MATH 710: TOPICS IN MODERN ANALYSIS II – L^2-METHODS

MATTIAS JONSSON*

Course Description. This course is about a set of techniques in complex geometry that go under the name of L^2-methods. They have their roots in Hörmander's work in function theory in several complex variables, and have in recent years seen a large range of applications to complex algebraic geometry.

Contents

*Notes were taken by Matt Stevenson, who is responsible for any and all errors. Special thanks to Takumi Murayama for taking notes for March 5-9, and to Sanal Shivaprasad for his notes for March 16th. Please email stevmatt@umich.edu with any corrections. Compiled on April 16, 2018.

1. January 3rd

The office hours for the class are 1-2 pm on Monday and Friday, and from 2-3pm on Wednesday. A goal for the course is to use Hilbert space methods to solve certain PDEs that are relevant for complex analytic/algebraic geometry and applications. References will be given as we go. The plan for today is to state a few examples of these results that will be covered in the class.

1.1. **Example 1: The Ohsawa–Takegoshi Theorem.**

This is an extension theorem, that comes in two versions: a function-theoretic version and a geometric one.

(A) *Function-theoretic version.* In this form, the result is due to Ohsawa–Takegoshi [OT87]. Let $\mathbf{D} \subseteq \mathbf{C}$ denote the open unit disc. Assume $\Omega \subseteq \mathbf{C}^{n-1} \times \mathbf{D}$ is a pseudoconvex domain, and set $\Omega' = \Omega \cap \{z_n = 0\}$. The basic picture is the following:

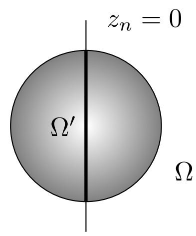


Figure 1. The (vertical) hyperplane $z_n = 0$ intersects the ball Ω in the domain Ω'.

Then, for every plurisubharmonic function $\varphi \in \mathrm{PSH}(\Omega)$ and every holomorphic function $f \in \mathcal{O}(\Omega')$ such that

$$\int_{\Omega'} |f|^2 e^{-2\varphi} d\lambda < +\infty,$$

there exists $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega'} = f$, and

$$\int_{\Omega} |F|^2 e^{-2\varphi} d\lambda \leq \pi \int_{\Omega'} |f|^2 e^{-2\varphi} d\lambda$$

where $d\lambda$ denotes the Lebesgue measure (on Ω or on Ω', as appropriate). The above integral inequality is called a 'weighted L^2-estimate'.

When one proves this theorem, one generally replaces the term π by a positive constant depending only on the domain Ω. With additional work, one can show (in the above setup) that the optimal such constant is π; see [Bo13].

(B) *Geometric version.* In this form, the result is due to Manivel [Man93] and Siu [Siu96]. Here, we have a fibration as pictured below:
More precisely, let $\pi\colon \mathfrak{X} \to \mathbf{D}$ be a proper holomorphic submersion (so each fibre of π is a compact complex manifold). Let $\mathcal{L}$ be a holomorphic line bundle on $\mathfrak{X}$, and ϕ a semipositive (singular) metric on $\mathcal{L}$. Write $L_0 \coloneqq \mathcal{L}|_{X_0}$. Then, given a section $s_0 \in H^0(X_0, K_{X_0} + L_0)$ (i.e. s_0 is an L_0-valued holomorphic n-form on X_0, where n is the dimension of any fibre of π), there exists $s \in H^0(\mathfrak{X}, K_{\mathfrak{X}} + \mathcal{L})$ such that "$s|_{X_0} = s_0$" (in the sense of adjunction) and

$$\int_{\mathfrak{X}} |s|^2 e^{-2\phi} \leq C \int_{X_0} |s_0|^2 e^{-2\phi|_{L_0}}$$

for some constant $C > 0$ independent of ϕ and f.

In the formalism that will be set up later in the class, the expression $|s|^2 e^{-2\phi}$ is a volume form on $\mathfrak{X}$, so we do not need to include the Lebesgue measure in the above integrals.

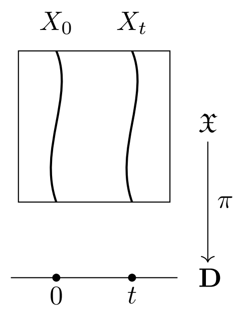


FIGURE 2. The fibre of π above a point $t \in \mathbf{D}$ is denoted by X_t.

The geometric version of the Ohsawa–Takegoshi theorem fits into a more general theme in complex geometry, where one takes an object on the central element of some family and extends it to the whole family in a controlled way. These techniques are very useful.

The method for solving this is a collection of techniques in several complex variables known as the Hörmander-Skoda theorems, namely solving the $\overline{\partial}$-equation, which we will discuss next.

1.2. **Example 2: The Hörmander(-Skoda) Theorem.** The idea is to solve the equation $\overline{\partial} u = f$ with estimates, where f is a given holomorphic (p,q)-form.

(A) *Function-theoretic version.* Let $\Omega \subseteq \mathbf{C}^n$ be a pseudoconvex (ψcx) domain (e.g. the unit ball) and $\varphi \in \mathrm{PSH}(\Omega) \cap C^\infty(\Omega)$. In addition, we require that φ is strictly plurisubharmonic: there exists a constant $c > 0$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \, \partial \overline{z}_k}(z) w_j \overline{w}_k \geq c \sum_{j=1}^{n} |w_j|^2$$

for $z \in \Omega$ and $w \in \mathbf{C}^n$, where the terms of the left-hand side form the complex Hessian of φ at z. If $q > 0$, then for any smooth (p,q)-form f on Ω with $\overline{\partial} f = 0$, there exists a smooth (p,q)-form u on Ω such that $\overline{\partial} u = f$ and

$$\int_\Omega |u|^2 e^{-2\varphi} d\lambda \leq \frac{1}{c} \int_\Omega |f|^2 e^{-2\varphi} d\lambda$$

provided the right-hand side is finite.

This is one of many different versions of this theorem, but it is the one with which we will begin. In principle, this is a PDE that we will solve using Hilbert space methods.

(B) *Geometric version.* Let (X, ω) be a Kähler manifold of dimension n (i.e. X is a complex manifold and ω is a closed, positive $(1,1)$-form on X). Let L be a holomorphic line bundle on X, and ϕ a (smooth) positive metric on L. Suppose that

$$dd^c \phi \geq c\omega$$

as $(1,1)$-forms, for some positive constant $c > 0$ (the form $dd^c\phi$ is some kind of curvature of the metric ϕ). If $q > 0$, then given any $\overline{\partial}$-closed (n,q)-form f with values in L such that $\overline{\partial} f = 0$, there exists an $(n, q-1)$-form u with values in L such that $\overline{\partial} u = f$ and

$$\int_X |u|^2 e^{-2\phi} \leq \frac{1}{cq} \int_X |f|^2 e^{-2\phi},$$

provided the right-hand side is finite.

As before, the formalism is set up in such a way that $|u|^2 e^{-2\phi}$ is a volume form on X.

Remark 1.1. If X is projective (and L is ample), then this can be viewed as a "quantitative" version of the Kodaira vanishing theorem, which says that $H^q(X, K_X + L) = 0$ for $q > 0$. Similarly, a suitable version of the Ohsawa–Takegoshi theorem can be viewed as a version of inversion of adjunction.

1.3. **Other Topics.** Once we have discussed Examples 1 and 2, we can move on to further topics, some of which are listed below.

- Berndtsson's theorem on the positivity of direct images[1] [Ber09];
- Siu's theorem on the invariance of plurigenera [Siu98, P̆07];
- Nadel's vanishing theorem [Nad90];
- Singularities of plurisubharmonic functions.

The exact material to be covered will depend on time and the interests of the audience.

1.4. **Review of Several Complex Variables.** Let $\Omega \subseteq \mathbf{C}^n$ be an open subset.

Definition 1.2. A *holomorphic function* on Ω is a function $f\colon \Omega \to \mathbf{C}$ that is complex differentiable: that is, for any $z \in \Omega$, there exists a complex-linear map $f'(z)\colon \mathbf{C}^n \to \mathbf{C}$ such that

$$f(z+w) = f(z) + f'(z)w + o(w)$$

for $w \in \mathbf{C}^n$ as $w \to 0$.

As in complex analysis, there is alternate definition in terms of power series.

Definition 1.3. A function $f\colon \Omega \to \mathbf{C}$ is *analytic* if for any $z \in \Omega$, there exists $\epsilon > 0$, $c_\alpha \in \mathbf{C}$ ($\alpha \in \mathbf{N}^n$) such that $\sum_\alpha |c_\alpha|\epsilon^{|\alpha|} < +\infty$ and

$$f(z+w) = \sum_\alpha c_\alpha w^\alpha$$

when $w \in \mathbf{C}^n$ satisfies $|w_j| \le \epsilon$ and $z + w \in \Omega$.

One of the miracles of several complex variables is that one can show the following, using Cauchy's formula for polydiscs.

Theorem 1.4. *Holomorphic and analytic functions coincide.*

As an aside, if one naively attempts to use these definitions over a non-Archimedean field (e.g. $\mathbf{Q}_p$), then these two definitions need not coincide.

2. January 5th

While later in the course we will move on to more geometric aspects, we must first work in $\mathbf{C}^n$ for the next few lectures. For this material, we are following Hörmander's book [Ḧ90], and there are also nice notes by Bo Berndtsson [Ber10].

2.1. **The $\overline{\partial}$-Equation in $\mathbf{C}^n$.** Fix coordinates $(z_1, \dots, z_n)$ on $\mathbf{C}^n$. For $0 \le p, q \le n$, a (p,q)-*form* is a differential form of the form

$$f = \sum_{|I|=p, |J|=q} f_{I,J} dz^I \wedge d\,\overline{z}^J$$

where $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are multi-indices, the $f_{I,J}$'s are functions, and

$$dz^I \wedge d\,\overline{z}^J := dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\,\overline{z}_{j_1} \wedge \dots d\,\overline{z}_{j_q}\,.$$

We may assume there are no repetitions in the multi-indices I or J, as otherwise the differential form $dz^I \wedge d\,\overline{z}^J$ is zero.

The usual exterior derivative operator d on differential forms can be decomposed as $d = \partial + \overline{\partial}$, where ∂ and $\overline{\partial}$ are defined as follows: if f is a (p,q)-form, then ∂f is the $(p+1,q)$-form given by

$$\partial f := \sum_{I,J} \partial f_{I,J} dz^I \wedge d\,\overline{z}^J$$

[1]The function-theoretic version of this result is known as the 'subharmonic variation of Bergman kernels' in the literature.

and $\overline{\partial} f$ is the $(p, q+1)$-form given by

$$\overline{\partial} f := \sum_{I,J} \overline{\partial} f_{I,J} dz^I \wedge d\overline{z}^J,$$

where

$$\partial f_{I,J} := \sum_{k=1}^{n} \frac{\partial f_{I,J}}{\partial z_k} dz_k \qquad \text{and} \qquad \overline{\partial} f_{I.J} := \sum_{k=1}^{n} \frac{\partial f_{I,J}}{\partial \overline{z}_k} d\overline{z}_k .$$

The relation $d^2 = 0$ implies that $\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0$.

Given an open set $U \subseteq \mathbf{C}^n$, set

$$C^\infty_{(p,q)}(U) := \{(p,q)\text{-forms on } U \text{ with } C^\infty\text{-coefficients}\}.$$

These give rise to the *Dolbeault cohomology group* $H^{p,q}(U)$ on U, which is given by

$$H^{p,q}(U) := \frac{\ker\left(\overline{\partial}\colon C^\infty_{(p,q)}(U) \to C^\infty_{(p,q+1)}(U)\right)}{\operatorname{im}\left(\overline{\partial}\colon C^\infty_{(p,q-1)}(U) \to C^\infty_{(p,q)}(U)\right)}.$$

In the above definition, we are implicitly assuming that $q \geq 1$. implicitly assuming here that $q \geq 1$. One can show the following comparison theorem:

Theorem 2.1. [Dolbeault's Theorem] *There is an isomorphism*

$$H^{p,q}(U) \simeq H^q(U, \Omega^p_U),$$

where the right-hand side is sheaf cohomology on U with values in the sheaf Ω^p_U of holomorphic p-forms on U.

The main ingredient in the proof of Dolbeault's theorem (modulo standard sheaf-theoretic yoga) is the following lemma.

Lemma 2.2. [$\overline{\partial}$-Poincaré Lemma] *If $U \subseteq \mathbf{C}^n$ is a polydisc or a ball, then $H^{p,q}(U) = 0$ for $0 \leq p \leq n$ and $1 \leq q \leq n$.*

This is a complex analogue of the usual Poincaré lemma for the d-operator in $\mathbf{R}^n$. One of the first goals of the class is to show the following generalization.

Goal 2.3. If $U \subseteq \mathbf{C}^n$ is a pseudoconvex domain, prove that $H^{p,q}(U) = 0$ using Hilbert space methods.

One should think of pseudoconvex domains as a kind of generalized convex domains with no interesting topology, and we will discuss them in more depth later.

Let $U \subseteq \mathbf{C}^n$ be an open set. Set

$$L^2(U, \mathrm{loc}) := \left\{ f\colon U \to \mathbf{C} \text{ (Lebesgue) measurable function such that } \int_K |f|^2 d\lambda < +\infty \text{ for all } K \Subset U \right\}.$$

Strictly speaking, we should mod out by those functions that are equal outside a set of measure zero, but we will ignore this issue. Fix a "weight" $\varphi \in C^0(U)$, and set

$$L^2(U, \varphi) := \left\{ f \in L^2(U, \mathrm{loc})\colon \int_U |f|^2 e^{-2\varphi} d\lambda < +\infty \right\}$$

This is a Hilbert space with the inner product defined by

$$\langle f, g\rangle := \int_U f\overline{g} e^{-2\varphi} d\lambda$$

for $f,g \in L^2(U,\varphi)$. There are versions of these spaces for forms: let $L^2_{(p,q)}(U,\mathrm{loc})$ denote the space of (p,q)-forms with coefficients in $L^2(U,\mathrm{loc})$, and let $L^2_{(p,q)}(U,\varphi)$ denote the space of (p,q)-forms with coefficients in $L^2(U,\varphi)$. These are both Hilbert spaces with norms defined as follows: if $f = \sum_{I,J} f_{I,J} dz^I \wedge d\overline{z}^J$, define the function

$$|f|^2 \coloneqq \sum_{I,J} |f_{I,J}|^2 \colon U \to [0,+\infty]$$

and declare

$$\|f\|^2 \coloneqq \int_U |f|^2 e^{-2\varphi} d\lambda$$

Both $L^2(U,\mathrm{loc})$ and $L^2(U,\varphi)$ are Hilbert spaces with respect to the above norm.

Furthermore, adopting the notation from the theory of distributions, set

$$D(U) \coloneqq C_0^\infty(U) \coloneqq \{C^\infty\text{-functions on } U \text{ with compact support}\}.$$

The functions in $D(U)$ are often called the *test functions* on U. Similarly, set

$$D_{(p,q)}(U) \coloneqq \{(p,q)\text{-forms on } U \text{ with coefficients in } \mathrm{D}(U)\},$$

and elements of this space are known as *test* (p,q)*-forms*.

Lemma 2.4. *For any weight* $\varphi \in C^0(U)$*,* $D_{(p,q)}(U)$ *is dense in* $L^2_{(p,q)}(U,\varphi)$*.*

Sketch. The idea is to use convolution to approximate an arbitrary function a smooth, compactly-supported one. By approximating the coefficients of a form, we may assume that $p = q = 0$. Given $f \in L^2(U,\varphi)$, we may assume that there is a compact set $K \Subset U$ such that $\mathrm{supp}(f) \subseteq K$ (indeed, replace f by $f \cdot \mathbf{1}_K$ for a sufficiently large compact set $K \Subset U$). Now, pick a test function $\chi \in D(\mathbf{C}^n)$ such that $\int_{\mathbf{C}^n} \chi d\lambda = 1$ (and one can also assume that $\chi \geq 0$, and $\chi \equiv 1$ in a neighbourhood of 0). For $\epsilon > 0$, set

$$\chi_\epsilon(z) \coloneqq \epsilon^{-2n}\chi(z/\epsilon).$$

That is, we have shrunk the support of χ but maintained that the integral be 1. For $0 < \epsilon \ll 1$, one can check that $f * \chi_\epsilon \in D(U)$ and $f * \chi_\epsilon \to f$ in $L^2(U,\varphi)$ as $\epsilon \to 0$. □

Now, fix 3 weights[2] $\varphi_1, \varphi_2, \varphi_3 \in C^0(U)$. Given (p,q) with $0 \leq p \leq n$ and $1 \leq q \leq n$, consider the following 3 Hilbert spaces:

$$\begin{cases} H_1 \coloneqq L^2_{(p,q-1)}(U,\varphi_1) \\ H_2 \coloneqq L^2_{(p,q)}(U,\varphi_2) \\ H_3 \coloneqq L^2_{(p,q+1)}(U,\varphi_3) \end{cases}$$

and there is a diagram

$$H_1 \xrightarrow{T:=\overline{\partial}} H_2 \xrightarrow{S:=\overline{\partial}} H_3$$

of maps between these Hilbert spaces, where one formally has $S \circ T = 0$.

Problem 2.5. The maps T and S are not everywhere defined! They are (as we will see) closed, densely-defined, unbounded linear maps.

[2]This is why Hörmander's method is often referred to as the "method of 3 weights".

2.2. **Hilbert Space Digression.** Let H_1 and H_2 be (complex) Hilbert spaces. A (possibly) unbounded linear map from H_1 to H_2 is a pair (D_T, T), where $D_T \subseteq H_1$ is a linear subspace and $T\colon D_T \to H_2$ is a linear map. Usually, we simply write T for the pair (D_T, T).

We say T is *densely-defined* if $D_T \subseteq H_1$ is dense, and say T is *closed* if the graph

$$\Gamma_T := \{(u, Tu)\colon u \in D_T\} \subseteq H_1 \times H_2$$

is closed. More concretely, if $u_n \in D_T$ and $u_n \to u \in H_1$ and $Tu_n \to f \in H_2$, then $u \in D_T$ and $Tu = f$.

For many purposes, closed and densely-defined linear maps behave similarly to bounded linear maps.

Theorem 2.6. [Definition of Adjoints] *If $T\colon H_1 \to H_2$ is closed and densely-defined, then there exists a unique closed and densely-defined linear map $T^*\colon H_2 \to H_1$ such that*

$$\langle u, T^* f\rangle_{H_1} = \langle Tu, f\rangle_{H_2} \tag{2.1}$$

for any $u \in D_T$ and $f \in D_{T^*}$.

Proof. We begin by defining the domain of the adjoint operator to be

$$D_{T^*} := \{f \in H_2\colon \exists\, C > 0 \text{ such that } |(Tu, f)| \leq C\|u\|_1 \text{ for all } u \in D_T\}\,.$$

For $f \in D_{T^*}$, the map $u \mapsto (Tu, f)$ is a bounded linear functional on D_T. As D_T is dense, this extends to all of H_1. By the Riesz representation theorem, there is a unique $v \in H_1$ such that $(Tu, f) = (u, v)$ for $u \in D_T$. Set $T^* f := v$. One must now check that T^* is closed and densily-defined. The idea is to work on the graph $\Gamma_T \subseteq H_1 \times H_2$ and use the closedness of T, which we will explain next time. □

3. January 8th

Today, we will continue to discuss the functional analysis necessary to solve the $\overline{\partial}$-equation in $\mathbf{C}^n$, i.e. to solve $\overline{\partial}\, u = f$ provided $\overline{\partial}\, f = 0$.

3.1. **The $\overline{\partial}$-Equation on Domains in $\mathbf{C}^n$.** If $U \subseteq \mathbf{C}^n$ is an open subset, and a weight $\varphi \in C^0(U)$, consider the Hilbert space

$$L^2_{(p,q)}(U, \varphi) := \left\{(p,q)\text{-forms } f \text{ with } L^2(U, \text{loc})\text{-coefficients such that } \int_U |f|^2 e^{-2\varphi} < +\infty\right\},$$

inside of which lies the dense subset $D_{(p,q)}(U)$ of test forms. Given two weights $\varphi_1, \varphi_2 \in C^0(U)$, consider the map

$$T := \overline{\partial}\colon L^2_{(p,q-1)}(U, \varphi_1) \to L^2_{(p,q)}(U, \varphi_2).$$

Last time, we asserted the following:

Lemma 3.1. *The map T is closed and densely-defined.*

Proof. The domain of T is

$$D_T = \left\{f \in L^2_{(p,q-1)}(U, \varphi_1)\colon\ \overline{\partial}\, f \in L^2_{(p,q)}(U, \varphi_2)\right\},$$

where $\overline{\partial}\, f$ is computed as a current (i.e. a form whose coefficients are distributions). It is clear that D_T contains $D_{(p,q-1)}(U)$, so we have that T is densely-defined.

It remains to show that the map T is closed, i.e. the graph Γ_T is closed. Suppose that $(u_n, Tu_n) \in \Gamma_T$ and $(u_n, Tu_n) \to (u, f) \in L^2_{(p,q-1)}(U, \varphi_1) \times L^2_{(p,q)}(U, \varphi_2)$. Then, $u_n \to u$ and $\overline{\partial}\, u_n \to f$ in the sense of currents. Differentiation is (almost by definition) made to be continuous on the space of currents, so it follows that $\overline{\partial}\, u = f$. Thus, $u \in D_T$ and $Tu = f$. □

3.2. **Functional Analysis Background.** It is not enough to deal with bounded maps between Hilbert spaces (because differentiation tends not to be a bounded operation), so we instead work with closed and densely-defined operators.

Theorem 3.2. [Definition of Adjoints] *If $T\colon H_1 \to H_2$ is a closed and densely-defined map between Hilbert spaces, then there exists a unique closed and densely-defined linear map $T^*\colon H_2 \to H_1$ such that*

$$\langle Tu, f\rangle_{H_2} = \langle u, T^*f\rangle_{H_1}$$

for $u \in D_T$ and $f \in D_{T^*}$. Furthermore,

(1) $\ker(T^*) = \operatorname{im}(T)^\perp$ and $\ker(T) = \operatorname{im}(T^*)^\perp$;
(2) $\overline{\operatorname{im}(T)} = \ker(T^*)^\perp$ and $\overline{\operatorname{im}(T^*)} = \ker(T)^\perp$;
(3) $T^{**} = T$.

More generally, to any closed and densely-defined linear map $X \to Y$ between Banach spaces, there is an adjoint $Y^* \to X^*$ between the corresponding dual spaces.

Proof. The domain of T^* is

$$D_{T^*} := \{f \in H_2\colon\ \exists C > 0 \text{ such that } |\langle Tu, f\rangle_{H_2}| \leq C\|u\|_1 \text{ for all } u \in D_T\}.$$

The density of D_T in H_1 (along with the Riesz representation theorem) implies that for all $f \in D_{T^*}$, there exists $v \in H_1$ such that $\langle Tu, f\rangle_{H_2} = \langle u, v\rangle_{H_1}$ for all $u \in D_T$. Set $T^*f := v$.

It is clear that $T^*\colon D_{T^*} \to H_1$ is linear, and we must show that T^* is closed and densely-defined. Define the isometry $J\colon H_1 \times H_2 \to H_2 \times H_1$ by

$$(u, f) \mapsto (-f, u).$$

Then, one can check that $\Gamma_{T^*} = J(\Gamma_T)^\perp$, which implies that T^* is closed. (In fact, since the orthogonal complement of a subspace is always closed, this argument shows that T^* is closed even without assuming that T is closed.)

To prove that D_{T^*} is dense in H_2, pick $g \in D_{T^*}^\perp$ and we must show that $g = 0$. Observe that

$$(g, 0) \in \Gamma_{T^*}^\perp = J(\Gamma_T) = \{(-Tu, u)\colon u \in D_T\}.$$

This implies that $g = 0$. It is now easy to prove (1-3) using the fact that $\Gamma_{T^*} = J(\Gamma_T)$ (and this is left as an exercise). □

Consider the following real-variable example.

Example 3.3. Consider $H_1 = H_2 = H = L^2(\mathbf{R}, x^2/4)$ and $T = \frac{d}{dx}\colon H_1 \to H_2$, with $D_T = \{u \in H\colon u' \in H\}$, where u' is computed in the sense of distributions. We want to compute both D_{T^*} and T^*.

The *formal adjoint* is defined using only test functions, i.e. we demand $\langle Tu, f\rangle = \langle u, T^*f\rangle$ for $u, f \in D(\mathbf{R})$. This can be rewritten as

$$\int_{-\infty}^{\infty} u' \cdot f \cdot e^{-x^2/2} dx = \int_{-\infty}^{\infty} u \cdot T^*f \cdot e^{-x^2/2} dx,$$

and, using integration by parts, the first integral is given by

$$-\int_{-\infty}^{\infty} u \frac{d}{dx}\left(f \cdot e^{-x^2/2}\right) dx$$

where the boundary term vanishes by construction. Thus, the formal adjoint is

$$T^*_{\text{formal}}(f) := -f' + xf,$$

where the right-hand side is computed in the sense of distributions. We claim that

$$D_{T^*} = \{f \in H\colon -f' + xf \in H\}$$

and $T^*f = -f' + xf$ for $f \in D_{T^*}$. The proof of the claim is an approximation argument: pick $\chi \in D(\mathbf{R})$ such that $\chi \geq 0$, $\chi \equiv 1$ near 0, and set $\chi_k(x) := \chi(x/k)$ for $k \in \mathbf{N}$. If $u \in D_T$ and $-f' + xf \in H$, then

$$\langle u, -f' + xf\rangle = \lim_{k\to+\infty} \Big\langle \underbrace{\chi_k u}_{\in D_T}, -f' + xf \Big\rangle = \lim_{k\to+\infty} \langle (\chi_k u)', f\rangle = \lim_{k\to+\infty} \Big\langle \underbrace{\chi_k'}_{\to 0} \cdot u + \chi_k \cdot u', f \Big\rangle = \langle u', f\rangle,$$

which completes the proof of the claim.

Remark 3.4. If we replace $\mathbf{R}$ by $(0, 1)$ in Example 3.3 (and keep all else the same), then the formal adjoint remains the same; however, the domain D_{T^*} is smaller than expected. For example, suppose $f \in C^\infty([0, 1])$; when is $f \in D_{T^*}$? One requires that there exists $C > 0$ such that

$$C \cdot \|u\| \geq |\langle Tu, f\rangle| = \left|\int_0^1 u' \cdot f \cdot e^{-x^2/2} dx\right| = \underbrace{u(1)f(1)e^{-1/2} - u(0)f(0)}_{=(*)} - \int_0^1 u \cdot (fe^{-x^2/2})' dx$$

for (at least) all $u \in C^\infty([0, 1])$. In order for this estimate to hold, the terms $(*)$ must vanish. Thus, a necessary condition for $f \in D_{T^*} \cap C^\infty([0, 1])$ is that $f(0) = f(1) = 0$.

The lesson to take away from Example 3.3 and Remark 3.4 is that boundary phenomena are very annoying to deal with when using these Hilbert space methods, but we will avoid them by picking weights that render the boundary irrelevant.

We will use functional analysis to prove existence of solutions using a priori estimates, such as the one appearing the lemma below.

Lemma 3.5. *Let $T\colon H_1 \to H_2$ be a closed and densely-defined linear map between Hilbert spaces. Then, the following are equivalent:*

(1) $\operatorname{im}(T)$ *is closed;*

(2) there exists $\delta > 0$ such that $\|T^*f\| \geq \delta\|f\|$ for all $f \in \overline{\operatorname{im}(T)} \cap D_{T^*}$.

We will later apply this lemma to the sequence of maps $H_1 \xrightarrow{\overline{\partial}} H_2 \xrightarrow{\overline{\partial}} H_3$ and for $f \in H_2$ such that $\overline{\partial} f = 0$. We then want to solve the equation $\overline{\partial} u = f$, and we will somehow know that $f \in \overline{\operatorname{im}(\overline{\partial})}$. Then, we will prove an estimate as in Lemma 3.5.

4. January 10th

There are some notes [Jon18] on the course website that contain more of the details of the functional analysis background that we have been discussing.

4.1. Functional Analysis Background (Continued).

Lemma 4.1. *Let $T\colon H_1 \to H_2$ be a closed and densely-defined linear map. Then, the following are equivalent:*

(a) $\operatorname{im}(T)$ *is closed;*

(b) there exists $\delta > 0$ such that $\|T^*f\|_{H_1} \geq \delta\|f\|_{H_2}$ for all $f \in \overline{\operatorname{im}(T)} \cap D_{T^*}$.

In this case, given $f \in \overline{\operatorname{im}(T)} = \operatorname{im}(T)$, there exists $u \in D_T$ such that $Tu = f$ and $\|u\| \leq \delta^{-1}\|f\|$.

Proof. For (b $\Rightarrow$ a), pick $g \in \overline{\operatorname{im}(T)}$, then one has the estimate

$$|\langle g, f\rangle_{H_2}| \leq \delta^{-1}\|g\|_{H_2}\|T^*f\|_{H_1} \qquad , \qquad f \in D_{T^*} \tag{4.1}$$

Indeed, decompose $H_2 = \overline{\operatorname{im}(T)} \oplus \overline{\operatorname{im}(T)}^\perp$, and (4.1) follows from (b) for $f \in D_{T^*} \cap \overline{\operatorname{im}(T)}$, and (4.1) is obvious for $f \in D_{T^*} \cap \overline{\operatorname{im}(T)}^\perp$. Now, (4.1) implies that the antilinear form $T^*f \mapsto \langle g, f\rangle_{H_2}$, defined on $\operatorname{im}(T^*)$, is bounded. By Hahn–Banach and the Riesz representation theorem, there exists $u \in H_1$ such that

$$\langle g, f\rangle_{H_2} = \langle u, T^*f\rangle_{H_1} \qquad , \qquad f \in D_{T^*}$$

This says that $g = T^{**}u = Tu$. Furthermore, $\|u\|_{H_1} \le \delta^{-1}\|g\|_{H_2}$.

Conversely, for (a $\Rightarrow$ b), we must prove that the set

$$B := \left\{ f \in \overline{\operatorname{im}(T)} \cap D_{T^*} : \|T^* f\|_{H_1} \le 1 \right\} \subseteq \overline{\operatorname{im}(T)} \subseteq H_2$$

is bounded. The uniform boundedness principle implies that it suffices to show that

$$\sup_{f \in B} |\langle f, g\rangle_{H_2}| < +\infty$$

for all $g \in \overline{\operatorname{im}(T)}$. However, the assumption (a) implies that $g = Tu$ for some $u \in D_T$, so

$$|\langle f, g\rangle_{H_2}| = |\langle f, Tu\rangle_{H_2}| = |\langle T^* f, u\rangle_{H_1}| \le \|u\|_{H_1} < +\infty,$$

where the final inequality follows from Cauchy–Schwarz. □

We will record, without proof, some further results that we will use; see [Jon18] for the proofs.

