frac{f_1}{g}, \ldots, \frac{f_n}{g}$. The subring $B^+ \subset B$ is open (because $B_0 \subset B^+$) and it is clearly integrally closed; therefore, $(B, B^+)$ is an affinoid Tate $(A, A^+)$-algebra. Set $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to be the completion of $(B, B^+)$.

By construction, the image of $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ lies inside $U$. Let $(C, C^+)$ be a complete affinoid Tate $(A, A^+)$-algebra such that $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ has image in $U$. To prove the universal property, we need to show that the map $(A, A^+) \to (C, C^+)$ factors uniquely through $(B, B^+)$.

For any $x \in \mathrm{Spa}(C, C^+)$, we have $|f_i(x)| \leq |g(x)| \neq 0$. Therefore, $|g(x)| \neq 0$ for all $x \in \mathrm{Spa}(C, C^+)$. By completeness and Proposition 16.1(4), $g$ is a unit of $C$, and hence we get a map $B \to C$ factoring $A \to C$. Also, we have $|f_i(x)| \leq |g(x)|$ for all $x \in \mathrm{Spa}(C, C^+)$, and so $\frac{f_i}{g} \in C^+$. Therefore, because $g$ is invertible in $C$, we get a unique map

[21] The specialization map is often called the *reduction map* in both Berkovich and rigid-analytic geometry.

$$\begin{tikzcd}
(B, B^+) \arrow[rr] & & (C, C^+) \\
& (A, A^+) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This verifies the universal property. Note that we needed $C$ to be complete only to conclude that $g$ was a unit, otherwise we could have worked with the non-complete rings.

Finally, it remains to check that $\operatorname{Spa}(B, B^+) \xrightarrow{\simeq} U \subseteq \operatorname{Spa}(A, A^+)$. The bijectivity follows from the universal property. For homeomorphy, see [Bha17, Theorem 4.1]. □

The universal property in Theorem 17.5 guarantees that $\mathcal{O}_X$ is a presheaf on the category of rational subsets.

**Definition 17.6.** Fix $x \in X = \operatorname{Spa}(A, A^+)$. The *stalks* of $\mathcal{O}_X$ and $\mathcal{O}_X^+$ at $x$ are

$$\mathcal{O}_{X,x} \coloneqq \operatorname*{colim}_{x \in U} \mathcal{O}_X(U) \text{ and } \mathcal{O}_{X,x}^+ \coloneqq \operatorname*{colim}_{x \in U} \mathcal{O}_X^+(U),$$

where the colimit is taken over all rational subsets $U \subseteq X$ containing $x$. The transition maps in the colimit are defined by the universal property, and the colimit is taken in the category of rings.

**Proposition 17.7.**

1. *The valuation $f \mapsto |f(x)|$ on $A$ extends to a valuation $f \mapsto |f(x)|$ on $\mathcal{O}_{X,x}$, and the value group is the same as the value group of $x$.*
2. *The stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x \coloneqq \{f \in \mathcal{O}_{X,x} \colon |f(x)| = 0\}$. In particular, the residue field of $\mathcal{O}_{X,x}$ is a valued field.*
3. *The stalk $\mathcal{O}_{X,x}^+$ is a local ring. In fact, $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} \colon |f(x)| \leq 1\}$. In particular, $\mathfrak{m}_x$ is a common ideal of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$.*
4. *The stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$ are henselian local rings.*

## 18. March 22nd

The notation for today is as usual: $(A, A^+)$ is an affinoid Tate ring, $(A_0, t)$ is a couple of definition with $t \in A_0 \subseteq A^+ \subseteq A^\circ \subseteq A$, and $X = \operatorname{Spa}(A, A^+)$ is the adic spectrum.

18.1. **The structure presheaf (continued).** Let $U = X(\frac{f_1, \ldots, f_n}{g})$ be a rational subset, so $f_1, \ldots, f_n, g \in A$ satisfy $(f_1, \ldots, f_n) = (1)$ as ideals of $A$; after scaling by powers of $t$, we may assume that $f_1, \ldots, f_n, g \in A_0$. Last time, we defined $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to be the completion of the pair

$$\left( A\left[\frac{1}{g}\right], \text{integral closure of the image of } A^+\left[\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right] \subseteq A\left[\frac{1}{g}\right] \right).$$

**Remark 18.1.** The universal property Theorem 17.5 implies that $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ depends only on the rational subset $U \subseteq X$, and not the choice of parameters $f_1, \ldots, f_n, g \in A$ (however, had we not taken a completion in the construction of $\mathcal{O}_X(U)$, this would no longer be the case).

If $x \in X$, we defined the local rings as $\mathcal{O}_{X,x} \coloneqq \operatorname{colim}_{x \in U} \mathcal{O}_X(U)$ and $\mathcal{O}_{X,x}^+ \coloneqq \operatorname{colim}_{x \in U} \mathcal{O}_X^+(U)$, where the colimits run over all rational subsets $U \subseteq X$ containing $x$. Since the rational subsets form a basis for the topology on $X$, if $\mathcal{O}_X$ is a sheaf, the above formulas give the correct stalks.

**Proposition 18.2.** *Fix $x \in X$.*

1. *The valuation $f \mapsto |f(x)|$ on $A$ extends to a valuation on $\mathcal{O}_{X,x}$, and $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} \colon |f(x)| \leq 1\}$.*
2. *The stalk $\mathcal{O}_{X,x}$ is local, with maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \colon |f(x)| = 0\}$.*
3. *The stalk $\mathcal{O}_{X,x}^+$ is local, with maximal ideal $\{f \in \mathcal{O}_{X,x}^+ \colon |f(x)| < 1\}$. In particular, $\mathfrak{m}_x$ is a common ideal of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$ (but it is not maximal in $\mathcal{O}_{X,x}^+$).*

(4) *Write $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ for the residue field, and $k(x)^+ := \mathcal{O}_{X,x}^+/\mathfrak{m}_x$. Then, $(k(x), k(x)^+)$ is an affinoid Tate $(A, A^+)$-algebra*[22]*; that is, $k(x)^+$ is a microbial valuation ring with pseudouniformizer $t$, and $k(x) = k(x)^+[1/t]$.*

Proposition 18.2(4) says that $k(x)$ is a non-Archimedean field (but it is not complete, in general); however, the continuous valuation on $k(x)$ need not be of rank 1 (it only is known to have a rank-1 generalization, because $k(x)^+$) is microbial).

*Proof.* For (1), attached to $x \in X$ is the valuation ring $R_x \subseteq \kappa(\mathfrak{p}_x)$. Set $\widehat{R_x}$ to be the $t$-adic completion of $R_x$, and set $\widehat{\kappa(\mathfrak{p}_x)} = \widehat{R_x}[1/t]$ to be the fraction field. The pair $(\widehat{\kappa(\mathfrak{p}_x)}, \widehat{R_x})$ is a complete affinoid Tate $(A, A^+)$-algebra. For each rational subset $U \ni x$, write $U = X\left(\frac{f_1,\ldots,f_n}{g}\right)$, then $|f_i(x)| \leq |g(x)|$ for all $i = 1, \ldots, n$; thus, in $\widehat{R_x}$, we have $g|f_i$. Therefore, by the universal property, there is a unique factorization

$$\begin{tikzcd}
(A, A^+) \arrow[rr] \arrow[dr] & & (\widehat{\kappa(\mathfrak{p}_x)}, \widehat{R_x}) \\
& (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \arrow[ur, dashed, "\exists!"] &
\end{tikzcd}$$

Passing to the limit over all rational subsets $U \ni x$, we obtain maps

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, dashed, "\exists"] & \widehat{\kappa(\mathfrak{p}_x)} \\
\mathcal{O}_{X,x}^+ \arrow[u, hook] \arrow[r, dashed, "\exists"] & \widehat{R_x} \arrow[u, hook]
\end{tikzcd}$$

Therefore, we get a valuation $f \mapsto |f(x)|$ on $\mathcal{O}_{X,x}$ from the valuation on $\widehat{\kappa(\mathfrak{p}_x)}$ by precomposition with $\mathcal{O}_{X,x} \to \widehat{\kappa(\mathfrak{p}_x)}$.

Moreover, we already have the containment $\mathcal{O}_{X,x}^+ \subseteq \{f \in \mathcal{O}_{X,x} : |f(x)| \leq 1\}$ by the above diagram. For the opposite inclusion, pick $f \in \mathcal{O}_{X,x}$ such that $|f(x)| \leq 1$. We may represent $f$ by some $f \in \mathcal{O}_X(U)$, for some rational subset $U \ni x$, so $x \in U\left(\frac{f}{1}\right)$. Note that $U\left(\frac{f,1}{1}\right)$ is a rational subset of $X$, because a rational subset of the rational subset $U$ of $X$ is again a rational subset of $X$ (see e.g. [Hub94, Lemma 1.5]). Hence, $f \in \mathcal{O}_X^+\left(U\left(\frac{f,1}{1}\right)\right)$ and, by passing to the direct limit, this implies that $f \in \mathcal{O}_{X,x}^+$.

For (2), it suffices to show that any $g \in \mathcal{O}_{X,x} \backslash \mathfrak{m}_x$ is invertible. For any such $g$, we have $|g(x)| \geq |t^N(x)|$ for some $N \gg 0$, by the continuity condition. We may choose a rational subset $U \ni x$ such that $g \in \mathcal{O}_X(U)$ and $|g(x)| \geq |t^N(x)|$; therefore, $x \in U\left(\frac{t^N}{g}\right) \subseteq U$. In $\mathcal{O}_X\left(U\left(\frac{t^N}{g}\right)\right)$, we have $g|t^N$, so $g$ is a unit (because $t$ is a unit in $A$, and hence in any $A$-algebra). Hence, $g$ is a unit in $\mathcal{O}_{X,x}$.

For (3), consider the ideal $I := \{f \in \mathcal{O}_{X,x}^+ : |f(x)| < 1\}$ of $\mathcal{O}_{X,x}^+$. It suffices to show that any $g \in \mathcal{O}_{X,x}^+ \backslash I$ is invertible. Any $g \in \mathcal{O}_{X,x}^+ \backslash I$ must have $|g(x)| = 1$. Represent $g$ by some $g \in \mathcal{O}_X^+(U)$, then we still have that $|g(x)| = 1$, so $x \in U\left(\frac{1}{g}\right) \subseteq U$. However, $g$ has an inverse in $\mathcal{O}_X^+\left(U\left(\frac{1}{g}\right)\right)$, and hence $g$ also has an inverse in the stalk $\mathcal{O}_{X,x}^+$.

For (4), we know that $\mathcal{O}_{X,x} = \mathcal{O}_{X,x}^+[1/t]$ because $\mathcal{O}_X(U) = \mathcal{O}_X^+(U)[1/t]$ for every rational subset $U \ni x$. It follows that $k(x) = k(x)^+[1/t]$. The valuation on $\mathcal{O}_{X,x}$ restricts to a valuation on $k(x)$ with valuation ring $k(x)^+$. The rest of the proof is left as an exercise. $\square$

[22] In [Hub93, Hub94], an affinoid Tate ring that is a field is called an *affinoid field*.

The upshot of the proof of Proposition 18.2 is that we have a diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}_x \arrow[r] \arrow[d, equal] & \mathcal{O}^+_{X,x} \arrow[r] \arrow[d] & k(x)^+ \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{m}_x \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & k(x) \arrow[r] & 0 \end{tikzcd}$$

In particular, the right-most square is Cartesian.

18.2. **Background on henselian pairs.** The references for this sections are [Sta17, Tag 09XD] and [Ray70]. Intuitively, a henselian ring is one where Hensel's lemma holds; this is made precise in the definition below.

**Definition 18.3.** A pair $(A, I)$, where $A$ is a (commutative) ring and $I \subseteq A$ is an ideal, is *henselian* if it satisfies any of the following equivalent conditions:

(1) The ideal $I$ is contained in the Jacobson radical[23] of $A$, and for any monic $f \in A[T]$ and any factorization $\overline{f} = \overline{g}\overline{h}$ in $(A/I)[T]$ with $\overline{g}, \overline{h}$ monic and $(\overline{g}, \overline{h}) = (1)$, there exists a factorization $f = gh$ in $A[T]$ with $g, h$ monic lifting $\overline{g}, \overline{h}$.
(2) For any étale map $A \to B$, the induced map $\mathrm{Hom}_{A\text{-alg}}(B, A) \to \mathrm{Hom}_{A\text{-alg}}(B, A/I)$ is surjective. Geometrically, this says that étale covers split if they split mod $I$.
(3) For every finite $A$-algebra $B$, there is a bijection $\mathrm{Idemp}(B) \xrightarrow{\simeq} \mathrm{Idemp}(B/IB)$ between sets of idempotents. Geometrically, this says that connected components of finite covers can be computed after reducing mod $I$.
(4) For every integral $A$-algebra $B$, there is a bijection $\mathrm{Idemp}(B) \xrightarrow{\simeq} \mathrm{Idemp}(B/IB)$ between sets of idempotents.

The properties of henselian pairs that we will need are listed below.

(1) If $A$ is $I$-adically complete, then the pair $(A, I)$ is henselian.

   *Proof.* Use Definition 18.3(2): the infinitesimal criterion for étaleness implies that $\mathrm{Hom}_{A\text{-alg}}(B, A/I) = \mathrm{Hom}_{A\text{-alg}}(B, A/I^n)$ for all $n \geq 1$, and the $I$-adic completeness of $A$ says that mapping compatibly into each quotient $A/I^n$ is equivalent to mapping to $A$. □

(2) If $\{(A_i, I_i)\}_i$ is a filtered collection of henselian pairs, then the pair $(\mathrm{colim}_i\, A_i, \mathrm{colim}_i\, I_i)$ is also henselian.

   *Proof.* Use Definition 18.3(1). □

(3) If $(A, I)$ is a henselian pair and $A \to B$ is an integral map, then $(B, IB)$ is also henselian.

   *Proof.* Use Definition 18.3(4) and the fact that the composite of two integral maps is again integral. □

(4) If $(A, I)$ is a pair, there is a universal map $(A, I) \to (A^h, I^h)$ into a henselian pair $(A^h, I^h)$. The pair $(A^h, I^h)$ is called the *henselization* of $(A, I)$.

   *Proof.* The idea is to force Definition 18.3(2) to hold: let

$$A^h := \operatorname*{colim}_{\substack{A \xrightarrow{\text{étale}} B \\ \downarrow \ \swarrow \\ A/I}} B$$

   and set $I^h := I \cdot A^h$. Note that the category of diagrams appearing in the colimit is filtered: if $A \to B$ and $A \to B'$ are two such étale $A$-algebras, then the corresponding diagrams both admit maps to the diagram corresponding to the étale $A$-algebra $B \otimes_A B'$. □

   *Remark.*
   (i) The map $A \to A^h$ is flat (even better, it is the inductive limit of étale maps, i.e. it is *ind-étale*).

[23] Recall that the Jacobson radical of a ring $A$ is the intersection of all maximal ideal of $A$.

(ii) The universal map $(A, I) \to (A^h, I^h)$ induces an isomorphism $A/I \xrightarrow{\simeq} A^h/I^h$.
(iii) If $A$ is a local ring with maximal ideal $I$, then $I^h$ is maximal by Property (2). As $I^h$ belongs to the Jacobson radical of $A^h$, it follows that $A^h$ must be local with maximal ideal $I^h$.
(iv) If $(A, I)$ is henselian and $J \subseteq I$ is a subideal, then $(A, J)$ is also henselian.

(5) [Gab94] If $(A, I)$ is henselian, then $A \to A/I$ induces an equivalence of categories $A_{\text{fét}} \xrightarrow{\simeq} (A/I)_{\text{fét}}$ of finite étale covers (geometrically, the étale fundamental groups of $A$ and $A/I$ coincide).
(6) If $(A, I)$ is any pair and $I$ is contained in the Jacobson radical of $A$, then $(A, I)$ is henselian iff $(\mathbf{Z} \oplus I, I)$ is henselian, where $\mathbf{Z} \oplus I$ is given a ring structure in the obvious way (that is, $\mathbf{Z} \oplus I$ is the smallest ring in which $I$ is an ideal). In particular, the property of $(A, I)$ being henselian only depends on $I$ as a non-unital ring.

**Remark 18.4.** Properties (2) and (3) are not true for complete rings, in general (and neither is (i) from the Remark).

**Proposition 18.5.** *Let $(A, A^+)$ be an affinoid Tate ring, $X = \mathrm{Spa}(A, A^+)$, and $x \in X$.*
*(1) $(\mathcal{O}^+_{X,x}, t)$ is henselian.*
*(2) $(\mathcal{O}^+_{X,x}, \mathfrak{m}_x)$ is henselian.*
*(3) $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is henselian.*

It is also true that the pair consisting of $\mathcal{O}^+_{X,x}$ and its maximal ideal $\{f \in \mathcal{O}^+_{X,x} : |f(x)| < 1\}$ is henselian, but we will not prove this.

*Proof.* For (1), we know that $\mathcal{O}^+_{X,x} = \mathrm{colim}_{U \ni x} \mathcal{O}^+_X(U)$; by Property (2), it sufffices to show $(\mathcal{O}^+_X(U), t)$ is henselian. By replacing $X$ with $U$, it suffices to show that $(A^+, t)$ is henselian for a complete affinoid Tate ring $(A, A^+)$. Now, $A^+$ is the direct limit of all rings of definition $A_i$ that it contains. Each such couple of definition $(A_i, t)$ is complete (in the algebraic sense), and hence henselian by Property (1). By Property (2), the same is thus true for $(A^+, t)$.

For (2), the ideal $\mathfrak{m}_x \subseteq \mathcal{O}^+_{X,x}$ is uniquely $t$-divisible, meaning that is has no $t$-torsion and if $f \in \mathfrak{m}_x$, then $f/t \in \mathfrak{m}_x$. In particular, $\mathfrak{m}_x = t \cdot \mathfrak{m}_x \subseteq (t)$ as ideals in $\mathcal{O}^+_{X,x}$. As $(\mathcal{O}^+_{X,x}, t)$ is henselian, $(\mathcal{O}^+_{X,x}, \mathfrak{m}_x)$ is henselian by (iv) of the Remark.

For (3), use Property (6) (one needs to check that $\mathfrak{m}_x$ lies in the Jacobson radical of $\mathcal{O}_{X,x}$, but this is trivial because it is the maximal ideal in a local ring). □

## 19. March 27th

Let $(A, A^+)$ be an affinoid Tate ring, and $x \in X = \mathrm{Spa}(A, A^+)$. To this data, we associated last time two henselian local rings: $\mathcal{O}_{X,x}$ and $\mathcal{O}^+_{X,x}$. Let $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ denote the maximal ideal, with associated residue field $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$; $k(x)$ is a microbial valued field (but it may not be complete); moreover, $\mathfrak{m}_x$ is also an ideal of $\mathcal{O}^+_{X,x}$ and $k(x)$ is $k(x)^+ := \mathcal{O}^+_{X,x}/\mathfrak{m}_x$ is the valuation ring of $k(x)$. They fit into the following picture:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}_x \arrow[r] \arrow[d, equal] & \mathcal{O}^+_{X,x} \arrow[r] \arrow[d, hook] & k(x)^+ \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathfrak{m}_x \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & k(x) \arrow[r] & 0 \end{tikzcd}$$

From this diagram, the upshot is that $\mathcal{O}^+_{X,x}$ can be reconstructed from $\mathcal{O}_{X,x}$ and the valuation on its residue field $k(x)$.

**Definition 19.1.** Let $\widehat{k(x)}$ denote the completion of $k(x)$ as a valued field; if $t \in A^+$ is a pseudouniformizer, let $\widehat{k(x)^+}$ be the completion of $k(x)^+$ in the $t$-adic topology. Then,

$$\widehat{k(x)^+}[1/t] = \widehat{k(x)}.$$

**Fact 19.2.** Let $R_x \subseteq \kappa(\mathfrak{p}_x)$ denote the valuation ring of $x$. If $t \in A^+$ is a pseudouniformizer, then

(1) $\widehat{k(x)^+}$ is the $t$-adic completion of the valuation ring $R_x$;
(2) $\widehat{k(x)} = \widehat{R_x}[1/t]$.