Lemma 4.2. *If $T\colon H_1 \to H_2$ is closed, then the following are equivalent:*

(a) $\operatorname{im}(T)$ *is closed;*

(b) there exists $\delta > 0$ such that $\|Tu\|_{H_2} \ge \delta\|u\|_{H_1}$ for all $u \in \ker(T)^\perp \cap D_T$.

If T were everywhere-defined and bounded, then it is easy to sketch out the idea of Lemma 4.2: replace H_2 by $\overline{\operatorname{im}(T)}$ to assume that the image is dense, and replace H_1 by the $\ker(T)^\perp$ to assume that T is injective. Then, assuming (a), the Banach open mapping theorem implies (b); the converse is easy. When T is not necessarily everywhere, one can employ a similar strategy, but one must instead work on the graph of T.

The upshot of Lemma 4.1 and Lemma 4.2 is that, when T is both closed and densely-defined, then there are estimates on both T and T^*.

An immediate corollary of Lemma 4.2 is the following result (which also holds for Banach spaces).

Corollary 4.3. [Banach Closed Range Theorem] *If T is a closed and densely-defined map between Hilbert spaces, then* $\operatorname{im}(T)$ *is closed iff* $\operatorname{im}(T^*)$ *is closed.*

Now, consider the situation where we have closed and densely-defined maps

$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$$

between Hilbert spaces and assume that this is a complex, in the sense that $\operatorname{im}(T) \subseteq \ker(S)$ In this case, write $S \circ T = 0$. It is easy to check that we also have $T^* \circ S^* = 0$.

Lemma 4.4. *In the setting above, there is an orthogonal decomposition*

$$H_2 = (\ker(S) \cap \ker(T^*)) \oplus \overline{\operatorname{im}(T)} \oplus \overline{\operatorname{im}(S^*)},$$

and the following are equivalent:

(a) $\operatorname{im}(T) = \ker(S)$ *and* $\operatorname{im}(S^*) = \ker(T^*)$*;*

(b) there exists $\delta > 0$ such that $\|Sf\|^2_{H_1} + \|T^* f\|_{H_3} \ge \delta^2\|f\|^2_{H_2}$ for all $f \in D_S \cap D_{T^*}$.

Lemma 4.4 follows formally from the previous results.

4.2. **The $\overline{\partial}$-Equation in Dimension 1.** The plan is to first discuss the solution of the $\overline{\partial}$-equation in the dimension 1 case (i.e. for domains in $\mathbf{C}$) following [Ber10], and then to proceed to the geometric version (i.e. extension of sections of line bundles on manifolds).

Let $U \subseteq \mathbf{C}$ be an open subset (we will see later that U is pseudoconvex). Let $\phi \in C^2(U) \cap \mathrm{SH}(U)$ be a "smooth", subharmonic function on U. In fact, we assume that ϕ is *strictly subharmonic*, in the sense that

$$\Delta\phi := 2\frac{\partial^2 \phi}{\partial z\, \partial \overline{z}} > 0.$$

A solution of the $\overline{\partial}$-equation with estimates, which is slightly different from the one discussed previously, is the following theorem.

Theorem 4.5. *If $f \in L^2_{(0,1)}(U, \text{loc})$, then there exists $u \in L^2(U, \text{loc})$ such that $\overline{\partial}\, u = f$ and*

$$\int_U |u|^2 e^{-2\phi} d\lambda \leq \int_U \frac{|f|^2}{\Delta\phi} e^{-2\phi} d\lambda,$$

provided the right-hand side is finite.

Theorem 4.5 admits the following (arguably slightly cleaner) corollary.

Corollary 4.6. *If $\Delta\phi \geq \delta > 0$, then for any $f \in L^2_{(0,1)}(U, \phi)$, there exists $u \in L^2(U, \phi)$ such that $\overline{\partial}\, u = f$ and*

$$\int_U |u|^2 e^{-2\phi} d\lambda \leq \frac{1}{\delta} \int_U |f|^2 e^{-2\phi} d\lambda,$$

provided the right-hand side is finite.

We will aim to prove Corollary 4.6, bypassing the proof of Theorem 4.5. One can make sense of Corollary 4.6 without the smoothness assumption on ϕ, where $\Delta\phi$ is now well-defined only as a positive measure on U, as opposed to a function (but we will ignore this complication for now).

We follow the Hilbert space approach:

$$H_1 = L^2(U, \phi) \xrightarrow{T = \overline{\partial}} H_2 = L^2_{(0,1)}(U, \phi) \xrightarrow{S=0} H_3 = 0.$$

By general results, it suffices to prove the estimate

$$\|T^* f\| \geq \delta \|f\| \tag{4.2}$$

for all $f \in D_{T^*}$. The proof of (4.2) is broken down into two steps:

(1) prove the estimate when $f \in D_{(0,1)}(U)$ is a test form;
(2) deduce the estimate in general via a suitable approximation technique.

To show (1), take a test form $f \in D_{(0,1)}(U)$ and write $f = \tilde{f} dz$ for some test function $\tilde{f} \in D(U)$. In this case, the Hilbert space adjoint agrees with the *formal adjoint* $\overline{\partial}^*_\phi$ on test forms. One can compute $\overline{\partial}^*_\phi$ by integration by parts, i.e. for all $u \in D(U)$ and $f \in D_{(0,1)}(U)$

$$\int_U u \overline{\overline{\partial}^*_\phi f} e^{-2\phi} = \int_U \overline{\partial}\, u \cdot \overline{f} e^{-2\phi}.$$

This implies that

$$\begin{aligned} \overline{\partial}^*_\phi f &= -e^{2\phi}\, \partial \left(e^{-2\phi} \tilde{f}\right) \\ &= -\frac{\partial \tilde{f}}{\partial z} + 2 \frac{\partial \phi}{\partial z} \tilde{f}. \end{aligned}$$

Proposition 4.7. [Basic Identity] *If $f = \tilde{f} dz \in D_{(0,1)}(U)$, then*

$$\int_U \Delta\phi \cdot |f|^2 e^{-2\phi} d\lambda + \int_U \left| \frac{\partial \tilde{f}}{\partial \overline{z}} \right|^2 e^{-2\phi} d\lambda = \int_U |\, \overline{\partial}^*_\phi f|^2 e^{-2\phi} d\lambda.$$

Granted Proposition 4.7, if one assumes that $\Delta\phi \geq \delta > 0$, then one obtains the estimate

$$\int_U |\, \overline{\partial}^*_\phi f|^2 e^{-2\phi} d\lambda \geq \delta \int_U |f|^2 e^{-2\phi} d\lambda,$$

which completes Step 1.

Remark 4.8. The proof of Proposition 4.7 is easy, but it has interesting generalizations in (higher-dimensional) geometric situations. These are often called Bochner-type identities.

5. January 12th

The goal for today is to finish the solution of the $\overline{\partial}$-equation in dimension 1, following Berndtsson's notes [Ber10] as well as Hörmander's original paper [H65].

5.1. The $\overline{\partial}$-Equation in Dimension 1 (Continued).

Let $U \subseteq \mathbf{C}$ be an open subset, let $\phi \in C^2(U)$ be strictly subharmonic (more precisely, there exists a constant $c > 0$ such that $\Delta\phi \geq c$ on U).

Theorem 5.1. *For any $f \in L^2_{(0,1)}(U, \phi)$, there exists $u \in L^2(U, \phi)$ such that $\overline{\partial}\, u = f$ and*

$$\int_U |u|^2 e^{-2\phi} d\lambda \leq \frac{1}{c} \int_U |f|^2 e^{-2\phi} d\lambda,$$

provided the right-hand side is finite.

If z is the coordinate on $\mathbf{C}$, then write $f = \tilde{f} dz$ (so $|f| := |\tilde{f}|$). By the functional-analytic results from last time, it suffices to prove an estimate of the form

$$\|T^* f\|^2 \geq c \|f\|^2, \tag{5.1}$$

for all $f \in D_{T^*}$, where T^* is the *Hilbert space adjoint* of the operator $T := \overline{\partial}$, which we view as a closed and densely-defined map $T\colon L^2(U, \phi) \to L^2_{(0,1)}(U, \phi)$.

There is also the *formal adjoint* $\overline{\partial}^*_\phi\colon D_{(0,1)}(U) \to D(U)$ defined by

$$\langle \overline{\partial}\, u, f \rangle = \langle u, \overline{\partial}^*_\phi\, f \rangle$$

for $u \in D(U)$ and $f \in D_{(0,1)}(U)$. By (e.g.) Stokes' theorem, one finds that

$$\overline{\partial}^*_\phi\, f = -\frac{\partial\, \tilde{f}}{\partial\, z} + 2 \frac{\partial\, \phi}{\partial\, z} \tilde{f}. \tag{5.2}$$

for $f \in D_{(0,1)}(U)$.

Remark 5.2.

(1) It is clear that $D_{(0,1)}(U) \subseteq D_{T^*}$ and $T^* = \overline{\partial}^*_\phi$ on $D_{(0,1)}(U)$.
(2) If, say, U is bounded and has C^1-boundary, then one can define $\overline{\partial}^*_\phi\, f$ for $f \in C^1_{(0,1)}(\overline{U})$ (i.e. f is the restriction to $\overline{U}$ of some $(0, 1)$-form defined on $\mathbf{C}$ with C^1-coefficients), but it is not clear that $f \in D_{T^*}$ because of boundary contributions in Stokes' formula.

Proposition 5.3. [Basic Identity] *If $f = \tilde{f} dz \in D_{(0,1)}(U)$ is a test form, then*

$$\int_U \Delta\phi \cdot |f|^2 e^{-2\phi} d\lambda + \int_U \left| \frac{\partial\, \tilde{f}}{\partial\, \overline{z}} \right|^2 e^{-2\phi} d\lambda = \int_U \left| \overline{\partial}^*_\phi\, f \right|^2 e^{-2\phi} d\lambda.$$

An immediate corollary of the basic identity, obtained by throwing away the second term on the left-hand side, is the following:

Corollary 5.4. *If $\Delta\phi \geq c > 0$, then $\|T^* f\|^2 \geq c\|f\|^2$ for all $f \in D_{(0,1)}(U)$.*

Lemma 5.5. *If $f = \tilde{f} dz \in D_{(0,1)}(U)$, then*

$$\frac{\partial}{\partial\, \overline{z}} \left(\overline{\partial}^*_\phi\, f \right) - \overline{\partial}^*_\phi \left(\overline{\partial}\, \tilde{f} \right) = \tilde{f} \Delta\phi.$$

The proof of Lemma 5.5 is an easy exercise, granted the formula (5.2).

Proof of the Basic Identity. Observe that

$$\begin{aligned}\int_U |\,\overline{\partial}_\phi^* f|^2 e^{-2\phi} d\lambda &= \int_U \overline{\partial}_\phi^* f \cdot \overline{\overline{\partial}_\phi^* f} 2^{-2\phi} d\lambda \\ &= \int_U \frac{\partial}{\partial \overline{z}} \left(\overline{\partial}_\phi^* f\right) \overline{\tilde{f}} e^{-2\phi} d\lambda \\ &\overset{\text{Lemma 5.5}}{=} \int_U \overline{\partial}_\phi^* \left(\frac{\partial \tilde{f}}{\partial \overline{z}}\right) \overline{\tilde{f}} e^{-2\phi} d\lambda + \int_U \tilde{f} \Delta\phi \overline{\tilde{f}} e^{-2\phi} d\lambda \\ &= \int_U \left|\frac{\partial \tilde{f}}{\partial \overline{z}}\right|^2 e^{-2\phi} d\lambda + \int_U \Delta\phi \cdot |f|^2 e^{-2\phi} d\lambda.\end{aligned}$$

□

In order to apply the functional analytic results, we require the estimate $\|T^* f\|^2 \geq c\|f\|^2$ for all $f \in D_{T^*}$, but we only have it so far for $f \in D_{(0,1)}(U)$. It is enough to show that $D_{(0,1)}(U) \subseteq D_{T^*}$ is dense for the graph norm

$$f \mapsto \left(\|f\|^2 + \|T^* f\|^2\right)^{1/2}.$$

Unfortunately, it is not clear whether or not this is true (in fact, Mattias suspects it is false).

Following Hörmander's paper, we first assume that U has C^2-boundary. Let $C^k(\overline{U})$ denote the image of the restriction map $C^k(\mathbf{C}) \to C^0(\overline{U})$, and let $\dot{C}^k(\overline{U}) \subseteq C^k(\overline{U})$ denote the functions vanishing outside of some compact set (equivalently, some disc). Similarly, one can define $C^k_{(0,1)}(U)$ and $C^k_{(0,1)}(\overline{U})$ and so on.

Lemma 5.6. [Approximation Lemma] *Assume $\phi \in C^1(\overline{U})$.*

(a) The set $\dot{C}^1_{(0,1)}(\overline{U})$ is dense in D_{T^*} for the graph norm.

(b) The set $\dot{C}^1(\overline{U})$ is dense in D_T for the graph norm.

We will not prove the approximation lemma now, but will see (and prove) other versions later.

The question is now: how to use the fact that U has C^2-boundary? One can pick $\rho \in C^2(\overline{U})$ such that

- $\rho < 0$ on U;
- $\rho = 0$ on ∂U;
- $|\nabla\rho| = 1$ on ∂U.

The last condition can be achieved using partitions of unity. One can use this function to control boundary integrals in Green's (or Stokes') formula. These calculations lead to the following:

Lemma 5.7. *If $f = \tilde{f} dz \in \dot{C}^1_{(0,1)}(\overline{U})$, then $f \in D_{T^*}$ iff $\tilde{f}|_{\partial U} \equiv 0$.*

Corollary 5.8. *The basic identity holds for $f \in \dot{C}^1_{(0,1)}(\overline{U}) \cap D_{T^*}$.*

The proof of Corollary 5.8 is identical to the proof of Proposition 5.3, because the boundary integrals vanish by Lemma 5.7.

Corollary 5.9. *There exists $c > 0$ such that $\|T^* f\|^2 \geq c\|f\|^2$ for all $f \in D_{T^*}$.*

Proof. The estimate is ok by basic identity for $f \in \dot{C}^1_{(0,1)}(\overline{U}) \cap D_{T^*}$, and it is true in general by density. □

Therefore, we have shown Hörmander's theorem (Theorem 5.1) is true when U has C^2-boundary. Now, consider a general open set $U \subseteq \mathbf{C}$, and the idea is to exhaust U from the inside by smooth, bounded domains: write $U = \bigcup_{j=1}^\infty U_j$, where $U_j \Subset U$ has C^2-boundary (and $U_j \subseteq U_{j+1}$). For all $j \geq 1$, there exists $u_j \in L^2(U_j, \phi)$ such that $\overline{\partial} u_j = f|_{U_j}$ and

$$\int_{U_j} |u_j|^2 e^{-2\phi} d\lambda \leq \frac{1}{c} \int_{U_j} |f|^2 e^{-2\phi} d\lambda \leq \frac{1}{c} \int_U |f|^2 e^{-2\phi} d\lambda.$$

Extend u_j to all of U, and take the weak limit u of the u_j's in $L^2(U,\phi)$, i.e. (after possibly passing to a subsequence) $u_j \to u$ weakly in $L^2(U,\phi)$, and hence $u_j \to u$ in the sense of distributions. Thus, (on any compact subset) $\overline{\partial}\, u = \lim_j \overline{\partial}\, u_j = \lim_j f_j = f$ on U, and

$$\|u\| \le \liminf_{j\to+\infty} \|u_j\| \le \frac{1}{c}\|f\|^2.$$

One must be careful to make this all precise.

Next time, we will begin aiming towards a version of Hörmanders theorem for metrics on line bundles on Kähler manifolds.

6. January 17th

Today, we will discuss Hörmander's theorem for $(0,1)$-forms in $\mathbf{C}^n$, the best reference for which are Berndtsson's notes [Ber95]. Most calculations will be skipped, and we will focus instead on the ingredients of the proof.

6.1. The $\overline{\partial}$-Equation in Higher Dimensions.

The goal is to solve the equation

$$\overline{\partial}\, u = f,$$

where $f \in L^2_{(0,1)}(U, \mathrm{loc})$ satisfies $\overline{\partial}\, f = 0$ and $u \in L^2(U, \mathrm{loc})$. Write $f = \sum_{j=1}^n f_j dz_j$. Consider a pseudoconvex open set $U \subseteq \mathbf{C}^n$ with C^2-boundary; that is.

- we can write $U = \{\rho < 0\}$ for some $\rho \in C^2(\mathbf{C}^n)$, where $\nabla\rho \ne 0$ on $\partial U = \{\rho = 0\}$;
- if $p \in \partial U$ and $a \in \mathbf{C}^n$ lies in the tangent space at p (i.e. $\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(p) a_j = 0$), then

$$\sum_{j,k=1}^n \rho_{j\overline{k}} a_j \overline{a_k} \ge 0, \tag{6.1}$$

  where $\rho_{j\overline{k}} = \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p)$ are the components of the complex Hessian. (Note that this is slightly weaker than saying ρ is plurisubharmonic, which would require (6.1) to hold for all $a \in \mathbf{C}^n$.)

Now, consider a weight $\phi \in C^2(\overline{U})$ and assume ϕ is strictly psh, i.e. the matrix $(\phi_{j\overline{k}})_{j,k=1}^n$ is a positive Hermitian (and we write $(\phi_{j\overline{k}})_{j,k=1}^n > 0$). Set $(\phi^{j\overline{k}}) \coloneqq (\phi_{j\overline{k}})^{-1}$ to be the inverse matrix.

Theorem 6.1. *Suppose $f \in L^2_{(0,1)}(U, \mathrm{loc})$ satisfies $\overline{\partial}\, f = 0$. Then, there exists $u \in L^2(U, \mathrm{loc})$ such that $\overline{\partial}\, u = f$ and*

$$\int_U |u|^2 e^{-2\phi} \le \frac{1}{2}\int_U \sum_{j,k=1}^n \phi^{j\overline{k}} f_j \overline{f_k} e^{-2\phi}$$

provided the right-hand side is finite.

In one variable, this right-hand side is simply $|f|^2$ divided by the Laplacian of ϕ, as in Theorem 4.5. Unless otherwise specified, all integrals are taken with respect to Lebesgue measure, so this added notation is often omitted.

The setup from before was to consider the "exact" sequence

$$L^2(U,\phi) \xrightarrow{T} L^2_{(0,1)}(U,\phi) \xrightarrow{S} L^2_{(0,2)}(U,\phi),$$

where both S and T are the operator $\overline{\partial}$, viewed as closed and densely-defined operators.

Lemma 6.2. *If $\alpha \in C^1_{(0,1)}(\overline{U}) \cap D_{T^*}$, then $\sum_{j=1}^n \alpha_j \frac{\partial \rho}{\partial z_j} = 0$ on ∂U.*

Informally, the above lemma says that one can view α as a complex tangent vector to the boundary ∂U.

Sketch. The proof uses the definition of D_{T^*} and the divergence theorem. □

The Bochner-type identity that is used in this setting is (once again) called the Basic Identity.

Theorem 6.3. [Basic Identity] *If $\alpha \in C^2_{(0,1)}(\overline{U}) \cap D_{T^*}$, then*

$$2\int_U \sum_{j,k=1}^n \phi_{j\overline{k}}\alpha_j\overline{\alpha_k}e^{-2\phi} + \int_U \sum_{j,k=1}^n \left|\frac{\partial\,\alpha_j}{\partial\,\overline{z}_k}\right|^2 e^{-2\phi} + \int_{\partial\,U} \sum_{j,k=1}^n \rho_{j\overline{k}}\alpha_j\overline{\alpha_k}e^{-2\phi}\frac{dS}{|\,\partial\,\rho|} = \int_U |\overline{\partial}^*_\phi\alpha|^2 e^{-2\phi} + \int_U |\,\overline{\partial}\,\alpha|e^{-2\phi},$$

where dS is the Lebesgue measure on $\partial\,U$.

The proof of the basic identity is analogous to the previous version. By bounding from below by zero the second two terms of the left-hand side of the basic identity, we get the basic inequality.

Corollary 6.4. [Basic Inequality] *If $\alpha \in C^2_{(0,1)}(\overline{U}) \cap D_{T^*}$, then*

$$\|T^*\alpha\|^2 + \|S\alpha\|^2 = \int_U |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} + \int_U |\,\overline{\partial}\,\alpha|^2 e^{-2\phi} \geq 2\int_U \sum_{j,k=1}^n \phi_{j\overline{k}}\alpha_j\overline{\alpha_k}e^{-2\phi}.$$

Observe that the right-hand side of the basic inequality is $\geq \delta\|\alpha\|^2$ for some $\delta > 0$, because of the strict plurisubharmonicity of ϕ.

One now requires an approximation argument in order to get the basic inequality for all $\alpha \in D_{T^*}$ (in which case, one can apply the Hilbert space machinery to conclude).

Theorem 6.5. [Approximation] *The subset $D_S \cap D_{T^*} \cap C^\infty_{(0,1)}(\overline{U})$ is dense in $D_S \cap D_{T^*}$ in the graph norm, which is given by*

$$\alpha \mapsto \left(\|\alpha\|^2 + \|S\alpha\|^2 + \|T^*\alpha\|^2\right)^{1/2}.$$

This is a bit tricky! It is written minimalistically in [H$\ddot{6}$5], and done in more detail in [Ber95].

Corollary 6.6. *If $\alpha \in D_S \cap D_{T^*}$, then there exists $\delta > 0$ such that*

$$\|T^*\alpha\|^2 + \|S\alpha\|^2 \geq 2\int_U \sum_{j,k=1}^n \phi_{j\overline{k}}\alpha_j\overline{\alpha_k}e^{-2\phi} \geq \delta\|\alpha\|^2.$$

Now, Corollary 6.6 and the Hilbert space machinery imply that $\ker(S) = \operatorname{im}(T)$, i.e. we can solve the $\overline{\partial}$-equation. However, for Theorem 6.1, we need improved estimates.

Proof of Theorem 6.1. Set

$$C := \left(\frac{1}{2}\int_U \sum_{j,k=1}^n \phi^{j\overline{k}} f_j\overline{f_k}e^{-2\phi}\right)^{1/2} < +\infty.$$

Lemma 6.7. *For any $\alpha \in D_{(0,1)}(U)$, we have*

$$|\langle f, \alpha\rangle| = \left|\int_U f\cdot\overline{\alpha}e^{-2\phi}\right| \leq C\|\,\overline{\partial}^*_\phi\,\alpha\|.$$

Granted Lemma 6.7, then the rule $\overline{\partial}^*_\phi\,\alpha \mapsto \langle f, \alpha\rangle$ defines a bounded, antilinear form L on the set

$$E := \left\{\overline{\partial}^*_\phi\,\alpha \colon \alpha \in D_{(0,1)}(U)\right\} \subseteq L^2(U, \phi).$$

By Hahn–Banach and the Riesz representation theorem, L can be extended to a bounded linear functional on all of $L^2(U, \phi)$ and there exists $u \in L^2(U, \phi)$ such that $\|u\| \leq C$ and $L(v) = \langle u, v\rangle$ for all $v \in L^2(U, \phi)$. Set $v = \overline{\partial}^*_\phi\,\alpha$ for some $\alpha \in D_{(0,1)}(U)$, so it follows that

$$\int_U u\overline{\overline{\partial}^*_\phi\,\alpha}e^{-2\phi} = L(\overline{\partial}^*_\phi\,\alpha) = \int_U f\overline{\alpha}e^{-2\phi}.$$

This means that $\overline{\partial}\, u = f$ in the sense of distributions (or really, in the sense of currents). One also has that

$$\int_U |u|^2 e^{-2\phi} = \|u\|^2 \le C^2 = \frac{1}{2}\int_U \sum_{j,k=1}^n \phi^{j\overline{k}} f_j \overline{f_k} e^{-2\phi},$$

which completes the proof of Theorem 6.1. □

Proof of Lemma 6.7. Use the decomposition $L^2_{(0,1)}(U,\phi) = \ker(S) \oplus \ker(S)^\perp$ to write $\alpha = \beta + \gamma$ for some $\beta \in \ker(S)$ and $\gamma \in \ker(S)^\perp \subseteq \operatorname{im}(T)^\perp = \ker(T^*) \subseteq D_{T^*}$. As $\alpha \in D_{(0,1)}(U) \subseteq D_{T^*}$, it follows that $\beta = \alpha - \gamma \in D_{T^*}$ and $T^*\beta = T^*\alpha$. Thus, using that $\langle f, \gamma\rangle = 0$ since $f \in \ker(S)$, we have

$$\begin{aligned}
|\langle f,\alpha\rangle|^2 &= |\langle f,\beta\rangle|^2 \\
&= \left|\int_U f\cdot\overline{\beta} e^{-2\phi}\right|^2 \\
&\le \left(2\int_U \sum_{j,k=1}^n \phi_{j\overline{k}} \beta_j \overline{\beta_k} e^{-2\phi}\right) \underbrace{\left(\frac{1}{2}\int_U \sum_{j,k=1}^n \phi^{j\overline{k}} f_j \overline{f_k} e^{-2\phi}\right)}_{=C^2},
\end{aligned}$$

where the inequality follows from Cauchy–Schwarz. By Corollary 6.6 (using that $S\beta = 0$), we get that

$$\left(2\int_U \sum_{j,k=1}^n \phi_{j\overline{k}} \beta_j \overline{\beta_k} e^{-2\phi}\right) \le \|T^*\beta\|^2 = \|T^*\alpha\|^2,$$

which completes the proof. □

Next time, we'll start moving towards the setting of metrics on line bundles on Kähler manifolds, and the analogue of Hörmander's theorem in that setting.

7. January 19th

Today, we will begin to discuss some background in complex geometry, the references for which are [Dem12, GH78, Voi02]. The goal, once we have finished with the preliminaries, is to have a version of Hörmander's theorem on compact Kähler manifolds.

7.1. Complex Geometry Background.

Let X be a (real) C^∞-manifold of dimension $2n$. For $x \in X$, we have $T_xX \simeq \mathbf{R}^{2n}$, where TX is the (real) tangent bundle of X.

Definition 7.1. We say X is a *complex manifold* if it admits an atlas $(U_\alpha, \varphi_\alpha)_\alpha$, where $U_\alpha \subseteq X$ is an open subset and $\varphi_\alpha \colon U_\alpha \to \mathbf{C}^n \simeq \mathbf{R}^{2n}$ is a homeomorphism such that the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic for all α, β.

Definition 7.2. We say X is an *almost complex manifold* if it admits an *almost complex structure*, i.e. an endomorphism $J\colon TX \to TX$ of the tangent bundle such that $J^2 = -\mathrm{id}$.

It is easy to see that if X is a complex manifold, then X is an almost complex manifold: if $(z_1,\ldots,z_n) \in \mathbf{C}^n$ are local coordinates with $z_j = x_j + iy_j$, then set

$$\begin{cases} J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \\ J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}. \end{cases}$$

The converse implication is addressed by a deep theorem of Newlander–Nirenberg: an *integrable* almost complex manifold is a complex manifold.

We will generally work with the complexified tangent bundle $T^{\mathbf{C}}X \coloneqq TX \otimes_{\mathbf{R}} \mathbf{C}$, which is a complex vector bundle of rank $2n$. Locally, $T^{\mathbf{C}}X$ is spanned by the vectors

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n},$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

The endomorphism J extends by $\mathbf{C}$-linearity to $J\colon T^{\mathbf{C}}X \to T^{\mathbf{C}}X$, and $T^{\mathbf{C}}X$ splits into eigenspaces

$$T^{\mathbf{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

where $T^{1,0}X = \operatorname{span}_j\left(\frac{\partial}{\partial z_j}\right)$ has eigenvalue $+i$, and $T^{0,1}X = \operatorname{span}_j\left(\frac{\partial}{\partial \overline{z}_j}\right)$ has eigenvalue $-i$.

Similarly, there is a decomposition

$$\Lambda^1_{\mathbf{C}}X = \Lambda^{1,0}X \oplus \Lambda^{0,1}X,$$

where $\Lambda^{1,0}X = \operatorname{span}(dz_1, \ldots, dz_n)$ and $\Lambda^{0,1}X = \operatorname{span}(d\overline{z}_1, \ldots, d\overline{z}_n)$. Recall that our conventions are

$$\begin{cases} dz_j = dx_j + i dy_j, \\ d\overline{z}_j = dx_j - i dy_j. \end{cases}$$

Consequently, we can decompose

$$\Lambda^r_{\mathbf{C}}X = \bigoplus_{p+q=r} \Lambda^{p,q}X,$$

where the (p,q)-forms $\Lambda^{p,q}X$ are those of the form $\sum_{|I|=p,|J|=q} \alpha_{I,J} dz^I \wedge d\overline{z}^J$. The operator $d\colon \Lambda^r X \to \Lambda^{r+1}X$ splits as $d = \partial + \overline{\partial}$, where we view ∂ and $\overline{\partial}$ as operators $\partial\colon \Lambda^{p,q}X \to \Lambda^{p+1,q}$ and $\overline{\partial}\colon \Lambda^{p,q}X \to \Lambda^{p,q+1}X$.

Definition 7.3. Write $d^c \coloneqq \frac{i}{2\pi}(\overline{\partial} - \partial)$, and hence $dd^c \coloneqq \frac{i}{\pi}\,\partial\,\overline{\partial}$.

The normalization in the definition of d^c is chosen so that the Poincaré–Lelong formula looks nice. For example, if $X = \mathbf{C}$, then this formula says that

$$dd^c \log|z| = \delta_0$$

in the sense of currents, where δ_0 is the Dirac mass at $0 \in \mathbf{C}$. Said differently, for any test function $\alpha \in D(\mathbf{C})$, we have

$$\int_{\mathbf{C}} \log|z| dd^c\alpha = \alpha(0).$$

Example 7.4. If $f\colon \mathbf{C} \to \mathbf{R}$ is a smooth function, then

$$\begin{aligned} dd^c f &= \frac{i}{4\pi}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f (dx + idy) \wedge (dx - idy) \\ &= \frac{1}{2\pi}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy. \end{aligned}$$

That is, $dd^c f$ agrees (up to scale) with the Laplacian of f, viewed as a measure (equivalently, a top form) on $\mathbf{C}$.

7.2. **Hermitian Metrics and Forms.** Let V be a $\mathbf{C}$-vector space of dimension n (which we will later take to be holomorphic tangent space at a point). We can also view V as an $\mathbf{R}$-vector space $V_{\mathbf{R}}$ of dimension $2n$, together with an $\mathbf{R}$-linear endomorphism $J\colon V_{\mathbf{R}} \to V_{\mathbf{R}}$ such that $J^2 = -\mathrm{id}$ (that is, J encodes how to multiply by i). As before, there is a decomposition

$$V_{\mathbf{C}} \coloneqq V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = V^{1,0} \oplus V^{0,1},$$

where $V^{1,0}$ and $V^{0,1}$ are the two eigenspaces of J. There is also conjugation on $V_{\mathbf{C}}$, under which $V^{0,1} = \overline{V^{1,0}}$.

Lemma 7.5. *The following objects are in 1-1 correspondence:*

(1) Hermitian forms $h\colon V \times V \to \mathbf{C}$ (i.e. sesquilinear forms that are conjugate symmetric);
(2) symmetric bilinear forms $g\colon V_{\mathbf{R}} \times V_{\mathbf{R}} \to \mathbf{R}$ such that $g(Jv, Jw) = g(v, w)$ for all $v, w \in V_{\mathbf{R}}$;
(3) alternating bilinear forms $\omega\colon V_{\mathbf{R}} \times V_{\mathbf{R}} \to \mathbf{R}$ of type $(1,1)$ (i.e. the complexification $\omega\colon V_{\mathbf{C}} \times V_{\mathbf{C}} \to \mathbf{C}$ satisfies $\omega \equiv 0$ on $V^{1,0} \times V^{1,0}$ and $V^{0,1} \times V^{0,1}$).

Sketch. The relation between the objects is as follows: $h = g - i\omega$, and $g(u, v) = \omega(u, Jv)$ for all $u, v \in V_{\mathbf{R}}$. ☐

We are mainly interested in the case when h is *positive*: $h(v, v) \geq 0$ for all $v \in V$, with equality iff $v = 0$. If this holds, the corresponding symmetric bilinear form g satisfies the same property.

7.3. **The $*$-Operator.** The $*$-operation is a construction in multilinear algebra, which we first explain in the real case and then extend to the complex case.

Let V be an $\mathbf{R}$-vector space of dimension n, and let $g\colon V \times V \to \mathbf{R}$ be an inner product (i.e. a positive, symmetric, bilinear form). For any $1 \leq k \leq n$, this induces a positive, symmetric, bilinear form

$$g_k\colon \Lambda^k V \times \Lambda^k V \to \mathbf{R}$$

as follows: if $e_1, \ldots, e_n$ is an orthonormal basis of V, then $(e_I)_{|I|=k}$ is an orthonormal basis for $\Lambda^k V$, where $e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$ for indices $i_1 < i_2 < \ldots < i_k$.

Also, pick an *orientation* of V. This determines an element $\mu \in \Lambda^n V$ such that $|\mu| = g_n(\mu, \mu) = 1$. Then, we have an $\mathbf{R}$-linear isometry

$$*\colon \Lambda^k V \to \Lambda^{n-k} V$$

such that $\alpha \wedge *\beta = g_k(\alpha, \beta)\mu$ for all $\alpha, \beta \in \Lambda^k V$. Note that $*e_I = \pm e_{I^c}$, where $I^c \coloneqq \{1, \ldots, n\} \setminus I$. See Brian Conrad's notes [Con] for all the details of the construction.

We will use a version of this in the complex case. Let X be a complex manifold of dimension n, let $x \in X$, and let h be a positive Hermitian metric on $T_x X$. If ω is the corresponding alternating bilinear form (in the sense of Lemma 7.5), then view $\omega \in \Lambda^{1,1} T_x^* X$.

Definition 7.6. For any $1 \leq p \leq n$, set $\omega_p \coloneqq \frac{\omega^p}{p!} \in \Lambda^{p,p} T_x^* X$.

Then, $\omega_n \in \Lambda^{n,n} T_x^* X$ determines an orientation at x. Thus, we can define the $*$-operator as before. The upshot of this is that we will be able to define the norm of a (p,q)-form, which we will come back to next class.

8. January 22nd

We continue with the preliminaries required to make sense of Hörmander's theorem in the geometric setting.

8.1. **Complex Geometry Background (Continued).** Let X be a complex manifold of dimension n, and let h be a (positive) Hermitian metric; that is, h determines a Hermitian metric on each complex tangent space $T_x X \simeq T_x^{1,0} X$ for all $x \in X$, varying smoothly with x.

By general nonsense, h induces a metric on (p,q)-forms on X. Fix an orthonormal basis $dz_1, \ldots, dz_n$ for $\Lambda_x^{1,0} = T_x^* X \simeq T_x^{*1,0} X$, which induces a choice of orthonormal basis $(dz_I \wedge d\,\overline{z}_J)'_{|I|=p,|J|=q}$ for $\Lambda_x^{p,q}$. (Note: the notation $(\cdot)'_{|I|=p,|J|=q}$ means that one should only allow I and J with increasing indices, so that this actually forms a basis.)

We would like to express norms of forms using the $(1,1)$-form ω induced by h; at $x \in X$, ω is given by

$$\omega = \sum_{j=1}^{n} i dz_j \wedge d\overline{z}_j \,.$$

For $1 \le p \le n$, define

$$\omega_p \coloneqq \frac{\omega^p}{p!} = \sum_{|J|=p}{}' \bigwedge_{j \in J} i dz_j \wedge d\overline{z}_j \,.$$

In particular, $\omega_n = \bigwedge_{j=1}^{n} i dz_j \wedge d\overline{z}_j$ is a volume form at x. Furthermore, for $1 \le p \le n$, set $c_p \coloneqq i^{p^2}$.

Lemma 8.1. *If η is a $(p,0)$-form at x, then*

$$|\eta|^2 \omega_n = c_p \eta \wedge \overline{\eta} \wedge \omega_{n-p}$$

as (n,n)-forms at x.

One can see [Ber95] for a proof of Lemma 8.1 that avoids the choice of an orthonormal basis.

Sketch. If $\eta = dz_1 \wedge \ldots \wedge dz_p$, then

$$\begin{aligned} c_p \eta \wedge \overline{\eta} \wedge \omega_{n-p} &= i^{p^2} dz_1 \wedge \ldots \wedge dz_p \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_p \wedge \left(\bigwedge_{j=p-1}^{n} i dz_j \wedge d\overline{z}_j \right) \\ &= \bigwedge_{j=1}^{n} i dz_j \wedge d\overline{z}_j \\ &= \omega_n = |\eta|^2 \omega_n, \end{aligned}$$

where the first equality holds since $\bigwedge_{j=p-1}^{n} i dz_j \wedge d\overline{z}_j$ is the only term of the sum ω_p that survives in the wedge product. The general case is left as an exercise. □

Corollary 8.2. *If ξ, η are $(p,0)$-forms at x, then*

$$\langle \xi, \eta \rangle \omega_n = c_p \xi \wedge \overline{\eta} \wedge \omega_{n-p}.$$

Proof. This is immediate from Lemma 8.1 and polarization. □

The same proof yields the analogous result for $(0,q)$-forms.

Corollary 8.3. *If ξ, η are $(0,q)$-forms at x, then*

$$\langle \xi, \eta \rangle \omega_n = c_q \overline{\xi} \wedge \eta \wedge \omega_{n-q}.$$

Lemma 8.4. *If η is a $(p,1)$-form at x, then*

$$i c_p (-1)^{p-1} \eta \wedge \overline{\eta} \wedge \omega_{n-p-1} = \left(|\eta|^2 - |\eta \wedge \omega_{n-p}|^2 \right) \omega_n$$

as (n,n)-forms at x.

The proof of Lemma 8.4 is a direct (but a bit painful) calculation in an orthonormal basis.

There is also a version of the $*$-operation (which differs from the previous one by complex conjugation): if η is an (n,q)-form, then there exists an $(n-q,0)$-form γ_η such that

$$\langle \xi, \eta \rangle \omega_n = c_{n-q} \xi \wedge \overline{\gamma_\eta} \tag{8.1}$$

for all (n,q)-forms ξ. This operation will be used to define and study the adjoint of the $\overline{\partial}$-operator on forms.

In an orthonormal basis, we have

$$\eta = \sum_{|J|=q}{}' \gamma_J dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_J \Longrightarrow \gamma_\eta = \sum_{|J|=q}{}' \epsilon_J dz_{J^c}$$

for some constants $|\epsilon_J| = 1$. In particular, $|\gamma_\eta| = |\eta|$.

An immediate consequence of (8.1) is that $|\eta|^2\omega_n = c_{n-q}\eta \wedge \overline{\gamma_\eta}$; however, one in fact can say more, as is demonstrated below.

Lemma 8.5. *If η is an (n, q)-form at x, then $\eta = \gamma_\eta \wedge \omega_q$.*

8.2. **Line Bundles.** Let X be a complex manifold of dimension n. One can study line bundles on X as locally free sheaves, but we instead consider the associated total space.

Definition 8.6. A *(holomorphic) line bundle* on X is a complex manifold L together with a holomorphic map $p\colon L \to X$ such that there exists an open covering $(U_\alpha)_\alpha$ of X and, for every α, there is a biholomorphism

$$\varphi_\alpha\colon L_{U_\alpha} \coloneqq p^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbf{C}$$

such that for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$(U_\alpha \cap U_\beta) \times \mathbf{C} \xrightarrow{\varphi_\alpha \circ \varphi_\beta^{-1}} (U_\alpha \cap U_\beta) \times \mathbf{C}$$

is of the form $(\varphi_\alpha \circ \varphi_\beta^{-1})(x, v) = (x, g_{\alpha\beta}(x)v)$, for some nonvanishing holomorphic function $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbf{C}^*$.

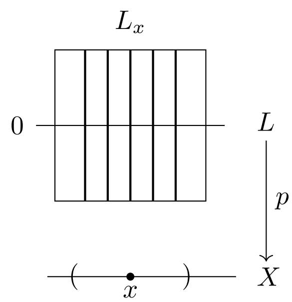


Figure 3. The line bundle L on X is locally of this form, where the horizontal lines denotes the zero section of p.

For every $x \in X$, the fibre $L_x \coloneqq p^{-1}(x)$ is isomorphic to $\mathbf{C}$ as a $\mathbf{C}$-vector space, but L_x does *not* have a canonical basis element.

Definition 8.7. A *(global, holomorphic) section* of L is a holomorphic map $s\colon X \to L$ such that $p \circ s = \mathrm{id}$.

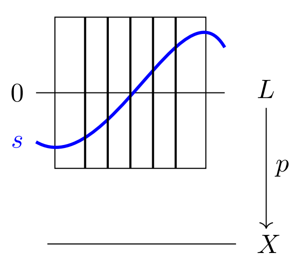


Figure 4. A local section s of L is drawn in blue.

Example 8.8. The *trivial line bundle* is $L = \mathcal{O}_X = X \times \mathbf{C}$. The sections of L are precisely the holomorphic functions on X.

Example 8.9. The *canonical bundle* is $L = \Omega^n_X$, whose sections are the global holomorphic n-forms (equivalently, $(n, 0)$-forms) on X.

Example 8.10. If $X = \mathbf{P}^n$ and $m \in \mathbf{Z}$, then there is a family of line bundles $L = \mathcal{O}(m)$ whose sections are (holomorphic) homogeneous polynomials of degree m (at least for $m \geq 0$).

8.3. **Metrics on Line Bundles.**

Definition 8.11. A *metric* on a line bundle L is a function

$$\|\cdot\| \colon L \to \mathbf{R}_+ := [0, +\infty)$$

such that for all $x \in X$, the restriction of $\|\cdot\|$ to L_x is a norm (on the $\mathbf{C}$-vector space L_x); that is, for all $v \in L_x$,

- $\|v\| = 0$ iff $v = 0$;
- for all $\lambda \in \mathbf{C}$, $\|\lambda v\| = |\lambda| \cdot \|v\|$.

Equivalently, $\|\cdot\|$ determines the unit circle in every fibre L_x of L (though it still does not specify a basis!).

This is a very weak definition, since there is no relation between the fibres. This is often referred to as a Finsler metric on the line bundle.

Definition 8.12. A metric $\|\cdot\|$ is *smooth/continuous/upper semicontinuous (usc)/lower semicontinuous (lsc)* if for any local nonvanishing section $s \colon U \to L$ (defined on an open subset $U \subseteq X$), the function

$$x \mapsto \|s(x)\|$$

is smooth/continuous/usc/lsc.

9. January 24th

9.1. **Metrics on Line Bundles (Continued).** Let X be a complex manifold, and let $p \colon L \to X$ be a line bundle. Last time, we introduced the multiplicative version of a metric on L: a function

$$\|\cdot\| \colon L \longrightarrow \mathbf{R}_+ = [0, +\infty)$$

such that the restriction $\|\cdot\||_{L_x}$ is a vector space norm for all $x \in X$.

For various purposes, it is more convenient to look at the *additive version* of a metric: $\phi := -\log\|\cdot\|$. That is, the metric ϕ is now thought of as a function

$$\phi \colon L^\times \longrightarrow \mathbf{R},$$

where $L^\times = L \backslash \{\text{zero section}\}$ (if one wishes to include the zero section, one must allow ϕ to take the value $+\infty$). The function ϕ has the property that

$$\phi(\lambda v) = \phi(v) - \log|\lambda|$$

for $v \in L$ and $\lambda \in \mathbf{C}^*$.

The choice of a metric $\|\cdot\|$ or ϕ on L is equivalent to the choice of a unit circle $\{\|\cdot\| = 1\} = \{\phi = 0\}$ in each fibre L_x of L.

Example 9.1. A metric on $\mathcal{O}_X = X \times \mathbf{C}$ is equivalent to the data of a $\mathbf{R}$-valued function on X: given a metric $\phi \colon \mathcal{O}_X^\times = X \times \mathbf{C}^* \to \mathbf{R}$, define a function $\chi \colon X \to \mathbf{R}$ by the formula

$$\chi(x) = \phi(x, 1).$$

Conversely, given a function χ, set

$$\phi(x, \lambda) = \chi(x) - \log|\lambda|.$$

Definition 9.2. A *singular metric* on L is a function $\phi\colon L^\times \to \mathbf{R}\cup\{-\infty\}$ such that

$$\phi(\lambda v) = \phi(v) - \log|\lambda|$$

for all $v \in L^\times$ and $\lambda \in \mathbf{C}^*$.

Example 9.3. A global section $s \in \Gamma(X, L)$ defines a singular metric $\phi = \log|s|$ such that

$$\begin{cases} \phi(s(x)) = 0 & \text{if } s(x) \neq 0, \\ \phi|_{L_x^\times} \equiv -\infty & \text{if } s(x) = 0. \end{cases}$$

The unit circle of ϕ in the fibre L_x is precisely the circle containing $s(x)$ (provided $s(x) \neq 0$).

Example 9.4. If $\phi_1, \dots, \phi_N$ are (singular) metrics on L, then $\max\{\phi_1, \dots, \phi_N\}$ and $\min\{\phi_1, \dots, \phi_N\}$ are again (singular) metrics on L.

Example 9.5. If $\|\cdot\|_1, \dots, \|\cdot\|_N$ are (singular) metrics on L, then the ℓ^p-average

$$\left(\sum_{j=1}^N \|\cdot\|_i^p\right)^{1/p},$$

for $1 \le p \le +\infty$, is again a metric on L

Example 9.6. Let $s_1, \dots, s_N$ be global sections of L without common zero. Then,

$$\phi = \max_{1\le j\le N} \log|s_j|$$

is a continuous metric on L (but it may not be smooth, since the max of two smooth functions need not be smooth). One can also perform the ℓ^2-version of this construction:

$$\phi = \frac{1}{2}\log\left(\sum_{j=1}^N |s_j|^2\right)$$

is a smooth metric on L.

Example 9.7. If $X = \mathbf{P}^n$ has homogeneous coordinates $z_0, \dots, z_n$ and $L = \mathcal{O}_X(1)$, then

$$\phi = \frac{1}{2}\log\left(\sum_{j=0}^n |z_j|^2\right)$$

is the *Fubini–Study metric* on L.

A more common description of metrics is using local trivializations: let $X = \bigcup_\alpha U_\alpha$ be an open covering such that $\varphi_\alpha\colon L_{U_\alpha} = p^{-1}(U_\alpha) \overset{\simeq}{\to} U_\alpha \times \mathbf{C}$ is a biholomorphism and the transition maps

$$(U_\alpha \cap U_\beta) \times \mathbf{C} \overset{\varphi_\alpha\circ\varphi_\beta^{-1}}{\longrightarrow} (U_\alpha \cap U_\beta) \times \mathbf{C}$$

are given by $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$ for some holomorphic transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbf{C}^*$. The transition functions satisfy the cocycles conditions

$$\begin{cases} g_{\alpha\alpha} = 1, \\ g_{\alpha\beta} g_{\beta\alpha} = 1, \\ g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1. \end{cases}$$

Now, consider a section $s\colon X \to L$, which induces functions $s_\alpha\colon U_\alpha \to \mathbf{C}$ given by

$$\varphi_\alpha(s(x)) = (x, s_\alpha(x))$$

for $x \in U_\alpha$. The compatibility/transition rules give that

$$s_\alpha = g_{\alpha\beta} s_\beta$$

on $U_\alpha \cap U_\beta$.

Define local sections $e_\alpha \colon U_\alpha \to L_{U_\alpha} = p^{-1}(U_\alpha)$ by the formula

$$\varphi_\alpha(e_\alpha(x)) = (x, 1)$$

for $x \in U_\alpha$. Now, similar to the case of sections, a metric ϕ can be described using functions $\phi_\alpha \colon U_\alpha \to \mathbf{R}$, where $\phi_\alpha \coloneqq \phi \circ e_\alpha$. One can check that

$$\begin{cases} e_\alpha = g_{\alpha\beta}^{-1} e_\beta, \\ \phi_\alpha - \phi_\beta = \log |g_{\alpha\beta}| \end{cases} \tag{9.1}$$

on $U_\alpha \cap U_\beta$. Thus, one can equivalently define a metric ϕ on L to be a family (ϕ_α) of $\mathbf{R}$-valued function satisfying the compatibility condition (9.1).

9.2. **Operations on Line Bundles and Metrics.**

9.2.1. *Mappings.* If X, Y are complex manifolds, $f \colon Y \to X$ is a holomorphic map, and L is a line bundle on X, then $M \coloneqq f^*L$ is a line bundle on Y.

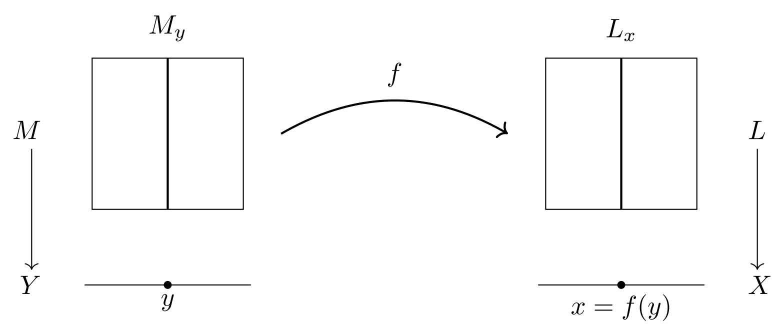


Figure 5. The pullback line bundle $M = f^*L$ of L.

Given a trivializing cover (U_α) of X and transition function $g_{\alpha\beta}$ of L, then $(f^{-1}(U_\alpha))$ is a trivializing cover of Y and transitions functions $g_{\alpha\beta} \circ f$ of M.

If ϕ is a metric on L corresponding to the family (ϕ_α) of $\mathbf{R}$-valued functions on the cover (U_α), then $\psi \coloneqq f^*\phi$ is a metric on L given by the family $(\psi_\alpha = \phi_\alpha \circ f)$ on the cover $(f^{-1}(U_\alpha))$ of Y.

Example 9.8. If $Y \hookrightarrow X$ is an open subset of a closed submanifold, then the pullback of the line bundle is simply the restriction of the line bundle.

Example 9.9. If $Y \hookrightarrow X = \mathbf{P}^n$ is a closed embedding, then we will often be interested in pulling back $L = \mathcal{O}_X(1)$ with the Fubini–Study metric.

9.2.2. *Tensor Products.* If L' and L'' are line bundles on X, then $L \coloneqq L' \otimes L''$ is a line bundle on X with fibres

$$L_x = L'_x \otimes_{\mathbf{C}} L''_x$$

as (1-dimensional) vector spaces over $\mathbf{C}$. In terms of transition functions, if $(g'_{\alpha\beta})$ and $(g''_{\alpha\beta})$ are the transition functions of L' and L'' respectively, then

$$g_{\alpha\beta} = g'_{\alpha\beta} \cdot g''_{\alpha\beta}$$

are the transition functions of L. We will sometimes use the additive notation $L' + L'' \coloneqq L' \otimes L''$ (this is not to be confused with the direct sum of line bundles, which would be a vector bundle of higher rank).

Now, if ϕ' and ϕ'' are metrics on L' and L'' respectively, then $\phi = \phi' + \phi''$ is a metric on L with local description given by

$$\phi_\alpha = \phi'_\alpha + \phi''_\alpha.$$

In particular, if ϕ is a metric on L and χ is a function on X (viewed as a metric on $\mathcal{O}_X$ in the sense of Example 9.1), then $\phi + \chi$ is a metric on L. (Conversely, any other metric on L can be obtained from ϕ from a function on X.)

9.2.3. *Inverses.* If L is a line bundle on X with transition functions $(g_{\alpha\beta})$, then $-L := L^{-1}$ is a line bundle on X with transition functions $(g_{\alpha\beta}^{-1})$. Similarly, if ϕ is a metric on L given by the local functions (ϕ_α), then $-\phi$ denotes the metric on $-L$ given by the local function $(-\phi_\alpha)$.

9.3. **The Curvature Form.** Let ϕ be a smooth (really, C^2) metric on a line bundle L on X, which is locally given by the function (ϕ_α) satisfying $\phi_\alpha - \phi_\beta = \log|g_{\alpha\beta}|$ on $U_\alpha \cap U_\beta$, and the transition function $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbf{C}^*$ are holomorphic and non-vanishing. This implies that $\log|g_{\alpha\beta}|$ is *pluriharmonic*, i.e.

$$\partial\overline{\partial}\log|g_{\alpha\beta}| = 0 \tag{9.2}$$

as a $(1,1)$-form on $U_\alpha \cap U_\beta$.

Definition 9.10. The *curvature form* of the metric ϕ is the $(1,1)$-form $dd^c\phi$ that is locally given by

$$dd^c\phi := dd^c\phi_\alpha = \frac{i}{\pi}\,\partial\overline{\partial}\,\phi_\alpha$$

on U_α. This is well-defined precisely because of (9.2).

Remark 9.11. The normalization in Definition 9.10 of $dd^c\phi$ is such that the de Rham cohomology class of $dd^c\phi$ in $H^2(X, \mathbf{R})$ is equal to the (first) Chern class $c_1(L)$ of L.

Definition 9.12. The metric ϕ is *positive* if $dd^c\phi$ is a positive form, which is the pointwise condition on the holomorphic tangent spaces $T_x^{1,0}X$ that

$$dd^c\phi(v, \overline{v}) > 0$$

for all $v \in T_x^{1,0}X$. Equivalently, this means that the local functions ϕ_α are strictly plurisubharmonic for all α: in local coordinates $(z_1, \dots, z_n)$, we have

$$\sum_{j,k=1}^n i\frac{\partial^2\,\phi_\alpha}{\partial\,z_j\,\partial\,\overline{z}_k}(x)w_j\overline{w_k} > 0$$

for all x and all $w \in \mathbf{C}^n\backslash\{0\}$.

Example 9.13. The Fubini–Study metric on $\mathbf{P}^n$ is positive.

10. January 26th

10.1. **Kähler Manifolds.** Let X be a complex manifold.

Definition 10.1. A *Kähler form* on X is a smooth, positive, closed $(1,1)$-form ω (where *closed* means that $d\omega = 0$, or equivalently $\partial\,\omega = 0$ and $\overline{\partial}\,\omega = 0$). We say that (X, ω) (or simply X) is a *Kähler manifold.*

Example 10.2. If L is a line bundle on X and ϕ is a smooth, positive, metric on L, then $\omega := dd^c\phi = \frac{i}{\pi}\,\partial\,\overline{\partial}\,\phi$ is a Kähler form.

Example 10.3. [The Fubini–Study metric/form on $\mathbf{P}^n$] If $X = \mathbf{P}^n$ has homogeneous coordinates $z_0, \dots, z_n$ and $L = \mathcal{O}_X(1)$, then the Fubini–Study metric on L is given by

$$\phi := \frac{1}{2}\log\left(\sum_{j=0}^n |z_j|^2\right).$$

In terms of the standard cover $X = \bigcup_{j=0}^{n} U_j$, where $U_j = \{z_j \neq 0\} \simeq \mathbf{C}^n$: on U_j, use coordinates $\xi_i = \frac{z_i}{z_j}$ for $i \neq j$ and trivializing sections $e_j = z_j$, then

$$\phi = \log|z_j| + \underbrace{\frac{1}{2}\log\left(1 + \sum_{i\neq j} |\xi_i|^2\right)}_{=\phi_j},$$

where we view $\log|z_j|$ as a metric on $\mathcal{O}(1)|_{U_j}$, and ϕ_j as a function on U_j. One can check that $\phi_i - \phi_j = \log|z_j/z_i|$, where $g_{ij} = z_j/z_i$ are the transition functions.

Claim 10.4. ϕ is a positive metric.

The claim implies that $\omega := dd^c\phi$ is a Kähler form on $X = \mathbf{P}^n$, called the Fubini–Study form.

Sketch. Consider the function $\Phi = \frac{1}{2}\log\left(\sum_{j=0}^{n}|z_j|^2\right)$ on $\mathbf{C}^{n+1}\backslash\{0\} \simeq L^\times$. It suffices to check that $dd^c\Phi > 0$ except along lines thru 0. By the $U(n+1)$-invariance of Φ, it suffices to look at the point $(z_0, \dots, z_n) = (1, 0, \dots, 0)$. Observe that

$$\frac{\partial\,\Phi}{\partial\, z_j} = \frac{1}{2}\frac{\overline{z}_j}{\sum_{i=0}^{n}|z_i|^2}$$

and

$$\frac{\partial^2\,\Phi}{\partial\, z_j\,\partial\,\overline{z}_k} = \begin{cases} -\frac{1}{2}\frac{\overline{z}_j\, z_k}{\left(\sum_{i=0}^{n}|z_i|^2\right)^2}, & j \neq k \\ \frac{1}{2}\frac{1-|z_j|^2}{\left(\sum_{i=0}^{n}|z_i|^2\right)^2}, & j = k \end{cases}$$

Thus, at $(1, 0, \dots, 0)$, we get that

$$dd^c\Phi = \frac{1}{2\pi}\sum_{j=1}^{n} i dz_j \wedge d\,\overline{z}_j\,.$$

This completes the proof of the claim. □

In particular, $\mathbf{P}^n$ is a Kähler manifold.

Example 10.5. As a consequence of Example 10.3, any quasiprojective manifold is Kähler: take an embedding $X \hookrightarrow \mathbf{P}^n$ and take ω to the pullback of the Fubini–Study form on $\mathbf{P}^n$. (The converse is not true: there are Kähler manifolds that do not arise as complex algebraic varieties, e.g. a 'generic' complex torus $\mathbf{C}^n/\Lambda$.)

Example 10.6. Continuing Example 10.3 in the case of $\mathbf{P}^1$, we can use the coordinate $\xi = \xi_1 = \frac{z_1}{z_0}$ on $\mathbf{C} \subseteq \mathbf{P}^1$ to write the Fubini–Study form as

$$\omega = \frac{1}{2}dd^c\log\left(1 + |\xi|^2\right) = \frac{i}{2\pi}\frac{d\xi \wedge d\overline{\xi}}{(1+|\xi|^2)^2}.$$

One can check (in polar coordinates) that $\int_{\mathbf{P}^1}\omega = 1$.

10.2. **Forms with Values in a Line Bundle.** Let X be a complex manifold, and let L be a line bundle on X with transitions functions g_{ij} corresponding to trivializations (φ_i, U_i, e_i).

Definition 10.7. A (p,q)*-form* η *with coefficients in* L is a section of the (complex[3]) vector bundle $\Lambda^{p,q} \otimes L$ on X. More concretely, η is given locally by $\eta_i \otimes e_i$, where η_i is a (p,q)-form on U_i, and they satisfy $\eta_i = g_{ij}\eta_j$.

[3]This is is a complex, but not holomorphic, vector bundle on X because the transition functions are only smooth in general.

Now, fix a (smooth) metric ϕ on L (corresponding to $\mathbf{R}$-valued functions ϕ_i on U_i), then

$$|\eta|e^{-\phi} := |\eta_i|e^{-\phi}$$

is a well-defined global function on X, since

$$\begin{cases} \phi_i - \phi_j = \log|g_{ij}|, \\ e_i = g_{ij}^{-1} \cdot e_j. \end{cases}$$

If ω is a Kähler (or any positive) form on X, then one can define the L^2-norms of (p,q)-forms with values in L:

$$\|\eta\|^2 := \int_X |\eta|^2 e^{-2\phi} dV_\omega,$$

where dV_ω is the volume form $\omega_n = \frac{\omega^n}{n!}$. (For example, if ϕ is a positive metric, then one could take $\omega = dd^c\phi$.)

There are additional global objects that one can associate to L-valued forms:

- If ξ, η are L-valued (p,q)-forms, then $\langle \xi, \eta \rangle e^{-2\phi}$ is a global function on X, where $\langle \xi, \eta \rangle$ is the pointwise inner-product.
- If ξ, η are L-valued forms, then $\xi \wedge \overline{\eta} e^{-2\phi}$ is a global form on X (but it is no longer L-valued); locally, it is given by
$$\xi_i \wedge \overline{\eta_i} e^{-2\phi_i},$$
and notice that there are no e_i's present in the above expression.

One can also define $\overline{\partial}$ on (p,q)-forms (or any forms, really) with values in L: locally, write such a form as $\eta = \eta_i \otimes e_i$, and set

$$\overline{\partial}\,\eta := \overline{\partial}\,\eta_i \otimes e_i.$$

This is well-defined since $\eta_i = g_{ij}\eta_j$, so

$$\overline{\partial}\,\eta_i = g_{ij}\,\overline{\partial}\,\eta_j$$

because the g_{ij}'s are holomorphic. However, there is no d-operator of ∂-operator, unless $\partial\, g_{ij} = 0$ (however, if $\overline{\partial}\, g_{ij} = 0$ and $\partial\, g_{ij} = 0$, then the g_{ij}'s are constant, which is not very interesting).