**Lemma 19.3.** *If $t \in A^+$ is a pseudouniformizer, then $\widehat{\mathcal{O}^+_{X,x}} \simeq \widehat{k(x)^+}$, where $\widehat{}$ denotes the $t$-adic completion.*

Lemma 19.3 says that, after completion, all local information at a point $x$ can be detected from the residue field $k(x)^+$. This is very different from the case of varieties, where there is a huge difference between the local ring at a point and its residue field (e.g. for a variety over $\mathbf{C}$, the residue field is always a copy of $\mathbf{C}$).

*Proof.* There is a short exact sequence

$$0 \to \mathfrak{m}_x \to \mathcal{O}^+_{X,x} \to k(x)^+ \to 0,$$

so it suffices to show that the $t$-adic completion of $\mathfrak{m}_x$ is zero. As $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is an ideal, $t$ acts invertibly on $\mathfrak{m}_x$ (as $t \in A$ is a unit), and hence $\mathfrak{m}_x$ is uniquely $t$-divisible. Therefore, the short exact sequence induces an isomorphism

$$\mathcal{O}^+_{X,x}/t^n \xrightarrow{\simeq} k(x)^+/t^n$$

for any $n \geq 1$. Taking inverse limit over $n$ gives the result. $\square$

For schemes, a closed immersion induces a surjection on local rings, but this is not necessarily an injection, otherwise the closed immersion would also be an open immersion. In the adic setting, a closed immersion does in fact induce an isomorphism on completed local rings.

**Proposition 19.4.** [Zariski closed subsets] *Let $I \subseteq A$ be an ideal and assume*[24] *that $I = (f_1, \dots, f_r)$.*

*(1) There is a universal affinoid Tate $(A, A^+)$-algebra $(R, R^+)$ such that $I \cdot R = 0$; in fact, $R = A/I$.*
*(2) The natural map $i \colon Z = \operatorname{Spa}(A/I, (A/I)^+) \to \operatorname{Spa}(A, A^+) = X$ is closed immersion (of topological spaces*[25]*). Moreover, there is an identification*

$$i(Z) = \bigcap_{n>0} \operatorname{Spa}(A, A^+)\left(\frac{f_1, \dots, f_r, t^n}{t^n}\right).$$

*In particular, $i(Z)$ is the intersection of rational subsets.*
*(3) For any $z \in Z$, there is a natural isomorphism $\widehat{\mathcal{O}^+_{X,i(z)}} \xrightarrow{\simeq} \widehat{\mathcal{O}^+_{Z,z}}$.*
*(4) If $U \subseteq X$ is an open set containing $Z$, then there exists $n > 0$ such that*

$$U \supseteq \operatorname{Spa}(A, A^+)\left(\frac{f_1, \dots, f_r, t^n}{t^n}\right).$$

*In particular, "$\epsilon$-tubes around $Z$ form a basis of open neighborhoods of $Z$".*

While the assertions (1) and (2) of Proposition 19.4 have analogues in the world of schemes, (3) is specific to the adic setting.

*Proof.* For (1), take $R \coloneqq A/I$. If $A_0 \subseteq A$ is a ring of definition, then we declare that $R_0 \coloneqq A_0/(I \cap A_0) \subseteq R$ is a ring of definition of $R$. If $t \in A$ is a pseudouniformizer, then the image of $t$ in $R$ is also a pseudouniformizer, and one can check that $R_0[1/t] = R$, hence $R$ is a Tate ring with couple of definition $(R_0, t)$. Set $R^+ = (A/I)^+$ to be the integral closure of the image of $A^+$ in $R$, and one can verify that $(R, R^+)$ satisfies the universal property.

For (2), the fact that $i$ is a closed immersion is left as an exercise. We now show that $Z$ can be written as the specified intersection of rational subsets. The inclusion $\subseteq$ is clear, as $|f_i(x)| = 0$ if $I \subseteq \operatorname{supp}(x)$. Conversely, $x \in X$ belongs to the specified intersection of rational subsets iff $|f_i(x)| \leq |t^n(x)|$ for all $n$. By the continuity condition, this is equivalent to saying that $|f_i(x)| = 0$, so $x \in Z$.

[24]Proposition 19.4 holds more generally for any ideal, not necessarily a finitely-generated one.

[25]A closed immersion of topological spaces is a closed map that is a homeomorphism onto its image.

For (3), if $z \in Z$, then

$$\begin{array}{ccc} \widehat{\mathcal{O}^+_{X,i(z)}} & \longrightarrow & \widehat{\mathcal{O}^+_{Z,z}} \\ \| & & \| \\ \widehat{k(i(z))^+} & & \widehat{k(z)^+} \end{array}$$

by Lemma 19.3, and moreover $\widehat{k(x)^+} = \widehat{R_x}$ by Fact 19.2. The desired equality thus follows from the fact that $R_z = R_{i(z)}$, as subrings of $\kappa(z) = \kappa(i(z))$.

Part (4) is left as an exercise (hint: use the quasi-compactness of spectral spaces). $\square$

19.1. **Adic spaces.**

**Definition 19.5.** Let $(A, A^+)$ be an affinoid Tate ring and let $X = \mathrm{Spa}(A, A^+)$.

(1) The *structure presheaf* $\mathcal{O}_X$ on $X$ is defined as follows: if $U \subseteq X$ is open, set
$$\mathcal{O}_X(U) := \varprojlim_{\substack{V \subset U \\ \text{rational}}} \mathcal{O}_X(V).$$
Similarly, one can define the presheaf $\mathcal{O}_X^+$.

(2) We say that $(A, A^+)$ is *sheafy* if $\mathcal{O}_X$ is a sheaf.

If $\mathcal{O}_X$ was indeed a sheaf, then since the rational subsets of $X$ form a basis, the formula for $\mathcal{O}_X(U)$ appearing in Definition 19.5(1) would have to hold. Furthermore, if $U \subseteq X$ is a rational subset, then $\mathcal{O}_X(U)$ in the old sense (of Theorem 17.5) coincides with $\mathcal{O}_X(U)$ in the new sense (of Definition 19.5).

**Remark 19.6.** If $U \subseteq X$ is an open set and $x \in X$, then we obtain a valuation $x\colon \mathcal{O}_X(U) \to \Gamma \cup \{0\}$ by choosing a rational open subset $V \subseteq U$ containing $x$ (but the valuation is independent of the choice, because the intersection of two rational subsets containing $x$ is again a rational subset containing $x$). Then,
$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \colon |f(x)| \leq 1 \text{ for all } x \in U\}.$$
If $\mathcal{O}_X$ is a sheaf, then $\mathcal{O}_X^+$ is a sheaf, precisely because of the above equality.

In general, the problem with sheafiness is that the sheaf axioms are not well behaved with respect to completion; nonetheless, there are large classes of examples where sheafiness can be shown.

**Fact 19.7.** The affinoid Tate ring $(A, A^+)$ is sheafy if any of the following are satisfied:

(1) [Hub94, Theorem 2.2] $A$ has a noetherian ring of definition;
(2) [Hub94, Theorem 2.2] $A$ is *strongly noetherian*, i.e. $A\langle X_1, \ldots, X_n\rangle$ is noetherian for all $n \geq 0$;
(3) $A$ is perfectoid;
(4) [BV14, Theorem 7] $A$ is *stably uniform*, i.e. for any rational subset $U \subseteq X$, $\mathcal{O}_X(U)$ is a uniform ring.

The assertion Fact 19.7(3) will be proved later. Note that Fact 19.7(4) disallows nilpotents in $A$, but in the reduced case it implies (1-3).

**Example 19.8.** Fact 19.7(2) implies that a Tate algebra (in the classical sense) over an algebraically closed non-Archimedean field $K$ is sheafy (even though the valuation ring of $K$ is non-noetherian).

Finally, we can define "global" adic spaces via Huber's category $\mathcal{V}$.

**Definition 19.9.** [Hub94, §2] The category $\mathcal{V}$ has as objects triples $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$, where

(a) $X$ is a topological space;
(b) $\mathcal{O}_X$ is a sheaf of complete topological rings;
(c) $(X, \mathcal{O}_X)$ is a locally ringed space (ignoring the topology on the rings $\mathcal{O}_X(U)$);

(d) for each $x \in X$, $v_x$ is a valuation on the local ring $\mathcal{O}_{X,x}$.

The morphisms in $\mathcal{V}$ are required to preserve all the structure.

**Remark 19.10.** If $(X, \mathcal{O}_X, \{v_x\}_{x\in X}) \in \mathcal{V}$, then one gets a sheaf $\mathcal{O}_X^+$ via the following formula: for any $U \subseteq X$ open, set

$$\mathcal{O}_X^+(U) := \{f \in \mathcal{O}_X(U) : |f(x)| := v_x(f) \leq 1 \forall x \in U\}.$$

Conversely, given $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ satisfying certain conditions, one can recover the valuations $v_x$; see [Con14, §14.4].

**Definition 19.11.** An *adic space* is an object $(X, \mathcal{O}_X, \{v_x\}_{x\in X}) \in \mathcal{V}$ such that there exists an open cover $\{U_i\}_{i\in I}$ of $X$ such that $(U_i, \mathcal{O}_X|_{U_i}, \{v_i\}_{x\in U_i})$ is an *affinoid adic space*, i.e. it is of the form

$$\left(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, \{v_x\}_{x\in \mathrm{Spa}(A,A^+)}\right),$$

where $v_x$ is the canonical valuation on $\mathcal{O}_{\mathrm{Spa}(A,A^+),x}$, and $(A, A^+)$ is a sheafy affinoid Tate ring.

There is a slight discrepancy in our terminology: we only developed the theory in the Tate case, so the adic spaces of Definition 19.11 are the analytic adic spaces of [Hub93, Hub94]. In order to construct Huber's adic spaces, one must look at the objects of $\mathcal{V}$ that are locally the adic spectrum of a sheafy affinoid Huber ring.

19.2. **Analytic aspects of perfectoid spaces.** Let $K$ be a perfectoid field, let $K^\flat$ be its tilt, and fix a pseudouniformizer $t \in K^{\flat\circ}$ such that if $\pi := t^\sharp \in K^\circ$, then $|\pi| \geq |p|$ (this is the condition that ensures that the Frobenius map is defined modulo $\pi$, i.e. $K^\circ/\pi$ has characteristic $p$). The basic picture is the following:

$$\begin{tikzcd}
\varprojlim_{x\mapsto x^p} K^\circ \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\simeq"'] & K^\circ \arrow[r, hook] \arrow[d] & K \\
K^{\flat,\circ} := (K^\circ/\pi)^{\mathrm{perf}} \arrow[r] \arrow[ur, "\sharp"] \arrow[d, hook] & K^\circ/\pi & \\
K^\flat & &
\end{tikzcd}$$

Note that, in the above diagram, the maps in red are only multiplicative, not additive. Let $\mathfrak{m} := K^{\circ\circ} \subseteq K^\circ$ denote the maximal ideal of the valuation ring $K^\circ$.

**Definition 19.12.** An affinoid $(K, K^\circ)$-algebra $(R, R^+)$ (also called a *affinoid $K$-algebras*[26]) is *perfectoid* if $R$ is perfectoid (in the sense of Definition 9.12).

**Example 19.13.** Examples of perfectoid affinoid $K$-algebras include:

(1) $(R, R^+) = (K, K^\circ)$;
(2) $(R, R^+) = \left(K\langle X^{1/p^\infty}\rangle, K^\circ\langle X^{1/p^\infty}\rangle\right)$; recall that $K^\circ\langle X^{1/p^\infty}\rangle := K^\circ[X^{1/p^\infty}]^\wedge$ (where $\wedge$ denotes the $\pi$-adic completion) and $K\langle X^{1/p^\infty}\rangle := K^\circ\langle X^{1/p^\infty}\rangle[1/\pi]$.

**Proposition 19.14.** [Properties of perfectoid algebras] *Let $(R, R^+)$ be a perfectoid affinoid.*

*(1) There are inclusions*

$$\mathfrak{m}R^{\circ\circ} = R^{\circ\circ} \subseteq R^+ \subseteq R^\circ.$$

*In particular, $R^+$ determines (and is determined by) an integrally closed subring*

$$\overline{R^+} := R^+/\mathfrak{m}R^\circ \subseteq R^\circ/\mathfrak{m}R^\circ.$$

*Therefore, there is a bijective, inclusion-preserving correspondence*

$$\{\textit{rings of integral elements } R^+\} \xleftrightarrow{\simeq} \{\textit{integrally closed subrings of } R^\circ/\mathfrak{m}R^\circ\}.$$

[26]This terminology is different than the standard usage in rigid-analytic or Berkovich geometry.

*(2) There is an equivalence of categories*

$$\{\text{perfectoid affinoid } K\text{-algebras}\} \xleftrightarrow{\simeq} \{\text{perfectoid affinoid } K^\flat\text{-algebras}\},$$

*written as* $(R, R^+) \mapsto (R^\flat, R^{+\flat})$.

*Proof.* The assertion (1) will be proved next class. For (2), we already saw that equivalence $R \mapsto R^\flat$ induces an equivalence of perfectoid algebras; thus, (2) follows from (1) and the fact that there is a canonical identification

$$R^\circ/\mathfrak{m}R^\circ \xrightarrow{\simeq} R^{\flat\circ}/\mathfrak{m}^\flat R^{\flat\circ},$$

where $\mathfrak{m}^\flat := R^{\flat\circ\circ}$ is the maximal ideal of $R^{\flat\circ}$. $\square$

## 20. March 29th

Let $K$ be a perfectoid field, $K^\circ$ be the valuation ring, and $\mathfrak{m} \subseteq K^\circ$ the maximal ideal. Let $K^\flat$ denote its tilt, and let $\mathfrak{m}^\flat \subseteq K^{\flat\circ}$. If $t \in K^\flat$ is a pseudouniformizer, take $\pi = t^\sharp$ such that $|\pi| \geq |p|$ (this implies $K^\circ/\pi$ has characteristic $p$). These are related by the following diagram:

$$\begin{tikzcd}
\varprojlim_{x\mapsto x^p} K^\circ \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\simeq"'] & K^\circ \arrow[r, hook] \arrow[d] & K \\
K^{\flat,\circ} := (K^\circ/\pi)^{\mathrm{perf}} \arrow[r] \arrow[ur, "\sharp"] \arrow[d, hook] & K^\circ/\pi & \\
K^\flat & &
\end{tikzcd}$$

Recall that if $R$ is a perfectoid $K$-algebra, then $K^\circ/\pi \to R^\circ/\pi$ is relatively perfect (in fact, this was part of one of the definitions of a perfectoid algebra). In particular, the base change $K^\circ/\mathfrak{m}K^\circ \to R^\circ/\mathfrak{m}R^\circ$ is relatively perfect; as $K^\circ/\mathfrak{m}K^\circ$ is perfect, it follows that $R^\circ/\mathfrak{m}R^\circ$ is a perfect ring of characteristic $p$.

Furthermore, under tilting, we have an isomorphism $R^\circ/\pi \simeq R^{\flat\circ}/t$, and hence $R^\circ/\mathfrak{m}R^\circ \simeq R^{\flat\circ}/\mathfrak{m}^\flat R^{\flat\circ}$.

**Definition 20.1.** A *perfectoid affinoid $K$-algebra* is an affinoid Tate $(K, K^\circ)$-algebra $(R, R^+)$ such that $R$ is perfectoid.

**Observation 20.2.** Let $R$ be a perfectoid affinoid $K$-algebra.

(1) There is an inclusion-preserving bijection

$$\{\text{rings of integral elements } R^+ \subseteq R^\circ\} \xrightarrow{\simeq} \{\text{integrally closed subrings } \overline{R^+} \subseteq R^\circ/\mathfrak{m}R^\circ\},$$

given by $R^+ \mapsto R^+/\mathfrak{m}R^\circ$ (and this quotient makes sense by Remark 14.11(3)). The inverse is given by taking the preimage of $\overline{R^+} \subseteq R^\circ/\mathfrak{m}R^\circ$ under the quotient map $R^\circ \to R^\circ/\mathfrak{m}R^\circ$.

(2) Using (1), one obtains an equivalence of categories

$$\{\text{perfectoid affinoid } K\text{-algebras}\} \simeq \{\text{perfectoid affinoid } K^\flat\text{-algebras}\},$$

denoted by $(R, R^+) \mapsto (R^\flat, R^{\flat+})$, where $R^\flat$ is obtained from the usual tilting correspondence, and $R^{\flat+}$ is the preimage of the image of $R^+$ in $R^{\flat\circ}/\mathfrak{m}R^{\flat\circ}$ (as in (1)). If the adjective 'affinoid' had been omitted, then this recovers the tilting correspondence. The proof is easy given (1) and the identification $R^\circ/\mathfrak{m}R^\circ \simeq R^{\flat\circ}/\mathfrak{m}R^{\flat\circ}$ under tilting.

(3) Any integrally closed subring of a perfect ring is perfect; therefore, $R^+/\mathfrak{m}R^\circ$ is perfect for any perfectoid affinoid $K$-algebra $(R, R^+)$. It follows that $R^{\flat+} \simeq (R^+)^\flat$, where $(R^+)^\flat$ is the tilt in the algebraic sense of Definition 2.9. Indeed, consider the system of Cartesian diagrams

$$\begin{tikzcd}
R^+/\pi \arrow[rr] \arrow[dd, hook] \arrow[dr, "\mathrm{Frob}"] & & R^+/\mathfrak{m}R^\circ \arrow[dd, hook] \arrow[dr, "\mathrm{Frob}", "\simeq"'] & \\
& R^+/\pi \arrow[rr] \arrow[dd, hook] & & R^+/\mathfrak{m}R^\circ \arrow[dd, hook] \\
R/\pi \arrow[rr] \arrow[dr, "\mathrm{Frob}"] & & R/\mathfrak{m}R^\circ \arrow[dr, "\mathrm{Frob}", "\simeq"'] & \\
& R/\pi \arrow[rr] & & R/\mathfrak{m}R^\circ
\end{tikzcd}$$

As taking the inverse limit commutes with Cartesian diagrams, $(R^+)^\flat$ is given by the Cartesian diagram

$$\begin{tikzcd}
(R^+)^\flat \arrow[r] \arrow[d] & R^+/\mathfrak{m}R^{\flat\circ} \arrow[d, hook] \\
R^\flat \arrow[r] & R^\flat/\mathfrak{m}R^\circ
\end{tikzcd}$$

and this last diagram also describes $R^{\flat+}$, hence $R^{\flat+} \simeq (R^+)^\flat$.

20.1. **Affinoid perfectoid spaces.** If $(R, R^+)$ is a perfectoid affinoid $K$-algebra, there is a multiplicative map $\sharp\colon R^\flat \to R$ and we saw that it induces a map

$$\{\text{continuous valuations on } R\} \to \{\text{continuous valuations on } R^\flat\},$$

given by $x \mapsto x^\flat := x \circ \sharp$. To see that $x \circ \sharp$ satisfies the non-Archimedean inequality on $R^\flat$, we needed an explicit description of the addition law on $R^{\flat\circ}$: given sequences $(a_n), (b_n) \in \varprojlim R^\circ \simeq R^{\flat\circ}$, then $(a_n) + (b_n) = (c_n)$, where

$$c_n := \lim_{k\to\infty} (a_{n+k} + b_{n+k})^{p^k}.$$

Granted this description, it is not hard to verify that $x \circ \sharp$ is a valuation.

From Observation 20.2(3), one sees that if $x(R^+) \le 1$, then $x^\flat(R^{\flat+}) \le 1$. Therefore, we get a map

$$X := \mathrm{Spa}(R, R^+) \xrightarrow{(\cdot)^\flat} \mathrm{Spa}(R^\flat, R^{\flat+}) =: X^\flat$$

on adic spectra. The map $(\cdot)^\flat$ is continuous and, in fact,

$$\left((\cdot)^\flat\right)^{-1}\left(X^\flat\left(\frac{f_1, \dots, f_n}{g}\right)\right) = X\left(\frac{f_1^\sharp, \dots, f_n^\sharp}{g^\sharp}\right).$$

As we may choose $f_n = t^N$, it follows that $f_n^\sharp = \pi^N$ is a unit, and so the $(\cdot)^\flat$-preimage of a rational subset is again a rational subset. In particular, $(\cdot)^\flat$ is continuous.