We can now state a (non-optimal version) of Hörmander's theorem that we will later try to prove.

Theorem 10.8. [Hörmander's Theorem – Non-Optimal Version] *Let X be a compact Kähler manifold of dimension n, let L be a line bundle with a positive metric ϕ, so $\omega := dd^c\phi$ is a Kähler form on X. Fix $q > 0$. Given a $\overline{\partial}$-closed (n,q)-form f with values in L, then there exists a $(n, q-1)$-form u with values in L such that $\overline{\partial}\, u = f$ and*

$$\int_X |u|^2 e^{-2\phi} dV_\omega = \|u\|^2 \le \frac{1}{q}\|f\|^2,$$

provided that the right-hand side is finite.

The method of proof will be similar to the more classical versions of Hörmander's theorem: we need to understand the adjoint of $\overline{\partial}$ and prove a 'basic identity'. In order to have an adjoint, we need an additional construction called the Chern connection.

10.3. **The Chern Connection.** Recall that we cannot define $\partial\,\eta$ when η is a form with values in a line bundle L. However, given a smooth metric ϕ on L, there is the *Chern connection*, which is a rule

$$D\colon \{L\text{-valued forms of degree } r\} \longrightarrow \{L\text{-valued forms of degree } r+1\}$$

satisfying some natural conditions. In our situation, there is a decomposition

$$D = \overline{\partial} + \delta,$$

where δ is an operator of bidegree $(1,0)$ (just as for ∂) and, locally, satisfies

$$\begin{aligned}\delta(\eta_i \otimes e_i) &= \left(e^{2\phi}\,\partial\, e^{-2\phi_i}\eta_i\right) \otimes e_i \\ &= (-2\,\partial\,\phi_i \wedge \eta_i + \partial\,\eta_i) \otimes e_i.\end{aligned}$$

Said differently, δ is a "twisted version" of ∂.

Exercise 10.9. The operator δ is globally-defined.

Exercise 10.10. If ξ, η are forms with values in L, then

$$\overline{\partial}\left(\eta \wedge \overline{\xi} e^{-2\phi}\right) = \overline{\partial}\,\eta \wedge \overline{\xi} e^{-2\phi} + (-1)^{\partial(\eta)}\eta \wedge \overline{\delta\xi} e^{-2\phi}.$$

Exercise 10.11. We have the identities $\overline{\partial}^2 = 0$ and $\delta^2 = 0$.

Exercise 10.12. The *curvature* D^2 of the Chern connection D can be expressed as

$$D^2 = \delta\,\overline{\partial} + \overline{\partial}\,\delta = 2\,\partial\,\overline{\partial}\,\phi = -2\pi i dd^c\phi,$$

i.e. we have $D^2\eta = 2\,\partial\,\overline{\partial}\,\phi \wedge \eta$ for any L-valued form η.

11. January 29th

The goal of today's class is to work towards the basic identity in the geometric setting. First, we recall the notation that was established in the previous class.

Let (X, ω) be a Kähler manifold of dimension n, let L be a line bundle on X, and let ϕ be a smooth metric on L. In terms of local data, if there is an open cover $X = \bigcup_i U_i$ over which L is trivialized with transition functions g_{ij}, then L has local sections e_i and ϕ is given by functions ϕ_i. These satisfy the transition rules

$$\begin{cases} e_i = g_{ij}^{-1} e_j, \\ \phi_i = \phi_j + \log|g_{ij}|. \end{cases}$$

Furthermore, an L-valued form η on X is given by the local data $\eta = \eta_i \otimes e_i$, where the η_i's are local forms satisfying $\eta_i = g_{ij}\eta_j$.

There are standard operations that can be performed on these forms with values in a line bundle, which are summarized below:

- if ξ, η are L-valued (p,q)-forms, then $\langle \xi, \eta \rangle e^{-2\phi}$ is a function on X;
- if ξ, η are L-valued forms, then $\xi \wedge \overline{\eta} e^{-2\phi}$ is a form on X;
- if η is an L-valued form, then $\overline{\partial}\,\eta \coloneqq \overline{\partial}\,\eta_i \otimes e_i$ and

$$\begin{aligned}\delta\eta &\coloneqq \left(e^{2\phi_i}\,\partial\, e^{-2\phi_i}\eta_i\right) \otimes e_i \\ &= (-2\,\partial\,\phi_i \wedge \eta_i + \partial\,\eta_i) \otimes e_i.\end{aligned}$$

There are two important formulas satisfied by the $\overline{\partial}$ and δ operators:

$$\overline{\partial}\left(\eta \wedge \overline{\xi} e^{-2\phi}\right) = \overline{\partial}\,\eta \wedge \overline{\xi} e^{-2\phi} + (-1)^{\deg(\eta)}\eta \wedge \overline{\delta\xi} e^{-2\phi}, \tag{11.1}$$

$$(\delta\,\overline{\partial} + \overline{\partial}\,\delta)\eta = 2\,\partial\,\overline{\partial}\,\phi \wedge \eta. \tag{11.2}$$

11.1. **The Formal Adjoint of $\overline{\partial}$.** The formal adjoint of $\overline{\partial}$, denote $\overline{\partial}_\phi^*$, is an operator that should satisfy the identity

$$\int_X \langle \overline{\partial}\,\eta, \xi\rangle e^{-2\phi} dV_\omega = \int_X \langle \eta, \overline{\partial}_\phi^*\,\xi\rangle e^{-2\phi} dV_\omega, \tag{11.3}$$

where η is an L-valued $(n, q-1)$-form of compact support, and ξ is an L-valued (n,q)-form. We will express $\overline{\partial}_\phi^*$ using the $*$-operator, which sends an (n,q)-form ξ to an $(n-q, 0)$-form γ_ξ. Recall that there was an equality

$$\langle \overline{\partial}\,\eta, \xi\rangle e^{-2\phi} dV_\omega = c_{n-q}\,\overline{\partial}\,\eta \wedge \overline{\gamma_\xi} e^{-2\phi}$$

of (n,n)-forms on X, which one can verify locally. In particular, the left-hand side of (11.3) can be expressed as

$$c_{n-q}\int_X \overline{\partial}\,\eta \wedge \overline{\gamma_\xi} e^{-2\phi} = c_{n-q}\int_X \overline{\partial}\left(\eta \wedge \overline{\gamma_\xi}\right) e^{-2\phi} - (-1)^{n-q+1} c_{n-q} \int_X \eta \wedge \overline{\delta\gamma_\xi} e^{-2\phi} \tag{11.4}$$

where the first equality holds by (11.1). Note that

$$\partial\left(\eta \wedge \overline{\gamma_\xi}\right) e^{-2\phi} = 0$$

for degree reasons (indeed, it is an $(n+1, n-1)$-form on the n-dimensional manifold X, and hence it is zero); thus, it follows that

$$\int_X \overline{\partial}\left(\eta \wedge \overline{\gamma_\xi}\right) e^{-2\phi} = 0$$

by Stokes' formula. Similarly, the integrand on the right-hand side of (11.3) can be written as

$$\langle \eta, \overline{\partial}_\phi^*\,\xi\rangle e^{-2\phi} dV_\omega = c_{n-q+1} \eta \wedge \overline{\gamma_{\overline{\partial}_\phi^*\,\xi}} e^{-2\phi}.$$

Now, one can check that

$$\left.\begin{aligned} c_{n-q+1} &= i^{(n-q+1)^2} \\ c_{n-q} &= i^{(n-q)^2} \end{aligned}\right\} \Longrightarrow c_{n-q+1} = (-1)^{n-q} c_{n-q} \cdot i$$

Putting these together, one finds that

$$i\int_X \eta \wedge \overline{\delta\gamma_\xi} e^{-2\phi} = \int_X \eta \wedge \overline{\gamma_{\overline{\partial}_\phi^*\,\xi}} e^{-2\phi} \tag{11.5}$$

for all test forms η. We conclude that

$$\gamma_{\overline{\partial}_\phi^*\,\xi} = i\delta_{\gamma_\xi}. \tag{11.6}$$

From this formula[4], one can get a local description for the formal adjoint $\overline{\partial}_\phi^*$, but really we will only require (11.6).

11.2. **Positivity of Forms.** The reference for this material is [Dem12, III.1]. Let V be a $\mathbf{C}$-vector space of dimension n (which we will later take to be the tangent space to X at a point). The exterior power of $V_{\mathbf{C}}^*$, which is a $2n$-dimensional $\mathbf{C}$-vector space, admits a decomposition

$$\Lambda V_{\mathbf{C}}^* = \bigoplus_{p,q\le n} \Lambda^{p,q} V^*,$$

where $\Lambda^{p,q}V^*$ is the vector subspace of (p,q)-forms.

[4]The formula (11.6) might be off by a minus sign.

11.2.1. *Positivity of (n,n)-Forms.* As $\mathbf{C}$-vector spaces, we have an isomorphism $\Lambda^{n,n}V^* \simeq \mathbf{C}$. Given (linear) coordinates $z = (z_1, \dots, z_n) \xrightarrow{\simeq} \mathbf{C}^n$, set

$$\begin{aligned}\tau(z) &\coloneqq idz_1 \wedge d\,\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\,\overline{z}_n \\ &= 2^n dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n,\end{aligned}$$

which is a (real) (n,n)-form. Note that, for another choice of coordinates w, τ transforms as

$$\tau(w) = \tau(z) \cdot \left|\det\left(\frac{\partial\, w_j}{\partial\, z_k}\right)\right|^2. \tag{11.7}$$

We say that an (n,n)-form u is *positive* if there exists $c \geq 0$ such that $u = c \cdot \tau(z)$. (In particular, with this definition, the zero (n,n)-form is positive.) By (11.7), this definition of positivity is independent of the choice of coordinates z.

11.2.2. *Positivity of (p,p)-Forms.* Given a (p,p)-form u, the following are equivalent:

(1) $u|_W \geq 0$ for any p-dimensional $\mathbf{C}$-linear subspace $W \subseteq V$;
(2) $u \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \wedge i\alpha_{n-p} \wedge \overline{\alpha_{n-p}} \geq 0$ for any $(1,0)$-forms $\alpha_1, \ldots, \alpha_{n-p}$.

We say u is *positive* if (1) and (2) hold.

Fact 11.1.

(1) A $(1,1)$-form $i\sum_{j,k} u_{jk} dz_j \wedge d\,\overline{z}_k$ is positive iff the corresponding Hermitian form

$$\xi \mapsto \sum_{j,k} u_{jk}\xi_j\overline{\xi_k}$$

is semipositive definite.
(2) If α is a $(p,0)$-form, then $c_p\alpha \wedge \overline{\alpha}$ is a positive (p,p)-form.
(3) If α is a positive (p,p)-form and β is a positive $(1,1)$-form, then $\alpha \wedge \beta$ is a positive $(p+1,p+1)$-form.

While (3) holds, be warned that it is not true in general that the wedge of two positive forms is again positive.

11.3. **The Basic Identity.** Given a smooth (n,q)-form α with values in L, set

$$T_\alpha \coloneqq c_{n-q}\gamma_{n-q} \wedge \overline{\gamma_{n-q}} \wedge \omega_{q-1} e^{-2\phi},$$

which is a positive $(n-1,n-1)$-form (with values in $\mathcal{O}_X$). Note that

$$\begin{aligned}T_\alpha \wedge \omega &= qc_{n-q}\gamma_{n-q} \wedge \overline{\gamma_{n-q}} \wedge \omega_q e^{-2\phi} \\ &= q|\gamma_\alpha|^2 e^{-2\phi} dV_\omega \\ &= q|\alpha|^2 e^{-2\phi} dV_\omega,\end{aligned}$$

where the final equality follows from the fact that $\alpha \mapsto \gamma_\alpha$ is an isometry.

Theorem 11.2. [Basic Identity] *For an (n,q)-form α with values in L, we have*

$$i\,\partial\,\overline{\partial}\,T_\alpha = \left(-2\mathrm{Re}\left\langle\overline{\partial}\,\overline{\partial}^*_\phi\,\alpha, \alpha\right\rangle + |\,\overline{\partial}\,\gamma_\alpha|^2 + |\,\overline{\partial}^*_\phi\,\alpha|^2 - |\,\overline{\partial}\,\alpha|^2\right) e^{-2\phi} dV_\omega + 2i\,\partial\,\overline{\partial}\,\phi \wedge T_\alpha \tag{11.8}$$

as (n,n)-forms on X.

The proof of the basic identity is not difficult - it is simply an integration by parts argument (which we may discuss next time). Integrating (11.8) gives the following, important corollary.

Corollary 11.3. *If α has compact support (e.g. if X is compact), then*

$$2\int_X i\,\partial\,\overline{\partial}\,\phi \wedge T_\alpha + \int_X |\,\overline{\partial}\,\gamma_\alpha|^2 e^{-2\phi} dV_\omega = \int_X |\,\overline{\partial}\,\alpha|^2 e^{-2\phi} dV_\omega + \int_X |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} dV_\omega. \tag{11.9}$$

Proof. By Stokes' formula, $\int_X \partial\,\overline{\partial}\,T_\alpha = 0$, and we have

$$\int_X \langle \overline{\partial}\,\overline{\partial}^*_\phi\,\alpha, \alpha\rangle e^{-2\phi} dV_\omega = \int_X |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} dV_\omega$$

by the definition of the adjoint. Thus, (11.9) follows just from integrating (11.8). □

Granted Corollary 11.3, if one knows that $2i\,\partial\,\overline{\partial}\,\phi \geq c\omega$, then one obtains an estimate of the form

$$\int_X |\,\overline{\partial}\,\alpha|^2 e^{-2\phi} dV_\omega + \int_X |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} dV_\omega \geq c\int_X T_\alpha \wedge \omega = cq\int_X |\alpha|^2 e^{-2\phi} dV_\omega,$$

which will be a good starting point in order to apply L^2-methods similar to the ones previously introduced.

12. January 31st

Recall the notation from last time: (X, ω) is a Kähler manifold of dimension n, L is a line bundle on X, and ϕ is a metric on L. If η is an L-valued (n, q)-form, then γ_η is an L-valued $(n-q, 0)$-form satisfying $\eta = \gamma_\eta \wedge \omega_q$, as well as the identities

$$\langle \eta, \xi\rangle e^{-2\phi} dV_\omega = c_{n-q}\eta \wedge \overline{\gamma_\xi} e^{-2\phi},$$

$$|\eta|^2 e^{-2\phi} dV_\omega = c_{n-q}\eta \wedge \overline{\gamma_\eta} e^{-2\phi} = c_{n-q}\gamma_\eta \wedge \overline{\gamma_\eta} \wedge \omega_q e^{-2\phi}.$$

There are further identities that depend on the type of the form:

- If ξ is an L-valued $(p, 0)$-form, then it satisfies
$$|\xi|^2 e^{-2\phi} dV_\omega = c_p \xi \wedge \overline{\xi} \wedge \omega_{n-p} e^{-2\phi}.$$
- If η is an L-valued (n, q)-form, then
$$|\eta|^2 e^{-2\phi} dV_\omega = c_{n-q}\gamma_\eta \wedge \overline{\gamma_\eta} \wedge \omega_q e^{-2\phi}.$$
- If ξ is an L-valued $(p, 1)$-form, then
$$\left(|\xi|^2 - |\xi \wedge \omega_{n-p}|^2\right) e^{-2\phi} dV_\omega = -c_{p+1}\xi \wedge \overline{\xi} \wedge \omega_{n-p-1} e^{-2\phi}.$$

Finally, there are operators $\overline{\partial}\colon \eta_i \otimes e_i \mapsto \overline{\partial}\,\eta_i \otimes e_i$ and $\delta\colon \eta_i \otimes e_i \mapsto \left(e^{2\phi_i}\,\partial\, e^{-2\phi_i}\eta_i\right) \otimes e_i$ satisfying the identity $(\overline{\partial}\,\delta + \delta\,\overline{\partial})\eta = \partial\,\overline{\partial}\,\phi \wedge \eta$ and the Leibniz rules

$$\overline{\partial}(\eta \wedge \overline{\xi} e^{-2\phi}) = \overline{\partial}\,\eta \wedge \overline{\xi} e^{-2\phi} + (-1)^{\deg(\eta)}\eta \wedge \overline{\delta\xi} e^{-2\phi},$$

$$\partial\left(\eta \wedge \overline{\xi} e^{-2\phi}\right) = \delta\eta \wedge \overline{\xi} e^{-2\phi} + (-1)^{\deg(\eta)}\eta \wedge \overline{\overline{\partial}\,\xi} e^{-2\phi}.$$

Finally, there is the relation $\gamma_{\overline{\partial}^*_\phi\,\xi} = i\delta_{\gamma_\xi}$ involving the formal adjoint $\overline{\partial}^*_\phi$; in particular, using that $\xi \mapsto \gamma_\xi$ is an isometry, we have

$$|\,\overline{\partial}^*_\phi\,\xi|^2 e^{-2\phi} = |\gamma_{\overline{\partial}^*_\phi\,\xi}|^2 e^{-2\phi} = |\delta\gamma_\xi|^2 e^{-2\phi}.$$

12.1. The Basic Identity (Continued).

Given an L-valued (n, q)-form α, consider the auxiliary $(n-1, n-1)$-form given by

$$T_\alpha := c_{n-q}\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1} e^{-2\phi}.$$

This is a positive $(n-1, n-1)$-form on X (it is no longer L-valued!).

Theorem 12.1. [The Basic Identity] *If α has compact support, then*

$$2\int_X i\,\partial\,\overline{\partial}\,\phi \wedge T_\alpha + \int_X |\,\overline{\partial}\,\gamma_\alpha|^2 e^{-2\phi} dV_\omega = \int_X |\,\overline{\partial}\,\alpha|^2 e^{-2\phi} dV_\omega + \int_X |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} dV_\omega. \tag{12.1}$$

Proof. Use the Leibnitz rules and that $\partial\omega = \overline{\partial}\omega = 0$ (because ω is a Kähler form) to write

$$\begin{aligned} i\,\partial\overline{\partial} T_\alpha &= ic_{n-q}\,\partial\left(\overline{\partial}\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} + (-1)^{n-q}\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi}\right) \\ &= ic_{n-q}\delta\,\overline{\partial}\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} + ic_{n-q}(-1)^{n-q+1}\,\overline{\partial}\gamma_\alpha \wedge \overline{\overline{\partial}\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} + d\left(ic_{n-q}(-1)^{n-q}\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi}\right) \\ &= (I) + (II) + (III) \end{aligned}$$

Now, Stokes' formula implies that

$$0 = \int_X i\,\partial\overline{\partial} T_\alpha = \int_X (III).$$

Collecting terms, we get that

$$\int_X \delta\,\overline{\partial}\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} + (-1)^{n-q+1}\int_X \overline{\partial}\gamma_\alpha \wedge \overline{\overline{\partial}\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} = 0. \tag{12.2}$$

Further:

$$\begin{aligned} d\left(\delta\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi}\right) &= \overline{\partial}\left(\delta\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi}\right) \\ &= \overline{\partial}\,\delta\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} + (-1)^{n-q+1}\delta\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi}. \end{aligned}$$

Integrating the above expression and using Stokes' formula implies that

$$\int_X \overline{\partial}\,\delta\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} + (-1)^{n-q+1}\int_X \delta\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} = 0. \tag{12.3}$$

Adding (12.2) and (12.3), multiplying by ic_{n-q}, and using the formulas

$$\begin{cases} ic_{n-q}(-1)^{n-q} = c_{n-q+1}, \\ (\delta\,\overline{\partial} + \overline{\partial}\,\delta)\gamma_\alpha = 2\,\partial\overline{\partial}\phi \wedge \gamma_\alpha, \end{cases}$$

yields the equation

$$2\int_X i\,\partial\overline{\partial}\phi \wedge T_\alpha - c_{n-q+1}\int_X \overline{\partial}\gamma_\alpha \wedge \overline{\overline{\partial}\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} - c_{n-q+1}\int_X \delta\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} = 0. \tag{12.4}$$

Now, as $\delta\gamma_\alpha$ is an $(n-q+1,0)$-form, we can use the formula

$$\begin{aligned} c_{n-q+1}\delta\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} &= |\delta\gamma_\alpha|^2 e^{-2\phi}dV_\omega \\ &= |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi}dV_\omega \end{aligned}$$

and, since $\overline{\partial}\gamma_\alpha$ is an $(n-q,1)$-form, we also have

$$\begin{aligned} c_{n-q+1}\,\overline{\partial}\gamma_\alpha \wedge \overline{\overline{\partial}\gamma_\alpha} \wedge \omega_{q-1}e^{-2\phi} &= |\,\overline{\partial}\gamma_\alpha \wedge \omega_q|^2 e^{-2\phi}dV_\omega - |\,\overline{\partial}\gamma_\alpha|^2 e^{-2\phi}dV_\omega \\ &= |\,\overline{\partial}(\gamma_\alpha \wedge \omega_q)|^2 e^{-2\phi}dV_\omega - |\,\overline{\partial}\gamma_\alpha|^2 e^{-2\phi}dV_\omega \\ &= |\,\overline{\partial}\,\alpha|^2 e^{-2\phi}dV_\omega - |\,\overline{\partial}\gamma_\alpha|^2 e^{-2\phi}dV_\omega \end{aligned}$$

where the last equality follows since $\gamma_\alpha \wedge \omega_q = \alpha$, and the second-to-last equality follows since $d\omega = 0$. Substituting these two formulas into (12.4) gives (12.1). □

By dropping the second term on the left-hand side of (12.1), one gets an obvious corollary of the basic identity.

Corollary 12.2. *If α is a smooth (n,q)-form with compact support, then*

$$2\int_X i\,\partial\overline{\partial}\phi \wedge T_\alpha \le \int_X |\,\overline{\partial}\,\alpha|^2 e^{-2\phi}dV_\omega + \int_X |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi}dV_\omega. \tag{12.5}$$

Corollary 12.3. *If $i\,\partial\overline{\partial}\phi \ge c\omega$ for some positive constant $c > 0$ (in particular, ϕ is a positive metric), then*

$$\int_X |\,\overline{\partial}\,\alpha|^2 e^{-2\phi}dV_\omega + \int_X |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi}dV_\omega \ge 2cq\int_X |\alpha|^2 e^{-2\phi}dV_\omega. \tag{12.6}$$

Proof. The inequality $i\,\partial\overline{\partial}\,\phi \geq c\omega$ means that one can write $i\,\partial\overline{\partial}\,\phi = c\omega + \beta$, where β is a positive $(1,1)$-form on X. The positivity of T_α and β (and that β is of bidegreee $(1,1)$) implies that $\beta \wedge T_\alpha$ is a positive (n,n)-form; in particular, $\int_X \beta \wedge T_\alpha \geq 0$. Also, we have

$$\begin{aligned}\int_X \omega \wedge T_\alpha &= \int_X \omega \wedge c_{n-q}\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_{q-1} e^{-2\phi} \\ &= q \int_X c_{n-q}\gamma_\alpha \wedge \overline{\gamma_\alpha} \wedge \omega_q e^{-2\phi} \\ &= q \int_X c_{n-q}\alpha \wedge \overline{\gamma_\alpha} e^{-2\phi} \\ &= q \int_X |\alpha|^2 e^{-2\phi} dV_\omega.\end{aligned}$$

Combining the above expression with (12.5) immediately gives (12.6). □

12.2. **Setup for the Hilbert Space Machinery.** Consider the sequence of closed and densely-defined operators

$$H_1 \xrightarrow{T=\overline{\partial}} H_2 \xrightarrow{S=\overline{\partial}} H_3,$$

where

$$H_2 = \left\{ L\text{-valued } (n,q)\text{-forms } \alpha \text{ with } L^2_{\text{loc}}\text{-coefficients such that } \|\alpha\|^2 := \int_X |\alpha|^2 e^{-2\phi} dV_\omega \right\},$$

and the Hilbert spaces H_1 and H_3 are defined similarly, with $(n,q-1)$ and $(n,q+1)$-forms respectively. Corollary 12.3 gives the bound

$$\|T^*\alpha\|^2 + \|S\alpha\|^2 \geq 2cq\|\alpha\|^2$$

for all smooth, compactly-support α, and we must extend the bound to all forms $\alpha \in D_{T^*} \cap D_S$. This does not work in general! (For example, if $X = U \subseteq \mathbf{C}^n$ is an open subset, $L = \mathcal{O}_X$, and ϕ is strictly psh, then we must assume that U is pseudoconvex!).

The methods works when

- X is compact;
- (X,ω) is a *complete* Kähler manifold (i.e. the associated Riemannian metric on X is complete);
- (X,ω) is Kähler, and X admits some other complete Kähler metric.

The method used to tackle the final condition is analogous to the approach we used to prove Hörmander's theorem for a pseudoconvex domain in $\mathbf{C}^n$.

The goal of next class is to discuss the approxmation result required to make the above method work.

Theorem 12.4. [Approximation Lemma] *Assume that X is compact, and α is an L-valued (n,q)-form.*

(1) if α is smooth, then $\alpha \in D_{T^*} \cap D_S$ and $T^*\alpha = \overline{\partial}^*_\phi\,\alpha$;
(2) if $\alpha \in D_{T^*} \cap D_S$, then there exists a sequence $(\alpha_k)_{k=1}^\infty$ of L-valued smooth forms such that

$$\|\alpha_k - \alpha\|, \|S\alpha_k - S\alpha\|, \|T^*\alpha_k - T^*\alpha\| \longrightarrow 0 \text{ as } k \to +\infty.$$

The idea of the proof of the approximation lemma is to use convolution, which we will outline next time.

13. February 2nd

13.1. **The Proof of Hörmander's Theorem on Compact Kähler Manifolds.** The goal of today's class is to complete the proof of Hörmander's theorem in the geometric setting, which we recall below.

Theorem 13.1. [Hörmander's Theorem] *Let (X,ω) be a compact Kähler manifold of dimension n, let L be a line bundle on X, and let ϕ be a metric on L such that*

$$2i\,\partial\,\overline{\partial}\,\phi \geq c\omega.$$

Given an (n,q)-form f with values in L such that $\overline{\partial}\,f = 0$, then one can solve $\overline{\partial}\,u = f$ with the estimate

$$\int_X |u|^2 e^{-2\phi} dV_\omega \leq \frac{1}{cq}\int_X |f|^2 e^{-2\phi} dV_\omega,$$

provided the right-hand side is finite.

Last time, we proved the basic identity, from which we deduced the corollary below.

Corollary 13.2. [Of the Basic Identity] *With notation as in Theorem 13.1, for any L-valued (n,q)-form α (with compact support), we have*

$$c\|\alpha\|^2 \leq \|\,\overline{\partial}\,\alpha\|^2 + \|\,\overline{\partial}^*_\phi\,\alpha\|^2.$$

Consider the seq uence

$$H_1 \xrightarrow{T=\overline{\partial}} H_2 \xrightarrow{S=\overline{\partial}} H_3$$

where H_1 consists of $(n,q-1)$ forms, H_2 consists of (n,q)-forms, and H_3 consists of $(n,q+1)$-forms.

Theorem 13.3. [Approximation Lemma] *Assume X is compact and $\alpha \in H_2$.*

(1) If α is smooth, then $\alpha \in D_{T^*} \cap D_S$ and $T^*\alpha = \overline{\partial}^*_\phi\,\alpha$.
(2) If $\alpha \in D_S \cap D_{T^*}$, then there exists a sequence $(\alpha_k)_{k=1}^\infty$ of smooth (n,q)-forms such that

$$\|\alpha_k - \alpha\|, \|S\alpha_k - S\alpha\|, \|T^*\alpha_k - T^*\alpha\| \longrightarrow 0 \text{ as } k \to +\infty.$$

Proof. For (1), it is clear that $\alpha \in D_S$. Moreover, the statement that $\alpha \in D_{T^*}$ and $T^*\alpha = \overline{\partial}^*_\phi\,\alpha$ means that

$$\int_X \langle \overline{\partial}\,u, \alpha\rangle e^{-2\phi} dV_\omega = \int_X \langle u, \overline{\partial}^*_\phi\,\alpha\rangle e^{-2\phi} dV_\omega$$

for all $u \in D_T$ (and a form u lies in D_T if $u \in H_1$ and $\overline{\partial}\,u$, computed in the sense of distributions, lies in H_2). This equality holds by the definition of the distributional derivative (because α is a test form).

For (2), we will only sketch the argument. We first prove that for $\alpha \in D_S \cap D_{T^*}$ and $\chi \in C^\infty(X)$ (either real or complex-valued), then $\chi\alpha \in D_S \cap D_{T^*}$. That $\chi\alpha \in D_S$ is easy, since

$$\overline{\partial}\,(\chi\alpha) = \overline{\partial}\,\chi \wedge \alpha + \chi\,\overline{\partial}\,\alpha$$

in the sense of distributions. Further, one must check that $\chi\alpha \in D_{T^*}$: this amounts to showing that there exists a constant $C > 0$ such that

$$|\langle \overline{\partial}\,u, \chi\alpha\rangle_{H_2}| \leq C\|u\|_{H_1}$$

for all $u \in D_T$. Observe that

$$\begin{aligned}\langle \overline{\partial}\,u, \chi\alpha\rangle_{H_2} &= \int_X \langle \overline{\partial}\,u, \chi u\rangle e^{-2\phi} dV_\omega \\ &= \langle \overline{\chi}\,\overline{\partial}\,u, \alpha\rangle_{H_2} \\ &= \langle \overline{\partial}(\overline{\chi}u), \alpha\rangle_{H_2} - \langle \overline{\partial}\,\overline{\chi} \wedge u, \alpha\rangle_{H_2}.\end{aligned}$$

As $u \in D_T$ (hence $\overline{\chi}u \in D_T$) and $\alpha \in D_{T^*}$, it follows from the above equality that there are constants $C, C' > 0$ such that

$$|\langle \overline{\partial}(\overline{\chi}u, \alpha\rangle_{H_2}| \leq C\|\overline{\chi}u\|_{H_1} \leq C'\|u\|_{H_1}$$

and

$$|\langle \overline{\partial}\,\chi \wedge u, \alpha\rangle_{H_2}| \leq C\|u\|_{H_1},$$

where the last equality follows by the Cauchy–Schwarz inequality. We conclude that $\chi\alpha \in D_S \cap D_{T^*}$.

Now, we can use a partition of unity (ψ_j) to write $\alpha = \sum_j \psi_j \alpha$ and we can approximate each term. Thus, we can assume WLOG that α has support in a coordinate chart $z\colon U \hookrightarrow \mathbf{C}^n$ where $L|_U$ is trivial. We can now use convolutions: pick $\chi\colon \mathbf{C}^n \to \mathbf{R}$ with compact support and $\int_{\mathbf{C}^n} \chi = 1$, and set $\chi_k(z) := k^{2n}\chi(kz)$ and $\alpha_k := \alpha * \chi_k$. Thus, it follows from the general properties of convolution that

$$\|\alpha_k - \alpha\| \longrightarrow 0 \text{ as } k \to +\infty.$$

However, we also need to show that $\|S\alpha_k - S\alpha\| \to 0$ and $\|T^*\alpha_k - T^*\alpha\| \to 0$ as $k \to +\infty$. For $S = \overline{\partial}$, this is ok because S has *constant coefficients*, i.e. we have

$$\overline{\partial}(\alpha * \chi_k) = \overline{\partial}\,\alpha * \chi_k.$$

Therefore, the same argument gives that $\|S\alpha_k - S\alpha\| \to 0$ as $k \to +\infty$. For T^*, one must be more careful because it does not have constant coefficents; indeed, when $n = 1$, recall that

$$\overline{\partial}^*_\phi(f dz) = \frac{\partial f}{\partial z} - 2\frac{\partial \phi}{\partial z} f.$$

To conclude the estimate for T^*, one must use Friedrich's lemma; see [Dem12, VIII,3.3]. This concludes the (sketch of the) proof of the approximation lemma. □

13.2. **Hörmander's Theorem on Noncompact Kähler Manifolds.** Let (X, ω) be a Kähler manifold of dimension n and we no longer assume that X is compact. Let L and ϕ be as before; in particular, we have the estimate

$$2i\,\partial\,\overline{\partial}\,\phi \geq c\omega$$

on the curvature of ϕ. Then, the statement of Hörmander's theorem is the same, assuming X admits some other complete Kähler metric ω'.

Example 13.4. This condition is satisfied when X is a Stein manifold (e.g. a pseudoconvex open subset of $\mathbf{C}^n$).

Granted this additonal assumption, the proof of Hörmander's theorem is as follows:

Step 1. Assume $\omega' = \omega$, so ω is complete. Use this to prove the density of test forms in $D_S \cap D_{T^*}$.

Step 2. Apply Step 1 for $(X, \omega + k^{-1}\omega')$ (which is possible since $\omega + k^{-1}\omega'$ remains complete for $k \in \mathbf{Z}_{>0}$), and solve $\overline{\partial}\,u_k = f$ with estimates. Then, one shows that $u_k \to u$, which is a solution with the required estimates.

13.3. **Relation to Kodaira's Theorems.**

Theorem 13.5. [The Kodaira Vanishing Theorem] *If X is compact of dimension n and L is a* positive *line bundle on X (i.e. there exists ϕ such that $i\,\partial\,\overline{\partial}\,\phi > 0$), then the Dolbeault cohomology satisfies*

$$H^{n,q}(X, L) = 0$$

for $q > 0$.

Remark 13.6. The algebro-geometric incarnation of Kodaira's vanishing theorem says that if X is projective and L is ample, then $H^q(X, K_X + L) = 0$ for $q > 0$.

Theorem 13.5 is almost a consequence of Hörmander's theorem, except the solution u to the equation $\overline{\partial}\,u = f$ is a priori not smooth. One can either modify the approach in the proof of Hörmander's theorem to guarantee u is smooth (but one loses the estimates), or one can use Hodge theory. Hodge theory shows that each element of $H^{n,q}(L)$ is represented by a harmonic form α, i.e. $\Delta\alpha = (\overline{\partial}^*_\phi\,\overline{\partial} + \overline{\partial}\,\overline{\partial}^*_\phi)\alpha = 0$. This occurs iff $\overline{\partial}\,\alpha = \overline{\partial}^*_\phi\,\alpha = 0$. The Laplacian $\Delta\alpha$ is a zeroth-order elliptic operator, so one can use strong results from the theory of elliptic PDE's; in particular, we can use the estimate

$$0 = \|\,\overline{\partial}\,\alpha\|^2 + \|\,\overline{\partial}^*_\phi\,\alpha\|^2 \geq \|\alpha\|^2$$

where the norms are taken with respect to the Kähler form $\omega := 2i\,\partial\,\overline{\partial}\,\phi$. It is thus clear that, if $\overline{\partial}\,\alpha = 0$ and $\overline{\partial}^*_\phi\,\alpha = 0$, then $\alpha = 0$.

From Theorem 13.5, we can deduce another important result due to Kodaira.

Theorem 13.7. [Kodaira Embedding Theorem] *If X is compact and there exists a positive line bundle L on X, then X is* projective*, i.e. there exists a closed embedding $X \hookrightarrow \mathbf{P}^N_{\mathbf{C}}$.*

Idea of Proof of Theorem 13.7. For $m \gg 0$, given a basis $s_0, \ldots, s_{N_m}$ of holomorphic sections of $\omega_X \otimes L^m$ (i.e. global holomorphic n-forms with values in L^m), define a meromorphic map

$$X \dashrightarrow \mathbf{P}\left(H^0(X, \omega_X \otimes L^m)^*\right)$$

given by $x \mapsto [s_0(x) \colon \ldots \colon s_{N_m}(x)]$, and one checks that this map is well-defined (i.e. holomorphic), injective, and that is separates 1-jets.

We will just explain the first property: given $x \in X$, we must find a (global) holomorphic n-form s with values in L such that $s(x) \neq 0$. The idea is to construct s as $s = \eta - u$, where η is a locally-defined smooth form at x (with some additional properties) and u is a solution to $\overline{\partial}\, u = \overline{\partial}\, \eta$ (so that $\overline{\partial}\, s = 0$) with estimates that guarantee $u(x) = 0$ and $\eta(x) \neq 0$. □

14. February 5th

Let (X, ω) be compact Kähler manifold of dimension n, let L be a line bundle on X, and let ϕ be a smooth metric on L such that $2i\,\partial\overline{\partial}\,\phi \geq \epsilon\omega$. In this setting, we completed the proof of Hörmander's theorem, which asserts the following: for any L-valued (n,q)-form f such that $\overline{\partial}\, f = 0$ and $\int_X |f|^2 e^{-2\phi} dV_\omega < +\infty$, then there exists an L-valued $(n, q-1)$-form u such that $\overline{\partial}\, u = f$ and with the estimate

$$\int_X |u|^2 e^{-2\phi} dV_\omega \leq \frac{1}{\epsilon q} \int_X |f|^2 e^{-2\phi} dV_\omega.$$

14.1. Smoothness of Solutions.

In Hörmander's theorem, one can always pick the solution u to have minimal norm (though there is not a unique such u), and this condition is equivalent to demanding that $u \in \ker(\overline{\partial})^\perp$. Now, assume that f is smooth and u is a minimal solution to $\overline{\partial}\, u = f$. One can consider the Laplace operator

$$\Delta \coloneqq \overline{\partial}^*_\phi\, \overline{\partial} + \overline{\partial}\, \overline{\partial}^*_\phi,$$

thought of as an operator on the space of $(n, q-1)$-forms with values in L. As $u \in \ker(\overline{\partial})^\perp = \overline{\operatorname{im}(\overline{\partial}^*_\phi)}$, we have $\overline{\partial}^*_\phi\, u = 0$; in particular,

$$\Delta u = (\overline{\partial}^*_\phi\, \overline{\partial} + \overline{\partial}\, \overline{\partial}^*_\phi) u = \overline{\partial}^*_\phi,$$

which is smooth. It follows from elliptic regularity that u is smooth.

14.2. The Kodaira Embedding Theorem (Continued).

We continue the proof of Kodaira's embedding theorem (Theorem 13.7), which we began last time. Let X be compact and let L be a *positive* line bundle. Fix $m \gg 1$. Given $x \in X$, construct global holomorphic L^m-valued $(n, 0)$-forms on X that do not vanish at x (where m is chosen independent of x). This gives rise to a holomorphic map

$$X \longrightarrow \mathbf{P}\left(H^0(X, \omega_X \otimes L^m)^*\right).$$

The construction of such sections is discussed below.

Pick a (smooth) positive metric ϕ on L, and consider the Kähler form $\omega \coloneqq 2i\,\partial\overline{\partial}\,\phi$. Pick open neighbourhoods $V \Subset W \Subset U$ of x such that there exists a coordinate chart $z \colon U \hookrightarrow \mathbf{C}^n$ with $z(x) = 0$ and $L|_U$ is trivial. Pick $\chi \in C^\infty(X)$ such that $0 \leq \chi \leq 1$, $\operatorname{supp}(\chi) \subseteq U$, and $\chi \equiv 1$ on W. Given $\delta > 0$, set

$$\psi_\delta(y) \coloneqq (n+1)\chi(y)) \cdot \frac{1}{2} \log\left(|z(y)|^2 + \delta^2\right).$$

Then, $\psi_\delta \in C^\infty(X)$, $\operatorname{supp}(\psi_\delta) \subseteq U$, $\delta \mapsto \psi_\delta$ is increasing, and

$$\lim_{\delta \to 0} \psi_\delta(y) = (n+1)\chi(y) \log|z(y)| = \psi_0(y)$$

has a singularity at $y = x$ (indeed, it is $-\infty$ there and we understand its growth as $y \to x$). Furthermore, $i\,\partial\overline{\partial}\,\psi_\delta \geq 0$ on W (i.e. on the locus where $\chi \equiv 1$). We want to "add" ψ_δ to the metric ϕ and apply Hörmander's theorem with the resulting metric.

Pick $m \gg 1$ and $\epsilon > 0$ such that

$$2i\,\partial\overline{\partial}\,(m\phi + \psi_\delta) \geq \epsilon\omega \tag{14.1}$$

for every $\delta \in (0, 1)$, where $m\phi + \psi_\delta$ is thought of as a metric on L^m. (In addition, the estimate (14.1) can be made uniform in x.) Pick a holomorphic section σ of $L|_U$ and assume that $\sigma(x) \neq 0$ (this is possible since $L|_U$ is trivial, so one can e.g. take a trivialization). Set

$$\eta \coloneqq \chi \cdot \sigma^m dz_1 \wedge \ldots \wedge dz_n,$$

which is a smooth (but not holomorphic) L^m-valued $(n, 0)$-form on X with $\mathrm{supp}(\eta) \subseteq U$ and $\overline{\partial}\,\eta = 0$ on W (i.e. on the locus where $\chi \equiv 1$). Set $f \coloneqq \overline{\partial}\,\eta$, which is a smooth L^m-valued $(n, 1)$-form on X such that $\mathrm{supp}(f) \subseteq U \backslash W$ and $\overline{\partial}\,f = 0$.

Now, use Hörmanders' theorem to get u_δ, which is an L^m-valued $(n, 0)$-form on X such that $\overline{\partial}\,u_\delta = f = \overline{\partial}\,\eta$ and with the estimate

$$\int_X |u_\delta|^2 e^{-2(m\phi+\psi_\delta)} dV_\omega \leq \frac{1}{\epsilon} \int_X |f|^2 e^{-2(m\phi+\psi_\delta)} dV_\omega, \tag{14.2}$$

The right-hand side of (14.2) is bounded above by some constant $C > 0$ that is independent of δ. We want to take $u = \lim_{\delta\to 0} u_\delta$, but we must be careful about the sense in which this limit is taken. If we set

$$H_\delta \coloneqq \left\{ v \colon \int_X |v|^2 e^{-2(m\phi+\psi_\delta)} dV_\omega < +\infty \right\},$$

then there are inclusions $H_\delta \hookrightarrow H_{\delta'}$ for $\delta < \delta'$. By a diagonal argument and weak compactness (by the Banach–Alaoglu theorem), there exists a sequence $\delta_j \searrow 0$ and $u \in \bigcap_j H_{\delta_j}$ such that $\|u\|_{H_{\delta_j}} \leq C$ for all j and $u_{\delta_j} \to u$ weakly in H_{δ_j} for all j; thus, $u_{\delta_j} \to u$ in the sense of distributions, hence $\overline{\partial}\,u_{\delta_j} \to \overline{\partial}\,u$ in the sense of distributions. It follows that $\overline{\partial}\,u = f$.

Set $s \coloneqq \eta - u$, then $\overline{\partial}\,s = \overline{\partial}\,\eta - \overline{\partial}\,u = f - f = 0$ in the sense of distributions. However, if the $(n, 0)$-form s with values in L^m satisfies $\overline{\partial}\,s = 0$ in the sense of distributions, then s is holomorphic. In particular, u is smooth (because η is smooth) and it is holomorphic on W. However, we also have the estimates

$$\int_X |u|^2 e^{-2(m\phi+\psi_{\delta_j})} dV_\omega \leq C \tag{14.3}$$

for all j, so the monotone convergence theorem implies that there is the estimate

$$\int_X |u|^2 e^{-2(m\phi+\psi_0)} dV_\omega \leq C. \tag{14.4}$$

Rewrite $e^{-2\psi_0(y)} = |z(y)|^{-(n+1)}$ for y in the neighbourhood W of x, which is *not* locally integrable at x. Therefore, the only way that (14.4) can hold is if $u(x) = 0$, and hence $s(x) \neq 0$, as required.

14.3. **The Ohsawa–Takegoshi Theorem.** The next goal of the class is to discuss the extension theorem of Ohsawa–Takegoshi, which originates in [OT87] and it has since been generalized in many different directions. Later, we will focus on the geometric version of the extension theorem, but we will begin with the function-theoretic version.

Consider a bounded, pseudoconvex domain $U \subseteq \mathbf{C}^n$ (so there is a "good supply" of holomorphic functions on U, for example a polydisc), and fix a hyperplane $H \subseteq \mathbf{C}^n$ such that $U \cap H \neq \emptyset$.

Question 14.1. Given a holomorphic function on f on $U \cap H$, does there exist F on U such that $F|_{U\cap H} = f$?

The Ohsawa–Takegoshi theorem asserts that the answer to Question 14.1 is yes, and this can be done *with estimates.*

Theorem 14.2. [Ohsawa–Takegoshi Theorem] *There exists a constant $C = C(U, H) > 0$ such that given any (smooth) $\phi \in \mathrm{PSH}(U)$ and $f \in \mathcal{O}(U \cap H)$ such that $\int_{U\cap H} |f|^2 e^{-2\phi} < +\infty$, then there exists $F \in \mathcal{O}(U)$ such that $F|_U = f$ and with the estimate*

$$\int_U |F|^2 e^{-2\phi} \leq C \int_{U\cap H} |f|^2 e^{-2\phi}.$$

The key point is that the constant C is independent of both ϕ and f. Note that all of the integrals appearing in Theorem 14.2 are taken with respect to Lebesgue measure on the appropriate subset of $\mathbf{C}^n$.

Geometrically, it is more natural to look at $(n, 0)$-forms, in which case "restriction" means to apply the Poincaré residue map (i.e. the map appearing in the adjunction formula "$K_X + H|_H \simeq K_H$").

15. February 7th

15.1. The Ohsawa–Takegoshi Extension Theorem in the Unit Disc.

The simplest, nontrivial case of the Ohsawa–Takegoshi extension theorem takes places in the unit disc, where we are attempting to construct a holomorphic function with a specific value at (say) the origin and satisfying an appropriate integral estimate.

Theorem 15.1. *There exists a universal constant $C > 0$ such that for all $\phi \in \mathrm{SH}(\mathbf{D}) \cap C^\infty(\mathbf{D})$, there exists $h \in \mathcal{O}(\mathbf{D})$ such that $h(0) = 1$ and*

$$\int_{\mathbf{D}} |h|^2 e^{-2\phi} dV \leq C e^{-2\phi(0)} \tag{15.1}$$

The assumption that ϕ be smooth is not necessary, but we add this first for simplicity. The main point is that C does not depend on ϕ, which means that one can derive the result for any $\phi \in SH(\mathbf{D})$, which is no longer assumed to be smooth.

Moreover, one can specify that h have any value at the origin 0 (or at any point in $\mathbf{D}$), provided one adds a factor of $|h(0)|^2$ to the right-hand side of (15.1).

There are alternate versions of Theorem 15.1, where one replaces (15.1) with the sharper result

$$\int_{\mathbf{D}} \frac{|h|^2}{(|z| \log |z|)^2} e^{-2\phi} dV \leq C |h(0)|^2 e^{-2\phi(0)}.$$

This inequality is still not optimal, but we will later proceed to the geometric setting, instead of optimizing the classical setting.

Remark 15.2. There is an "adjoint version" of Theorem 15.1, where one replaces h by $h(z)\frac{dz}{z}$, but it is the same estimate.

We will follow Berndtsson's approach in [Ber10]. The idea is to write $h = u \cdot z$, where u solves (in the sense of distributions) the equation

$$\frac{\partial u}{\partial z} = \delta_0$$

with suitable estimates, where δ_0 denotes the Dirac mass at 0.

15.2. Distributions & Currents in C.

Use the coordinate $z = x + iy$ on $\mathbf{C}$, and pick a Kähler form

$$\omega := \frac{i}{2} dz \wedge d\,\overline{z} = dz \wedge dy = dV.$$

Consider the space[5] $D := D(\mathbf{C}) = C_0^\infty(\mathbf{C})$ of test functions on $\mathbf{C}$, and the space D' of distributions on $\mathbf{C}$ is the set of continuous linear functional on D. There is an obvious embedding $D \subseteq L^1_{\mathrm{loc}}$, as well as an embedding $L^1_{\mathrm{loc}} \hookrightarrow D'$ given by

$$\alpha \mapsto \left(\beta \mapsto \int_{\mathbf{C}} \alpha\beta dV\right).$$

Operations on D' are defined to commute with the embedding $D \hookrightarrow D'$, and a few of these are illustrated below:

[5] The topology on D is the Fréchet topology where we demand that all possible derivatives converge on all compact subsets.

- *Differentiation*: if $u \in D'$ and $\alpha \in D$, then $\langle \frac{\partial u}{\partial z}, \alpha\rangle := -\langle u, \frac{\partial \alpha}{\partial z}\rangle$, and similarly for $\frac{\partial u}{\partial \overline{z}}$.
- *Multiplication*: if $u \in D'$ and $\alpha, \beta \in D$, then $\langle \alpha u, \beta\rangle := \langle u, \alpha\beta\rangle$.

These are the main two operations that we will use.

For $0 \leq p, q \leq 1$, let $D_{(p,q)}$ be the set of test (p,q)-forms on $\mathbf{C}$, and the space $D'_{(p,q)}$ of currents of *bidegree* (p,q) is defined to be the set of continuous linear functions on $D_{(1-p,1-q)}$ (often, we say that the elements of $D'_{(p,q)}$ are of *bidimension* (p,q)). Similar to the case of distributions, there is an embedding $D_{(p,q)} \hookrightarrow D'_{(p,q)}$ given by

$$\alpha \mapsto \left(\beta \mapsto \int_{\mathbf{C}} \alpha \wedge \beta\right).$$

There are natural operations that one can define on currents:

- *Wedge Product*: if $T \in D'_{(1-p-r,1-q-s)}$, $\alpha \in D_{(p,q)}$, $\beta \in D_{(r,s)}$, then $\langle T \wedge \alpha, \beta\rangle := \langle T, \alpha \wedge \beta\rangle$.
- *Differentiation*: define $\partial \colon D'_{(p,q)} \to D'_{(p+1,q)}$ by the formula
$$\langle \partial T, \alpha\rangle := (-1)^{p+q+1}\langle T, \partial \alpha\rangle,$$
and similarly one defines $\overline{\partial}\colon D'_{(p,q)} \to D'_{(p,q+1)}$. From ∂ and $\overline{\partial}$, set $d := \partial + \overline{\partial}$ and $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$, and the Laplacian-type operator $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

One can realize currents as "forms with distribution-coefficients" as follows: consider the isomorphism

$$D'_{(0,0)} \xrightarrow{\simeq} D'$$

given by

$$T \mapsto (\alpha \mapsto \langle T, \alpha dV\rangle).$$

Similarly, $D'_{(1,0)} \simeq D'dz$, $D'_{(0,1)} \simeq D'd\overline{z}$, and $D'_{(1,1)} \simeq D'dz \wedge d\overline{z}$ via analogous isomorphism.

Lemma 15.3. *Computed in D', we have the equalities*

$$\frac{\partial}{\partial \overline{z}}\frac{1}{z} = \frac{\partial}{\partial z}\frac{1}{\overline{z}} = \pi\delta_0.$$

Proof. Use the Cauchy integral formula: if $\Omega \subseteq \mathbf{C}$ is open and bounded with $\partial\Omega$ consisting of finitely-many Jordan curves (i.e. we are able to integrate along the boundary), then

$$u(\xi) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{u(z)}{z-\xi}dz + \frac{1}{2\pi i}\int_{\Omega}\frac{\frac{\partial u}{\partial \overline{z}}}{z-\xi}dz \wedge d\overline{z} \tag{15.2}$$

for all $u \in C^1(\overline{\Omega})$ (i.e. u is the restriction of a C^1-function on an open neighbourhood of $\overline{\Omega}$); see the first chapter of [H$\ddot{9}$0]. Now, if $\alpha \in D$, then

$$\begin{aligned}\langle \frac{\partial}{\partial \overline{z}}\frac{1}{z}, \alpha\rangle &= -\langle \frac{1}{z}, \frac{\partial \alpha}{\partial \overline{z}}\rangle \\ &= -\int_{\mathbf{C}}\frac{1}{z}\frac{\partial \alpha}{\partial \overline{z}} \\ &= -\frac{i}{2}\int_{\mathbf{C}}\frac{1}{z}\frac{\partial \alpha}{\partial \overline{z}}dz \wedge d\overline{z} \\ &= -\frac{i}{2}\cdot 2\pi i\alpha(0) \\ &= \pi\alpha(0),\end{aligned}$$

where the second-to-last equality follows from Cauchy's integral formula (15.2). Notice that, since α has compact support, we can take all of the integrals to be over some large disc containing the origin. □

Corollary 15.4. *Computed in $D'_{(1,1)}$, we have $dd^c \log|z| = \delta_0$.*

For most purposes (as in Corollary 15.4), it is convenient to think of measures as currents of maximal bidegree. In fact, Corollary 15.4 is a special case of the more general Poincaré–Lelong formula.

Proof. Write $\log|z| = \frac{1}{2}\log(z\,\overline{z})$ so that

$$dd^c \log|z| = \frac{i}{2\pi}\,\partial\overline{\partial}\log(z\,\overline{z}) = \frac{i}{2\pi}\,\partial\left(\frac{1}{z}d\,\overline{z}\right) = \frac{i}{2\pi}\pi\delta_0 dz\wedge d\,\overline{z} = \delta_0\cdot dV.$$

□

Fact 15.5. If $u \in D'$ satisfies $\frac{\partial\, u}{\partial\, \overline{z}} = 0$, then u is holomorphic (in particular, it is a C^∞-function).

This is a simple case of a more general phenomenon known as elliptic regularity, and it also follows from Weyl's lemma (which is the analogous fact for (real) harmonic functions).

15.3. **The Ohsawa–Takegoshi Extension Theorem in the Unit Disc (Continued).** Recall that we must construct $h \in \mathcal{O}(\mathbf{D})$ such that $h(0) = 1$ and satisfying the L^2-estimate

$$\int_{\mathbf{D}} |h|^2 e^{-2\phi} dV \le Ce^{-2\phi(0)}.$$

Consider the following heuristics: suppose h exists and write $h = u\cdot z$ for some $u \in L^1_{\mathrm{loc}} \subseteq D'$, which implies that $\frac{\partial\, u}{\partial\, \overline{z}} = \pi\delta_0$ in D'. If $\alpha \in D(\mathbf{D})$, then this formally gives

$$\begin{aligned}\pi\overline{\alpha(0)}e^{-2\phi(0)} &= \int_{\mathbf{D}} \frac{\partial\, u}{\partial\, \overline{z}}\overline{\alpha}e^{-2\phi}dV \\ &= \int_{\mathbf{D}} u\overline{\left(\frac{\partial}{\partial\,\overline{z}}\right)^*\alpha}e^{-2\phi}dV \\ &= \int_{\mathbf{D}} h\frac{\overline{\left(\frac{\partial}{\partial\,\overline{z}}\right)^*\alpha}}{\overline{z}}e^{-2\phi}dV,\end{aligned}$$

where $\left(\frac{\partial}{\partial\,\overline{z}}\right)^*$ denotes a suitable adjoint operator. Using the Cauchy–Schwarz inequality to estimate this last integral, we find that

$$\pi^2|\alpha(0)|^2e^{-4\phi(0)} \le \underbrace{\left(\int_{\mathbf{D}} |h|^2e^{-2\phi}dV\right)}_{\le Ce^{-2\phi(0)}}\left(\int_{\mathbf{D}}\frac{\left|\left(\frac{\partial}{\partial\,\overline{z}}\right)^*\alpha\right|^2}{|z|^2}e^{-2\phi}dV\right).$$

Simplifying, this implies that

$$|\alpha(0)|^2e^{-2\phi(0)} \le \frac{C}{\pi^2}\int_{\mathbf{D}}\frac{\left|\left(\frac{\partial}{\partial\,\overline{z}}\right)^*\alpha\right|^2}{|z|^2}e^{-2\phi}dV \tag{15.3}$$

for all $\alpha \in D(\mathbf{D})$. There is one problem with this, namely that $|z|^{-2}$ is not locally integrable, so this last integral will often be $+\infty$, and so (15.3) may not have much content to it.

The idea is to prove a modified version of the estimate (15.3), and then use the Hahn–Banach theorem and the Riesz representation theorem in order to reconstruct a holomorphic function h with the desired properties.

16. February 9th

16.1. **The Ohsawa–Takegoshi Extension Theorem in the Unit Disc (Continued).** Recall that the goal is to show the following: there exists a constant $C > 0$ such that for all $\phi \in \mathrm{SH}(\mathbf{D}) \cap C^\infty(\mathbf{D})$, there exists $h \in \mathcal{O}(\mathbf{D})$ such that $h(0) = 1$ and with the estimate

$$\int_{\mathbf{D}} |h|^2 e^{-2\phi} dV \leq C e^{-2\phi(0)}. \tag{16.1}$$

We will prove this with $C = 8\pi$, which is larger than the optimal constant $C = \pi$.

The proof, like the proof of Hörmander's theorem, uses the Hilbert space approach. We will work with ($\mathcal{O}_X$-valued) forms, as opposed to functions. Fix the Kähler form

$$\omega := \frac{i}{2} dz \wedge d\bar{z} = dV,$$

so the norms of dz and $d\bar{z}$ with respect to this metric are $|dz| = |d\bar{z}| = \sqrt{2}$. For $\alpha \in D_{(1,1)}(\mathbf{D}) = D_{(1,1)}$, write $\alpha = \gamma \wedge \omega$, where $\gamma = \gamma_\alpha \in D$ is such that $\alpha = \frac{i}{2}\gamma dz \wedge d\bar{z}$. As before, we have operators $\bar{\partial}$ and $\delta = \delta_\phi = e^{2\phi}\, \partial\, e^{-2\phi}$ on such forms.

Proposition 16.1. *There exists $r < 1$ and $C = C_r > 0$ such that for all test functions $\gamma \in D$,*

$$|\gamma(0)|^2 e^{-2\phi(0)} \leq C \int_{\mathbf{D}} \frac{|\delta\gamma|^2}{|z|^{2r}} e^{-2\phi} dV. \tag{16.2}$$

Proposition 16.1 implies the Ohsawa–Takegoshi Theorem. Set

$$V := \left\{ \frac{\delta\gamma}{|z|^r} : \gamma \in D \right\} \subseteq L^2_{(1,0)}(\phi),$$

and Proposition 16.1 implies that $\frac{\delta\gamma}{|z|^r} \mapsto \overline{\gamma(0)} e^{-\phi(0)}$ is a bounded antilinear functional on V of norm $\leq \sqrt{C}$. By the Hahn–Banach theorem and the Riesz representation theorem, there exists $\eta = \eta_r \in L^2_{(1,0)}(\phi)$ such that

$$\int_{\mathbf{D}} |\eta|^2 e^{-2\phi} dV \leq C \tag{16.3}$$

and

$$\overline{\gamma(0)} e^{-\phi(0)} = \int_{\mathbf{D}} \langle \eta, \frac{\delta\gamma}{|z|^r} \rangle e^{-2\phi} dV = \int_{\mathbf{D}} \eta \wedge \overline{\frac{\delta\gamma}{|z|^r}} e^{-2\phi} \tag{16.4}$$

for all $\gamma \in D$. Set $u = u_r := \frac{\eta}{|z|^r} e^{-\phi(0)} \in L^1_{(1,0)}(\mathrm{loc}) \subseteq D'_{(1,0)}$. Now, (16.3) and (16.4) imply that

$$\begin{cases} \bar{\partial} u = \delta_0, \\ \int_{\mathbf{D}} |u|^2 |z|^{2r} e^{-2\phi} \leq C e^{-2\phi(0)}. \end{cases} \tag{16.5}$$

Let us check the equality $\overline{\partial}\, u = \delta_0$ of currents more carefully: any test function can be written as $\overline{\gamma} e^{-2\phi}$ for some test function γ, and so we can check

$$\begin{aligned}\langle \overline{\partial}\, u, \overline{\gamma} e^{-2\phi}\rangle &= -\langle u, \overline{\partial}\left(\overline{\gamma} e^{-2\phi}\right)\rangle \\ &= -\int_{\mathbf{D}} u \wedge \overline{\partial}(\overline{\gamma} e^{-2\phi}) \\ &= -\int_{\mathbf{D}} u \wedge \overline{\partial(\gamma e^{-2\phi})} \\ &= -\int_{\mathbf{D}} u \wedge \overline{\delta\gamma} e^{-2\phi} \\ &= -e^{-\phi(0)} \int_{\mathbf{D}} \eta \wedge \frac{\overline{\delta\gamma}}{|z|^r} e^{-2\phi} \\ &= \overline{\gamma(0)} e^{-2\phi(0)},\end{aligned}$$

and hence $\overline{\partial}\, u = \delta_0$ as currents on $\mathbf{D}$.

Now, set $\beta = 2\pi i z u \in D'_{(1,0)}$, then $\overline{\partial}\, \beta = 2\pi i z\, \overline{\partial}\, h = 2\pi i z \cdot \delta_0 = 0$ in the sense of distributions, so Fact 15.5 implies that β is holomorphic; thus, we can write $\beta = h dz$ for some holomorphic function h. Observe that $\overline{\partial}\left(h \frac{dz}{z}\right) = 2\pi i \delta_0$, so $h(0) = 1$ because $\overline{\partial}\left(\frac{dz}{z}\right) = 2\pi i \delta_0$. Furthermore,

$$\begin{aligned}\int_{\mathbf{D}} |h|^2 e^{-2\phi} dV = \int_{\mathbf{D}} |\beta|^2 e^{-2\phi} dV &= 4\pi^2 \int_{\mathbf{D}} |z|^2 |u|^2 e^{-2\phi} dV \\ &\leq 4\pi^2 \int_{\mathbf{D}} |z|^{2r} |u|^2 e^{-2\phi} dV \\ &\leq 4\pi^2 C e^{-2\phi(0)},\end{aligned}$$

where the last inequality follows from (16.5). □

We will prove Proposition 16.1 with the constant $C = C_r = \frac{2}{\pi r}$, which gives the final constant of 8π in the Ohsawa–Takegoshi estimate (16.1).