**Theorem 20.3.** [Sch12, Theorem 6.3]

*(1) The map $(\cdot)^\flat$ is a homeomorphism that preserves rational subsets. In particular, given a rational subset $U \subseteq X$, write $U^\flat := (\cdot)^\flat(U)$, and we call it the* tilt *of $U$.*

*(2) For any rational subset $U \subseteq X$ with tilt $U^\flat \subseteq X^\flat$, the complete affinoid Tate $(R, R^+)$-algebra $(\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$ is perfectoid affinoid and tilts to $\left(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}^+_{X^\flat}(U^\flat)\right)$.*

*Strategy of proof.*

(i) Prove the theorem when $\mathrm{char}(K) = p$: in this case, (1) is vacuous because the spaces $X$ and $X^\flat$ are literally the same, and the only contentful statement in (2) is the assertion that $\mathcal{O}_X(U)$ is a perfectoid algebra.

(ii) Prove the theorem for rational subsets $U \subseteq X$ of the form $\left((\cdot)^\flat\right)^{-1}(V)$, where $V \subseteq X^\flat$ is a rational subset; that is, for those rational subsets in $X$ defined by inequalities coming from characteristic $p$, or more precisely, which can be written as

$$U = X\left(\frac{f_1, \ldots, f_n}{g}\right),$$

where $f_i = a_i^\sharp$ and $g = b^\sharp$, for some $a_i, b \in R^\flat$.

(iii) Reduce the general case to (ii): one must argue that every rational subset in $X$ is of the form in (ii). This is difficult because $\sharp$ need not be surjective, so one must use an approximation result for elements of $R$ in terms of those coming from $R^\flat$.

Let us begin with (i), Huber's presheaf in characteristic $p$.

**Lemma 20.4.** [Sch12, Lemma 6.4] *Assume char$(K) = p$. Fix a rational subset $U = X\left(\frac{f_1,\ldots,f_n}{g}\right)$ and assume (by scaling) that $f_i, g \in R^+$ and $f_n = \pi^N$.*

*(1) Let $R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle$ be the $\pi$-adic completion of the subring*[27] *$R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \subseteq R\left[\frac{1}{g}\right]$. Then, $R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle$ is a perfectoid $K^{\circ a}$-algebra.*

*(2) The canonical $R^+$-algebra map*

$$\psi\colon R^+\left[X_i^{1/p^\infty}\right] \to R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]$$

*given by $X_i^{1/p^m} \mapsto \left(\frac{f_i}{g}\right)^{1/p^m}$ is surjective and the kernel is almost equal to*

$$I \coloneqq \left(g^{1/p^m} X_i^{1/p^m} - f_i^{1/p^m} : m \geq 0\right).$$

*(3) The Tate $K$-algebra $\mathcal{O}_X(U)$ is a perfectoid $K$-algebra. Moreover,*

$$\mathcal{O}_X(U)^{\circ a} \simeq R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle^a.$$

From the rigid-analytic perspective, Lemma 20.4(3) is surprising; in the rigid setting, one can never hope to describe $\mathcal{O}_X(U)^\circ$ explicitly in general (because one must take integral closures).

*Proof.* For (1), the ring $R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \subseteq R\left[\frac{1}{g}\right]$ is visibly perfect and $\pi$-torsion free. Therefore, its $\pi$-adic completion $R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle$ gives a perfectoid $K^{\circ a}$-algebra.

For (2), the map $\psi$ is clearly surjective and $I \subseteq \ker(\psi)$. We must now show that the induced map

$$\overline{\psi}\colon P \coloneqq R^+[X_i^{1/p^\infty}]/I \to R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]$$

is an almost isomorphism. Observe that $I[1/\pi] = \ker(\psi[1/\pi])$. Indeed, there is a commutative diagram

[27] The definition of the subring $R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]$ already uses that the characteristic of $K$ is $p$, so all rings are perfect and there is a unique compatible system of $p$-power roots of $f_i/g$.

$$\begin{tikzcd} R^+\left[X_i^{1/p^\infty}\right] \arrow[r, "\psi"] \arrow[d, hook] & R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \arrow[d, hook] \\ R\left[X_i^{1/p^\infty}\right] \arrow[r, "\psi[1/\pi]"] & R\left[\frac{1}{g}\right], \end{tikzcd}$$

from which it follows that $\ker(\psi[1/\pi])$ is generated by $I$ and the $g^\infty$-torsion of $R\left[X_i^{1/p^\infty}\right]$. As $gX_n = f_n = \pi^N$, inverting $\pi$ inverts $g$ in $P$, so there is no $g^\infty$-torsion in $P[1/\pi]$. Therefore, $P[1/\pi] \to R^+[(f_i/g)^{1/p^\infty}][1/\pi]$ is an isomorphism; in particular, $I[1/\pi] = \ker(\psi[1/\pi])$.

It follows that $\ker(P \xrightarrow{\overline{\psi}} R^+[(f_i/g)^{1/p^\infty}])$ is contained in the $\pi^\infty$-torsion of $P$, so we may use the following lemma to conclude:

**Lemma 20.5.** *Let $A$ be a perfect ring and $f \in A$. If $\alpha \in A$ is killed by $f^N$ for some $N > 0$, then $\alpha$ is killed by $f^{1/p^n}$ for all $n$.*

*Proof.* As $f^N\alpha = 0$, $f^N\alpha^{p^m} = 0$ for all $m$. Equivalently (using that $A$ is perfect), $f^{N/p^m}\alpha = 0$ for all $m$, and hence $f^{1/p^m}\alpha = 0$ for all $m$. □

For (3), consider the inclusions

$$R^+\left[\frac{f_i}{g}\right] \overset{\alpha}{\hookrightarrow} R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \subseteq R\left[\frac{1}{g}\right].$$

The completion of $R^+\left[\frac{f_i}{g}\right]$ is a ring of definition of $\mathcal{O}_X(U)$.

**Claim 20.6.** $\operatorname{coker}(\alpha)$ is killed by $\pi^{nN}$.

Granted Claim 20.6, we see that the perfectoid $K^{\circ a}$-algebra $R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle$ and the $\pi$-adic completion $\widehat{R^+\left[\frac{f_i}{g}\right]}$ give the same Tate $K$-algebra upon inverting $\pi$. The first presentation gives a perfectoid $K$-algebra, while the second gives $\mathcal{O}_X(U)$; in particular, $\mathcal{O}_X(U)$ is perfectoid.

To see the isomorphism $\mathcal{O}_X(U)^{\circ a} \simeq R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle^a$, use that there is an equivalence of categories $\mathrm{Perf}_K \xrightarrow{\simeq} \mathrm{Perf}_{K^{\circ a}}$, given by $R \mapsto R^\circ$ with inverse $A[1/\pi] \leftarrow\!\shortmid A$. Both $\mathcal{O}_X(U)^{\circ a}$ and $R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle^a$ give objects of $\mathrm{Perf}_{K^{\circ a}}$ with the same image in $\mathrm{Perf}_K$, hence they are almost isomorphic.

*Proof of Claim 20.6.* It suffices to show that $\pi^{nN}\cdot\prod_{i=1}^n\left(\frac{f_i}{g}\right)^{1/p^{a_i}} \in R^+\left[\frac{f_i}{g}\right]$ for any $a_1, \dots, a_n > 0$. This expression can be rewritten as

$$\prod_{i=1}^n \pi^N\left(\frac{f_i}{g}\right)^{1/p^{a_i}} = \prod_{i=1}^n \frac{f_n f_i^{1/p^{a_i}} g^{1-1/p^{a_i}}}{g} = \prod_{i=1}^n \left(f_i^{1/p^{a_i}} g^{1-1/p^{a_i}}\right)\frac{f_n}{g} \in R^+\left[\frac{f_n}{g}\right] \subseteq R^+\left[\frac{1}{g}\right].$$

□

This concludes the proof of Lemma 20.4. □

## 21. April 3rd

21.1. **Affinoid perfectoid spaces (continued).** Last time, we were in the midst of proving the tilting correspondence for perfectoid affinoid algebras. The notation is as follows: let $K$ be a perfectoid field, $K^\circ$ be the valuation ring, and $\mathfrak{m} = K^{\circ\circ}$ the maximal ideal. Let $K^\flat$ denote its tilt, and let $\mathfrak{m}^\flat = K^{\flat\circ\circ}$. If $t \in K^\flat$ is a pseudouniformizer, take $\pi = t^\sharp$ such that $|\pi| \geq |p|$.

**Theorem 21.1.** [Sch12, Theorem 6.3] *Let $(R, R^+)$ be perfectoid affinoid algebra over $K$.*

*(1) The map $(\cdot)^\flat \colon X \coloneqq \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R^\flat, R^{+\flat}) =: X^\flat$ is a homeomorphism, preserving rational subsets.*
*(2) For any $U \subseteq X$ rational, $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$ is perfectoid affinoid, and it tilts to $\left(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\right)$, where $U^\flat = (\cdot)^\flat(U)$.*

Last time, we showed Theorem 21.1 under the additional hypothesis that $K$ has characteristic $p$. In fact, in this case, if $U = X\left(\frac{f_1, \ldots, f_n}{g}\right)$ is a rational subset of $X$ with $f_n = \pi^N$ for some $N > 0$, then we showed that

$$\mathcal{O}_X(U)^\circ \overset{a}{\simeq} R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]^\wedge =: R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle.$$

**Proposition 21.2.** *Let $U = X\left(\frac{f_1, \ldots, f_n}{g}\right)$ be a rational subset of $X$ with $f_i, g \in R^+$, $f_n = \pi^N$, and $f_i = a_i^\sharp$ and $g = b^\sharp$ for some $a_i, b \in R^{\flat +}$.*[28]

*(1) Let $R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle$ be the $\pi$-adic completion of the subring $R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \subseteq R^+\left[\frac{1}{g}\right]$. Then, $R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle$ is a perfectoid $K^{\circ a}$-algebra.*[29]
*(2) The Tate $K$-algebra $\mathcal{O}_X(U)$ is a perfectoid $K$-algebra. Moreover,*

$$\mathcal{O}_X(U)^{\circ a} \simeq R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle^a.$$

*(3) The perfectoid affinoid $K$-algebra $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$ tilts to $\left(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\right)$.*

The description in Proposition 21.2(2) of the ring of power bounded elements of $\mathcal{O}_X(U)$ is a something one gains in the perfectoid world, but that is not present in the finite-type setting.

*Proof.* Write $P_0 \coloneqq R^+\left[X_i^{1/p^\infty}\right]/I$, where $I = \left(g^{1/p^m}X_i^{1/p^m} - f_i \colon m \geq 0\right)$. There is an obvious map

$$P_0 \xrightarrow{a_0} R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]$$

given by $X_i^{1/p^m} \mapsto \left(\frac{f_i}{g}\right)^{1/p^m}$. Also, we have a map

$$R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \xrightarrow{b_0} \mathcal{O}_X^+(U),$$

by the construction of the pair $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$. Write $(S, S^+)$ for the untilt of $\left(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\right)$, and thus we get a map $(R, R^+) \to (S, S^+)$ by untilting the structure map. Furthermore, we have $b|a_i$ in $\mathcal{O}_{X^\flat}^+(U^\flat)$ because

[28]The hypotheses of Proposition 21.2 imply that $U = \left((\cdot)^\flat\right)^{-1}(U^\flat)$, where $U^\flat = X^\flat\left(\frac{a_1, \ldots, a_n}{b}\right)$.

[29]The compatible system of $p$-power roots $\left(\frac{f_i}{g}\right)^{1/p^\infty}$ is the one lifted from characteristic $p$, i.e. obtained as the image of $\left(\frac{a_i}{b}\right)^{1/p^\infty}$ under the $\sharp$-map. In characteristic 0, there is no distinguished system of $p$-power roots.

$U^\flat = X^\flat\left(\frac{a_1,\ldots,a_n}{b}\right)$, and by untilting we get that $g|f_i$ in $S^+$ (using also that the $\sharp$-map is multiplicative). By the universal property of Huber's presheaf, there is a factorization

$$\mathcal{O}_X^+(U) \xrightarrow{c} S^+$$

of $R^+ \to S^+$ (these are really maps of pairs). Write $a$ for the $\pi$-adic completion of $a_0$, and $d$ for the $\pi$-adic completion of $d_0 \coloneqq c \circ b_0$. The picture so far is as follows:

$$\begin{tikzcd}
P_0 \arrow[r, "a_0"] \arrow[d, "\text{completion}"] & R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right] \arrow[r, "d_0"] \arrow[d, "\text{completion}"] & S^+ \arrow[d, equal, "\text{id}"] \\
P \coloneqq \widehat{P_0} \arrow[r, "a"] & R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle \arrow[r, "d"] & S^+
\end{tikzcd}$$

Here, we are using the fact that $S^+$ is complete.

Now, $a_0$ is surjective (by the construction of $P_0$) and therefore $a$ is surjective (essentially by a general version of Nakayama's lemma; see [Sta17, Tag 0315, Tag 07RC]). Also, $d\circ a$ is an almost isomorphism modulo $\pi$: indeed, there is a commutative diagram

$$\begin{tikzcd}
P = \widehat{P_0} \arrow[rrr, "d\circ a"] \arrow[d] & & & S^+ \arrow[d] \\
P/\pi = \frac{(R^+/\pi)\left[X_i^{1/p^\infty}\right]}{(g^{1/p^m}X_i^{1/p^m} - g_i \colon m\geq 0)} \arrow[r] & \frac{R^{+\flat}\left[X_i^{1/p^\infty}\right]}{(g^{1/p^m}X_i^{1/p^m} - g_i \colon m\geq 0)} = \mathcal{O}_{X^\flat}^+(U^\flat)/t \arrow[rr, equal] & & S^+/\pi
\end{tikzcd}$$

Therefore, $d\circ a$ is almost surjective (again by Nakayama's lemma). Now, we may use the following general fact:

**Lemma 21.3.** *If $\alpha\colon M \to N$ is an almost surjective map of $K^\circ$-modules and assume:*

*(1) $M$ is $\pi$-adically separated;*
*(2) $N$ is $\pi$-torsion free;*
*(3) $\alpha \bmod \pi$ is an almost isomorphism.*

*Then, $\alpha$ is an almost isomorphism.*

Applying Lemma 21.3 to $d\circ a$ shows that it is an almost isomorphism. As $a$ is surjective, it follows that $a$ and $d$ are both almost isomorphisms. This proves (1).

For (2), literally the same proof from characteristic $p$ works (recall that the key point is that $\pi^{nN}$ annihilates the cokernel of $R^+\left[\frac{f_i}{g}\right] \hookrightarrow R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]$, which completely general). Therefore, $\mathcal{O}_X(U)$ is perfectoid, and thus uniform; in particular, $\mathcal{O}_X^+(U)$ is a ring of definition of $\mathcal{O}_X(U)$, hence it is $\pi$-adically complete.

For (3), recall that (1) implies that $R^+\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\rangle$ tilts to $\mathcal{O}_{X^\flat}^+(U^\flat)^a$, and moreover (2) asserts that

$$R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle\left[\frac{1}{\pi}\right] = \mathcal{O}_X(U).$$

Therefore, $\mathcal{O}_X(U)$ is the untilt of $\mathcal{O}_{X^\flat}(U^\flat)$. Now, observe that there is a map $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right) \xrightarrow{\alpha} (S, S^+)$ coming from (1). Conversely, in the proof (1), we also found a map

$$\left(R\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle\left[\frac{1}{\pi}\right], R\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle\right) \longrightarrow \left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$$

Here, we are using that $\mathcal{O}_X^+(U)$ is $\pi$-adically complete. Therefore, upon tilting, we get a map

$$\left(R^\flat, R^{\flat+}\right) \longrightarrow \left(R\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle\left[\frac{1}{\pi}\right], R\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle\right)^\flat \longrightarrow \left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)^\flat.$$

It follows that $b|a_i$ in $\left(\mathcal{O}_X^+(U)\right)^\flat$. By Huber's universal property, we get a map

$$\left(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\right) \xrightarrow{\beta} \left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$$

It is not hard to verify that the untilt of $\beta$ gives an inverse of $\alpha$. This completes the proof of Proposition 21.2. $\square$

To complete the proof of Theorem 21.1, we need the following "approximation lemma" (in fact, this same lemma plays a crucial role in Scholze's proof of the weight-monodromy conjecture for hypersurfaces).

**Lemma 21.4.** [Sch12, Lemma 6.5] *Assume $R = K\langle T_0^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$. Let $f \in R^\circ$ be homogeneous of degree $d \in \mathbf{N}[1/p]$. For any rational number $c \geq 0$ and $\epsilon > 0$, there exists a homogeneous element*

$$g_{c,\epsilon} \in R^{\flat\circ} = K^{\flat\circ}\langle T_0^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$$

*of degree $d$ such that*

$$|f(x) - g_{c,\epsilon}^\sharp(x)| \leq |\pi|^{1-\epsilon} \max\{|f(x)|, |\pi|^c\} \qquad \textit{for all } x \in \operatorname{Spa}(R, R^\circ).$$

*In particular, choosing $\epsilon < 1$, we have that*

$$\max\{|f(x)|, |\pi|^c\} = \max\left\{|g_{c,\epsilon}^\sharp(x)|, |\pi|^c\right\} \qquad \textit{for all } x \in \operatorname{Spa}(R, R^\circ).$$

*Proof.* We will not discuss the proof of the main assertion (it is a long computation), but we will quickly indicate how to get the second assertion from the first. If $x \in \operatorname{Spa}(R, R^\circ)$, assume that $|f(x)| \leq |\pi|^c$, then $\max\{|f(x)|, |\pi|^c\} = |\pi|^c$. We want to show that $|g_{c,\epsilon}^\sharp(x)| \leq |\pi|^c$. If this is not the case, $|g_{c,\epsilon}^\sharp(x)| > |\pi|^c$, so the first assertion of the lemma plus the strict non-Archimedean inequality implies that

$$|f(x) - g_{c,\epsilon}^\sharp(x)| = |g_{c,\epsilon}^\sharp(x)| > |\pi|^c$$

but $|\pi|^{1-\epsilon} \max\{|f(x)|, |\pi|^c\} < |\pi|^c$ as $0 < \epsilon < 1$, a contradiction. There are further cases, all of which are similar. $\square$

Before proceeding to the proof of Theorem 21.1, we require one further lemma.

**Lemma 21.5.** *A perfectoid $K$-algebra $R$ is a perfectoid field iff $R^\flat$ is a perfectoid field.*

*Proof.* It is always true that, if $R$ is a perfectoid field, then $R^\flat$ is a perfectoid field. Conversely, assume $R^\flat$ is a perfectoid field. We must show that $R$ is a perfectoid field, so it suffices to show that it is a non-Archimedean field (because we already know that Frobenius on the ring of integers modulo $p$ is surjective, as $R$ is a perfectoid algebra).

Consider the *spectral norm*

$$\|f\|_R \coloneqq \inf\{|s| \colon s \in K, f \in tR^\circ\}.$$

The spectral norm defines the topology on $R$, and its unit ball is precisely $R^\circ$ (but it is only submultiplicative, in general). We must show the following:

(a) $\|\cdot\|_R$ is a non-Archimedean valuation on $R$; that is, $\|\cdot\|_R$ satisfies the non-Archimedean inequality and it is multiplicative.
(b) $R$ is a field.

The rest of the proof will be discussed next class. $\square$

## 22. April 5th

22.1. **Affinoid perfectoid spaces (continued).** The notation is as usual: let $K$ be a perfectoid field, $K^\circ$ be the valuation ring, and $\mathfrak{m} = K^{\circ\circ}$ the maximal ideal. Let $K^\flat$ denote its tilt, and let $\mathfrak{m}^\flat = K^{\flat\circ\circ}$. Take $t \in K^\flat$ a pseudouniformizer, and $\pi = t^\sharp$, such that $|\pi| \geq |p|$. Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra.

The goal of today is to complete the proof of the tilting correspondence for affinoid perfectoid spaces.

**Theorem 22.1.** *Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra.*

*(1) The map $(\cdot)^\flat \colon X \coloneqq \operatorname{Spa}(R, R^+) \to \operatorname{Spa}(R^\flat, R^{+\flat}) =: X^\flat$ is a homeomorphism, preserving rational subsets.*
*(2) If $U \subseteq X$ rational, with tilt $U^\flat$, then $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$ is perfectoid affinoid with tilt $\left(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\right)$.*

Last time, we showed the following: if $U = X\left(\frac{f_1,\ldots,f_n}{g}\right)$ is a rational subset of $X$ with $f_i = a_i^\sharp, g = b^\sharp \in R^+$, then
$$\mathcal{O}_X(U)^\circ \overset{a}{\simeq} R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]^\wedge$$
and furthermore
$$\mathcal{O}_X(U) \simeq R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]^\wedge\left[\frac{1}{\pi}\right].$$
That is, we have a "generators and relations"-style description for $\mathcal{O}_X(U)^{\circ a}$. This description will be used later when we discuss the direct summand conjecture.