Proof of Proposition 16.1. We will use the "basic identity": set $T := c_0 \gamma \wedge \overline{\gamma} \wedge \omega_0 e^{-2\phi} = |\gamma|^2 e^{-\phi}$, which is a function on $\mathbf{D}$, where $c_0 = 1$ and $\omega_0 = 1$. Then, the smooth $(1,1)$-form $i\, \partial\, \overline{\partial}\, T$ can be written as

$$i\, \partial\, \overline{\partial}\, T = 2\mathrm{Im}\left(\left(\overline{\partial}\, \delta\gamma\right) \cdot \overline{\gamma} e^{-2\phi}\right) + \underbrace{2i\left(\partial\, \overline{\partial}\, \phi\right) \cdot T}_{\geq 0} + \underbrace{\left(|\, \overline{\partial}\, \gamma|^2 + |\delta\gamma|^2\right) e^{-2\phi} dV}_{\geq 0},$$

where the second term is non-negative by the subharmonicity of ϕ. Thus, we will use the inequality

$$i\, \partial\, \overline{\partial}\, T \geq 2\mathrm{Im}\left(\left(\overline{\partial}\, \delta\gamma\right) \cdot \overline{\gamma} e^{-2\phi}\right). \tag{16.6}$$

Now, set $w = -r \log|z|$ and $W = 1 - e^{-w}$, so $w \geq 0$ and $0 \leq W \leq 1$. Multiplying (16.6) by w, integrating, using the Cauchy–Schwarz inequality gives one estimate; repeating the same thing for W gives another estimate; finally, combining these two estimates completes the proof. This proof will be explained in more detail next time. □

17. February 12th

There will be no class on Friday the 16th and Monday the 19th.

17.1. **The Ohsawa–Takegoshi Extension Theorem in the Unit Disc (Continued).** Recall the version of the Ohsawa–Takegoshi theorem in the unit disc $\mathbf{D}$ that we are in the midst of proving:

Theorem 17.1. [Ohsawa–Takegoshi] *For any $\phi \in \mathrm{SH}(\mathbf{D}) \cap C^\infty(\mathbf{D})$, there exists $h \in \mathcal{O}(\mathbf{D})$ such that $h(0) = 1$ and satisfying the estimate*

$$\int_X |h|^2 e^{-2\phi} dV \leq 8\pi e^{-2\phi(0)}.$$

Last time, we reduced the proof of Theorem 17.1 to the following proposition.

Proposition 17.2. *For all $r < 1$ and all $\gamma \in D := D(\mathbf{D})$, we have*

$$|\gamma(0)|^2 e^{-2\phi(0)} \leq \frac{2}{\pi r} \int_X \frac{|\delta\gamma|^2}{|z|^{2r}} e^{-2\phi} dV$$

Proposition 17.2 implies Theorem 17.1. By the Hahn–Banach theorem and the Riesz representation theorem, there exists $\eta = \eta_r \in L^1_{(1,0)}(\phi)$ such that

$$\int_X |\eta|^2 e^{-2\phi} dV \leq \sqrt{\frac{2}{\pi r}}, \tag{17.1}$$

and

$$\overline{\gamma(0)} e^{-\phi(0)} = \int_X i\eta \wedge \frac{\delta\gamma}{|\eta|^r} e^{-2\phi}. \tag{17.2}$$

Now, (17.2) implies that $u := \frac{ie^{-\phi(0)}}{|z|^r}\eta$ satisfies $\overline{\partial}\, u = \delta_0$ in $D'_{(1,1)}$; this means that, when paired against any test function (which can be written as $\overline{\gamma}e^{-2\phi}$), we have

$$\begin{aligned}
\langle \overline{\partial}\, u, \overline{\gamma}e^{-2\phi} \rangle &= \langle u, \overline{\partial}\left(\overline{\gamma}e^{-2\phi}\right) \rangle \\
&= \langle u, \overline{\partial(\gamma e^{-2\phi})} \rangle \\
&= \langle u, \overline{\delta\gamma} e^{-2\phi} \rangle \\
&= \int_X u \wedge \overline{\delta\gamma} e^{-2\phi} \\
&= e^{-\phi(0)} \int_X i \frac{\eta}{|z|^r} \wedge \overline{\delta\gamma} e^{-2\phi} \\
&= \overline{\gamma(0)} e^{-2\phi(0)},
\end{aligned}$$

where the final equality follows from (17.2). Granted the formula $\overline{\partial}\, u = \delta_0$, if we set $\beta := 2\pi i z u = h dz \in D'_{(1,0)}$, then it satisfies $\overline{\partial}\, \beta = 0$, so $\overline{\partial}\, h = 0$ and $h(0) = 1$. Now, (17.1) implies that

$$\int_X |h|^2 e^{-2\phi} dV \leq 4\pi^2 \frac{2}{\pi r} e^{-2\phi(0)},$$

and one can send $r \to 1^-$ to conclude. □

Recall that the pointwise inner product on $(1,0)$-forms satisfies the formula $\langle \xi, \eta \rangle e^{-2\phi} = i\xi \wedge \overline{\eta} e^{-2\phi}$; that is, $\gamma_\xi = \xi$ in this case.

Proof of Proposition 17.2. The first step is to write the 'basic identity': if $T := |\gamma|^2 e^{-2\phi}$, then

$$i\,\partial\,\overline{\partial}\, T = 2\mathrm{Im}\left(\left(\overline{\partial}\,\partial\,\gamma\right) \cdot \overline{\gamma} e^{-2\phi}\right) + 2i\,\partial\,\overline{\partial}\,\phi \cdot T + \left(|\,\overline{\partial}\,\gamma|^2 + |\delta\gamma|^2\right) e^{-2\phi} dV, \tag{17.3}$$

which, when combined with the subharmonicity of ϕ, gives the inequality

$$i\,\partial\,\overline{\partial}\, T \geq 2\mathrm{Im}\left(\left(\overline{\partial}\,\partial\,\gamma\right) \cdot \overline{\gamma} e^{-2\phi}\right). \tag{17.4}$$

Multiplying (17.4) by the non-negative function $w \coloneqq -r \log|z|$ and integrating over $\mathbf{D}$ yields the inequality

$$\int_X i\,\partial\,\overline{\partial}\,T \cdot w \geq 2\mathrm{Im} \int_X \overline{\partial}\,\delta\gamma \cdot \overline{\gamma w} e^{-2\phi}. \tag{17.5}$$

By thinking of w as a current, the left-hand side of (17.5) can be rewritten as the pairing of currents

$$\langle w, i\,\partial\,\overline{\partial}\,T\rangle = \langle i\,\partial\,\overline{\partial}\,w, T\rangle = -\pi r T(0) = -\pi r|\gamma(0)|^2 e^{-2\phi(0)}.$$

To compute the right-hand side of (17.5), we want to use Stokes' theorem, so observe that

$$d\left(\delta\gamma \cdot \overline{\gamma w} e^{-2\phi}\right) = \overline{\partial}\left(\delta\gamma \cdot \overline{\gamma w} e^{-2\phi}\right) = \overline{\partial}\,\delta\gamma \cdot \overline{\gamma w} e^{-2\phi} - \delta\gamma \wedge \overline{\delta(\gamma w)} e^{-2\phi} \tag{17.6}$$

where the first equality follows for degree reasons. Using the Leibnitz formula, the second term on the right-hand side of (17.6) can be computed as

$$\delta(\gamma w) e^{-2\phi} = \delta(\gamma \overline{w}) e^{-2\phi} = \delta\gamma \cdot \overline{w} e^{-2\phi} + \gamma(\overline{\overline{\partial}\,w}) e^{-2\phi}.$$

Thus, the right-hand side of (17.5) can be rewritten as

$$\begin{aligned} 2\mathrm{Im} \int_X \overline{\partial}\,\delta\gamma \cdot \overline{\gamma w} e^{-2\phi} &= 2\mathrm{Im} \int_X \delta\gamma \wedge \overline{\delta(\gamma w)} e^{-2\phi} \\ &= 2\mathrm{Im} \int_X \delta\gamma \wedge \overline{\delta\gamma} w e^{-2\phi} + 2\mathrm{Im} \int_X \delta\gamma \wedge \overline{\gamma}(\overline{\partial}\,w) e^{-2\phi} \\ &= -2 \int_X |\delta\gamma|^2 w e^{-2\phi} dV + 2\mathrm{Im} \int_X \delta\gamma \wedge \overline{\gamma}(\overline{\partial}\,w) e^{-2\phi} \end{aligned}$$

where the first equality follows from Stokes' formula. Therefore, (17.5) implies that

$$\pi r|\gamma(0)|^2 e^{-2\phi(0)} \leq 2 \int_X |\delta\gamma|^2 w e^{-2\phi} dV - 2\mathrm{Im} \int_X \delta\gamma \wedge \overline{\gamma}(\overline{\partial}\,w) e^{-2\phi} \tag{17.7}$$

Now, applying the Cauchy–Schwarz inequality to the last term of (17.7), we get that

$$\begin{aligned} -2\mathrm{Im} \int_X \delta\gamma \wedge \overline{\gamma}(\overline{\partial}\,w) e^{-2\phi} &= -2\mathrm{Im} \int_X e^w \delta\gamma \wedge e^{-w} \overline{\gamma}(\overline{\partial}\,w) e^{-2\phi} \\ &\leq \int_X e^{2w} |\delta\gamma|^2 e^{-2\phi} dV + \int_X e^{-2w} |\gamma|^2 |\,\overline{\partial}\,w|^2 e^{-2\phi} dV. \end{aligned}$$

Combining the above inequality with (17.7) yields

$$\pi r|\gamma(0)|^2 e^{-2\phi(0)} \leq \int_X \left(e^{2w} + 2w\right) |\delta\gamma|^2 e^{-2\phi} dV + \int_X e^{-2w} |\gamma|^2 |\,\overline{\partial}\,w|^2 e^{-2\phi} dV. \tag{17.8}$$

The first term of the right-hand side of (17.8) is essentially want we want, because $e^{2w} = \frac{1}{|z|^{2r}}$, so we must deal with the second term. This is done via a trick, where one redoes the same arguments as above, but with a different weight function.

Consider the function $W \coloneqq 1 - e^{-2w}$, so $0 \leq W \leq 1$. Then,

$$\begin{cases} \overline{\partial}\,W = e^{-2w}\,\overline{\partial}\,w \text{ in } D'_{(0,1)}, \\ i\,\partial\,\overline{\partial}\,W = -4e^{-2w} |\,\overline{\partial}\,w|^2 dV \text{ in } D'_{(1,1)}. \end{cases} \tag{17.9}$$

The proof of these equalities is left as an exercise. Now, we want to repeat the previous procedure with W instead of w: this gives the inequality

$$\int_X i\,\partial\,\overline{\partial}\,T \cdot W \geq 2\mathrm{Im} \int_X \overline{\partial}\,\partial\,\gamma \cdot \overline{\gamma W} e^{-2\phi} \tag{17.10}$$

and (17.9) implies that the left-hand side of (17.10) is

$$-4\int_X e^{-2w}|\gamma|^2|\overline{\partial}\,w|^2 e^{-2\phi}dV,$$

whereas the right-hand side of (17.10) can (using Stokes's theorem) be written as

$$-2\int_X |\delta\gamma|^2 W e^{-2\phi}dV + 2\mathrm{Im}\int_X \delta\gamma\wedge\overline{\gamma}(\overline{\partial}\,W)e^{-2\phi}.$$

Thus, (17.10) can be rewritten as

$$4\int_X e^{-2w}|\gamma|^2|\overline{\partial}\,w|^2 e^{-2\phi}dV \le 2\int_X |\delta\gamma|^2 W e^{-2\phi}dV - 2\mathrm{Im}\int_X \delta\gamma\wedge\overline{\gamma}(\overline{\partial}\,W)e^{-2\phi}. \tag{17.11}$$

Now, use the Cauchy–Schwarz inequality on the last term of (17.11) to get

$$\begin{aligned}-2\mathrm{Im}\int_X \delta\gamma\wedge\overline{\gamma}(\overline{\partial}\,W)e^{-2\phi} &\le \int_X |\delta\gamma|^2 e^{-2\phi}dV + \int_X |\gamma|^2|\overline{\partial}\,W|^2 e^{-2\phi}dV\\ &= \int_X |\delta\gamma|^2 e^{-2\phi}dV + \int_X |\gamma|^2 e^{-2w}|\overline{\partial}\,w|^2 e^{-2\phi}dV,\end{aligned}$$

where the last equality follows from (17.9). Combining this inequality with (17.11) gives the inequality

$$3\int_X e^{-2w}|\gamma|^2|\overline{\partial}\,w|^2 e^{-2\phi}dV \le \int_X \underbrace{(2W+1)}_{\le 3}|\delta\gamma|^2 e^{-2\phi}dV. \tag{17.12}$$

Finally, combining (17.8) and (17.12) gives the inequality

$$\pi r|\gamma(0)|^2 e^{-2\phi(0)} \le \int_X (e^{2w}+2w+1)|\delta\gamma|^2 e^{-2\phi}dV \le \int_X 2e^{2w}|\delta\gamma|^2 e^{-2\phi}dV,$$

where the second inequality follows from the convexity of the exponential function, i.e. $e^x \ge x+1$. Rearranging, we find that

$$\pi r|\gamma(0)|^2 e^{-2\phi(0)} \le 2\int_X e^{2w}|\delta\gamma|^2 e^{-2\phi}dV = 2\int_X \frac{|\delta\gamma|^2}{|z|^{2r}}e^{-2\phi}dV,$$

as required. □

17.2. **Preview of the Geometric Version of the Ohsawa–Takegoshi Theorem.** Let us quickly explain the ingredients and players involved in the geometric version of the Ohsawa–Takegoshi extension theorem, which will be our next goal.

Let X be a compact Kähler manifold of dimension n (or perhaps one does not assume compactness), let $Y\subseteq X$ be a smooth hypersurface, L is a positive line bundle on X, and let ϕ be a smooth, positive metric on L. Given an L-valued $(n-1,0)$-form u on Y, we want to find an $L+\mathcal{O}_X(Y)$-valued $(n,0)$-form U on X such that $\mathrm{PR}_Y(U)=u$ and some estimates hold, where PR_Y denotes the Poincaré residue map on Y.

To formulate this precisely, one needs a metric ψ on the line bundle $\mathcal{O}_X(Y)$ and the assumptions in the theorem will involve e.g. the usual semipositivity condition $i\,\partial\,\overline{\partial}\,\phi > 0$ and an estimate of the form

$$i\,\partial\,\overline{\partial}\,\phi \ge c\cdot i\,\partial\,\overline{\partial}\,\psi.$$

This second estimate is a new *geometric* restriction on the extension problem (for example, it depends on the positivity of the normal bundle of Y in X).

Next time, we will aim to formulate one of the many versions of the geometric Ohsawa–Takegoshi theorem.

18. February 14th

There will be no class on Friday the 16th and Monday the 19th.

18.1. **Geometric Version of the Ohsawa–Takegoshi Theorem.** Let X be a complex manifold of dimension n, $Y \subseteq X$ be a smooth hypersurface, and L be a holomorphic line bundle on X. Let ω_X be the canonical line bundle on X, i.e. the line bundle whose local sections are holomorphic $(n,0)$-forms. We are interested in the adjoint line bundles $\omega_X \otimes L$, whose local sections are L-valued $(n,0)$-forms, as well as the line bundle $\omega_X \otimes L \otimes \mathcal{O}_X(Y)$, whose local sections are L-valued $(n,0)$-forms with at at most simple pole along Y.

There is the *Poincaré residue map* $\mathrm{Res}_Y : \omega_X \otimes L \otimes \mathcal{O}_X(Y) \to \omega_Y \otimes L$, which is a higher-dimensional version of taking the residue of a differential form in the complex plane. We want to consider the an extension problem related to the residue map:

Problem 18.1. Does the Poincaré residue map Res_Y induce a surjective map

$$H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y)) \to H^0(Y, \omega_Y \otimes L|_Y)$$

on global sections?

This extension problem can be considered in the algebro-geometric setting: X is a projective manifold and we identify $\omega_X = \mathcal{O}_X(K_X)$ for a canonical divisor K_X on X, then there is an exact sequence

$$0 \to \mathcal{O}_X(K_X + L) \to \mathcal{O}_X(K_X + L + Y) \to \mathcal{O}_Y(K_Y + L|_Y) \to 0$$

of sheaves in either of the Zariski or analytic topologies, and the associated long exact sequence on cohomology is of the form

$$\ldots \to H^0(X, \mathcal{O}_X(K_X + L + Y)) \to H^0(Y, \mathcal{O}_Y(K_Y + L|_Y)) \to H^1(X, \mathcal{O}_X(K_X + L)) \to \ldots$$

Thus, if $H^1(X, \mathcal{O}_X(K_X + L))$ vanishes, then the extension problem has a positive solution; for example, the Kodaira vanishing theorem implies that $H^1(X, \mathcal{O}_X(K_X + L)) = 0$ when L is ample.

18.2. **The L^2-Extension Problem.** We are interested in the following L^2-version of Problem 18.1.

Problem 18.2. Given $u \in H^0(Y, \omega_Y \otimes L|_Y)$, can we find $U \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\mathrm{Res}_Y(U) = u$ and with the estimate

$$\|U\| \leq C\|u\|,$$

where C is some 'controlled' constant? Here, $\|U\|$ and $\|u\|$ are measured in some L^2-norms.

We will consider the L^2-extension problem in various geometric situations:

(1) X and Y are compact.
(2) $X \subseteq \mathbf{C}^n$ is a (bounded) pseudoconvex domain (e.g. a ball); in this case, $\mathcal{O}_X(Y)$ is a trivial line bundle (as is ω_X), so the statement simplifies.
(3) $p\colon X \to \mathbf{D}$ is a proper submersion (so p is locally diffeomorphic to a product, but not holomorphically-so) and the fibres $X_t \coloneqq p(t)$, for $t \in \mathbf{D}$, are compact complex manifolds and we set $Y \coloneqq X_0$; that is, we are extending sections from the central fibre to the whole family.

The situation (3) is relevant for Siu's proof of the deformation invariance of plurigenera.

To proceed further, we must specify the data needed to define L^2-norms on the various spaces of sections, i.e. we want norms on $H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ and $H^0(Y, \omega_Y \otimes L|_Y)$. This is ok if we have the following data:

- a Kähler form ω on X, which gives rise to a volume form $dV_\omega = \omega_n = \frac{\omega^n}{n!}$;
- a metric ϕ on L;
- a metric ψ on $\mathcal{O}_X(Y)$;
- a metrics χ on ω_X and χ' on ω_Y.

Given this data, we can define

$$\|u\|^2 = \int_Y |u|^2 e^{-2(\chi' + \phi)} dV_\omega$$

and

$$\|U\|^2 = \int_X |U|^2 e^{-2(\chi + \phi + \psi)} dV_\omega.$$

However, we do *not* need χ, χ' nor ω, for the reason that we are working with $(n,0)$-forms valued in various line bundles (from such forms, we can 'cook up' a volume form).

Remark 18.3. [Metrics on the Canonical Bundle] If X is a complex manifold of dimension n, there is a one-to-one correspondence

$$\{\text{metrics on } \omega_X\} \leftrightarrow \{\text{volume forms on } X\},$$

where here we think of volume forms on X as positive (n,n)-forms on X. Given a volume form dV on X, we define a metric on ω_X as follows: given a local nowhere-vanishing section η of ω_X (i.e. an $(n,0)$-form that is a local frame of ω_X), then $c_n\eta \wedge \overline{\eta}$ is a positive (n,n)-form, so it can be written as

$$c_n \eta \wedge \overline{\eta} = h \cdot dV,$$

where $h > 0$ is a (local) function on X (h is defined on the same domain as η), and we set

$$\|\eta\| := \sqrt{h}.$$

Here, we adopt the multiplicative notation for metrics, i.e. the metric is a function $\|\cdot\| \colon \omega_X \to \mathbf{R}_{\geq 0}$. Conversely, given a metric $\|\cdot\|$ on ω_X, define a volume form dV by the formula

$$dV = \frac{c_n \eta \wedge \overline{\eta}}{\|\eta\|^{1/2}}$$

for any local frame η of ω_X.

Now, if η is a (local) L-valued $(n,0)$-form on X (not necessarily holomorphic), it is locally given by

$$\eta = \eta_i \otimes e_i$$

where the η_i's transform as $\eta_i = g_{ij}\eta_j$. If ϕ is a metric on L, then it is locally given by functions ϕ_i that transform as $\phi_i - \phi_j = \log |g_{ij}|$. Then, set

$$|\eta|^2 e^{-2\phi} := c_n \eta \wedge \overline{\eta} e^{-2\phi} \overset{\text{loc}}{=} c_n \eta_i \wedge \overline{\eta_i} e^{-2\phi_i}.$$

This is a well-defined volume form on X. Therefore, we only need a metric ϕ on L in order to define an L^2-norm $\|\cdot\|$ on $H^0(X, \omega_X \otimes L)$: if $\eta \in H^0(X, \omega_X \otimes L)$, then set

$$\|\eta\|^2 := \int_X |\eta|^2 e^{-2\phi}.$$

To define an L^2-norm on $H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$, it suffices to have a metric ϕ on L and a metric ψ on $\mathcal{O}_X(Y)$. With this data, if $\eta \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$, then set

$$\|\eta\|_X := \int_X |\eta|^2 e^{-2(\phi+\psi)},$$

where $\phi + \psi$ is viewed as a metric on $L \otimes \mathcal{O}_X(Y)$.

We can now formulate a version of the Ohsawa–Takegoshi extension theorem in the geometric setting.

Theorem 18.4. [Ohsawa–Takegoshi Theorem] *Suppose X is a complex manifold (with some restrictions*[6]*), Y is a smooth, closed complex hypersurface, $s \in H^0(Y, \mathcal{O}_X(Y))$ is the canonical section such that $Y = \{s = 0\}$, L is a holomorphic line bundle on X, ϕ is a smooth metric on L, and ψ is a smooth metric on $\mathcal{O}_X(Y)$. Assume*

$$\begin{cases} \|s\|_\psi \leq e^{-1}, \\ dd^c\phi > 0 \text{ (i.e. } \phi \text{ is a positive metric)}, \\ dd^c\phi > dd^c\psi. \end{cases}$$

[6]The restrictions (to be added later) say that X is 'almost' a Stein manifold; more precisely, we require that X becomes Stein after removing a hypersurface.

Then, for any holomorphic section $u \in H^0(Y, \omega_Y \otimes L|_Y)$ such that $\int_Y |u|^2 e^{-2\phi} < +\infty$, there exists $U \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\mathrm{Res}_Y(U) = u$ and satisfying the estimate

$$\int_X |U|^2 e^{-2(\phi+\psi)} \leq C \int_Y |u|^2 e^{-2\phi}, \tag{18.1}$$

where $C > 0$ is a universal constant. In fact, one can achieve the stronger estimate

$$\int_X \frac{|U|^2 e^{-2(\phi+\psi)}}{\left(\|s\|_\psi \log \frac{1}{\|s\|_\psi}\right)^2} \leq C \int_Y |u|^2 e^{-2\phi}. \tag{18.2}$$

The condition that $\|s\|_\psi$ be bounded is an important technical condition, but the choice of e^{-1} as upper bound is simply a choice of normalization. Furthermore, the condition $dd^c\phi > dd^c\psi$ says roughly that "Y is positively embedded in X".

The function $\left(\|s\| \log \frac{1}{\|s\|}\right)^{-2}$ on X has a pole along $Y = \{s = 0\}$, but it is integrable in the sense that

$$\int_\Omega \left(\|s\| \log \frac{1}{\|s\|}\right)^{-2} dV < +\infty$$

for any choice of relatively-compact neighbourhood Ω of Y, i.e $Y \subseteq \Omega \Subset X$ and any choice of volume form dV on X. The important point of this discussion is that it is harder to satisfy (18.2) than it is to satisfy (18.1).

19. February 21st

19.1. **Geometric Version of the Ohsawa–Takegoshi Theorem (Continued).** We recall the rough statement of the Ohsawa–Takegoshi theorem that we will aim to prove: let X be a complex manifold (with some restrictions), $Y \subseteq X$ a (smooth) complex hypersurface, L a holomorphic line bundle on X, ϕ a smooth metric on L, and ψ a smooth metric on $\mathcal{O}_X(Y)$. Assume

$$\begin{cases} |s|_\psi \leq e^{-1} \text{ on } X, \\ dd^c\phi > 0, \\ dd^c\phi > dd^c\psi, \end{cases}$$

where s is the canonical section of $\mathcal{O}_X(Y)$. Then, for any $u \in H^0(Y, \omega_Y \otimes L|_Y)$ such that $\int_Y |u| e^{-2\phi} < +\infty$, there exists $U \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\mathrm{Res}_Y(U) = u$ and satisfying

$$\int_X \frac{|U|^2 e^{-2(\phi+\psi)}}{\left(|s|_\psi \log \frac{1}{|s|_\psi}\right)^2} \leq C \int_Y |u|^2 e^{-2\phi},$$

where C is a purely numeric constant.

Said differently, we want to extend sections of $\omega_Y \otimes L|_Y$ to sections of $\omega_X \otimes L \otimes \mathcal{O}_X(Y)$ (with estimates).

We will first discuss the necessary assumptions that we must place on X (and on Y): there exists a (possibly singular, possibly empty) hypersurface $Z \subseteq X$ such that

- $X \backslash Z$ is Stein;
- Z contains no connected component of Y.

The definition of Stein manifolds, which are the complex geometric analogue of affine varieties, is given below.

Definition 19.1. A complex manifold is *Stein* if it is biholomorphic to a closed submanifold of $\mathbf{C}^N$, with $N \geq 1$.

Example 19.2. Pseudoconvex domains in $\mathbf{C}^n$ are Stein (e.g. the unit ball).

Example 19.3. Below we give various examples of X (and Y) satisfying the above assumptions:

- if X is a Stein manifold;

- if X is a projective manifold, i.e. there is a closed embedding $X \hookrightarrow \mathbf{P}^N$ (in this case, one takes $Z = H \cap X$ for a (very) general hyperplane $H \subseteq \mathbf{P}^N$, which exists by a Bertini theorem).

In the proof of the Ohsawa–Takegoshi theorem (with this assumption on X, Y), then one can reduce to the case when X is Stein by passing to $X \backslash Z$. This is a standard argument (for example, it appears in more general versions of Hörmander's theorem), but to do so we must discuss another extension problem, namely extension across subvarieties. More precisely, we require an L^2-version of the Riemann's Hebbarkeitssatz (i.e. the Riemann extension theorem).

19.2. **Extension Across Subvarieties.** Let X be a complex manifold, let $Z \subseteq X$ be a (possibly singular) subvariety, and let L be a holomorphic line bundle on X.

Lemma 19.4. *Suppose u, f are L-valued forms on X with coefficients in $L^2(\mathrm{loc})$ such that $\overline{\partial}\, u = f$ on $X \backslash Z$ (in the sense of currents). Then, $\overline{\partial}\, u = f$ on X (also in the sense of currents).*

Remark 19.5. For applications to the Ohsawa–Takegoshi theorem, we want to take $f = 0$ in Lemma 19.4; however, for more general versions of Hörmander's theorem, f will be taken to be more general.

Proof. First, assume that Z is smooth. The statement is local, so WLOG we may assume $L = \mathcal{O}_X$, X is the unit ball in $\mathbf{C}^n$, and $Z = \{z' = 0\}$, where $z = (z', z'')$ are coordinates $\mathbf{C}^n$ with $z' = (z_1, \dots, z_m)$ and $z'' = (z_{m+1}, \dots, z_n)$. Pick a smooth function $\chi \colon \mathbf{C} \to [0, 1]$ such that

$$\begin{cases} \chi(w) = 0 & |w| \leq \frac{1}{2}, \\ \chi(w) = 1 & |w| \geq 1. \end{cases}$$

For $0 < \delta < 1$, set $\xi_\delta := \chi\left(\frac{\|z'\|}{\delta}\right)$, where $\|z'\|$ is (say) the L^2-norm on the first m coordinates. The function ξ_δ is smooth and it approximates the characteristic function $\mathbf{1}_{X \backslash Z}$. Note that

- $\operatorname{Supp}\left(\overline{\partial}\, \xi_\delta\right) \subseteq \left\{\frac{1}{2}\delta \leq \|z'\| \leq \delta\right\}$;
- $\xi_\delta \to 1$ as $\delta \to 0^-$ in the sense of distributions; in particular, $\xi_\delta \cdot u \to u$ and $\xi_\delta \cdot f \to f$ in the sense of currents.

Now, we have

$$\begin{aligned} \overline{\partial}\, (\xi_\delta \cdot u) &= \overline{\partial}\, \xi_\delta \wedge u + \xi_\delta \cdot \overline{\partial}\, u \\ &= \overline{\partial}\, \xi_\delta \wedge u + \xi_\delta \cdot f, \end{aligned}$$

because $\xi_\delta \equiv 0$ near Z and $\overline{\partial}\, u = f$ on $X \backslash Z$. As differentiation is continuous on currents, the convergence $\xi_\delta \cdot u \to u$ implies that $\overline{\partial}(\xi_\delta \cdot u) \to \overline{\partial}\, u$; therefore, it suffices to prove that $\overline{\partial}\, \xi_\delta \wedge u \to 0$ in the sense of currents as $\delta \to 0$.

This has no reason to be true in general, since u could blow up very fast as one approaches Z, but now we finally use the condition that these forms have $L^2(\mathrm{loc})$-coefficients. We must prove that the coefficients of $\overline{\partial}\, \xi_\delta \wedge u$ tend to zero in $L^1(\mathrm{loc})$. Granted that

$$\begin{cases} \operatorname{Supp}\left(\overline{\partial}\, \xi_\delta\right) \subseteq \{\|z'\| \leq \delta\}, \\ \int_{\|z'\| \leq \delta} |u|^2 dV \to 0, \end{cases}$$

where the convergence of the integal follows since u has $L^2(\mathrm{loc})$-coefficients, we can use the Cauchy–Schwartz inequality to say

$$\left(\int_X |\,\overline{\partial}\, \xi_\delta \wedge u|^2 dV\right)^2 \leq \left(\int_{\|z'\| \leq \delta} |u|^2 dV\right)^2 \left(\int_{\|z'\| \leq \delta} |\,\overline{\partial}\, \xi_\delta|^2 dV\right)^2,$$

so it suffices to prove that $\sup_\delta \int_{\|z'\|\le\delta} |\overline{\partial}\,\xi_\delta|^2 dV < +\infty$. This is fine, since $|\overline{\partial}\,\xi_\delta| \le \frac{C}{\delta}\mathbf{1}_{\|z'\|\le\delta}$, and hence

$$\int_{\|z'\|\le\delta} |\overline{\partial}\,\xi_\delta|^2 dV \le \frac{C^2}{\delta^2}\mathrm{vol}\,(\{\|z'\|\le\delta\}) \le \frac{C'}{\delta^2}\delta^{2m} \le C'.$$

This completes the proof when Z is smooth (in fact, if one does the estimates more carefully, this same argument can be used to tackle the singular case).