We will later relate the completed residue field of a point of $X$ with the completed residue field of the corresponding point of $X^\flat$, for which we will require the following lemma.

**Lemma 22.2.** *If $R \in \mathrm{Perf}_K$, then $R$ is a perfectoid field*[30] *iff $R^\flat$ is so.*

*Proof.* We must show that if $R^\flat$ is a perfectoid field, then $R$ is so. Consider the spectral norm on $R$, given by
$$\|f\|_R \coloneqq \inf\left\{|s| \colon s \in K, f \in sR^\circ\right\},$$
which measures "how far an element is from being power-bounded". The spectral norm $\|\cdot\|_R$ always defines the topology on $R$ (because $R$ is uniform). We need to check that $\|\cdot\|_R$ is multiplicative (a priori, it is just submultiplicative).

Note that $\|fg\|_R = \|f\|_R\|g\|_R$ if $f \in K$, because $R$ is a Banach $K$-algebra. To show multiplicativity of $\|\cdot\|_R$, it suffices to show the following (by scaling by elements of $K$): given $f, g \in R^\circ \backslash \pi^{1/p}R^\circ$, $\|fg\|_R = \|f\|_R\|g\|_R$. Choose $a, b \in R^{\flat\circ}$ such that $a^\sharp \equiv f \bmod \pi R^\circ$ and $b^\sharp \equiv f \bmod \pi R^\circ$.

**Claim 22.3.** We have $\|f\|_R = \|a\|_{R^\flat}$, $\|g\|_R = \|b\|_{R^\flat}$, and $\|fg\|_R = \|ab\|_{R^\flat}$.

As $R^\flat$ is a non-Archimedean field by hypothesis (so the spectral norm on $R^\flat$ is multiplicative), the claim clearly implies that $\|\cdot\|_R$ is a non-Archimedean valuation on $R$.

*Proof of Claim 22.3.* We know that $a^\sharp = f + \pi h$ for some $h \in R^\circ$. As $f \in R^\circ\backslash\pi^{1/p}R^\circ$, we have $f \in sR^\circ$ for some $s \in K^*$ with $|s| > |\pi^{1/p}|$, and this occurs iff $a^\sharp \in sR^\circ$ for the same $s \in K^*$. Thus, $\|f\|_R = \|a^\sharp\|_R = \|a\|_{R^\flat}$; the equality $\|a^\sharp\|_R = \|a\|_{R^\flat}$ holds because of the following fact: if $c \in R^\flat$, then $c \in R^{\flat\circ} \iff c^\sharp \in R^\circ$.

Similarly, one gets that $\|g\|_R = \|b\|_{R^\flat}$.

Finally, we have $fg = (ab)^\sharp + \pi\ell$ for some $\ell \in R^\circ$, and $a^\sharp, b^\sharp \notin \pi^{1/p}R^\circ$ because $f, g \notin \pi^{1/p}R^\circ$. Tilt this statement to get that $a, b \notin t^{1/p}R^{\flat\circ}$. Now, as $R^\flat$ is a non-Archimedean field, $ab \notin t^{2/p}R^{\flat\circ}$ and therefore $ab \notin tR^{\flat\circ}$, as $p \geq 2$. Thus, $(ab)^\sharp \notin \pi R^\circ$, and hence $fg \notin \pi R^\circ$. Now argue as before, using the strict non-Archimedean inequality. $\square$

[30] A priori, the term 'perfectoid field' is ambiguous, but it is a nontrivial theorem (due to Kedlaya) that a perfectoid ring that is abstractly a field is a perfectoid field in the sense of Definition 3.1; see [Ked17, Corollary 2.3.11].

To complete the proof of Lemma 22.2, it remains to check that $R$ is a field, which is left as an exercise. $\square$

**Proposition 22.4.** *Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra with adic spectrum $X = \mathrm{Spa}(R, R^\flat)$, and let $(R^\flat, R^{\flat+})$ be the tilt, which is a perfectoid affinoid $K^\flat$-algebra with adic spectrum $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$.*

*(1) Given $f \in R$ and a real number $c \geq 0$, there exists $g_c \in R^\flat$ such that*

$$|f(x) - g_c^\sharp(x)| < \max\{|f(x)|, |\pi|^c\} \qquad \text{for all } x \in \mathrm{Spa}(R, R^+).$$

*(2) Given $f, g \in R$ and a real number $c$ such that $|\pi|^c \in |K^*|$, there exist $a, b \in R^\flat$ such that*

$$X\left(\frac{f, \pi^c}{g}\right) = X\left(\frac{a^\sharp, \pi^c}{b^\sharp}\right)$$

*as subsets of $X$.*[31]

*(3) Every rational subset of $X$ is the preimage of a rational subset of $X^\flat$. In particular, $(\cdot)^\flat \colon X \to X^\flat$ is injective.*

*(4) For every rational subset $U \subseteq X$, the affinoid Tate $K$-algebra $\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big)$ is perfectoid affinoid with tilt $\big(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\big)$.*

*(4.5) For each $x \in X$, the completed residue field $\widehat{k(x)}$ is perfectoid.*

*(5) The map $(\cdot)^\flat \colon X \to X^\flat$ is a homeomorphism, preserving rational subsets.*

*Proof.* For (1), by scaling, it suffices to solve the problem for $f \in R^+$. Furthermore, we may assume that $c$ is an integer, because

$$\max\{|f(x)|, |\pi|^c\} \geq \max\{|f(x)|, |\pi|^{\lceil c \rceil}\}.$$

We may write $f$ as

$$f = g_0^\sharp + \pi g_1^\sharp + \pi^2 g_2^\sharp + \ldots + \pi^c g_c^\sharp + \pi^{c+1} h$$

for some $g_0, \ldots, g_c \in R^{\flat+}$ and $h \in R^+$. Set $f_0 \coloneqq \sum_{i=0}^c g_i^\sharp \pi^i$ to be the truncation of $f$, so that $f = f_0 + \pi^{c+1} h$. One can easily check that solving the problem for $f$ is the same as solving the problem for $f_0$, using the strict non-Archimedean inequality. Now, consider the map

$$\mu \colon K^\circ\langle T_0^{1/p^\infty}, \ldots, T_c^{1/p^\infty}\rangle \to R^+$$

given by $T_i \mapsto g_i^\sharp$; this is well defined because any element in the image of the $\sharp$-map has a distinguished compatible system of $p$-power roots. Let $f' = \sum_{i=0}^c \pi^i T_i \in K^\circ\langle T_0^{1/p^\infty}, \ldots, T_c^{1/p^\infty}\rangle$, a homogeneous polynomial of degree 1. By the "homogeneous approximation lemma" Lemma 21.4, there exists $g' \in K^\circ\langle T_0^{1/p^\infty}, \ldots, T_c^{1/p^\infty}\rangle^\flat$ such that for all $x \in \mathrm{Spa}(K\langle T_0^{1/p^\infty}, \ldots, T_c^{1/p^\infty}\rangle, K^\circ\langle T_0^{1/p^\infty}, \ldots, T_c^{1/p^\infty}\rangle)$,

$$|f'(x) - (g')^\sharp(x)| < \max\{|f'(x)|, |\pi|^c\}. \tag{$*$}$$

Set $g_c \coloneqq \mu(g') \in R^{\flat+}$, then the desired inequality follows from $(*)$.

For (2), use (1) to find $a, b \in R^{\flat+}$ "approximating" $f, g$ in the sense of (1); that is,

$$\begin{cases} |g(x) - b^\sharp(x)| < \max\{|g(x)|, |\pi|^c\}, \\ \max\{|f(x)|, |\pi|^c\} = \max\{|a^\sharp(x)|, |\pi|^c\}. \end{cases}$$

Now, use the strict non-Archimedean inequality (at least 4 times) to get that

$$X\left(\frac{f, \pi^c}{g}\right) = X\left(\frac{a^\sharp, \pi^c}{b^\sharp}\right),$$

as required.

[31]If $c \in |K^*|$ is not an integer, the notation $\pi^c$ refers to a nonzero element of $K^{\circ\circ}$ of absolute value $|\pi|^c$ that is also the image of a pseudouniformizer of $K^\flat$ under the $\sharp$-map.

For (3), let $U = X\left(\frac{f_1,\ldots,f_n,\pi^c}{g}\right) \subseteq X$ be a rational subset, then

$$U = \bigcap_{i=1}^{n} X\left(\frac{f_i, \pi^c}{g}\right).$$

One can now apply (2) $n$ times to conclude. The injectivity of $(\cdot)^\flat$ follows from the fact that a basis for the topology on $X$ is pulled-back from a basis for $X^\flat$ and the fact that $X$ is $T_0$.

For (4), use (3) and Proposition 21.2, which we recalled at the start of class.

For (4.5), recall that

$$\widehat{k(x)^+} = \left(\operatorname{colim}_{\substack{U \ni x \\ \text{rational}}} \mathcal{O}_X^+(U)\right)^\wedge.$$

Each $\mathcal{O}_X^+(U)$ is perfectoid over $K^{\circ a}$ and, as completed filtered colimits of perfectoid $K^{\circ a}$-algebras are perfectoid, it follows that $\widehat{k(x)^+}$ is a perfectoid $K^{\circ a}$-algebra. Therefore, $\widehat{k(x)}$ is a perfectoid field.

For (5), we have so far shown that $(\cdot)^\flat$ is injective and, as rational subsets of $X$ are pulled back from rational subsets of $X^\flat$, it suffices to prove surjectivity. Choose $x \in X^\flat$, then we get a map $(R^\flat, R^{\flat+}) \to \left(\widehat{k(x)}, \widehat{k(x)^+}\right)$, and the target is a perfectoid field by (4.5). Untilt to get a map $(R, R^+) \to (L, L^+)$, where the target is again a perfectoid field by Lemma 22.2. To finish, we need to show that $L^+ \subseteq L^\circ$ is an open valuation ring (a priori, it is only an integrally closed subring of $L^\circ$). For this, note that the correspondence

$$\{\text{ring of integral elements in } L\} \simeq \{\text{rings of integral elements in } L^\flat\}$$

preserves valuation rings (in fact, $L^+ \subseteq L^\circ$ is a valuation ring iff $L^+/\mathfrak{m}L^\circ \subseteq L^\circ/\mathfrak{m}L^\circ$ is so, which is a general commutative algebra fact). Therefore, $(L, L^+)$ is an affinoid field and the map $(R, R^+) \to (L, L^+)$ gives a point of $X$ lifting $x \in X^\flat$. $\square$

Next time, we will prove the Tate acyclicity theorem for perfectoid affinoid spaces via noetherian approximation.

## 23. April 10th

The notation is as usual: let $K$ be a perfectoid field, $K^\flat$ its tilt, $\pi \in \mathfrak{m}$ a pseudouniformizer, and $t \in \mathfrak{m}^\flat$ a pseudouniformizer such that $t^\sharp = \pi$ and $|\pi| \geq |p|$. Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra, with adic spectrum $X = \operatorname{Spa}(R, R^+)$. One can tilt it to a perfectoid affinoid $K^\flat$-algebra $(R^\flat, R^{\flat+})$ with adic spectrum $X^\flat = \operatorname{Spa}(R^\flat, R^{\flat+})$.

Last time, we proved that there is a homeomorphism $X \simeq X^\flat$, preserving both rational subsets and functions on rational subsets, i.e. for a rational subset $U \subseteq X$, $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)^\flat = \left(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\right)$. Thus, any reasonable problem about $X$ can be translated into a problem about $X^\flat$, which lives in characteristic $p$.

### 23.1. Tate acyclicity.

Our goal for today is to prove the following:

**Theorem 23.1.** [Tate acyclicity] *Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra and let $X = \operatorname{Spa}(R, R^+)$.*

*(1) The pair $(R, R^+)$ is sheafy (i.e. $\mathcal{O}_X$ is a sheaf).*
*(2) The global sections of $\mathcal{O}_X^+$ are $\mathcal{O}_X^+(X) = R^+$, and the higher cohomology is $H^i(X, \mathcal{O}_X^+) \overset{a}{=} 0$ for $i > 0$.*
*(3) The global sections of $\mathcal{O}_X$ are $\mathcal{O}_X(X) = R$, and the higher cohomology is $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

There is an analogous statement that holds in rigid-analytic geometry, due to Tate: there, only (1) and (3) are true, but (2) is generally false for classical Tate algebras. It is (2) that will allow us to carry out certain Čech covering arguments.

*Strategy of proof.* The goal is to show that the Čech complex

$$\mathcal{O}_X^+(X) \to \prod_i \mathcal{O}_X^+(U_i) \rightrightarrows \prod_{i<j} \mathcal{O}_X^+(U_i \cap U_j) \Rrightarrow \dots$$

is almost acyclic for any cover $X = \bigcup_i U_i$ by rational subsets. To do this, we proceed in 3 steps:

(i) Handle the case of perfectoid affinoid algebras in characteristic $p$ that are obtained by perfectifying classical Tate algebras (this will boil down to the classical Tate acyclicity theorem, as in [BGR84, §8.2]).
(ii) Handle all perfectoid affinoid algebras in characteristic $p$ via "noetherian approximation" (of course, the algebras involved are not noetherian, but this refers to the name of the technique).
(iii) The general case is obtained by tilting to characteristic $p$.

To prove (i), we introduce the following terminology (which is not standard).

**Definition 23.2.** Set $L \coloneqq \mathbf{F}_p[[t]]^{\wedge}_{\mathrm{perf}}[1/t]$, which is a perfectoid field of characteristic $p$.[32]

(1) An $\mathbf{F}_p[t]$-algebra $A^+$ is *algebraically admissible* if it is finitely-presented, reduced, $t$-torsion free, and integrally closed in $A^+[1/t]$.
(2) A perfectoid affinoid $L$-algebra $(R, R^+)$ is *$p$-finite* if there is an algebraically admissible algebra $A^+$ such that $(R, R^+)$ is the completed perfection of $(A^+[1/t], A^+)$.

**Remark 23.3.**

(1) Given an algebraically admissible algebra $A^+$, the $t$-adic completion $\widehat{A^+}$ is admissible in the sense of Raynaud [Ray74]. In particular, the algebraically admissible algebras attempt to isolate the class of finite type objects whose completions give the classical objects of rigid geometry.
(2) The definition of $p$-finiteness in this class is stronger than [Sch12, Definition 6.9] (there, Scholze does not require the algebraicity condition and demands that the algebras be $t$-adically complete), however Definition 23.2(2) will suffice for our purposes.

**Example 23.4.** Some examples of algebraically admissible $\mathbf{F}_p[t]$-algebras $A^+$ are as follows:

(1) $A^+ = \mathbf{F}_p[t]$.
(2) $A^+ = \mathbf{F}_p[t, x_1, \dots, x_n]$.
(3) $A^+ = \frac{\mathbf{F}_p[t,x,y]}{(y^2 - x^3)}$.

In each example, there is an associated completed perfection (which is $p$-finite by definition), but they are not easy to write down explicitly, e.g. if $A^+ = \mathbf{F}_p[t]$, then $\widehat{A^+}_{\mathrm{perf}} = \mathbf{F}_p[t^{1/p^\infty}]^\wedge$.

The Tate acyclicity theorem, specialized to the case of algebraically admissible rings, is as follows:

**Proposition 23.5.** *Let $A^+$ be an algebraically admissible ring. Set $A = A^+[1/t]$, so the pair $(A, A^+)$ is a uniform affinoid Tate ring. Set $X = \mathrm{Spa}(A, A^+)$.*

*(1) For any rational $U \subseteq X$, the pair $\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big)$ is also uniform (i.e. $A$ is stably uniform). Moreover, $\mathcal{O}_X^+(U)$ is the $t$-adic completion of an algebraically admissible ring, and $\mathcal{O}_X^+(U) = \mathcal{O}_X(U)^\circ$.*
*(2) For any covering $X = \bigcup_i U_i$ by rational subsets, the Čech complex*

$$M(X, \{U_i\}) \coloneqq \left( \mathcal{O}_X^+(X) \to \prod_i \mathcal{O}_X^+(U_i) \rightrightarrows \prod_{i<j} \mathcal{O}_X^+(U_i \cap U_j) \Rrightarrow \dots \right).$$

[32] Any perfectoid field of characteristic $p$ contains a copy of $L$ but not uniquely so—one needs to first specify a pseudouniformizer. In fact, there is no initial object in the category of perfectoid algebras of characteristic $p$, but $L$ is a *weakly initial object* of the category, in the sense that it maps to every other object of the category, but not uniquely so.

*has homology killed by $t^N$ for some $N \gg 0$.*[33]

(3) *The pair $(A, A^+)$ is sheafy, and the higher cohomology $H^i(X, \mathcal{O}_X^+)$ is $t^\infty$-torsion for $i > 0$. In particular, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

*Proof.* For (1), write $U = X\left(\frac{f_1,\ldots,f_n}{g}\right)$ for $f_1, \ldots, f_n, g \in A^+$ and $f_n = t^N$. Then, $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$ is the completion of $(B, B^+)$, where $B = A[1/g]$. A ring of definition of $B$ is $B_0 = A^+[f_i/g] \subseteq A[1/g] = B$, so $B = B_0[1/t]$ is a Tate ring, and $B^+$ is the integral closure of $B_0$ in $B$.

**Claim 23.6.** $(B, B^+)$ is uniform; equivalently $B^+/B_0$ is killed by $t^m$ for some $m > 0$.[34]

*Proof of Claim 23.6.* As $B_0$ is a finitely-presented and reduced $\mathbf{F}_p$-algebra (indeed, $B_0$ is finitely-generated over $\mathbf{F}_p$, so it is finitely-presented by the Hilbert basis theorem), it follows that $B^+$ is finite over $B_0$; however,

$$(B^+/B_0)[1/t] = B/B = 0.$$

Thus, since it is finitely-generated, $B^+/B_0$ is killed by $t^m$ for some $m \gg 0$ (pick the exponent $m$ that works for each of the generators). □

Therefore, $(B, B^+)$ is uniform, and so too is its completion $\left(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\right)$. This description also gives that $\mathcal{O}_X^+(U) = (B^+)^\wedge$; in particular, $\mathcal{O}_X^+(U)$ is the completion of an algebraically admissible ring. This already implies that $X$ is stably uniform (at this point, we can invoke [BV14] to get sheafiness; instead, we proceed with Scholze's argument as in [Sch12, Proposition 6.10], which will give us finer information).

For the assertion that $\mathcal{O}_X^+(U) = \mathcal{O}_X(U)^\circ$, observe that $\mathcal{O}_X^+(U)$ is noetherian (by construction, it is the completion of a noetherian ring) and $\mathcal{O}_X(U)^\circ$ is always (for any affinoid Tate ring) the total integral closure of $\mathcal{O}_X^+(U)$ in $\mathcal{O}_X(U) = \mathcal{O}_X^+(U)[1/t]$, more or less by definition. For noetherian rings, integral closure coincides with total integral closure, so it follows that $\mathcal{O}_X^+(U) = \mathcal{O}_X(U)^\circ$. In particular, this shows that there is a unique ring of integral elements in the setting of Proposition 23.5.

For (2), we want to show that the complex $M = M(X, \{U_i\})$ has homology killed by $t^N$ for $N \gg 0$. Tate showed that the complex $M[1/t]$ is acyclic (see e.g. [BGR84, §8.2]). Now, the Banach open mapping theorem (see [BGR84, §2.8]) implies that the homology $H^i(M)$ is killed by $t^N$ for some $N \gg 0$.

**Remark 23.7.** The Banach open mapping implies the following result in commutative algebra:

**Theorem 23.8.** *Let $R$ be a commutative ring, $f \in R$ a nonzero-divisor, and $K^\bullet$ a complex of $f$-adically complete $R$-modules. If $H^i(K^\bullet)[1/f] = 0$, then $f^N \cdot H^i(K^\bullet) = 0$ for $N \gg 0$.*

In fact, Bhargav does not know how to prove Theorem 23.8 by other means. One problem that one runs into is that any homology group of the complex $K^\circ$ is an $R$-module that need not be $f$-adically complete (really, it is the lack of $f$-adically separatedness that is the problem).

For (3), $(A, A^+)$ is sheafy, precisely because the complex $M(X, \{U_i\})[1/t]$ being acyclic formally implies the sheaf axiom for the cover $\{U_i\}$. For $i > 0$, we have

$$H^i(X, \mathcal{O}_X^+) = \varinjlim_{\{U_i\}} H^i(M(X, \{U_i\})).$$

This is isomorphic to $H^i(\varinjlim_{\{U_i\}} M(X, \{U_i\}))$, as taking homology commutes with colimits. Thus, every element of $H^i(X, \mathcal{O}_X^+)$ is killed by some finite power of $t$ by (2), so the colimit is $t^\infty$-torsion. Finally, to see that

[33] The exponent $N$ can taken to be uniform for each homology group. Indeed, if $\{U_i\}$ is not a finite cover, then it can be replaced by a finite subcover (because $X$ is quasi-compact) and the associated Čech complexes are homotopy equivalent. Assume now that $\{U_i\}$ is a finite cover, then we can take $N$ to be the maximum of the exponents that work for each of the finitely-many nonzero homology groups.

[34] The pair $(B, B^+)$ is uniform if $B^\circ$ is bounded. As $B^\circ$ and $B^+$ always differ by a bounded amount and $B_0$ is always bounded, it suffices to show that $B^+$ and $B_0$ differ by a bounded amount.

$H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, use that $H^i(X, -)$ commutes with filtered colimits (because $X$ is spectral), so $H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{O}_X^+)[1/t] = 0$ by (2). □

**Example 23.9.** The exponent $N$ appearing in Proposition 23.5(2) cannot be chosen independently of the cover $\{U_i\}$. To see this, it suffices to find a pair $(A, A^+)$ such that $H^i(X, \mathcal{O}_X^+)$ has unbounded $t$-power torsion. Let $A^+ = \frac{\mathbf{F}_p[t,z,y]}{(y^2 - z^3)}$, $A = A^+[1/t]$, and $X = \mathrm{Spa}(A, A^+)$. There is a short exact sequence

$$0 \to \mathcal{O}_X^+ \xrightarrow{t^n} \mathcal{O}_X^+ \to \mathcal{O}_X^+/t^n \to 0.$$

By chasing long exact sequences, it suffices to find functions $f_n \in H^0(X, \mathcal{O}_X^+/t^n)$ that are compatible in $n$, and do not come from $H^0(X, \mathcal{O}_X^+)$.

Let $0 \in X$ be the point defined by $z = y = 0$ (this corresponds to the map $(A, A^+) \to (\mathbf{F}_p((t)), \mathbf{F}_p[[t]])$ defined by $z, y \mapsto 0$). On $X \backslash \{0\}$, we have a well-defined function $y/z \in H^0(X\backslash\{0\}, \mathcal{O}_X)$. Note that $(y/z)^2 = z$, so $y/z \in H^0(X\backslash\{0\}, \mathcal{O}_X^+)$, because $z$ is. Let $U_n = \{x \in X : |z(x)| \leq |t|^{2n}\}$, where $n = 2m$.

**Claim 23.10.** The function $y/z$ maps to zero in $H^0(U_n\backslash\{0\}, \mathcal{O}_X^+/t^n)$.

Granted Claim 23.10, we can glue $y/z \in H^0(X\backslash\{0\}, \mathcal{O}_X^+/t^n)$ to $0 \in H^0(U_n, \mathcal{O}_X^+/t^n)$ to get a well-defined function $f_n \in H^0(X, \mathcal{O}_X^+/t^n)$. It does not come from a global function, because $y/z$ does not.

*Proof of Claim 23.10.* Both $y$ and $z$ are nonzero on $U_n\backslash\{0\}$, so it suffices to show that $|y(x)| \leq |z(x)| \cdot |t^n|$ for all $x \in U_n\backslash\{0\}$. On $U_n$, we have $|z(x)| \leq |t|^{2n}$, so $|z(x)|^{1/2} \leq |t|^n$ and multiplying by $|z(x)|$ gives the inequality

$$|y(x)| = |z(x)| \cdot |z(x)|^{1/2} \leq |z(x)| \cdot |t^n|,$$

which is the equality we want. □

## 24. April 12th

Let $K$ be a perfectoid field with tilt $K^\flat$, let $t \in K^\flat$ be a pseudouniformizer such that $\pi = t^\sharp \in K$ satisfies $|\pi| \geq |p|$. Let $L = \mathbf{F}_p[[t]]^\wedge_{\mathrm{perf}}[1/t]$, a perfectoid field of characteristic $p$, and then the choice of pseudouniformizer $t$ gives an embedding $L \hookrightarrow K^\flat$.

24.1. **Tate acyclicity (continued).** Last time, we proved the Tate acyclicity for pairs $(A, A^+)$, where $A^+$ is an algebraically admissible $\mathbf{F}[t]$-algebras, and $A = A^+[1/t]$.

**Construction**. If $(A, A^+)$ is a uniform affinoid Tate ring over $\mathbf{F}_p$ (in particular, $A$ is a ring of characteristic $p$), then $(A_{\mathrm{perf}}, A^+_{\mathrm{perf}})$ is also a uniform affinoid Tate ring over $\mathbf{F}_p$ (the topology on $A_{\mathrm{perf}}$ is specified by uniformity: $A^+_{\mathrm{perf}}$ is a ring of definition and the element that is inverted to pass to $A_{\mathrm{perf}}$ is a pseudouniformizer). The map

$$(A, A^+) \to (A_{\mathrm{perf}}, A^+_{\mathrm{perf}})$$

is the universal map from $(A, A^+)$ to a uniform pair $(B, B^+)$ such that $B$ perfect.

**Exercise 24.1.** The pullback map $\mathrm{Spa}(A_{\mathrm{perf}}, A^+_{\mathrm{perf}}) \xrightarrow{\simeq} \mathrm{Spa}(A, A^+)$ on adic spectra is a homeomorphism, preserving rational subsets. (This is analogous to the fact that the prime spectra of $A$ and $A_{\mathrm{perf}}$ coincide.)

Note that the terms 'uniform affinoid Tate ring' and 'uniform affinoid pair' are used interchangeably.

**Definition 24.2.** For a uniform affinoid pair $(A, A^+)$ over $\mathbf{F}_p$,

(1) the *completed perfection* is $\left(\widehat{A_{\mathrm{perf}}}, \widehat{A^+_{\mathrm{perf}}}\right)$; that is, the map $(A, A^+) \to \left(\widehat{A_{\mathrm{perf}}}, \widehat{A^+_{\mathrm{perf}}}\right)$ is universal among maps from $(A, A^+)$ to complete perfect uniform affinoid pairs.
(2) a perfectoid $L$-algebra is *$p$-finite* if it is the completed perfection of a uniform affinoid $(A, A^+)$, where $A^+$ is algebraically admissible over $\mathbf{F}_p[t]$.

The uniform affinoid pair $(A, A^+)$ appearing in Definition 24.2(2) is not uniquely determined by the associated $p$-finite perfectoid $L$-algebra, because there are different rings that have the same perfection.

**Corollary 24.3.** *Let $(R, R^+)$ be a $p$-finite perfectoid $L$-algebra; say $(R, R^+) = (\widehat{A_{\text{perf}}}, \widehat{A^+_{\text{perf}}})$, where $A^+$ is algebraically admissible over $\mathbf{F}_p[t]$.*

*(1) The pullback map $X = \operatorname{Spa}(R, R^+) \to \operatorname{Spa}(A, A^+) = Y$ is a homeomorphism, preserving rational subsets.*
*(2) For any rational subset $V \subseteq Y$ with preimage $U \subseteq X$, the completed perfection of $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V))$ is $(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$.*
*(3) For any covering $X = \bigcup_i U_i$ by rational subsets, the Čech complex*

$$M(X, \{U_i\}) := \left( \mathcal{O}^+_X(X) \to \prod_i \mathcal{O}^+_X(U_i) \rightrightarrows \prod_{i<j} \mathcal{O}^+_X(U_i \cap U_j) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \dots \right)$$

*is almost acyclic.*
*(4) The pair $(R, R^+)$ is sheafy, and $H^i(X, \mathcal{O}^+_X) \overset{a}{=} 0$ and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Corollary 24.3 completes the proof of the Tate acyclicity theorem for $p$-finite perfectoid $L$-algebras.

*Proof.* (1) is clear, because neither the completion nor the perfection change the adic spectrum (see Proposition 16.1(1) and Exercise 24.1).

For (2), if $(S, S^+)$ is a perfectoid affinoid $L$-algebra, then a map $(R, R^+) \to (S, S^+)$ factors over $(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ iff $\operatorname{Spa}(S, S^+) \to X$ has image inside $U$, by the universal property of Huber's presheaf. By (1), this is equivalent to the assertion that $\operatorname{Spa}(S, S^+) \to X \to Y$ has image in $V$. Now unravel this same logic: this is equivalent to saying that $(A, A^+) \to (S, S^+)$ factors over $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V))$, again using the universal property of Huber's presheaf. As $(S, S^+)$ is complete, this is the same as giving a map $(R, R^+) \to (S, S^+)$ that factors over $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V))$. Finally, as $(S, S^+)$ is a complete and perfect affinoid Tate ring, this occurs iff $(R, R^+) \to (S, S^+)$ factors over the completed perfection of $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V))$. Therefore, $(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ satisfies the universal property of the completed perfection of $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V))$.

For (3), suppose the cover $\{U_i\}$ of $X$ comes from an open cover $\{V_i\}$ of $Y$ via pullback, then the corresponding Čech complex $M(Y, \{V_i\})$ has homology killed by $t^N$ for some $N \gg 0$. We can now apply the following two lemmas:

**Lemma 24.4.** *If a complex $M$ of $t$-torsion free $L^\circ$-modules is almost acyclic, so is its completion $\widehat{M}$.*

The proof of Lemma 24.4 is left as an exercise (briefly, an almost acyclic complex maps to an almost acyclic complex under an $L^\circ$-linear functor; in particular, this holds for the completion functor).

**Lemma 24.5.** *Let $A$ be a commutative (perfect[35]) ring of characteristic $p$. Fix $t \in A$ and an $A$-module $M$ equipped with a map $\beta_M \colon M \to F_*M$ to its Frobenius pushforward[36]. Assume that $M$ is killed by $t^N$ for some $N > 0$. Then,*

$$\operatorname{colim}\left( M \overset{\beta_M}{\to} F_*M \overset{F_*(\beta_M)}{\to} F^2_*M \to \dots \right)$$

*is killed by $t^{1/p^n}$ for all $n \geq 0$. In particular, the colimit is almost zero if one performs almost mathematics with respect to the element $t$.*

*Proof.* As $M$ is killed by $t^N$, $F^e_*M$ is killed by $t^{N/p^e}$; thus, the colimit is killed by $t^{N/p^e}$ for all $e \geq 0$. ☐

[35] In fact, the colimit appearing in Lemma 24.5 is naturally a module over $A_{\text{perf}}$, so it makes sense to speak of the compatible system $\{t^{1/p^n}\}$ of $p$-power roots of $t$, without having to assume that $A$ is perfect.

[36] The map $\beta_M$ satisfies the "$p$-linearity" condition: $\beta_M(am) = a^p\beta_M(m)$ for $a \in A$ and $m \in M$).

To complete the proof of (3), observe that $M(X, \{U_i\})$ is the completed perfection of $M(Y, \{V_i\})$, and so it can be written as

$$M(X, \{U_i\}) = \operatorname{colim}\Big(M(Y, \{V_i\}) \stackrel{\mathrm{Frob}_*}{\longrightarrow} F_* M(Y, \{V_i\}) \stackrel{\mathrm{Frob}_*}{\longrightarrow} F_*^2 M(Y, \{V_i\}) \longrightarrow \dots\Big).$$

As filtered colimits are exact and $F_*$ is exact, it follows that $H^i(M(X, \{U_i\}))$ is the completion of

$$\operatorname{colim}\Big(H^i(M(Y, \{V_i\})) \stackrel{\mathrm{Frob}_*}{\to} F_* H^i(M(Y, \{V_i\})) \stackrel{\mathrm{Frob}_*}{\to} F_*^2 H^i(M(Y, \{V_i\})) \to \dots\Big).$$

The Tate acyclicity theorem on $Y$ implies that $t^N$ kills $H^i(M(Y, \{V_i\}))$, and so Lemma 24.5 implies that the above colimit is almost zero. Then, use Lemma 24.4.

The assertion (4) formally follows from (3), just as in Proposition 23.5. □

24.2. **"Noetherian approximation" for perfectoid algebras.**

**Lemma 24.6.** *Assume $K$ is a perfectoid field of characteristic $p$, and let $t \in K$ be a pseudouniformizer; using this choice of $t$, we can view $K$ as an extension of $L$. Fix a $t$-adically complete, $t$-torsion free, perfect $K^\circ$-algebra $A$ such that $A$ is integrally closed in $A[1/t]$ (that is, $A[1/t]$ is a perfectoid algebra, and $A$ is a ring of integral elements).*

*(1) We can write*

$$A = (\operatorname{colim}_{i \in I} B_i)^\wedge,$$

*where $I$ is filtered and $B_i$ is $p$-finite (that is, each $B_i$ is the completed perfection of an algebraically admissible $\mathbf{F}_p[t]$-algebra).*

*(2) The compatible system of maps $(B_i[1/t], B_i) \to (A[1/t], A)$ of uniform affinoid Tate rings induces a homeomorphism*

$$X = \operatorname{Spa}(A[1/t], A) \xrightarrow{\simeq} \varprojlim_{i \in I} X_i,$$

*where $X_i = \operatorname{Spa}(B_i[1/t], B_i)$. Moreover, every rational subset $U \subseteq X$ is the preimage of some rational subset $U_i \subseteq X_i$ for some $i \in I$.*

*(3) Fix a rational subset $U_i \subseteq X_i$ with preimages $U_j \subseteq X_j$ for $j \geq i$, and $U \subseteq X$. Then,*

$$\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big) = \operatorname{colim}_{j \geq i}\Big(\mathcal{O}_{X_j}(U_j), \mathcal{O}_{X_j}^+(U_j)\Big)$$

*as perfectoid affinoid $L$-algebras.*

This is a direct analogue of the corresponding statement for schemes: any commutative ring is the colimit of its finitely-generated subrings, which induces a homeomorphism on prime spectra, and one has a similar description of functions on Zariski open subsets. In fact, Lemma 24.6 fits into a general framework on noetherian approximation techniques.

*Proof.* For (1), we always have an isomorphism $A \simeq \operatorname{colim}_i A_i$, where $A_i \subseteq A$ runs through all finitely-presented $\mathbf{F}_p[t]$-subalgebras of $A$. By passing to a cofinal subsystem (or "corefining", if you like), we may assume that each $A_i$ is algebraically admissible (indeed, any finitely-generated subalgebra of $A$ is reduced and $t$-torsion free because $A$ is so, and one obtains admissibility by replacing $A_i$ with its integral closure in $A[1/t]$, which is again a finitely-generated subalgebra of $A$). Pass to perfections on both sides to get that $A \simeq A_{\mathrm{perf}} \simeq \operatorname{colim}_i (A_i)_{\mathrm{perf}}$, where $A \simeq A_{\mathrm{perf}}$ because $A$ is perfect. Passing to $t$-adic completions, one gets that $A \simeq \widehat{A} \simeq \Big(\operatorname{colim}_i \widehat{(A_i)_{\mathrm{perf}}}\Big)^\wedge$, and set $B_i = \widehat{(A_i)_{\mathrm{perf}}}$ to get (1). Each $B_i$ is $p$-finite, by construction.

For (2), (1) yields an isomorphism

$$\operatorname{colim}_i (B_i[1/t], B_i) \simeq (A[1/t], A)$$

as perfectoid affinoid $L$-algebras (this is even true as uniform complete Tate rings). Now, it is a general fact that the adic spectrum of a colimit of uniform Tate rings is the inverse limit of the adic spectra of the constituents of the system; moreover, as the adic spectrum is a spectral space, every quasi-compact open is pulled back from a quasi-compact at some level of the inverse limit.

For (3), when $j \geq i$, recall that there are maps $X \to X_j \to X_i$. The universal property of Huber's presheaf gives pushout squares

$$\begin{array}{ccc} (B_i[1/t], B_i) & \longrightarrow & (B_j[1/t], B_j) \\ \downarrow & & \downarrow \\ \big(\mathcal{O}_{X_i}(U_i), \mathcal{O}^+_{X_i}(U_i)\big) & \longrightarrow & \Big(\mathcal{O}_{X_j}(U_j), \mathcal{O}^+_{X_j}(U_j)\Big) \end{array}$$

in the category of (uniform) complete affinoid Tate rings; similarly, there are pushout squares

$$\begin{array}{ccc} (B_i[1/t], B_i) & \longrightarrow & (A[1/t], A) \\ \downarrow & & \downarrow \\ \big(\mathcal{O}_{X_i}(U_i), \mathcal{O}^+_{X_i}(U_i)\big) & \longrightarrow & \big(\mathcal{O}_X(U), \mathcal{O}^+_X(U)\big) \end{array}$$

Using that $(A[1/t], A) = \mathrm{colim}_j(B_j[1/t], B_j)$, chase diagrams to get conclusion. $\square$

The general Tate acyclicity theorem now follows as a corollary of Corollary 24.3 and Lemma 24.6. Below, we no longer assume that $K$ has characteristic $p$.

**Corollary 24.7.** [Tate acyclicity] *Let $X = \mathrm{Spa}(R, R^+)$ for any $(R, R^+)$ perfectoid affinoid $K$-algebra.*

*(1) If $X = \bigcup_i U_i$ is a rational covering, then the Čech complex $M(X, \{U_i\})$ is almost acyclic.*
*(2) The pair $(R, R^+)$ is sheafy, and $H^i(X, \mathcal{O}^+_X) \overset{a}{=} 0$ for $i > 0$.*

*Proof.* It is enough to prove (1), because (1) formally implies (2), as before. For (1), one first handles the case when $K$ has characteristic $p$, by using Lemma 24.6. In general, it is enough to show that $M(X, \{U_i\})/\pi$ is almost acyclic (indeed, if a $\pi$-adically complete complex is almost acyclic $\mod \pi$, then it is almost acyclic $\mod \pi^n$ for all $n \geq 1$, and by completeness, it follows that the complex is almost acyclic). However, $M(X, \{U_i\})/\pi \simeq M(X^\flat, \{U_i^\flat\})/t$, and the latter is almost acyclic because of the characteristic $p$ case. $\square$

## 25. April 17th

The notation is as usual: let $K$ be a perfectoid field with tilt $K^\flat$, let $t \in K^\flat$ be a pseudouniformizer such that $\pi = t^\sharp \in K$ satisfies $|\pi| \geq |p|$.

25.1. **Tate acyclicity (continued).** Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra, $X = \mathrm{Spa}(R, R^+)$ its adic spectrum, and $\{U_i\}$ is a cover of $X$ by rational subsets. We are interested in proving the Tate acyclicity theorem for this cover; that is,

**Theorem 25.1.** [Tate acyclicity] *The Čech complex*

$$M(X, \{U_i\}) = \left( \mathcal{O}^+_X(X) \to \prod_i \mathcal{O}^+_X(U_i) \to \prod_{i,j} \mathcal{O}^+_X(U_i \cap U_j) \to \dots \right)$$

*is almost acyclic.*

Theorem 25.1 immediately implies that $(R, R^+)$ is sheafy, by inverting the pseudouniformizer $\pi$ to get that the Čech complex associated to the sheaf $\mathcal{O}_X$ and the cover $\{U_i\}$ is acyclic.

*Proof.* Last time, we proved the theorem in the case where $(R, R^+)$ has characteristic $p$: the strategy was to reduce to the $p$-finite case by "noetherian approximation" (as in Lemma 24.6), which in turn reduced to the classical Tate acyclicity theorem.