Finally, suppose that Z is singular. Consider the stratification

$$Z = Z_1 \sqcup Z_2 \sqcup \dots,$$

where $Z_1 := Z_{\mathrm{reg}}$ and $Z_{j+1} = (Z_j)_{\mathrm{reg}}$. (Here, we are using a strong result that says that the regular part of an analytic set is again analytic.) Now, successively prove that $\overline{\partial}\,u = f$ on $X\backslash Z_j$ using the argument above (one must be careful because the Z_j's are no longer closed, but only locally closed). □

Thus, in the Ohsawa–Takegoshi theorem, we may assume that the complex manifold X is Stein (in fact, the same holds if we assume that the metrics ϕ and ψ are slightly singular). This begs the question: why is this useful? The answer is in two parts.

(1) In the proof, we start with *some* extension of u with no estimates and we must modify it to produce U.

Theorem 19.6. [Cartan's Theorem B] *For any analytic coherent sheaf $\mathcal{F}$ on a Stein manifold X, the Čech cohomology $H^p(X,\mathcal{F})$ vanishes for $p \ge 1$.*

We will apply Cartan's Theorem B to $\mathcal{F} = \omega_X \otimes L$, which will give the surjectivity of the restriction map

$$H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y)) \twoheadrightarrow H^0(Y, \omega_Y \otimes L|_Y).$$

(2) If X is Stein, we can exhaust X by relatively compact Stein subdomains $(X_n)_{n=1}^\infty$, i.e. satisfying $X_n \Subset X_{n+1} \Subset X$. (For example, if $X \subseteq \mathbf{A}^N$, then take $X_n = X \cap B(0,n)$.)

20. February 23rd

20.1. **Geometric Version of the Ohsawa–Takegoshi Theorem (Continued).** We recall the rough statement of the Ohsawa–Takegoshi theorem that we will aim to prove: let X be a complex manifold (with some restrictions), $Y \subseteq X$ a (smooth) complex hypersurface and X, Y is "Stein-able" (in the sense discussed last class), L a holomorphic line bundle on X, ϕ a smooth metric on L, and ψ a smooth metric on $\mathcal{O}_X(Y)$. Assume

$$\begin{cases} |s|_\psi \le e^{-1} \text{ on } X, \\ dd^c\phi > 0, \\ dd^c\phi > dd^c\psi, \end{cases}$$

where s is the canonical section of $\mathcal{O}_X(Y)$. Then, for any $u \in H^0(Y, \omega_Y \otimes L|_Y)$ such that $\int_Y |u|e^{-2\phi} < +\infty$, there exists $U \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\mathrm{Res}_Y(U) = u$ and satisfying

$$\int_X \frac{|U|^2 e^{-2(\phi+\psi)}}{\left(|s|_\psi \log \frac{1}{|s|_\psi}\right)^2} \le C\int_Y |u|^2 e^{-2\phi},$$

where C is a universal constant.

Reductions. Below we note the reductions that we can make in order to prove the Ohsawa–Takegoshi theorem.

(1) Assume X is Stein (this was explained last time, using the extension of holomorphic forms across subvarieties). Thus, Cartan's Theorem B implies that there exists $\widetilde{U} \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\mathrm{Res}_Y(\widetilde{U}) = u$ (but with no estimates!).

(2) Assume that

$$\int_X \frac{|\widetilde{U}|^2 e^{-2(\phi+\psi)}}{\left(|s|_\psi \log \frac{1}{|s|_\psi}\right)^2} < +\infty.$$

To do this, write $X = \bigcup_{n=1}^\infty X_n$, where X_n is a Stein open subset of X, and $X_n \Subset X_{n+1}$. Then,

$$\int_{X_n} \frac{|\widetilde{U}|^2 e^{-2(\phi+\psi)}}{\left(|s|_\psi \log \frac{1}{|s|_\psi}\right)^2} < +\infty$$

for all $n \geq 1$. Set $Y_n := Y \cap X_n$. Suppose we can find (using some modification of $\widetilde{U}$) some forms $U_n \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\mathrm{Res}_{Y_n}(U_n) = u|_{Y_n}$ and with the estimates

$$\int_{X_n} \frac{|U_n|^2 e^{-2(\phi+\psi)}}{\left(|s|_\psi \log \frac{1}{|s|_\psi}\right)^2} \leq C \int_{Y_n} |u|^2 e^{-2\phi} \leq C \int_Y |u|^2 e^{-2\phi}.$$

Then, one can extract a limit (uniformly on compacts) $U = \lim_{j\to+\infty} U_{n_j}$ that solves the problem on X.

The idea now is to modify $\widetilde{U}$ using a solution to the $\overline{\partial}$-equation (this is similar to the part of the proof of the Kodaira Embedding Theorem, where we cook up a holomorphic section that does not vanish at a specified point). To that end, pick a smooth function $\chi\colon \mathbf{R} \to [0,1]$ such that

$$\begin{cases} \chi = 0 & \text{on } [1, +\infty), \\ \chi = 1 & \text{on } (-\infty, \frac{1}{2}]. \end{cases}$$

Set $\chi_\epsilon := \chi\left(\frac{|s|_\psi^2}{\epsilon^2}\right)$, where s is the canonical section of Y; χ_ϵ is a smooth function on X and it is an approximation of the indicator function $\mathbf{1}_Y$ of Y. Now, solve the equation $\overline{\partial}\, V_\epsilon = \overline{\partial}(\chi_\epsilon \widetilde{U})$ with suitable estimates so that $V_\epsilon = 0$ on Y (this can be arranged by using a non-integrable weight, as in the proof of the Kodaira Embedding Theorem[7]). Then, we will set $U := \chi_\epsilon \widetilde{U} - V_\epsilon$ and this function will be the solution to the extension problem.

20.2. **Remarks on Hörmander's Theorem.** In order to make the aforementioned strategy work, we will need a slight variant of the version of Hörmander's theorem that we proved. Recall the version that we proved:

Theorem 20.1. [Hörmander's Theorem] *If X is (say) Stein of dimension n, L is a (positive) line bundle, ω is a Kähler form, ϕ is a metric on L such that*

$$2i\, \partial \overline{\partial}\, \phi \geq c\omega,$$

then for any L-valued (n,q)-form f, there exists an L-valued $(n, q-1)$-form u such that $\overline{\partial}\, u = f$ and

$$\int_X |u|^2 e^{-2\phi} dV_\omega \leq \frac{1}{qc} \int_X |f|^2 e^{-2\phi} dV_\omega. \tag{20.1}$$

We will only use Hörmander's theorem in the case $q = 1$. In this case, one can remove the dependence on ω! First, in the left-hand side of (20.1), we have

$$|u|^2 e^{-2\phi} dV_\omega = c_n u \wedge \overline{u} e^{-2\phi}$$

because u is an L-valued $(n, 0)$-form. For the right-hand side of (20.1), we require a slight modification of this, which is similar to the dimension-1 estimate of the form

$$\int_{\mathbf{D}} |u|^2 e^{-2\phi} \leq \int_{\mathbf{D}} \frac{|f|^2}{\Delta\phi} e^{-2\phi} \tag{20.2}$$

[7]In the proof of the Kodaira Embedding Theorem, the strategy was to show that $\int_X |f|^2 e^{-2n \log \|z - z_0\|} dV < +\infty$, which implies $f(z_0) = 0$, since the weight function is non-integrable near z_0.

on the unit disc $\mathbf{D}$ (or the corresponding estimate on $\mathbf{C}^n$ that we saw).

Recall the basic identity (for $q = 1$): if α is a compactly-supported L-valued $(n, 1)$-form, then

$$\int_X 2i\,\partial\,\overline{\partial}\,\phi \wedge T_\alpha + \int_X |\,\overline{\partial}\,\gamma_\alpha|^2 e^{-2\phi} dV_\omega = \int_X |\,\overline{\partial}\,\alpha|^2 e^{-2\phi} dV_\omega + \int_X |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} dV_\omega, \tag{20.3}$$

where γ_α is now an L-valued $(n-1, 0)$-form and $T_\alpha = c_{n-1}\gamma_\alpha \wedge \overline{\gamma_\alpha} e^{-2\phi}$ is an L-valued $(n-1, n-1)$-form. Here, the second term on the left-hand side of (20.3) is ≥ 0 and we will ignore it, and we will work with α such that $\overline{\partial}\,\alpha = 0$ (so the first term on the right-hand side of (20.3) disappears). Now, the first term of the left-hand side of (20.3) does *not* depend on the choice of Kähler form ω, and

$$\begin{aligned} |\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} dV_\omega &= |\gamma_{\overline{\partial}^*_\phi\,\alpha}|^2 e^{-2\phi} dV_\omega \\ &= |\delta\gamma_\alpha|^2 e^{-2\phi} dV_\omega \\ &= c_n \delta\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} e^{-2\phi}, \end{aligned}$$

so the term $|\,\overline{\partial}^*_\phi\,\alpha|^2 e^{-2\phi} dV_\omega$ is also independent of ω! Modifying the L^2-method using these observations yields the following result:

Theorem 20.2. *Given X, L, ϕ as above, one can solve $\overline{\partial}\,u = f$ with the estimate*

$$\int_X |u|^2 e^{-2\phi} dV_\omega \leq \int_X \{f\}_\theta \cdot e^{-2\phi}, \tag{20.4}$$

provided the right-hand side is finite, and θ is the Kähler form $\coloneqq 2i\,\partial\,\overline{\partial}\,\phi$. (Further, the left-hand side of (20.4) is independent of ω.)

The right-hand side of (20.4) is the analogue of the 1-dimesional estimate (20.2). To see this is just some Hermitian algebra: working pointwise, the Kähler form $\theta \coloneqq 2i\,\partial\,\overline{\partial}\,\phi$ is a positive $(1, 1)$-form and it defines a Hermitian inner product on $\Lambda^{p,q}$ for all p, q (in particular, on $\Lambda^{1,1}$). Taking the pairing with θ gives a linear function

$$\Lambda^{1,1} \xrightarrow{(\cdot,\theta)} \mathbf{C}.$$

This induces a map

$$\Lambda^{n,1} \otimes \Lambda^{1,n} \xrightarrow{\simeq} \Lambda^{1,1} \otimes \Lambda^{n,n} \xrightarrow{(\cdot,\theta)} \Lambda^{n,n},$$

i.e. the target consists of volume forms! There is also an L-valued version, given a metric ϕ: given an L-valued $(n, 1)$-form f, write $\{f\}_\theta \cdot e^{-2\phi}$ for the image of $if \otimes \overline{f} e^{-2\phi}$ under the map above.

Remark 20.3. [Minimal Solutions] If u is a solution to $\overline{\partial}\,u = f$ and $u \perp \ker(\overline{\partial})$, then u is the solution with minimum norm (indeed, up to these only being densely-defined, any solution can be decomposed as the minimal solution plus an element of $\ker(\overline{\partial})$, and the norms only increase upon adding an element of $\ker(\overline{\partial})$). Thus, u must satisfy (20.4).

21. March 5th

21.1. **Geometric Version of the Ohsawa–Takegoshi Theorem (Continued).** The plan is to finish the proof of this version of the Ohsawa–Takegoshi theorem by the next class. Afterwards, we will probably start talking about invariance of log plurigenera, which will be an application of this theorem.

Theorem 21.1. [Ohsawa–Takegoshi] *Let X be a complex manifold and let $Y \subset X$ be a smooth analytic hypersurface (and the pair (X, Y) is "Steinable", in the sense of the previous class). Let L be a holomorphic line bundle on X, and let ϕ and ψ be smooth metrics on L and $\mathcal{O}_X(Y)$, respectively. Assume that $|s|_\psi \leq e^{-1}$ on X,* *and that* $dd^c\phi > 0$ *and* $dd^c\phi > dd^c\psi$*. Then, for all* $u \in H^0(Y, \omega_Y \otimes L|_Y)$ *such that* $\int_Y |u|^2 e^{-2\phi} < \infty$*, there exists* $U \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ *such that* $\operatorname{Res}_Y U = u$ *and*

$$\int_X \frac{|U|^2 e^{-2(\phi+\psi)}}{(|s|_\psi \log \frac{1}{|s|_\psi})^2} \le C \int_Y |u|^2 e^{-2\phi}$$

where C *is a universal constant and* $|s|_\psi = |s|e^{-\psi}$*.*

The statement says that you can extend an L-valued top form on Y to an L-valued top form on X, with some estimates. The argument is a bit technical; the statement and proof is from lecture notes [Bou17] by Boucksom.

21.1.1. *Reductions So Far.* We have reduced to the case when X is Stein, in which case we already know that there exists $\widetilde{U} \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(-Y))$ such that $\operatorname{Res}_Y \widetilde{U} = u$ and

$$\int_X \frac{|\widetilde{U}|^2 e^{-2(\phi+\psi)}}{(|s|_\psi \log \frac{1}{|s|_\psi})^2} < \infty.$$

Note, however, that we have no estimates. The point is then to "correct" $\widetilde{U}$ using the $\overline{\partial}$-equation, as follows. Pick a C^∞ function $\chi\colon \mathbf{R} \to [0,1]$ such that

$$\begin{cases} \chi \equiv 0 & \text{on } [1,\infty] \\ \chi \equiv 1 & \text{on } [-\infty, \frac{1}{2}] \end{cases}$$

Then, $\chi_\varepsilon := \chi(\frac{|s|_\psi^2}{\varepsilon^2})$ is a C^∞ function on X that approximates $\mathbf{1}_Y$, where we recall that s is the canonical section of $\mathcal{O}_X(Y)$ such that $\{s = 0\} = Y$. We then want to solve $\overline{\partial} V_\varepsilon = \overline{\partial}(\chi_\varepsilon \widetilde{U})$ with estimates on V_ε that force $V_\varepsilon|_Y = 0$. If $U_\varepsilon := \chi_\varepsilon \widetilde{U} - V_\varepsilon$, then suitable estimates will show that $\lim_{\varepsilon\to 0} U_\varepsilon = U$ works.

21.1.2. *Reduction To A Twisted Hörmander Estimate.* Suppose we can solve the $\overline{\partial}$-equation

$$\overline{\partial} V_\varepsilon = \overline{\partial}(\chi_\varepsilon \widetilde{U}) = \varepsilon^{-2}\chi'\Big(\frac{|s|_\psi^2}{\varepsilon^2}\Big)\, \overline{\partial}(|s|_\psi^2) \wedge \widetilde{U}, \tag{21.1}$$

with an estimate

$$\int_X \frac{|V_\varepsilon|^2 |s|_\psi^{-2}}{(\log \frac{1}{|s|_\psi + \varepsilon})^2}\, e^{-2(\phi+\psi)} \le \int_{\operatorname{supp} \overline{\partial} V_\varepsilon} |\widetilde{U}|^2 |s|_\psi^{-2} e^{-2(\phi+\psi)} \tag{21.2}$$

where the constant C is a purely numerical constant. Then, the fact that the left-hand side of (21.2) is finite forces $V_\varepsilon = 0$ on Y (i.e., $\operatorname{Res}_Y V_\varepsilon = 0$). Now using (21.1), we have

$$\operatorname{supp} \overline{\partial} V_\varepsilon \subset \Big\{\frac{\varepsilon}{2} \le |s|_\psi \le \varepsilon\Big\} \tag{21.3}$$

and the volume of this set is roughly ε^2. This would imply that the right-hand side of (21.2) satisfies

$$\varlimsup_{\varepsilon\to 0} \int_{\operatorname{supp} \overline{\partial} V_\varepsilon} |\widetilde{U}|^2 |s|_\psi^{-2} e^{-2(\phi+\psi)} \le C \int_Y |u|^2 e^{-2\phi} \tag{21.4}$$

by the fact that $\operatorname{Res}_Y \widetilde{U} = u$. Using (21.2) and (21.4), we can take a weak limit (in L^2) as $\varepsilon \to 0$: Setting $U_\varepsilon := \chi_\varepsilon \widetilde{U} - V_\varepsilon$, we have that $\overline{\partial} U_\varepsilon = 0$ hence $U_\varepsilon \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$, and $\operatorname{Res}_Y U_\varepsilon = u$. Now

$$\lim_{\varepsilon\to 0} \int_X \frac{|\chi_\varepsilon \widetilde{U}|^2 |s|_\psi^{-2}}{(\log \frac{1}{|s|_\psi+\varepsilon})^2}\, e^{-2(\phi+\psi)} = 0$$

since the factor $|s|_\psi^{-2}/(\log \frac{1}{|s|_\psi+\varepsilon})^2$ is already integrable, hence you can use dominated convergence and the fact that $\chi_\varepsilon \widetilde{U} \to 0$. Thus, a solution to (21.1) satisfying the estimate in (21.2) would give the estimate in the Ohsawa–Takegoshi theorem 21.1.

We will try to use Hörmander's theorem to get a solution $\overline{\partial}\,V_\varepsilon = \overline{\partial}(\chi_\varepsilon \widetilde{U})$ with the estimate in (21.2). By the slightly stronger version that we stated last time (specialized to the case of $(n,0)$-forms), we have an estimate

$$\int_X |V_\varepsilon|^2 e^{-2(\phi+\psi+\kappa)} \leq \int_X \{\overline{\partial}(\chi_\varepsilon \widetilde{U})\}_\theta \, e^{-2(\phi+\psi+\kappa)} \tag{21.5}$$

where κ is a smooth function on X that we add for flexibility so that $\theta \coloneqq i\,\partial\,\overline{\partial}(\phi+\psi+\kappa) > 0$. Note that $\{\overline{\partial}(\chi_\varepsilon \widetilde{U})\}_\theta$ is an (n,n)-form. We will use the following choice of κ:

$$\kappa \coloneqq \log|s|_\psi = \log|s| - \psi$$

where on the right-hand side, we interpret the two terms as metrics on L. Then, $\phi + \psi + \kappa = \phi + \log|s|$, which is a *singular* semipositive metric. We can approximate it by smooth positive metrics roughly of the form $\frac{1}{2}\log(|s|^2 + \varepsilon^2)$, in which case we can still use Hörmander's theorem as above. The left-hand side of (21.5) then becomes

$$\int_X |V_\varepsilon|^2 |s|_\psi^{-2} \, e^{-2(\phi+\psi)},$$

which is smaller than the right-hand side of (21.2), and so it suffices to bound this integral.

On the other hand, the right-hand side of (21.5) is not quite what we want. We have that $\overline{\partial}(\chi_\varepsilon \widetilde{U}) = \overline{\partial}\,\chi_\varepsilon \wedge \widetilde{U}$ is the wedge product of a $(0,1)$ form and an $(n,0)$ form, hence

$$\{\overline{\partial}(\chi_\varepsilon \widetilde{U})\}_\theta e^{-2(\phi+\psi+\kappa)} = |\overline{\partial}\,\chi_\varepsilon|_\theta^2 |\widetilde{U}|^2 e^{-2(\phi+\psi+\kappa)}$$

is an (n,n)-form, where now $|\overline{\partial}\,\chi_\varepsilon|_\theta^2$ is a function and $|\widetilde{U}|^2 e^{-2(\phi+\psi)}$ is an (n,n)-form. The right-hand side of (21.5) is bounded above by

$$C \int_{\{\frac{\varepsilon}{2} \leq |s|_\psi \leq \varepsilon\}} \varepsilon^{-4} |\overline{\partial}(|s|_\psi^2)|^2 |\widetilde{U}|^2 |s|_\psi^{-2} \, e^{-2(\phi+\psi)}.$$

We then want to compare this with the right-hand side of (21.2). The extra factor here is $\varepsilon^{-4}|\overline{\partial}(|s|_\psi^2)|^2$, but there is a problem: $\varepsilon^{-2}|\overline{\partial}(|s|_\psi^2)|$ has no reason to be bounded when $|s|_\psi \sim \varepsilon$. For example, in dimension 1 when $s = z$, we have $\overline{\partial}(|z|^2) = z\,d\overline{z}$ has norm roughly equal to ε, so $\varepsilon^{-2}|\overline{\partial}(|z|^2)| \sim \varepsilon^{-1} \gg 0$. So the right-hand side of (21.5) is not bounded above by the right-hand side of (21.2).

The solution to this is to use a "twisted" version of the Hörmander estimates. We will sketch how this works next time. To summarize, the basic idea is that since you want to solve an extension problem for a hypersurface, and Hörmander's estimates work on X itself, you want to try to develop a version of Hörmander's estimates when working on a tubular neighborhood of Y.

22. March 7th

22.1. **Geometric Version of the Ohsawa–Takegoshi Theorem (Continued).** Recall the setup of the problem: we have a hypersurface $Y \subset X$ and we want to extend a section $u \in H^0(Y, \omega_Y \otimes L|_Y)$ to a section $U \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ with the estimate

$$\int_X \frac{|u|^2 e^{-2(\phi+\psi)}}{(|s|_\psi \log|s|_\psi^{-1})^2} \leq C \int_Y |u|^2 e^{-2\phi}$$

where ϕ and ψ are metrics on L and $\mathcal{O}_X(Y)$, respectively.

We mention that a good reference for the Ohsawa–Takegoshi theorem (at least for $X = \mathbf{C}^n$) is a short preprint [Che11] by Bo-Yong Chen. Boucksom's lecture notes [Bou17] mentioned last time are an adaptation of Chen's proof to a more geometric context, although it has some typos.

Today's goal is to explain the "twist" needed in the application of Hörmander's estimates that we started to explain last time.

22.1.1. *Reductions.* We first review the reductions performed so far. Without loss of generality, we can assume that X is Stein, in which case there exists $\widetilde{U} \in H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\operatorname{Res}_Y \widetilde{U} = u$ and

$$\int_X \frac{|\widetilde{U}|^2 e^{-2(\phi+\psi)}}{(|s|_\psi \log|s|_\psi^{-1})^2} < \infty.$$

We then wanted to "correct" $\widetilde{U}$ by solving the $\overline{\partial}$-problem. Let $\chi\colon \mathbf{R} \to [0,1]$ be a C^∞ function such that $0 \le \chi \le 1$, and

$$\begin{cases} \chi \equiv 0 & \text{on } [1,\infty) \\ \chi \equiv 1 & \text{on } (-\infty, \frac{1}{2}] \end{cases}$$

We set $\chi_\varepsilon = \chi(\frac{|s|_\psi^2}{\varepsilon^2})$, which approximates $\mathbf{1}_Y$.

We want to solve

$$\overline{\partial}\, V_\varepsilon = \overline{\partial}(\chi_\varepsilon \widetilde{U}) = \overline{\partial}\, \chi_\varepsilon \wedge \widetilde{U} \tag{22.1}$$

with the estimate

$$\int_X \frac{|V_\varepsilon|^2 |s|_\psi^{-2} e^{-2(\phi+\psi)}}{\left(-\frac{1}{2}\log(|s|_\psi^2 + \varepsilon^2)\right)^2} \le C \int_{\operatorname{supp} \overline{\partial}\, V_\varepsilon} |\widetilde{U}|^2 |s|_\psi^{-2} e^{-2(\phi+\psi)}. \tag{22.2}$$

Then, we can take $U = \lim_{\varepsilon\to 0} U_\varepsilon$, where $U_\varepsilon = \chi_\varepsilon \widetilde{U} - V_\varepsilon$. The estimate (22.2) forces $V_\varepsilon = 0$ on Y, since the denominator blows up there, and the right-hand side converges to the estimate we wanted in the Ohsawa–Takegoshi theorem by the fact that

$$\operatorname{supp} \overline{\partial}\, V_\varepsilon \subset \left\{\frac{\varepsilon}{2} \le |s|_\psi \le \varepsilon\right\}.$$

How can we get (22.1)–(22.2)? The idea is to use Hörmander's theorem with a twist. Recall the following:

Theorem 22.1 (Hörmander, slightly improved)**.** *Let X be a complex manifold, and let L be a line bundle on X. Let η be a Kähler form on X, and let ϕ be a (smooth) metric on L such that $2i\,\partial\overline{\partial}\,\phi \ge \eta$. Then, we can solve $\overline{\partial}\, u = f$ for an $(n,1)$-form f and an $(n,0)$-form u, with the estimate*

$$\int_X |u|^2 e^{-2\phi} \le \int_X \{f\}_\eta e^{-2\phi}.$$

The main difference in (22.2) is that there is a denominator that we need to deal with. The idea is that we have the freedom to choose η and ϕ in the statement of Hörmander's theorem. We will then consider *minimal* solutions satisfying Hörmander's estimate for a suitable choice of η and ϕ; the fact that they will be orthogonal to $\overline{\partial}$ can be exploited to show that the stronger estimate (22.2) holds.

Set $\kappa := \log|s|_\psi$, which is a function on X (that is singular along Y), and let V_ε be the *minimal* solution to (22.1) in $L^2(\phi+\psi+\kappa)$, i.e., such that

$$\int_X |V_\varepsilon|^2 e^{-2(\phi+\psi+\kappa)}$$

is minimal among solutions to (22.1). Note that such solutions exist by Hörmander's theorem. Now $V_\varepsilon \perp \ker \overline{\partial}$ in $L^2(\phi+\psi+\kappa)$ by the usual Hilbert space machinery. Let $\tau = \tau_\varepsilon \in C^\infty(X)$ be another auxiliary function. Then, we have that $V_\varepsilon e^{2\tau} \perp \ker \overline{\partial}$ in $L^2(\phi+\psi+\kappa+\tau)$, since

$$\int_X x V_\varepsilon e^{2\tau} \cdot \overline{q}\, e^{-2(\phi+\psi+\kappa+\tau)} = 0$$

by canceling out the factors involving τ's cancel out. We therefore have a minimal solution to a different $\overline{\partial}$-equation: $V_\varepsilon e^{2\tau}$ is the minimal solution in $L^2(\phi+\psi+\kappa+\tau)$ to the equation

$$\overline{\partial}(V_\varepsilon e^{2\tau}) = (\overline{\partial}\, \chi_\varepsilon \wedge \widetilde{U} + 2\,\overline{\partial}\, \tau \wedge V_\varepsilon) e^{2\tau}.$$

The Hörmander estimates imply

$$\int_X |V_\varepsilon|^2 e^{4\tau} e^{-2(\phi+\psi+\kappa+\tau)} \le \int_X x\{(\overline{\partial}\,\chi_\varepsilon \wedge \widetilde{U} + 2\,\overline{\partial}\,\tau \wedge V_\varepsilon)e^{2\tau}\}_\eta\, e^{-2(\phi+\psi+\kappa+\tau)} \tag{22.3}$$

for every Kähler form η such that

$$2i\,\partial\,\overline{\partial}(\phi+\psi+\kappa+\tau) \ge \eta. \tag{22.4}$$

We now choose η and τ to satisfy (22.4) in such a way that we can deduce the inequality in (22.2). Recall that $\kappa = \log|s|_\psi = \log|s| - \psi$, hence

$$\phi+\psi+\kappa+\tau = \phi+\tau+\log|s|,$$

which is a singular metric on $\mathcal{O}_X(Y)$. As a current, we have $i\,\partial\,\overline{\partial}\log|s| \ge 0$. Thus, to satisfy (22.4), it suffices to pick τ such that $\eta \coloneqq 2i\,\partial\,\overline{\partial}(\phi+\tau)$ is a Kähler form. The magic choice of $\tau = \tau_\varepsilon$ is the following:

$$\tau = -\log(-\log(|s|_\psi^2+\varepsilon^2) + \log(-\log(|s|_\psi^2+\varepsilon))).$$

Near Y, we have that $s=0$, hence $|s|_\psi = 0$, so $0 < |s|_\psi^2 + \varepsilon^2 \ll 1$. Taking logarithms, we have that $\log(|s|_\psi^2+\varepsilon^2) \ll 0$. Thus, $\tau \approx -\log(-\log(|s|_\psi^2+\varepsilon^2)) \ll 0$ near Y, but has slow growth as we approach Y.

Aside 6. In $\mathbf{C}$, near $0 \in \mathbf{C}$, a subharmonic function u can satisfy $u(0) = -\infty$, but the maximal growth towards $-\infty$ is $u \sim c\log|z|$. On the other hand, one can have slower growth, e.g., $u = -\log(-\log|u|)$ is subharmonic (Exercise). This shows where one might get the idea for how to define τ from.

The point is that now the choice of τ is okay. A straightforward (but painful) calculation (using $i\,\partial\,\overline{\partial} > 0$ and $i\,\partial\,\overline{\partial}\,\phi > i\,\partial\,\overline{\partial}\,\psi$) implies that $i\,\partial\,\overline{\partial}(\phi+\tau) > 0$ (using the chain rule and some clever estimates, where ψ complicates some things and one must use the hypotheses mentioned; this is where the triple log gets used). So $\eta \coloneqq 2i\,\partial\,\overline{\partial}(\phi+\tau)$ is a Kähler form, and Hörmander's theorem applies to the equation

$$\overline{\partial}(V_\varepsilon e^{2\tau}) = (\overline{\partial}\,\chi_\varepsilon \wedge \widetilde{U} + 2\,\overline{\partial}\,\tau \wedge V_\varepsilon)e^{2\tau}.$$

Since $V_\varepsilon e^{2\tau}$ is the *minimal* solution in $L^2(\phi+\psi+\kappa+\tau)$, it must satisfy the estimates in Hörmander's theorem in (22.3) with the τ, η that we have chosen.