In general, $X$ tilts to $X^\flat := \mathrm{Spa}(R^\flat, R^{\flat +})$. Using the tilting correspondence for rational subsets and the structure presheaf, it follows that $M(X, \{U_i\})/\pi \simeq M(X^\flat, \{U_i^\flat\})/t$ at the level of complexes (that is, the complexes are isomorphic in each degree). The complex $M(X^\flat, \{U_i^\flat\})$ is almost acyclic and $t$-torsion free, hence $M(X^\flat, \{U_i^\flat\})/t$ is almost acyclic: indeed, use the long exact sequence in cohomology associated to the short exact sequence of complexes

$$0 \to M^\flat \xrightarrow{t} M^\flat \to M^\flat/t \to 0.$$

Thus, $M(X, \{U_i\})/\pi$ is almost acyclic. By performing induction on the $\pi$-adic filtration, one concludes that $M(X, \{U_i\})/\pi^n$ is almost acyclic for $n \geq 1$ (use that this complex has a filtration whose graded pieces are of the form $M(X, \{U_i\})/\pi$, which are almost acyclic by assumption). Passing to the limit, we get that $M(X, \{U_i\}) = \varprojlim_n M(X, \{U_i\})/\pi^n$ is almost acyclic. Here, we are using the following homological algebra fact: if $\{M_n\}$ is a projective system of complexes with surjective transition maps (indexed by the natural numbers), then there is the Milnor short exact sequence[37]

$$0 \longrightarrow \varprojlim_n{}^1 H^{i-1}(M_n) \longrightarrow H^i\left(\varprojlim_n M_n\right) \longrightarrow \varprojlim_n H^i(M_n) \longrightarrow 0.$$

In our setting, both "outer" objects of the short exact sequence are almost acylic, hence the middle object is also. $\square$

25.2. **Perfectoid spaces.** The upshot of the Tate acyclicity theorem is that it shows that perfectoid affinoid $K$-algebras are sheafy! Thus, we can finally define a perfectoid space.

**Definition 25.2.** If $(R, R^+)$ is a perfectoid affinoid $K$-algebra, then the associated adic space $\mathrm{Spa}(R, R^+)$ is called an *affinoid perfectoid space over* $K$ (this is an object of Huber's category $\mathcal{V}$, as in Definition 19.9).

A *perfectoid space over* $K$ is an adic space $X$ over $\mathrm{Spa}(K, K^\circ)$ that is locally isomorphic to an affinoid perfectoid space over $K$.

**Example 25.3.** If $K$ has characteristic $p$ and $X/\mathrm{Spf}(K^\circ)$ is a $\pi$-adic formal scheme that is flat over $K^\circ$ (e.g. the $\pi$-adic formal completion of a scheme over $K^\circ$), then there exists a "completed perfection" $X_\eta^{\mathrm{perf}}$ of $X$ as a perfectoid space over $K$. If $X = \mathrm{Spf}(R)$, then $X_\eta^{\mathrm{perf}} = \mathrm{Spa}\left(A[1/\pi], A\right)$, where $A = \widehat{R_{\mathrm{perf}}}$. In general, one constructs $X_\eta^{\mathrm{perf}}$ by gluing (in this theory, all gluing takes place along open subsets, so this is not so hard).

**Remark 25.4.** If $X = \widehat{\mathcal{X}}$ for a scheme $\mathcal{X}/K^\circ$, then there exists a natural map

$$\left(X_\eta^{\mathrm{perf}}, \mathcal{O}_{X_\eta^{\mathrm{perf}}}\right) \to (\mathcal{X}_K, \mathcal{O}_{\mathcal{X}_K})$$

of locally ringed spaces, i.e. the "perfectified generic fiber maps to the scheme-theoretic generic fiber", whence the notation.

The tilting correspondence globalizes as follows:

[37]Recall that $\varprojlim_n^1$ denotes the first (and only) derived functor of $\varprojlim_n$, when working with abelian groups or modules indexed by the natural numbers. If $\{A_n\}$ is a projective system of abelian groups indexed by the natural numbers, with transition maps $f_n$, then one can compute $\varprojlim_n{}^1 A_n$ as

$$\varprojlim_n{}^1 A_n = \mathrm{coker}\left(\prod_n A_n \xrightarrow{f_n - \mathrm{id}} \prod_n A_n\right).$$

This simple description is only valid for inverse limits indexed over the natural numbers; for an arbitrary directed set, the higher direct functors of $\varprojlim$ can be "arbitrarily complicated", e.g. any global sections functor can always be realized as the inverse limit over some diagram.

**Theorem 25.5.** *Fix a perfectoid field $K$.*

*(1) For any perfectoid space $X/K$, there exists a unique (up to unique isomorphism) perfectoid space $X^\flat/K^\flat$ such that the functors of points*[38] *satisfy*
$$X(R,R^+) \simeq X^\flat(R^\flat, R^{\flat +}).$$
*functorially in the perfectoid affinoid $K$-algebra $(R,R^+)$ (it follows that it is also functorial in $X$).*

*(2) Tilting $X \mapsto X^\flat$ gives an equivalence of categories*
$$\{\textit{perfectoid spaces over } K\} \xrightarrow{\simeq} \{\textit{perfectoid spaces over } K^\flat\}.$$

*(3) If $X = \mathrm{Spa}(R,R^+)$ is affinoid perfectoid over $K$, then $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$ is affinoid perfectoid over $K^\flat$.*

**Warning 25.6.** If $X$ is a perfectoid space over $K$ that is affinoid as an adic space, we do not know that $X$ is affinoid perfectoid; that is, if $X$ is a perfectoid space (so it admits an open cover by affinoid perfectoid spaces) and moreover $X$ is the adic spectrum of some pair $(A,A^+)$, then we do not a priori know that $A$ is perfectoid.

**Corollary 25.7.** *The category of perfectoid spaces over $K$ admits fibre products.*

The corresponding fact for adic spaces is not known to be true, because of issues of sheafiness.

*Proof.* We may assume $K$ has characteristic $p$ (the statement is categorical, hence we may replace $K$ with $K^\flat$ by Theorem 25.5(2)). As in the case of schemes, we may assume that everything in sight is affinoid perfectoid. Then, consider a diagram of perfectoid affinoid $K$-algebras
$$\begin{tikzcd} (A,A^+) \arrow[r] \arrow[d] & (C,C^+) \\ (B,B^+) & \end{tikzcd}$$

It suffices to construct the pushout in the category of perfectoid affinoid $K$-algebras (in particular, this is the pushout in the category of uniform affinoid Tate $K$-algebras). Ignoring completeness, the first guess for the pushout is $D_0 = B \otimes_A C$, and $D_0^+$ is the integral closure of the image of
$$B^+ \otimes_{A^+} C^+ \longrightarrow D_0.$$
Then, $(D_0, D_0^+)$ is a uniform affinoid Tate $K$-algebra. Set $(D,D^+)$ to be the completion of $(D_0,D_0^+)$. General nonsense about universal properties implies that $(D,D^+)$ is the pushout of the diagram in the category of complete uniform affinoid Tate $K$-algebras.

**Claim 25.8.** The pair $(D,D^+)$ is perfectoid affinoid.

Note that the perfectoid-ness is really needed: even if all of $(A,A^+)$, $(B,B^+)$, and $(C,C^+)$ are stably uniform, it is not clear that $(D,D^+)$ is sheafy.

One can check that $D_0^+$ is $\pi$-torsion free (this is clear) and perfect (this comes from the fact that all of $A,B,C$ are perfect), hence $D^+$ is $\pi$-adically complete and $\pi$-torsion free and perfect. This grants the claim. Therefore, $(D,D^+)$ is sheafy, and hence there is a diagram of affinoid perfectoid spaces
$$\begin{tikzcd} \mathrm{Spa}(D,D^+) \arrow[r] \arrow[d] & \mathrm{Spa}(B,B^+) \arrow[d] \\ \mathrm{Spa}(C,C^+) \arrow[r] & \mathrm{Spa}(A,A^+) \end{tikzcd}$$

Finally, one must verify that this gives a fiber square; this is left as an exercise. □

[38] Recall that $X(R,R^+) \coloneqq \mathrm{Hom}_{\mathrm{Adic}/K}\left(\mathrm{Spa}(R,R^+), X\right)$.

**Exercise 25.9.** Given a pushout diagram

$$\begin{tikzcd} (A, A^+) \arrow[r] \arrow[d] & (B, B^+) \arrow[d] \\ (C, C^+) \arrow[r] & (D, D^+) \end{tikzcd}$$

of perfectoid affinoid $K$-algebras (in any characteristic), the map $B^+ \otimes_{A^+} C^+ \to D^+$ is an almost isomorphism, after $\pi$-adic completion; thus,

$$D^+ \overset{a}{\simeq} \left(B^+ \otimes_{A^+} C^+\right)^\wedge .$$

For classical Tate algebras, a statement such as Exercise 25.9 has no hope of being true.

25.3. **The étale topology.** The goal is to prove the following perfectoid version of Faltings' almost purity theorem from $p$-adic Hodge theory (see [Fal02]):

**Theorem 25.10.** *Fix a perfectoid affinoid $K$-algebra $(R, R^+)$.*

*(1)* [Almost purity theorem] *If $S \in R_{\mathrm{f\acute{e}t}}$, then the integral closure $S^+$ of $R^+$ in $S$ lies in $R^{+a}_{a\,\mathrm{f\acute{e}t}}$.*
*(2) Inverting $\pi$ gives an equivalence of categories $R^{+a}_{a\,\mathrm{f\acute{e}t}} \xrightarrow{\simeq} R_{\mathrm{f\acute{e}t}}$.*

**Remark 25.11.** This theorem is highly nontrivial, even in the simplest examples, e.g. for $R^+ = K^\circ[T^{1/p^\infty}]^\wedge$ with $K$ a perfectoid field. (The case of $R^+ = K^\circ$ is also nontrivial, but it was proven in the 1970s.)

Recall that we have seen certain parts of Theorem 25.10:

(1) The inclusion $R^+ \to R^\circ$ is an almost isomorphism, hence $R^{+a}_{a\,\mathrm{f\acute{e}t}} \xrightarrow{\simeq} R^{\circ a}_{a\,\mathrm{f\acute{e}t}}$.
(2) There is a diagram

$$\begin{tikzcd} R_{\mathrm{f\acute{e}t}} & R^{\circ a}_{a\,\mathrm{f\acute{e}t}} \arrow[l] \arrow[r, "\simeq"] & (R^{\circ a}/\pi)_{a\,\mathrm{f\acute{e}t}} \arrow[d, "\simeq"] \\ R^\flat_{\mathrm{f\acute{e}t}} & R^{\flat\circ a}_{a\,\mathrm{f\acute{e}t}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \left(R^\flat/t\right)_{a\,\mathrm{f\acute{e}t}} \end{tikzcd}$$

The isomorphisms $R^{\circ a}_{a\,\mathrm{f\acute{e}t}} \xrightarrow{\simeq} (R^{\circ a}/\pi)_{a\,\mathrm{f\acute{e}t}}$ and $R^{\flat\circ a}_{a\,\mathrm{f\acute{e}t}} \xrightarrow{\simeq} \left(R^\flat/t\right)_{a\,\mathrm{f\acute{e}t}}$ came from deformation-theoretic arguments, and the isomorphism $R^{\flat\circ a}_{a\,\mathrm{f\acute{e}t}} \xrightarrow{\simeq} R^\flat_{\mathrm{f\acute{e}t}}$ was the almost purity theorem in characteristic $p$ (see Theorem 7.10). The hard part, which is the content of Theorem 25.10, is showing that $R_{\mathrm{f\acute{e}t}} \leftarrow R^{\circ a}_{a\,\mathrm{f\acute{e}t}}$ is essentially surjective (we have already seen that it is fully faithful).
(3) We also proved Theorem 25.10 when $R$ is a perfectoid field (as in the proof of Theorem 13.4).

The strategy of the proof of Theorem 25.10 is to reduce to (3) by localizing on $\mathrm{Spa}(R, R^+)$.

**Definition 25.12.**

(1) A map $(A, A^+) \to (B, B^+)$ of affinoid Tate rings is *finite étale* if $A \to B$ is finite étale, and $B^+$ is the integral closure of $A^+$ in $B$.
(2) A map $f\colon X \to Y$ of adic spaces is *finite étale* if there exists an open cover of $Y$ by affinoids $U \subseteq Y$ such that $V = f^{-1}(U) \subseteq X$ is affinoid, and $\left(\mathcal{O}_Y(U), \mathcal{O}_Y^+(U)\right) \to \left(\mathcal{O}_X(V), \mathcal{O}_X^+(V)\right)$ is finite étale.

When one makes the analogous definition for schemes, it is a theorem that the condition of Definition 25.12(2) holds for any open affine cover. The analogous fact will be true for noetherian adic spaces and for perfectoid spaces, but it is not known whether it holds for general adic spaces.

To prove the theorem, we make the following (eventually redundant) definition:

**Definition 25.13.** Let $K$ be a perfectoid field.

(1) A map $(A, A^+) \to (B, B^+)$ of perfectoid affinoid $K$-algebras is *strongly finite étale* if it is finite étale, and $B^+$ is almost finite étale over $A^+$.

(2) A map $f\colon X \to Y$ of perfectoid spaces over $K$ is *strongly finite étale* if there exists an open cover of $Y$ by affinoid perfectoids $U \subseteq Y$ such that $V = f^{-1}(U) \subseteq X$ is affinoid perfectoid, and
$$\big(\mathcal{O}_Y(U), \mathcal{O}_Y^+(U)\big) \to \big(\mathcal{O}_X(V), \mathcal{O}_X^+(V)\big)$$
is strongly finite étale.

Next time, we will check that strongly finite étale maps $X \to Y$ are finite étale, by localizing on $Y$.

## 26. April 19th

26.1. **The étale topology (continued).** Last time, we declared a map $(A, A^+) \to (B, B^+)$ of perfectoid affinoid $K$-algebras to be *strongly finite étale* if

(i) $A \to B$ is finite étale;
(ii) $B^+$ is the integral closure of $A^+$ in $B$;
(iii) $A^+ \to B^+$ is almost finite étale.

There was a similar global definition for maps of perfectoid spaces. We would like to (eventually) prove that condition (iii) is redundant, which has already been done in characteristic $p$ (it follows from the almost purity theorem).

By the almost purity theorem (or, the parts that we have already shown), if $Y = \mathrm{Spa}(A, A^+)$ is an affinoid perfectoid space, one can check that
$$\mathrm{sF\acute{e}t}(Y) \coloneqq \{\text{strongly finite étale maps to } Y\} \simeq \{\text{strongly finite étale maps to } Y^\flat\} \eqqcolon \mathrm{sF\acute{e}t}(Y^\flat),$$
where the maps occurring in $\mathrm{sF\acute{e}t}(Y)$ are maps of perfectoid spaces over $K$, and those occurring in $\mathrm{sF\acute{e}t}(Y^\flat)$ are maps of perfectoid spaces over $K^\flat$. To see this equivalence, use first the statement that $A^{+a}_{a\,\mathrm{f\acute{e}t}} \simeq A^{\flat+a}_{a\,\mathrm{f\acute{e}t}}$, and the fact that affinoid perfectoids are preserved under tilting (more precisely, affinoid perfectoid subspaces of $Y$ correspond to affinoid perfectoid subspaces of $Y^\flat$ under tilting — see [Sch12, Proposition 6.17]).

Recall that a finite morphism to an affine scheme is determined by the corresponding map on global sections (a similar statement holds in the rigid-analytic setting). We will show an analogous result for strongly finite étale maps to affinoid perfectoid spaces, but we first require some preparatory results.

**Theorem 26.1.** [GR03, Proposition 5.4.53] *Let $A$ be a flat $K^\circ$-algebra that is henselian along $(\pi)$ (in the sense of Definition 18.3). Then,*
$$A[1/\pi]_{\mathrm{f\acute{e}t}} \xrightarrow{\simeq} \widehat{A}[1/\pi]_{\mathrm{f\acute{e}t}}.$$
*where $\widehat{A}$ is the $\pi$-adic completion of $A$.*

In the finite-type case, Theorem 26.1 is due to Elkik: see [Elk73]. The theorem is hard, but we will go ahead and use it without proof.

**Remark 26.2.** This is also a "higher-dimensional" analogue of Krasner's lemma.

**Corollary 26.3.** *Let $\{(A_i, A_i^+)\}$ be a filtered system of complete uniform affinoid $K$-algebras, and let $(A, A^+)$ be the colimit (as complete rings[39]). Then,*
$$\mathrm{colim}_i\, (A_i)_{\mathrm{f\acute{e}t}} \xrightarrow{\simeq} A_{\mathrm{f\acute{e}t}}.$$

For the pedantic reader, the colimit appearing in Corollary 26.3 is really a 2-colimit, as this is a colimit of categories.

*Proof.* Set $A^+_{\mathrm{nc}} \coloneqq \mathrm{colim}_i\, A_i^+$, where the colimit is computed as abstract rings. Then, $A^+_{nc}$ is henselian along the ideal $(\pi)$, because each $A_i^+$ is complete (hence henselian) and the filtered colimit of henselian rings is again henselian. Thus, $\widehat{A^+}_{nc} \simeq A^+$. Again applying Theorem 26.1, we see that

[39] To compute $(A, A^+)$, take the colimit of the $A_i^+$'s as rings, and its $\pi$-adic completion is $A^+$. Then, $A = A^+[1/\pi]$.

$$\begin{tikzcd} (A^+_{nc}[1/\pi])_{\text{fét}} \arrow[r, "\simeq"] & (A^+[1/\pi])_{\text{fét}} \arrow[d, equal] \\ \operatorname{colim}_i (A_i)_{\text{fét}} \arrow[u, "\simeq"] & A_{\text{fét}} \end{tikzcd}$$

The left-hand isomorphism follows from the general fact that taking categories of finite étale algebras commutes with filtered colimits. ☐

**Proposition 26.4.** *If $Y$ is affinoid perfectoid and $f\colon X \to Y$ is strongly finite étale, then $X$ is affinoid perfectoid, and moreover*

$$\big(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)\big) \longrightarrow \big(\mathcal{O}_X(X), \mathcal{O}_X^+(X)\big)$$

*is also strongly finite étale.*

*Sketch.* The strategy is to again use noetherian approximation, as described in Lemma 24.6. As strongly finite étale maps correspond under tilting, we may assume that the base field $K$ has characteristic $p$. Therefore, we can write $Y = \operatorname{Spa}(A, A^+)$, where $(A, A^+)$ is a perfectoid affinoid algebra $K$-algebra.

Set $L = \mathbf{F}_p((t))^{\wedge}_{\text{perf}} \hookrightarrow K$, where the embedding is via the choice of pseudouniformizer $t$. We can effectively forget about $K$, and instead work over $L$. By Lemma 24.6(1), we can write

$$(A, A^+) = \operatorname{colim}(B_i, B_i^+),$$

where $(B_i, B_i^+)$ are $p$-finite perfectoid $L$-algebras. Using Corollary 26.3 and the noetherian approximation results for rational subsets (i.e. by parts (2) and (3) of Lemma 24.6), we may assume that $(A, A^+)$ is itself $p$-finite. Thus, $(A, A^+) = \operatorname{colim}_\phi(\widehat{R}, \widehat{R^+})$ (as complete uniform rings), where $R^+$ is algebraically admissible over $\mathbf{F}_p[t]$, and the colimit is taken over copies of the Frobenius map $\phi$. Using similar arguments as before plus the invariance of finite étale maps under Frobenius (more precisely, a purely inseparable ring map induces an equivalence on finite étale covers), it follows that any strongly finite étale map $Z \to \operatorname{Spa}(A, A^+)$ arises via base change from a finite étale map $Z_0 \to \operatorname{Spa}(\widehat{R}, \widehat{R^+})$ (here, the "strongness" is not needed because we are working in characteristic $p$). Now, we may invoke a classical theorem, as in [Hub96, Example 1.6.6(ii)], which says that if $S$ is a classical Tate algebra, then

$$\{\text{finite étale maps to } \operatorname{Spa}(S, S^+)\} \simeq S_{\text{fét}}.$$

☐

The upshot of Proposition 26.4 is that strongly finite étale maps to affinoid are easy to describe: they are just given by strongly finite étale maps on global sections.

**Corollary 26.5.** *Let $Y = \operatorname{Spa}(A, A^+)$ be an affinoid perfectoid space.*

*(1) There is an equivalence of categories*

$$\operatorname{sFét}(Y) := \{\textit{strongly finite étale maps to } Y\} \xrightarrow{\simeq} A^{+a}_{a\,\text{fét}}.$$

*(2) The forgetful functor $\operatorname{sFét}(Y) \overset{\simeq}{\to} A^{+a}_{\text{fét}} \to A_{\text{fét}}$ is fully faithful.*

*(3) For rational subsets $U \subseteq Y$, the assignment $U \mapsto \operatorname{sFét}(U)$ is a sheaf of categories.*

Corollary 26.5 says that strongly finite étale covers give a stack on $Y$ (or equivalently, on the rational subsets of $Y$).

*Proof.* The assertion (1) is immediate from Proposition 26.4, and (2) follows from the fact that $A^{+a}_{a\,\text{fét}} \to A_{\text{fét}}$ is fully faithful.

For (3), recall what it means for sFét to form a sheaf of categories: this means that, given an open cover $\{V_i\}$ of $Y$ and a strongly finite etale cover of each $V_i$, isomorphisms on overlaps satisfying the cocycle condition on triple intersections, then this data glues to a strongly finite etale cover of $Y$. By using this definition, one can check that (3) is equivalent to Proposition 26.4. ☐

**Lemma 26.6.** *Let $X$ be a topological space with a basis $\mathcal{R}_X$. Let $F$ and $G$ be presheaves of categories on $\mathcal{R}_X$. Suppose there is a morphism $\eta\colon F \to G$ such that*

*(a) $F$ is a sheaf;*
*(b) $\eta$ is an equivalence on all stalks;*
*(c) for all $U \in \mathcal{R}_X$, $\eta(U)\colon F(U) \to G(U)$ is fully faithful;*
*(d) $G$ is separated*[40].

*Then, $\eta$ is an equivalence (and thus $G$ is a sheaf).*

*Proof.* If $H$ denotes the sheafification of $G$, then there are morphisms $F \xrightarrow{\eta} G \xrightarrow{\mu} H$, where $\mu$ is the sheafification map. The composition $\mu\circ\eta$ is a map of sheaves that is an equivalence on every stalk, hence $\mu\circ\eta$ is an equivalence and $F \xrightarrow{\simeq} H$. Now, the separatedness of $G$ implies that $G(U) \to H(U)$ is fully faithful for any $U \in \mathcal{R}_X$. Also, (c) says that $F(U) \to G(U)$ is fully faithful, but the composition $F \to G \to H$ is an equivalence, and hence both $G \to H$ and $F \to G$ are equivalences. □

26.2. **Proof of the Almost Purity Theorem.**

*Proof of the Almost Purity Theorem.* If $(A, A^+)$ is a perfectoid affinoid $K$-algebra, then it remains to show that $A^{+a}_{a\,\text{fét}} \xrightarrow{\simeq} A_{\text{fét}}$. Set $X = \operatorname{Spa}(A, A^+)$ and let $\mathcal{R}_X$ to be the collection of rational subsets of $X$. Define presheaves $F$ and $G$ on $\mathcal{R}_X$ by the formulas

$$F(U) = \mathcal{O}_X^+(U)_{a\,\text{fét}} = \text{sFét}(U),$$

and

$$G(U) = \mathcal{O}_X(U)_{\text{fét}}.$$

There is a morphism $\eta\colon F \to G$, given by inverting $\pi$.

**Claim 26.7.** The morphism $\eta$ satisfies the hypotheses of Lemma 26.6, and therefore $\eta$ is an equivalence.

*Proof of Claim 26.7.* There are 4 conditions to check:

(a) $F$ is a sheaf: this is Corollary 26.5(3).
(b) For any $U \in \mathcal{R}_X$, $\eta(U)\colon F(U) \to G(U)$ is fully faithful: this was done in the proof of Theorem 13.2.
(c) The morphism $\eta$ is an equivalence on stalks: for any $x \in X$, we must show that

$$\operatorname{colim}_{U\ni x} \mathcal{O}_X^+(U)_{a\,\text{fét}} \xrightarrow{\simeq} \operatorname{colim}_{U\ni x} \mathcal{O}_X(U)_{\text{fét}}.$$

Observe that, via tilting and the almost purity theorem in characteristic $p$, there are isomorphisms

$$\operatorname{colim}_{U\ni x} \mathcal{O}_X^+(U)_{a\,\text{fét}} \simeq \operatorname{colim}_{U^\flat\ni x^\flat} \mathcal{O}_{X^\flat}^+(U^\flat)_{a\,\text{fét}} \simeq \operatorname{colim}_{U^\flat\ni x^\flat} \mathcal{O}_{X^\flat}(U^\flat)_{\text{fét}}.$$

Note that $\operatorname{colim}_{U^\flat\ni x^\flat} \mathcal{O}_{X^\flat}^+(U^\flat)$ is henselian along $t$ and it has completion equal to $\widehat{k(x^\flat)^+}$ Thus, Theorem 26.1 implies that

$$\operatorname{colim}_{U^\flat\ni x^\flat} \mathcal{O}_{X^\flat}(U)_{\text{fét}} \xrightarrow{\simeq} \widehat{k(x^\flat)}_{\text{fét}}.$$

By the same reasoning, $\operatorname{colim}_{U\ni x} \mathcal{O}_X(U)_{\text{fét}} \simeq \widehat{k(x)}_{\text{fét}}$. Now, we can apply the almost purity theorem for a perfectoid field, which says that

$$\widehat{k(x)}_{\text{fét}} \simeq \widehat{k(x^\flat)}_{\text{fét}}.$$

Therefore, $\eta$ is an equivalence on stalks.
(d) $G$ is separated: we will ignore this condition.

□

This completes the proof of the almost purity theorem. □

[40] A presheaf $G$ on a topological space $X$ is *separated* if for any open set $U \subseteq X$, the map $G(U) \to \prod_{x\in U} G_x$ is injective. This definition holds more generally for a presheaf on a site (in particular, it makes sense in the setting of Lemma 26.6.

## 27. April 25th*

We will spend today and tomorrow talking about the direct summand conjecture, which will illustrate how the perfectoid machinery we have developed so far is used in practice. We will need a bit more machinery, which we will state once we give some motivation and the argument for a simpler case.

### 27.1. The Direct Summand Conjecture.

The Direct Summand Conjecture was a longstanding open problem first conjectured by Mel Hochster in (or before) 1969. The problem spurred a lot of research in what are now called $F$-singularities. The conjecture was settled last summer by Yves André, and so our goal is to present a (simplified) proof of the following resolution of the conjecture:

**Theorem 27.1.** [And16, Theorem 0.2.1] *Let $A_0$ be a regular ring, and let $f_0\colon A_0 \hookrightarrow B_0$ be a finite and injective ring map. Then, $f_0$ splits as an $A_0$-module map.*

Recall that a regular ring is noetherian, and that by a finite ring map $A_0 \to B_0$, we mean that the induced $A_0$-module structure on $B_0$ realizes $B_0$ as a finitely generated module over $A_0$. It is easy to convince yourself that you cannot have a splitting as a ring homomorphism; the conjecture was that the splitting as a map of $A_0$-modules exists.

We give a short description of the history behind this conjecture, beyond the easy case when $A_0$ contains $\mathbf{Q}$, which we will see soon (Example 27.3):

- In 1973, Mel Hochster proved Theorem 27.1 when $A_0$ is of equicharacteristic $p$, i.e., when $A_0$ contains $\mathbf{F}_p$ for some prime $p$ [Hoc73, Theorem 2].
- In 2002, Ray Heitmann proved Theorem 27.1 when $\dim A_0 = 3$ [Hei02].
- In September 2016, Yves André proved Theorem 27.1 in general [And16, Theorem 0.2.1]. The main difficult is when $A_0$ is of mixed characteristic, i.e., when $A_0$ does not contain a field, and instead is only a $\mathbf{Z}$- or $\mathbf{Z}_p$-algebra, for example.

### 27.2. Reductions and easy cases.

**Remark 27.2.**

(1) We may always replace $B_0$ with a ring over it: given ring homomorphisms

$$A_0 \longrightarrow B_0 \longrightarrow C_0,$$

if the composition $A_0 \to C_0$ admits a splitting, then so does $A_0 \to B_0$. Thus, letting $C_0 = B_0/\mathfrak{p}$ where $\mathfrak{p}$ is a prime lying above $(0) \subseteq A_0$, we may assume that $B_0$ is a domain.

(2) The conclusion "$f_0$ is split" is local on $\operatorname{Spec}(A_0)$, hence we may assume that $A_0$ is a complete regular local ring. This is why usually when stating Theorem 27.1, it is assumed that $A_0$ is a complete regular local ring.

*Proof.* The map $f_0$ is split if and only if the short exact sequence

$$0 \longrightarrow A_0 \longrightarrow B_0 \longrightarrow Q_0 \longrightarrow 0$$

is split as a sequence of $A_0$-modules. Let $\alpha \in \operatorname{Ext}^1_{A_0}(Q_0, A_0)$ be the class of this extension; it suffices to show $\alpha = 0$. Since localization is flat, the formation of this Ext group commutes with localization, and so it is enough to show $\alpha = 0$ locally. Completion is also flat, hence we may also pass to the completion to assume $A_0$ is a complete regular local ring. □

(3) Now consider the integral closure $\bar{A}_0$ of $A_0$ in $\operatorname{Frac}(B_0)$. Note $\operatorname{Frac}(B_0)$ is a finite field extension of $\operatorname{Frac}(A_0)$, and that $A_0$ is excellent since it is a noetherian complete local ring. Thus, replacing $B_0$ by $\bar{A}_0$, we may assume that $B_0$ is the integral closure of $A_0$ in a finite field extension of $\operatorname{Frac}(A_0)$.

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* Thanks to Takumi Murayama for the notes for the class on April 25th.

If you weaken the assumption in Theorem 27.1 from "finite" to "integral," then you can still conclude that $f_0$ is pure. The idea is to realize $B_0$ as a filtered direct limit of finite extensions, for which we have splittings, and thereby deduce that the direct limit is pure.

As mentioned before, we will explain a simplified argument that appears in [Bha16]. The proof is not completely distinct from André's, since we will need a crucial construction from [And16].

**Example 27.3.** Some special cases of Theorem 27.1 are easy to prove.

(1) Suppose $A_0$ contains $\mathbf{Q}$, i.e., the case when $A_0$ is of equicharacteristic zero, and assume the weaker condition that $A_0$ is a normal domain. Let $A_0 \to B_0$ be as in Remark 27.2(3). Then, at the level of fields, there is a trace map

$$\mathrm{Tr}\colon \mathrm{Frac}(B_0) \longrightarrow \mathrm{Frac}(A_0).$$

Since integral elements over $B_0$ get sent to integral elements over $A_0$ (the trace just takes the sum over all Galois conjugates), we see that the trace map restricts to a map

$$\mathrm{Tr}\colon B_0 \longrightarrow A_0$$

such that for all $a \in A_0$,

$$\mathrm{Tr}(a) = a \cdot \mathrm{Tr}(1) = a \cdot [\mathrm{Frac}(B_0) : \mathrm{Frac}(A_0)].$$

Since $A_0$ is of characteristic zero, the degree of the field extension $[\mathrm{Frac}(B_0) : \mathrm{Frac}(A_0)]$ is invertible in $A_0$, hence the trace map is surjective, and the composition

$$B_0 \xrightarrow{\mathrm{Tr}} A_0 \xrightarrow{\cdot \frac{1}{[\mathrm{Frac}(B_0):\mathrm{Frac}(A_0)]}} A_0$$

provides a splitting. Alternatively, just using that the trace map is surjective, we know that there exists $b \in B_0$ such that $\mathrm{Tr}(b) = 1$, and so the map $B_0 \to A_0$ defined by $c \mapsto \mathrm{Tr}(bc)$ is a splitting since the trace is $A_0$-linear.

This alternative description will be useful when we know that the trace map is surjective, even if the degree $[\mathrm{Frac}(B_0) : \mathrm{Frac}(A_0)]$ is not invertible in $A_0$.

(2) Suppose $A_0 \to B_0$ is finite étale and injective. Then, $A_0 \to B_0$ is faithfully flat, since the map on spectra is both dominant and closed. In this case, the trace map $\mathrm{Tr}\colon B_0 \to A_0$ is surjective, and so Theorem 27.1 holds as in the argument of (1).

*Proof.* We may work étale locally on $A_0$ (by fpqc descent, for example). Since $A_0 \to B_0$ is finite étale, after étale base change we have that $B_0 \cong \prod_{i=1}^n A_0$, since $\mathrm{Spec}\, B_0$ is étale locally a disjoint union of copies of $\mathrm{Spec}\, A_0$. In this case, the trace map is obviously surjective: the trace map is $A_0$-linear, and the element $(1, 0, 0, \ldots, 0) \in B_0$ maps to $1 \in A_0$. □

In mixed characteristic, we will try to reduce to this case by passing to a huge extension, which gets rid of the ramification.

(3) Suppose that $A_0$ is a regular local ring with $\dim A_0 \leq 2$, e.g., a DVR or a power series ring in one variable over a DVR. Also assume that $B_0$ is as in Remark 27.2(3). Then, $B_0$ is finite free as an $A_0$-module.

*Proof.* This cannot be true in general, since $B_0$ can have singularities. However, in small dimensions we can use some commutative algebra to prove that $B_0$ is free. Since $B_0$ is normal ( Remark 27.2(3)), we know that $B_0$ satisfies Serre's condition 2 (S2), i.e., $\mathrm{depth}(B_0) = 2$. The Auslander–Buchsbaum formula then says that

$$\mathrm{pd}(B_0) + \mathrm{depth}(B_0) = \dim(A_0)$$

hence $\mathrm{pd}(B_0) = 0$, i.e., $B_0$ is a projective $A_0$-module. Since $A_0$ is local, this implies $B_0$ is actually free as an $A_0$-module. Finally, one may check that $f_0\colon A_0 \to B_0$ extends to an isomorphism

$$A_0 \oplus A_0^{n+1} \xrightarrow{\sim} B_0.$$

Thie idea is to work modulo the maximal ideal in $A_0$ and use Nakayama's lemma. This isomorphism gives a splitting for $f_0$. □

27.3. **The Direct Summand Conjecture for unramified maps in characteristic zero.** We now explain how to think about Theorem 27.1 in mixed characteristic, albeit for special kinds of maps $A_0 \to B_0$. Even in this simplified setting, we will see that almost purity is crucial.

**Assumptions 1.** We make the following simplifying assumptions:

- Our base ring will be $A_0 = W[\![x_1, \ldots, x_d]\!]$, where $W = W(k)$ is the ring of Witt vectors for a perfect field $k$ of characteristic $p$. The easiest example for this is when $k = \mathbf{F}_p$, in which case $W = \mathbf{Z}_p$. This setting in fact suffices to show Theorem 27.1 in general, although this requires a non-trivial argument; see [Hoc83, Theorem 6.1].
- The actual simplifying assumption is that $f_0\colon A_0 \to B_0$ is a finite injective map of normal rings, such that
$$f_0[\tfrac{1}{p}]\colon A_0[\tfrac{1}{p}] \longrightarrow B_0[\tfrac{1}{p}]$$
is étale. Geometrically, this says that the ramification is "vertical" and only happens along the divisor $p = 0$; see Fig. 2.

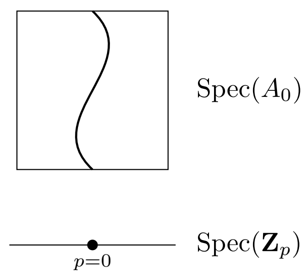


FIGURE 2. The ramification locus of $f_0$ in $\mathrm{Spec}(A_0)$ lies over $p = 0$.

**Example 27.4.** Let $B_0$ be the integral closure of $A_0$ in $A_0[\frac{1}{p}, u^{1/p^e}]$, where $u \in A_0[\frac{1}{p}]$ is a unit, e.g., a power series with constant coefficient 1. Then, $f_0[\frac{1}{p}]\colon A_0[\frac{1}{p}] \to B_0[\frac{1}{p}]$ is étale, and the inclusion $A_0 \hookrightarrow B_0$ satisfies Assumptions 1.

**Theorem 27.5.** *In the situation of Assumptions 1, the inclusion $f_0$ is split.*

*Proof.* We will use the formulation using obstruction classes, and study this class using perfectoid theory. Let $Q_0 = B_0/A_0$, and so
$$\alpha_{A_0} \in \mathrm{Ext}^1_{A_0}(Q_0, A_0)$$
is the obstruction to splitting. We want to show that $\alpha_{A_0} = 0$. For any $A_0$-algebra $R$, we denote
$$\alpha_R \in \mathrm{Ext}^1_R(Q_0 \otimes^{\mathbf{L}}_{A_0} R, R)$$
for the corresponding class obtained via base change. Note that one has to be slightly careful about this base change operation: unless $R$ is flat over $A_0$, one must take the *derived* tensor product over $A_0$ in the definition of $\alpha_R$.

The strategy is now pass to a huge extension $A_0 \hookrightarrow A_\infty$ where one can use perfectoid machinery to show $\alpha_{A_\infty} = 0$, and then go back down to $A_0$ to show $\alpha_{A_0} = 0$, i.e., that a splitting of $f_0\colon A_0 \to B_0$ exists. Recall that $A_0 = W[\![x_1, \ldots, x_d]\!]$, and so consider the ring extension
$$A_\infty = \bigl(A_0\bigl[p^{1/p^\infty}, x_i^{1/p^\infty}\bigr]\bigr)^\wedge.$$
Then, we claim $(A_\infty[\frac{1}{p}], A_\infty)$ is a perfectoid affinoid algebra over $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$, which is a perfectoid field[41]:

[41] Note that the ground field $K$ does not really change the argument. For example, instead of taking $p$-power roots of $p$, one may take $p$-power roots of 1 to use $K = \mathbf{Q}_p(\mu^{1/p^\infty})^\wedge$ as a base instead.

- Frobenius modulo $p$ is surjective by choice of $A_\infty$.
- One can check $A_\infty[\frac{1}{p}]$ is uniform with couple of definition $(A_\infty, p)$.
- It easy to check that $A_\infty$ is open.
- To check that $A_\infty$ is integrally closed in $A_\infty[\frac{1}{p}]$, note that the ring
$$A_0\big[p^{1/p^\infty}, x_i^{1/p^\infty}\big] \tag{27.1}$$
is integrally closed, and the completion of an integrally closed ring is integrally closed. Here, we are using that if $(A, A^+)$ is uniform, then
$$\widehat{(A, A^+)} = (\widehat{A^+}[\tfrac{1}{t}], \widehat{A^+})$$
for a pseudouniformizer $t \in A^+$.

We then use the following:

**Fact 27.6.** $A_0 \to A_\infty$ is faithfully flat.

Before completion, the ring (27.1) is obtained from $A_0$ just by adjoining $p$-power roots, hence the ring in (27.1) is faithfully flat (actually free) over $A_0$. Taking the completion to obtain $A_\infty$ does not affect faithful flatness by an argument that will be in the notes. Therefore, there is a base change isomorphism
$$\operatorname{Ext}^1_{A_0}(Q_0, A_0) \otimes_{A_0} A_\infty \xrightarrow{\sim} \operatorname{Ext}^1_{A_\infty}(Q_0 \otimes_{A_0} A_\infty, A_\infty) \tag{27.2}$$
obtained by base changing the resolution of $Q_0$ used in computing Ext to $A_\infty$, such that $\alpha_{A_0} \otimes 1 \mapsto \alpha_{A_\infty}$ under this isomorphism[42]. We then claim the following:

**Claim 27.7.** $\alpha_{A_\infty}$ is almost zero, i.e., $p^{1/p^n} \in \operatorname{Ann}(\alpha_{A\infty})$ for all $n \geq 0$.

We first explain why Claim 27.7 implies Theorem 27.5. We want to show that $\alpha_{A_0} = 0$, for which it suffices to show that that $\operatorname{Ann}(\alpha_{A_0}) = A_0$. First, Claim 27.7 implies that
$$p \in \operatorname{Ann}(\alpha_{A_\infty})^{p^n} \tag{27.3}$$
for all $n \geq 0$, and by faithful flatness of $A_0 \to A_\infty$ plus (27.2), we see that
$$\operatorname{Ann}(\alpha_{A_0}) = \operatorname{Ann}(\alpha_{A_\infty}) \cap A_0,$$
and similarly for powers of these ideals[43]. Now by Krull's intersection theorem, either

- $\bigcap_n \operatorname{Ann}(\alpha_0)^{p^n} = 0$; or
- $\operatorname{Ann}(\alpha_0) = (1)$.

Since we are not in the first case by (27.3), we see that $\operatorname{Ann}(\alpha_0) = (1)$, hence $\alpha_0 = 0$.

This argument illustrates how to use faithful flatness to get back down from a perfectoid statement to the original noetherian setting.

We now turn to the proof of Claim 27.7, which is how we get some control of the situation on the perfectoid level. Note we did not use any of the assumptions so far about étaleness, etc.. This will be used when we invoke the almost purity theorem.

To show Claim 27.7, note that
$$\alpha_{A_\infty} \overset{a}{=} 0 \iff (f_0 \colon A_0 \to B_0) \otimes_{A_0} A_\infty \text{ almost splits},$$
where the latter by definition means that for every $\varepsilon \in K^{\circ\circ}$, there exists $b_\varepsilon \colon B_0 \otimes_{A_0} A_\infty \to A_\infty$ such that
$$\begin{tikzcd} A_\infty \arrow[r, "f_0\otimes\mathrm{id}"] \arrow[rr, bend right, "\text{mult by } \varepsilon"'] & B_0 \otimes_{A_0} A_\infty \arrow[r, "b_\varepsilon"] & A_\infty \end{tikzcd} \tag{27.4}$$

[42]This is just unraveling the correspondence between elements in Ext and Yoneda extensions.

[43]Here, we are using the commutative algebraic fact that faithfully flat maps satisfy $\mathfrak{a}^{ec} = \mathfrak{a}$

By assumption, $A_0[\frac{1}{p}] \to B_0[\frac{1}{p}]$ is finite étale, and so by base change, we know that

$$A_\infty[\tfrac{1}{p}] \longrightarrow (B_0 \otimes_{A_0} A_\infty)[\tfrac{1}{p}]$$

is also finite étale. By the almost purity theorem, denoting

$$B_\infty := \text{integral closure of } A_\infty \text{ inside } (B_0 \otimes_{A_0} A_\infty)[\tfrac{1}{p}],$$

we have that $A_\infty \to B_\infty$ is almost finite étale. Thus, even though the original map could have lots of ramification over the divisor $p = 0$, the almost purity theorem says that you can make most of the ramification go away in the perfectoid world. Now an almost version of the argument in Example 27.3(2) implies that the trace map for $A_\infty \to B_\infty$ is almost surjective, i.e., every small $\varepsilon$ is in the image. This implies $A_\infty \to B_\infty$ is almost split, hence we can make appropriate choices of $b_\varepsilon$'s to give the diagram (27.4) by using Remark 27.2(1). □

The point was that faithful flatness allows one to go back and forth without losing information. In this version of the argument, almost purity says that finite étaleness on the rational level gives an almost splitting for integral models.

The extension $A_0 \to A_\infty$ is harder to construct in the general case of Theorem 27.1, and we will need a slightly stronger result in place of almost purity.

27.4. **Strategy for the general case of Theorem 27.1.** We will need to develop a few more tools from perfectoid theory. Before, we crucially used the étaleness of the map $A_0[\frac{1}{p}] \to B_0[\frac{1}{p}]$. In general, however, there will be some horizontal ramification that we will have to deal with.

The strategy of the proof of Theorem 27.1 in general is as follows:

(1) Choose $g \in A_0$ such that

$$A_0[\tfrac{1}{g}] \longrightarrow B_0[\tfrac{1}{g}]$$

is étale. This is possible since the extension $\mathrm{Frac}(A_0) \hookrightarrow \mathrm{Frac}(B_0)$ is separable (it is an extension of characteristic zero fields).

(2) Construct a (huge) extension

$$A_0 \longrightarrow A_\infty,$$

where $A_\infty$ is the ring of integral elements of a perfectoid algebra over some base, and contains a compatible system of $p$-power roots of $g \in A_\infty$.

(3) Show that the base change of $A_0 \to B_0$ along $A_0 \to A_\infty$ is almost split, where now almost mathematics is with respect to powers of $p \cdot g$. Geometrically, there is horizontal ramification, and so one must do almost mathematics away from $p \cdot g = 0$; see Fig. 3.

*Idea.* If $g \mid p^N$ in $A_\infty$, then the argument in § 27.3 works, since the map will be étale after inverting $g$, hence $p$. In general, this argument still applies over

$$U_N := \mathrm{Spa}(A_\infty[\tfrac{1}{p}], A_\infty)(\tfrac{p^N}{g}) \subseteq \mathrm{Spa}(A_\infty[\tfrac{1}{p}], A_\infty) =: X,$$

on which we force $g \mid p^N$ to hold. There is therefore a splitting over each of these $U_N$'s by the almost purity theorem.

Now we want the $U_N$'s to cover $X$, but this does not quite hold, since

$$U_\infty := \bigcup_N U_N = \{x \in X \mid |g(x)| \neq 0\} \subsetneq X.$$

We then take inspiration from complex geometry: the Riemann extension theorem says that if you have a bounded holomorphic function on a Zariski open subset, then you can extend the function to the complement. We want a similar statement that says that the commutative algebra over this open set $U_\infty$

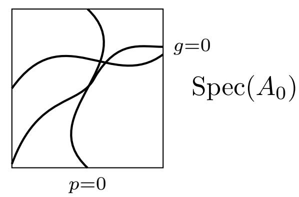


Figure 3. The ramification locus of $f_0$ in $\mathrm{Spec}(A_0)$ is $g = 0$.

looks a lot like the commutative algebra over the whole space $X$. Such a statement appears in [Sch15, Proposition II.3.2], and we will in fact prove a stronger version of it [Bha16, Theorem 4.2], which says

$$A_\infty = \mathcal{O}_X^+(X) \longrightarrow \varprojlim_N \mathcal{O}_X^+(U_N)$$

is an almost isomorphism. This holds pretty much for "diagrammatic" or "formal" reasons. This will be enough since we get an almost isomorphism

$$\mathrm{Ext}^1_{A_\infty}(-, \mathcal{O}_X^+(X)) \overset{a}{\approx} \varprojlim_N \mathrm{Ext}^1_{A_\infty}(-, \mathcal{O}_X^+(U_N)),$$

but on each $U_N$, we know that the obstruction class is zero. □

(4) Finally, we need almost faithful flatness of $A_0 \to A_\infty$ to descend the almost splitting over $A_\infty$ to actual splitting over $A_0$, using Krull's intersection theorem like before.

We note that under Assumptions 1, we could let $g = p$ in (1), we had an obvious choice of $A_\infty$ in (2), and we did not have to worry about the Riemann extension theorem. Thus, (1) and (4) are very similar to what we showed before, while (2) and (3) are the remaining steps we have to prove.

## 28. April 26th

Today, we will finish up the proof of the direct summand conjecture. To do so, we must first prove two theorems about perfectoid spaces. The goals are the following:

(1) Construct certain almost faithfully flat extensions of perfectoids algebras, which extract a compatible system of $p$-power roots of a given element.
(2) Prove a "finer version" of the Riemann extension theorem for Ext groups.
(3) Prove the direct summand conjecture, using the previous results.

28.1. **Extracting $p$-power roots.** The setup is as follows: let $K$ be a perfectoid field and let $(A, A^+)$ be a perfectoid affinoid $K$-algebra (this works more generally in the absolute setting, i.e. working with a perfectoid ring instead of with a perfectoid algebra over a perfectoid field; however, for the applications, it suffices to work in the relative setting). Fix some $g \in A^+$.

It is easy to construct a faithfully flat extension of $A$ as in (1) by formally adjoining roots of $g$, but this construction does not live in the perfectoid setting. Instead, we use André's construction from [And16], which is quite elegant.

**Theorem 28.1.** *There exists a map $(A, A^+) \to (B, B^+)$ of perfectoid affinoid $K$-algebras such that:*

*(1) the image of $g$ in $B^+$ has a compatible system of $p$-power roots;*
*(2) for any $m > 0$, the map $A^+ \to B^+$ is almost faithfully flat modulo $\pi^m$.*

Theorem 28.1(1) can be expressed more compactly by saying that $g$ lies in the image of $B^{\flat +} \xrightarrow{\sharp} B^+$. Moreover, while one probably cannot take $A^+ \to B^+$ to be faithfully flat in Theorem 28.1(2), (2) is equivalent to the statement that $A^+/\pi^\delta \to B^+/\pi^\delta$ is almost faithfully flat for some $\delta > 0$ (the only restriction on $\delta$ is that $\pi^\delta$ must make sense).

In the noetherian situation, if a module were almost faithfully flat modulo $\pi^m$ for all $m$, then it must be actually faithfully flat. However, this is false in this non-noetherian setup.

*Proof.* Consider the map $(A, A^+) \to (R, R^+)$, where $R^+ = A^+[T^{1/p^\infty}]^\wedge$. This is a map of perfectoid affinoid $K$-algebras. Now, we would like to "set $T = g$" (if one ignored issues of perfectoid-ness, then the minimal thing to do would be to mod out by the ideal $(T - g)$). To preserve the perfectoid property, a different strategy is used.

Let $X = \mathrm{Spa}(R, R^+)$, and let $Z \subseteq X$ be the Zariski-closed subset defined by the ideal $I = (T - g) \subseteq R^+$. To this data, we associate a pair $(R_Z, R_Z^+)$, where $R_Z$ is the affinoid ring of functions attached to $Z$. Recall that we can write $Z = \bigcap_N U_N$, where $U_N \coloneqq X(\frac{T-g}{\pi^N})$, and thus

$$(R_Z, R_Z^+) \coloneqq \mathrm{colim}_N \left(\mathcal{O}_X(U_N), \mathcal{O}_X^+(U_N)\right),$$

where the colimit is computed in perfectoid affinoid $K$-algebras.

**Remark 28.2.** Neither map $R \to R_Z$ or $R^+ \to R_Z^+$ need be surjective (if they were surjective, then the ideal $I$ could be defined by elements that are in the image of the sharp map). For example, one can check that neither map is surjective when $g = 1$. However, for our purposes, the surjectivity of these maps is irrelevant.

**Claim 28.3.** The ring $B^+ = R_Z^+$ satisfies the conclusion of Theorem 28.1.

*Proof of Claim 28.3.* It is clear that $g$ has a compatible system of $p$-power roots in $R_Z^+$, by construction (because $T = g$ in $R_Z^+$). Therefore, we must verify that $A^+/\pi^\delta \to R_Z^+/\pi^\delta$ is almost faithfully flat for some $\delta > 0$. In fact, we will show the stronger statement that $A^+/\pi^\delta \to \mathcal{O}_X^+(U_N)/\pi^\delta$ is almost faithfully flat for some $\delta > 0$ (where $\delta$ does not depend on $N$).

Fix some $N > 0$. The perfectoid theory (or more precisely, the "approximation lemma" Proposition 22.4(1) with $\epsilon = 1 - 1/p$) implies that there exists $f \in R^{\flat+}$ such that

(a) $U_N \coloneqq X\left(\frac{T-g}{\pi^N}\right) = X\left(\frac{f^\sharp}{\pi^N}\right)$;

(b) $f^\sharp = T - g \bmod \pi^{1/p}$.

In fact, (a) says that $U_N \subseteq X$ is the preimage of $U_N^\flat \coloneqq X^\flat\left(\frac{f}{t^N}\right) \subseteq X^\flat$ under the homeomorphism $X \xrightarrow{\simeq} X^\flat$. Therefore,

$$\mathcal{O}_X^+(U_N)/\pi^\delta \simeq \mathcal{O}_{X^\flat}^+(U_N^\flat)/t^\delta,$$

using the tilting correspondence. Now, it suffices to find some $\delta > 0$ such that $\mathcal{O}_{X^\flat}^+(U_N^\flat)/t^\flat$ is almost faithfully flat over $A^{\flat+}/t^\delta$, i.e. we have reduced to the characteristic $p$ case. In this case, we have an explicit description of the ring of functions on $U_N^\flat$.

Recall that $\mathcal{O}_{X^\flat}^+(U_N^\flat)$ is almost isomorphic to the $t$-adic completion of the perfection of

$$B_N^+ \coloneqq A^{\flat+}[T^{1/p^\infty}][X]/(t^N X - f),$$

where recall that $R^{\flat+}$ is equal to $A^{\flat+}[T^{1/p^\infty}]$, up to taking $t$-adic completions. Thus, it suffices to show $B_N^+/t^\delta$ is $t$-torsion free and almost faithfully flat over $A^{\flat+}/t^\delta$ for $\delta = 1/p$.

In fact, the pair of elements $(t^{1/p}, t^N X - f)$ form a Koszul regular sequence (i.e. the Koszul complex $\mathrm{Kos}(A^{\flat+}[T^{1/p^\infty}, X], t^{1/p}, t^N X - f)$ on these two elements is acyclic, outside of degree-0). The proof is very simple: as $t^{1/p}$ is a nonzero-divisor on $A^{\flat+}[T^{1/p^\infty}, X]$, we get a quasi-isomorphism

$$\begin{aligned}\mathrm{Kos}(A^{\flat+}[T^{1/p^\infty}, X], t^{1/p}, t^N X - f) &\simeq \mathrm{Kos}(A^{\flat+}[T^{1/p^\infty}, X]/t^{1/p}, t^N X - f)\\ &\simeq \mathrm{Kos}(A^{\flat+}[T^{1/p^\infty}, X]/t^{1/p}, f)\end{aligned}$$

But, $f = T - g \bmod t^{1/p}$, so this last complex is isomorphic to $\mathrm{Kos}(A^{+\flat}[T^{1/p^\infty}, X]/t^{1/p}, T - g)$, which is clearly acyclic outside of degree-0 because $T - g$ is a nonzero-divisor.

This presentation also shows that
$$B_N^+/t^{1/p} \simeq A^{\flat+}[T^{1/p^\infty}, X]/(t^{1/p}, T-g),$$
which is clearly faithfully flat over $A^{\flat+}/t^{1/p}$. In fact, it is free with basis $\{T^i : 0 \leq i < 1, i \in \mathbf{Z}[1/p]\}$. □

This concludes the proof of Theorem 28.1. □

**Remark 28.4.** A similar argument can be used to show the following: for any perfectoid affinoid $K$-algebra $(A, A^+)$, there exists a map $(A, A^+) \to (B, B^+)$ of perfectoid affinoid $K$-algebras such that

(1) $A^+ \to B^+$ is almost faithfully flat mod $\pi^\delta$ for any $\delta > 0$;
(2) any monic polynomial over $B^+$ has a root.

In fact, the map $(A, A^+) \to (B, B^+)$ can be constructed to be functorial in $(A, A^+)$. The argument relies on the fact that a countably-filtered colimit of $\pi$-adically complete rings is $\pi$-adically complete.

28.2. **The quantitative Riemann extension theorem.** In this section, $K$ is a perfectoid field, $t \in K$ is a fixed pseudouniformizer that admits a compatible system of $p$-power roots (note that this is slightly different from previous sections), and $(A, A^+)$ is a perfectoid affinoid $K$-algebra. Choose $g \in A^+$ that admits a compatible system of $p$-power roots. Let $X = \mathrm{Spa}(A, A^+)$, and set
$$U_\infty := \{x \in X : |g(x)| \neq 0\} = \bigcup_n U_n,$$
where $U_n := X\left(\frac{t^n}{g}\right)$.

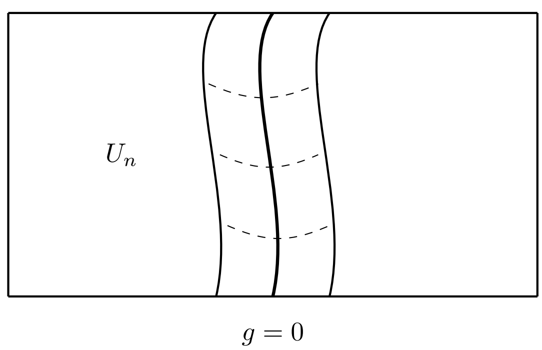


FIGURE 4. The rational subset $U_n$ of $X$ is the complement of the "tube of radius $|t|^n$" around the locus $\{g = 0\}$.

In this context, Scholze proved the following version of the Riemann extension theorem:

**Theorem 28.5.** [Sch15, Proposition II.3.2] *The map $\mathcal{O}_X^+(X) \to \mathcal{O}_X^+(U_\infty) = \varprojlim_n \mathcal{O}_X^+(U_n)$ is an almost isomorphism with respect to the ideal $(tg)$.*

This is the classical version, which says bounded functions on the open set $U_\infty$ extend to bounded functions on the whole space $X$. We need a "quantitative" variant.

We adopt the following notation: $\mathcal{O}_X^+(U_n) = A^+\langle \frac{t^n}{g} \rangle$.

**Theorem 28.6.** [Quantitative Riemann Extension Theorem] *Fix $m \geq 0$. Assume $g \in A^+$ is a nonzero-divisor modulo $t^m$. Then, the projective systems of maps*
$$\left\{ A^+/t^m \xrightarrow{f_n} \mathcal{O}_X^+(U_n)/t^m = A^+\langle \frac{t^n}{g} \rangle /t^m \right\}_{n \geq 1}$$

*has "almost-pro-zero" kernels and cokernels, i.e.*

(1) *for each* $n \geq 0$, *the kernel* $\ker(f_n)$ *is almost zero;*
(2) *for each* $k \geq 0$ *and* $c \geq p^k m$, *the transition map* $\mathrm{coker}(f_{n+c}) \to \mathrm{coker}(f_n)$ *has image almost annihilated by the element* $g^{1/p^k}$.

Theorem 28.6 implies that the limits of the two projective systems are isomorphic (as in [Sch15, Proposition II.3.2]), but it gives finer information, e.g. this isomorphism passes through certain functors (such as Ext groups). In order to refomulate Theorem 28.6, let us make the following definition.

**Definition 28.7.**

(1) A projective system $\{M_n\}$ of $A^+$-modules is *almost-pro-zero* if for all $k \geq 0$ and for all $n \geq 0$, there exists $c > 0$ such that $\mathrm{im}(M_{n+c} \to M_n)$ is killed by $(tg)^{1/p^k}$.
(2) A map $\{M_n \xrightarrow{f_n} N_n\}$ of projective systems of $A^+$-modules is an *almost-pro-isomorphism* if $\{\ker(f_n)\}$ and $\{\mathrm{coker}(f_n)\}$ are both almost-pro-zero.

The notion of an almost-pro-isomorphism is much weaker than the property of being a pro-almost-isomorphism (in the sense that it is an isomorphism of pro objects in the almost category).

**Remark 28.8.** The quantitative Riemann extension theorem (Theorem 28.6) implies that $\left\{A^+/t^n \to A^+\langle\frac{t^n}{g}\rangle/t^n\right\}$ is an almost-pro-isomorphism.

A formal consequence of the definition is that certain functors preserve the limits of almost-pro-isomorphic systems, as below.

**Proposition 28.9.** *If* $\{M_n \xrightarrow{f_n} N_n\}$ *is an almost-pro-isomorphism of projective systems, then for any* $A$*-linear functor* $F$, *there is a natural isomorphism*

$$\varprojlim_n F(M_n) \xrightarrow{\simeq} \varprojlim_n F(N_n).$$

A proof of Proposition 28.9 can be found in Bhargav's notes, but it is just chasing the definitions.

Combining Theorem 28.6 and Proposition 28.9, we obtain the following:

**Corollary 28.10.** *For any* $A^+$*-module* $Q$ *and for any* $m \geq 0$, *there is an almost isomorphism*

$$\mathrm{Ext}^i_{A^+}(Q, A^+/t^m) \overset{a}{\simeq} \varprojlim_n \mathrm{Ext}^i_{A^+}\left(Q, A^+\langle\frac{t^n}{g}\rangle/t^m\right)$$

*with respect to the ideal* $(tg)^{1/p^\infty}$.

*Proof of Theorem 28.6.* Set $M_n \coloneqq A^+\langle\frac{t^n}{g}\rangle/t^m$. We know that $A^+\langle\frac{t^n}{g}\rangle$ is the $t$-adic completion of

$$\frac{A^+[X_n^{1/p^\infty}]}{(g^{1/p^k}X_n^{1/p^k} - t^{n/p^k} : k \geq 0)}, \tag{28.1}$$

hence (28.1) modulo $t^m$ is almost isomorphic to $M_n$ (by using the "generators-and-relations"-style description of functions on a rational subset).

Now, using the fact that $g$ is a nonzero-divisor modulo $t^m$, one can check that $\ker\left(A^+/t^m \to M_n\right)$ is almost zero. For the assertion about the transition maps between cokernels, one must check that $g^{1/p^k}X_{n+c}^{i/p^\ell} \in M_{n+c}$ maps into $A^+/t^m \subseteq M_n$ for $c \geq p^k m$. One considers 2 cases:

(1) If $p^\ell \leq ip^k$, then it is clear $X_{n+c}^{i/p^\ell}$ maps to zero in $M_n$.
(2) if $p^\ell \geq ip^k$, one can check this at home.

∎

Granted Corollary 28.10, the proof of the direct summand conjecture proceeds as was outlined in yesterday's class.

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