We now expand (22.3) using these choices of τ and η. The left-hand side of (22.3) is

$$\int_X |V_\varepsilon|^2 e^{4\tau} e^{-2(\phi+\psi+\kappa+\tau)} = \int_X |V_\varepsilon|^2 |s|_\psi^{-2} e^{-2(\phi+\psi)} e^{2\tau}$$

and $e^{2\tau}$ is the form that we wanted to get the denominator in the Ohsawa–Takegoshi theorem (up to a factor of $\frac{1}{2}$, maybe). We estimate the right-hand side of (22.3) using Cauchy–Schwarz, where we use the fact that $\{\cdot\}_\eta$ is a quadratic form, and use Cauchy–Schwarz in the form $(a+b)^2 \le \frac{5}{4}a^2 + 5b^2$. The right-hand side of (22.3) is then less than or equal to

$$\begin{aligned}&\int_X \{(\overline{\partial}\,\chi_\varepsilon \wedge \widetilde{U} + 2\,\overline{\partial}\,\tau \wedge V_\varepsilon)e^{2\tau}\}_\eta\, e^{-2(\phi+\psi+\kappa+\tau)} \\ &\qquad \le 5\int_X |V_\varepsilon|^2 |\overline{\partial}\,\tau|_\eta |s|_\psi^{-2}\, e^{-2(\phi+\psi)} e^{2\tau} + 5\int_{\operatorname{supp}\overline{\partial}\,\chi_\varepsilon} |\overline{\partial}\,\chi_\varepsilon|^2 |\widetilde{U}|^2 |s|_\psi^{-2}\, e^{-2(\phi+\psi)} e^{2\tau}.\end{aligned}$$

Combining the first term of the right-hand side of (22.3) with the left-hand side of (22.3), one can show that $|\overline{\partial}\,\tau|_\eta$ is small, so this is okay. Now we are essentially done since one can show

$$e^{2\tau} \ge C^{-1}\Big(\frac{1}{\frac{1}{2}\log(|s|_\psi^2+\varepsilon^2)^{-1}}\Big)^2$$

and

$$e^{2\tau}|\overline{\partial}\,\chi_\varepsilon|_\eta \le C \quad \text{on } \operatorname{supp}\overline{\partial}\,\chi_\varepsilon \subset \Big\{\frac{\varepsilon}{2} \le |s|_\psi \le \varepsilon\Big\}.$$

This yields the estimate (22.2). ☐

We close by noting that the proof of the Ohsawa–Takegoshi theorem only really needed Hörmander's theorem, which in turn only relied on the basic identity. The difficulty of the proof lies in how to cleverly apply these results.

Since we have done technicalities for a while, we will move on to Siu's invariance of plurigenera (following Păun's paper [P07̆]), using the Ohsawa–Takegoshi theorem and some other nontrivial ingredients.

23. March 9th

Today we will start discussing an application of the technical material covered so far.

23.1. **Invariance of Plurigenera.** Our goal in the next few lectures is to give a proof of Siu's theorem on invariance of plurigenera.

We first describe the setup. Let X be a compact complex manifold, and let K_X denote both its canonical line bundle and its canonical divisor (we will not be too careful with line bundles vs. divisors). We define the following:

Definition 23.1. The *genus* of X is $g(X) \coloneqq h^0(X, K_X)$. For every $m \geq 1$, the natural numbers $g_m(X) \coloneqq h^0(X, mK_X)$ are the *plurigenera* of X.

Definition 23.2. We say that X is of *general type* if there exists $c > 0$ such that

$$g_m(X) \geq c \cdot m^{\dim X}$$

for all $m \gg 0$. Note that we need "$\gg$" here since we could have $g_m(X) = 0$ for small m.

We also note the following fact, although we won't need it in the sequel.

Fact 23.3. There exists $\kappa \in \{-\infty, 0, 1, \ldots, \dim X\}$ such that $g_m(X) \sim m^\kappa$ as $m \to \infty$. This number κ is called the *Kodaira dimension* of X.

We now consider a one-parameter family

$$\begin{array}{c} X \\ \big\downarrow{\scriptstyle p} \\ \mathbf{D} \end{array}$$

over a disc $\mathbf{D}$, where X is a complex manifold and p is a proper submersion. Properness implies that for every t, the fiber $X_t \eqqcolon p^{-1}(t)$ is compact, and submersiveness implies X_t is even a complex manifold. The fibers X_t are all diffeomorphic, but they can have different holomorphic structures. We can therefore ask:

Question 23.4. What can be said about the functions

$$\begin{aligned} \mathbf{D} &\longrightarrow \mathbf{N} \\ t &\longmapsto g_m(t) \end{aligned} \tag{23.1}$$

mapping each $t \in \mathbf{D}$ to the plurigenus of X_t?

It is not too hard that the plurigenera are semicontinuous in one direction:

Fact 23.5. The function in (23.1) is always upper semicontinuous, i.e., the sublevel sets $\{t \mid g_m(t) < \lambda\}$ are open for every $\lambda \in \mathbf{R}$. (Use an argument with normal families to obtain limits of sections.)

The proof of Fact 23.5 is left as an exercise. We therefore consider the question:

Question 23.6. Is the function in (23.1) lower semicontinuous, in which case Fact 23.5 implies that the function is constant as well?

Remark 23.7. In this setting, adjunction simply says $K_{X_0} = K_X|_{X_0}$ since $\mathcal{O}_X(X_0)|_{X_0} \cong \mathcal{O}_{X_0}$. Thus, lower semicontinuity of g_m becomes an extension problem: you want to extend sections on a fiber to other fibers nearby.

In the most general transcendental setting, there is not much known. We can say something under some assumptions, which bring us closer to an algebraic setting:

Theorem 23.8. [Siu98] *Assume $p\colon X \to \mathbf{D}$ is projective. Then, for every $m \geq 1$ and every $u \in H^0(X_0, mK_{X_0})$, there exists $U \in H^0(X, mK_X)$ such that $U|_X = u$.*

This implies what we sought after:

Corollary 23.9. [Invariance of Plurigenera] *The plurigenera $g_m(X_t)$ are independent of t.*

Recall that $p\colon X \to \mathcal{D}$ is *projective* if there is a factorization

$$\begin{array}{ccc} X & \hookrightarrow & \mathbf{P}^N \times \mathbf{D} \\ & {}_{p}\searrow \quad \swarrow_{\pi_2} & \\ & \mathbf{D} & \end{array}$$

Alternatively, we assume that there exists a *relatively ample* line bundle A on X, i.e., a line bundle such that $A|_{X_t}$ is ample for all t. (Strictly speaking, these conditions are not completely equivalent, but will be okay for us since we can replace $\mathbf{D}$ with a smaller ball.)

Corollary 23.9 may be false for completely general X and p, but Siu has conjectured the following:

Conjecture 23.10. [Siu] Invariance of pliurigenera also holds when X is Kähler.

The idea behind the proof of Theorem 23.8 is to use a new version of the Ohsawa–Takegoshi theorem. This new version of the Ohsawa–Takegoshi theorem is a bit easier to show than the previous version, but we need a new statement since the proof of Theorem 23.8 requires us to consider singular metrics. We first define this notion.

Definition 23.11. A *singular metric* on a line bundle L on X is a function $\phi\colon L^\times \to \mathbf{R} \cup \{-\infty\}$, where $L^\times := L \smallsetminus \{\text{zero section}\}$. We say that ϕ is *plurisubharmonic* (abbreviated *psh*) or *semipositive* if for every section $s\colon V \to L^\times$ on an open set $V \subset X$, the composition $\phi \circ s\colon V \to \mathbf{R} \cup \{-\infty\}$ is plurisubharmonic, or equivalently if $\phi\colon L^\times \to \mathbf{R} \cup \{-\infty\}$ is plurisubharmonic.

We can now state our new version of the Ohsawa–Takegoshi theorem.

Theorem 23.12 (Ohsawa–Takegoshi)**.** *Let $p\colon X \to \mathbf{D}$ as above, and let L be a line bundle on X. Let ϕ be a plurisubharmonic singular metric on L. Then, for every $u \in H^0(X_0, K_{X_0} + L|_{X_0})$ such that*

$$\int_{X_0} |u|^2 \, e^{-2\phi} < \infty,$$

there exists $U \in H^0(X, K_X + L)$ such that $U|_{X_0} = u$ and

$$\int_X |U|^2 \, e^{-2\phi} \leq C \cdot \int_{X_0} |u|^2 \, e^{-2\phi},$$

where C is a universal constant.

We will reduce the Ohsawa–Takegoshi theorem 23.12 to the case when ϕ is smooth and $i\,\partial\overline{\partial} > 0$ using regularization (convolutions). This convolution process is okay on Stein manifolds.

23.2. **Idea of Proof of Siu's Theorem.** We take the Ohsawa–Takegoshi theorem 23.12 for granted for now, and explain how one would try to prove Siu's theorem.

If $m = 1$, then this follows immediately from the Ohsawa–Takegoshi theorem 23.12 by setting $L = 0$ (i.e. $\mathcal{O}_X$) and $\phi = 0$.

If $m \geq 2$, then we write

$$mK_X = K_X + (m-1)K_X.$$

Stting $L = (m-1)K_X$, we can then try to apply the Ohsawa–Takegoshi theorem 23.12. We therefore need to find a plurisubharmonic metric ϕ on L that satisfies

$$\int_{X_0} |u|^2 \, e^{-2\phi} < \infty.$$

Note that since ϕ can attain the value $-\infty$, integrability is not automatic!

We will first construct such a metric on $L|_{X_0}$. In general, if M is a line bundle and $s \in H^0(X, M)$ is a global section, then $\phi = \log|s|$ is a (singular) plurisubharmonic metric on M by defining $\phi\colon M^\times \to \mathbf{R} \cup \{-\infty\}$ to be the function defined by

$$\begin{cases} \phi(s(x)) = 0 & \text{if } s(x) \neq 0, \\ \phi|_{M_x^\times} \equiv -\infty & \text{if } s(x) = 0. \end{cases}$$

In our setting, we are given a section $u \in H^0(X_0, mK_{X_0})$, in which case $\log|u|$ is a plurisubharmonic metric on mK_{X_0}, and $(1-m^{-1})\log|u|$ is a plurisubharmonic metric on $(1-m^{-1})mK_{X_0} = (m-1)K_{X_0} = L$. Futhermore,

$$\int_{X_0} |u|^2 \, e^{-2\phi} = \int_{X_0} |u|^{2/m} < \infty$$

where we note $|u|^{2/m}$ is a well-defined volume form.

To apply the Ohsawa–Takegoshi theorem 23.12, however, we need to extend the metric $\phi = (1-m^{-1})\log|u|$ on mK_{X_0} to a plurisubharmonic metric on mK_X. It turns out to be easier to extend the metric ϕ than the section u. This is where the projectivity assumption is used: a relatively ample line bundle A on X over $\mathbf{D}$ gives rise to sections of $mK_X + \ell A$, which will allow us to solve this extension problem for metrics.

This approach to Siu's theorem 23.8 and invariance of plurigenera (Corollary 23.9) is due to Mihai Paŭn [P˘07]. We note that there is an algebraic proof of invariance of plurigenera that only works for fibers of general type; see [Laz04, §11.5].

24. March 12th

24.1. **Invariance of Plurigenera (Continued).** Consider a proper (holomorphic) submersion $p\colon X \to \mathbf{D}$, so the fibre $X_t \coloneqq p^{-1}(t)$ is a compact complex manifold and adjunction says that $K_X|_{X_0} \simeq K_{X_0}$.

Theorem 24.1. [Siu] *Assume* $p\colon X \to \mathbf{D}$ *is* projective. *Then, for any* $m \geq 1$ *and any* m*-canonical form* $u \in H^0(X_0, mK_{X_0})$*, there exists* $U \in H^0(X, mK_X)$ *such that* $U|_{X_0} = u$.

The proof of Siu's theorem will involve using the Ohsawa–Takegoshi theorem and it is crucial that we use the version with uniform estimates.

Siu's theorem shows that the function $t \mapsto h^0(X_t, mK_{X_t})$ is lower semicontinuous, and we already know that it is upper semicontinuous. As this function is integer-valued, we find that:

Corollary 24.2. *The plurigenera* $h^0(X_t, mK_{X_t})$ *are independent of* t.

The idea of the proof of Siu's theorem, following [Ber10], is to use the Ohsawa–Takegoshi theorem repeatedly in the following form:

Theorem 24.3. [Ohsawa–Takegoshi] *Let $p\colon X \to \mathbf{D}$ be as above, and let L be a line bundle on X. Let ϕ be a (possibly) singular psh metric on L. Then, for any $\sigma \in H^0(X_0, K_{X_0} + L|_{X_0})$ such that $\int_{X_0} |\sigma|^2 e^{-2\phi} < +\infty$, then there exists $\widetilde{\sigma} \in H^0(X, K_X + L)$ such that $\widetilde{\sigma}|_{X_0} = \sigma$ and satisfying*

$$\int_X |\widetilde{\sigma}|^2 e^{-2\phi} \leq C \int_{X_0} |\sigma|^2 e^{-2\phi},$$

where $C > 0$ is a universal constant.

More precisely, when viewed as functions on the total space of L minus the zero section, $\phi\colon L^\times \to \mathbf{R} \cup \{-\infty\}$ should be psh.

Example 24.4. Consider examples of such singular psh metrics ϕ on L:

(1) The metric $\phi = \log|s|$ for $s \in H^0(X, L)\backslash\{0\}$ is singular at/above the zero locus of s, and it is smooth (even real-analytic) everywhere else.
(2) The metric $\phi = \frac{1}{2} \log \sum_j |s_j|^2$, for $s_j \in H^0(X, L)$, is smooth if the s_j's have no common zero.
(2') The metric $\phi = \max_j \log|s_j|$ is continuous if the s_j's are as before.
(3) If $(\phi_j)_j$ is a *decreasing* sequence of psh metrics on L, then either
  (a) $\phi_j \searrow -\infty$ locally uniformly on $L^\times$;
  (b) or $\phi_j \searrow \phi$ a psh metric on L.
  By convention, we do not count the function/metric $-\infty$ as being psh. This process is analogous to taking the decreasing limit of convex functions $\mathbf{R} \to \mathbf{R}$.
(4) If $(\phi_\alpha)_\alpha$ is a family of psh metrics on L, locally uniformly bounded from above, then $(\sup_\alpha \phi_\alpha)^*$ is a psh metric on L, where u^* denotes the smallest upper semicontinuous function $\geq u$.

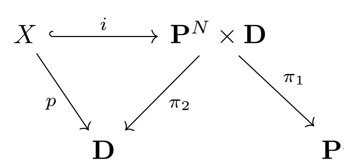
24.2. **Proof of Siu's Theorem.** Without loss of generality, assume $m \geq 2$.

Claim 24.5. There exists a (positive) line bundle B on X such that if $0 \leq p \leq m$, then:

(1) the restriction map $H^0(X, pK_X + B) \to H^0(X_0, pK_{X_0} + B|_{X_0})$ is surjective;
(2) the global sections of $pK_{X_0} + B|_{X_0}$ have no common zeros on X_0.

From the point of view of algebraic geometry, the corresponding claim is clear (one takes B to be a very large multiple of a relatively ample line bundle for p).

Indeed, by assumption, there is a diagram



Set $A := i^*\pi_1^*\mathcal{O}(1)$, which is a relatively ample line bundle on X, and set $B := qA$ for $q \gg 1$. For (1), one can use Theorem 24.3 with $L = (p+1)K_X + B$ (one can cook up a smooth psh metric on L); for (2), one uses the proof of the Kodaira embedding theorem.

We would like to remove the B-factor involved in Claim 24.5 to recover Siu's theorem. For $p \leq m-1$, let $(s_j^{(p)})_j$ be a basis of $H^0(X_0, pK_{X_0} + B|_{X_0})$, and by Claim 24.5(2), these have no common zero. For any $\ell \geq 0$, write $\ell = km + p$, where $0 \leq p \leq m$, and set

$$\sigma_j^{(\ell)} := u^k s_j^{(p)} \in H^0(X_0, \ell K_{X_0} + B|_{X_0}).$$

These $\sigma_j^{(\ell)}$'s are the sections that we wish to extend to all of X.

Lemma 24.6. *For any ℓ and any j, there exists an extension $\widetilde{\sigma_j^{(\ell)}} \in H^0(X, \ell K_X + B)$ of $\sigma_j^{(\ell)}$.*

One should think of the parameter ℓ as being very large, in which case the perturbation B becomes negligible.

Proof. The proof proceeds by induction on ℓ. We already know the result for $\ell < m$ by Claim 24.5(1), i.e. $k = 0$, $\ell = p$, and $\sigma_j^{(\ell)} = s_j^{(p)}$. Assume $\ell \geq m$. To use Theorem 24.3, we need a metric $\phi_{\ell-1}$ on $(\ell-1)K_X + B$, which we can cook up using the sections we already have: by induction, we have extensions $\widetilde{\sigma_i^{(\ell-1)}}$ of $\sigma_i^{(\ell-1)}$ to X. Set

$$\phi_{\ell-1} \coloneqq \frac{1}{2}\log\sum_i \left|\widetilde{\sigma_i^{(\ell-1)}}\right|^2.$$

The metric $\phi_{\ell-1}$ is psh, but one must check that

$$\int_{X_0} |\sigma_j^{(\ell)}|^2 e^{-2\phi_{\ell-1}} < +\infty$$

in order to apply Theorem 24.3. To check this, first assume $p \geq 1$, in which case $\ell - 1 = mk + (p-1)$. Then, on X_0, have

$$|\sigma_j^{(\ell)}|^2 e^{-2\phi_{\ell-1}} = \frac{|\sigma_j^{(\ell)}|^2}{\sum_i |\sigma_i^{(\ell-1)}|^2} = \frac{|u|^{2mk}|s_j^{(\ell)}|^2}{|u|^{2mk}\sum_i |s_i^{(p-1)}|^2} = \frac{|s_j^{(\ell)}|^2}{\sum_i |s_i^{(p-1)}|^2}$$

This is integrable over X_0 since the $s_i^{(\ell-1)}$'s have no common zero. The case $p = 0$ is similar (even better), and is left as an exercise. By Theorem 24.3, there exists an extension $\widetilde{\sigma_j^{(\ell)}}$ of $\sigma_j^{(\ell)}$, which completes the proof of the lemma. □

In the end of the proof of Lemma 24.6, we also get the estimate

$$\int_X |\widetilde{\sigma_j^{(\ell)}}|^2 e^{-2\phi_{\ell-1}} \leq C \int_{X_0} |\sigma_j^{(\ell)}|^2 e^{-2\phi_{\ell-1}}. \tag{24.1}$$

Recall that, in the proof of Lemma 24.6, we construct psh metrics

$$\phi_\ell = \frac{1}{2}\log\sum_j \left|\widetilde{\sigma_j^{(\ell)}}\right|^2$$

on $\ell K_X + B$. Now, one lets $\ell \to +\infty$ (in fact, we will write $\ell = mk$ and send $k \to +\infty$). The idea is that formally $\frac{1}{\ell}\phi_\ell$ is a metric on $K_X + \frac{1}{\ell}B$, where $\frac{1}{\ell}B$ is now only a $\mathbf{Q}$-line bundle, and take $\ell \to +\infty$ to get a metric on K_X. If we can show that $\frac{1}{\ell}\phi_\ell \to \phi$, where ϕ is a psh metric on K_X, and we can use this metric to extend the section u to all of X. More precisely, $\psi = (m-1)\phi$ is a psh metric on $(m-1)K_X$, and we will use this metric with Theorem 24.3 to extend the section u of $mK_{X_0} = K_{X_0} + (m-1)K_{X_0}$ to all of X (with some estimates).

25. March 14th

25.1. **Proof of Siu's Theorem (Continued).** Today, we will finish up the proof of Siu's theorem, as stated below.

Theorem 25.1. [Siu] *Let $p\colon X \to \mathbf{D}$ be a* projective *submersion, and write $X_0 = p^{-1}(0)$. Then, for any $m \geq 1$ and any $u \in H^0(X_0, mK_{X_0})$, there exists $U \in H^0(X, mK_X)$ such that $U|_{X_0} = u$.*

The main tool in the proof has been the following version of the Ohsawa–Takegoshi extension theorem.

Theorem 25.2. [Ohsawa–Takegoshi] *Let L be a line bundle on X, and let ϕ be a psh metric on L. Then, for any $v \in H^0(X_0, K_{X_0} + L|_{X_0})$ such that $\int_{X_0} |v|^2 e^{-2\phi} < +\infty$, there exists $V \in H^0(X, K_X + L)$ such that $V|_{X_0} = v$ and satisfying*

$$\int_X |V|^2 e^{-2\phi} \leq C \int_{X_0} |v|^2 e^{-2\phi},$$

where $C > 0$ is a universal constant.

The idea is to use the Ohsawa–Takegoshi theorem with $L = (m-1)K_X$, but to do so we need a psh metric on L! Moreover, we must introduce a "perturbation" by a relatively ample line bundle B on X.

More precisely, there exists a line bundle B on X such that for all $0 \le p \le m-1$, we have

(1) the restriction map $H^0(X, pK_X + B) \to H^0(X, pK_{X_0} + B|_{X_0})$ is surjective;
(2) the global sections of $H^0(X_0, pK_{X_0} + B|_{X_0})$ have no common zero (in particular, $pK_{X_0} + B|_{X_0}$ is globally generated).

Let $(s_j^{(p)})_j$ be a basis of $H^0(X_0, pK_{X_0} + B|_{X_0})$. For $\ell \ge 0$, set

$$\sigma_j^{(\ell)} \coloneqq u^k s_j^{(p)} \in H^0(X_0, \ell K_{X_0} + B|_{X_0}),$$

where we write $\ell = mk + p$ for $0 \le p \le m-1$.

Lemma 25.3. *For any $\ell \ge 0$ and any j, the section $\sigma_j^{(\ell)}$ admits an extension $\widetilde{\sigma_j^{(\ell)}} \in H^0(X, \ell K_X + B)$.*

These extensions $\widetilde{\sigma_j^{(\ell)}}$ are constructed by induction on ℓ using the Ohsawa–Takegoshi theorem. Along the way, we construct/use the psh metrics

$$\phi_\ell \coloneqq \frac{1}{2} \log \sum_j \left| \widetilde{\sigma_j^{(\ell)}} \right|^2 \tag{25.1}$$

on the line bundle $\ell K_X + B$. More precisely, we apply Theorem 25.2 with $L = (\ell-1)K_X + B$ and psh metric $\phi_{\ell-1}$ on L, and hence we have the estimates

$$\int_X \left| \widetilde{\sigma_j^{(\ell)}} \right|^2 e^{-2\phi_{\ell-1}} \le C \int_{X_0} |\sigma_j^{(\ell)}| e^{-2\phi_{\ell-1}} < +\infty. \tag{25.2}$$

We want to construct a psh metric ϕ on K_X as a limit of the psh metrics $\frac{1}{\ell}\phi_\ell$ on $K_X + \frac{1}{\ell}B$. Essentially, ϕ is an extension of the metric $\frac{1}{m}\log|u|$ (as a metric on K_{X_0}), and we can then apply the Ohsawa–Takegoshi theorem one last time to conclude.

Note that $\phi_\ell - \phi_{\ell-1}$ is a metric on $(\ell K_X + B) - ((\ell-1)K_X + B) = K_X$, so $e^{2(\phi_\ell - \phi_{\ell-1})}$ is a volume form on X (possibly with some zeros and poles).

Lemma 25.4. *For $\ell \ge 1$, the volume forms $e^{2(\phi_\ell - \phi_{\ell-1})}$ have uniformly bounded total mass on X; that is,*

$$\sup_{\ell \ge 1} \int_X e^{2(\phi_\ell - \phi_{\ell-1})} < +\infty.$$

Proof. From the estimate (25.2), we have

$$\begin{aligned}
\int_X e^{2(\phi_\ell - \phi_{\ell-1})} &= \sum_j \int_X \left| \widetilde{\sigma_j^{(\ell)}} \right|^2 e^{-2\phi_{\ell-1}} \\
&\le C \sum_j \int_{X_0} |\sigma_j^{(\ell)}|^2 e^{-2\phi_{\ell-1}} \\
&= \sum_j \int_{X_0} \frac{\left|\sigma_j^{(\ell)}\right|^2}{\sum_i |\sigma_i^{(\ell-1)}|^2}.
\end{aligned}$$

Write $\ell = km + p$ for some $0 \le p \le m-1$. If $p \ge 1$, then

$$\frac{\left|\sigma_j^{(\ell)}\right|^2}{\sum_i |\sigma_i^{(\ell-1)}|^2} = \frac{|u|^{2k} |s_j^{(p)}|^2}{|u|^{2k} \sum_i |s_i^{(p-1)}|^2} = \frac{|s_j^{(p)}|^2}{\sum_i |s_i^{(p-1)}|^2},$$

and this is a volume form on X_0 without any poles, since the the sections $(s_i^{(p-1)})_i$ have no common zeros on X_0. In particular, the integral $\int_X e^{2(\phi_\ell - \phi_{\ell-1})}$ is uniformly bounded above in this case, because there are only finitely-many p's and j's. The case $p = 0$ is similar. □

Now, $\phi_\ell - \phi_0$ is a metric on $(\ell K_X + B) - B = \ell K_X$, so $\frac{1}{\ell}(\phi_\ell - \phi_0)$ is a metric on K_X, and as before $e^{\frac{2}{\ell}(\phi_\ell - \phi_0)}$ is a volume form on X.

Lemma 25.5. *For $\ell \geq 1$, the volume forms $e^{\frac{2}{\ell}(\phi_\ell - \phi_0)}$ have uniformly bounded total mass on X; that is,*

$$\sup_{\ell \geq 1} \int_X e^{\frac{2}{\ell}(\phi_\ell - \phi_0)} < +\infty.$$

Proof. Consider the telescoping sum of metrics

$$\phi_\ell - \phi_0 = \sum_{r=1}^{\ell} (\phi_r - \phi_{r-1}),$$

and we wish to apply Hölder's inequality; to do so, we need to be working with functions instead of metrics. Pick a smooth (not necessarily psh) reference metric on K_X, and write

$$\phi_r - \phi_{r-1} = \psi + f_r$$

for some function f_r on X. Thus, $\frac{1}{\ell}(\phi_\ell - \phi_0) = \psi + \frac{1}{\ell}\sum_{r=1}^{\ell} f_r$, and hence

$$\int_X e^{\frac{2}{\ell}(\phi_\ell - \phi_0)} = \int_X \prod_{r=1}^{\ell} e^{\frac{2}{\ell} f_r} e^{2\psi}, \tag{25.3}$$

where $e^{2\psi}$ is viewed as a measure on X. By Hölder's inequality (with many factors), the equation (25.3) is bounded above by

$$\prod_{r=1}^{\ell} \left(\int_X \left(e^{\frac{2}{\ell} f_r}\right)^{\ell} e^{2\psi} \right)^{1/\ell} = \left(\prod_{r=1}^{\ell} \int_X e^{2(\phi_r - \phi_{r-1})} \right)^{1/\ell}. \tag{25.4}$$

By Lemma 25.4, $\int_X e^{2(\phi_r - \phi_{r-1})} \leq C'$ for some $C' > 0$ independent of r, so (25.4) is bounded above by $(C'^{\ell})^{1/\ell} = C'$ as well. □

Now on to the proof of Siu's theorem: pick $\ell = km$ (so $p = 0$). Write

$$\frac{1}{km}(\phi_{km} - \phi_0) = \psi + g_k,$$

where ψ is some reference metric on K_X as before, and g_k is a function on X. In fact, the functions g_k are "almost" psh, in the sense that

$$dd^c g_k = \frac{1}{km} \underbrace{dd^c \phi_{km}}_{\geq 0} - \frac{1}{km} dd^c \phi_0 - dd^c \psi \geq -\frac{1}{km} dd^c \phi_0 - dd^c \psi,$$

and this last curvature form is uniformly bounded from below (in k). (The correct terminology is that g_k is ω-psh for some fixed Kähler form ω, i.e. $\omega + dd^c g_k \geq 0$.) Thus, for all intents and purposes, we can think of the g_k's as actually being psh functions on X.

By Lemma 25.5, we have

$$\sup_{k \geq 1} \int_X e^{2g_k} e^{2\psi} = \sup_{k \geq 1} \int_X e^{\frac{2}{km}(\phi_{km} - \phi_0)} < +\infty,$$

where we think of $e^{2\psi}$ is a fixed volume form on X. In particular, this gives that $\sup_{k \geq 1} \int_X g_k e^{2\psi}$ is bounded above as well.

If the g_k's were psh, then they would be subharmonic, so the sub-mean-value principle gives uniform pointwise estimates on the values of g_k; more precisely,

$$\sup_{k\geq 1}\sup_{K\Subset X} g_k < +\infty.$$

This can be done by using the compactness principle for psh functions (which is also referred to as Hartog's theorem). The compactness principle asserts that either (a) or (b) holds:

(a) $g_k \to -\infty$ uniformly on compact subsets;
(b) there exists a subsequence $g_{k_j} \to g$ in L^1_{loc} and $\psi + g$ is a psh metric on K_X. In this case,

$$g = \left(\limsup_{j\to+\infty} g_{k_j}\right)^*,$$

where $(-)^*$ denotes the upper semicontinuous regularization.

(Note that L^1_{loc}-convergence is the natural one for psh functions.)

Here, (a) cannot happen because on X_0 (which we take as the compact subset $K \Subset X$), then we have

$$\psi + g_k = \frac{1}{km}(\phi_{km} - \phi_0) = \frac{1}{m}\log|u|,$$

which is independent of k and in particular it is $\not\equiv -\infty$. Thus, (b) holds. Then, $g_{k_j} = \frac{1}{m}\log|u| - \psi$ on X_0 and so

$$g \geq \frac{1}{m}\log|u| - \psi$$

on X_0. This implies that $\phi \geq \frac{1}{m}\log|u|$ on X_0. Thus, we have a psh metric ϕ on K_X such that $\phi \geq \frac{1}{m}\log|u|$ on K_{X_0} (i.e. ϕ is a "superextension" of the metric $\frac{1}{m}\log|u|$ on K_{X_0} to K_X). Now, apply the Ohsawa–Takegoshi theorem with $L = (m-1)K_X$ and the metric $(m-1)\phi$: the hypothesis of the extension theorem are satisfied since

$$\int_{X_0} |u|^2 e^{-2(m-1)\phi} = \int_{X_0} |u|^{2/m} < +\infty,$$

because $|u|^{2/m}$ is some volume form on X_0, possibly with zeros but without poles; thus, there exists an extension $U \in H^0(X, mK_X)$ of u to all of X and moreover we have the estimate

$$\int_X |U|^2 e^{-2(m-1)\phi} \leq C\int_{X_0} |u|^2 e^{-2(m-1)\phi} \leq \int_{X_0} |u|^{2/m} < +\infty.$$

This completes the proof of Siu's theorem.

26. March 16th

There will be no class on Wednesday, March 21st and Friday, March 23rd.

26.1. **Bergman Kernels: Motivation.** Suppose X is a compact complex manifold of dimension n, L is a holomorphic line bundle on X, and ϕ is a smooth positive metric on L (so X is projective and L is ample, by the Kodaira embedding theorem).

Question 26.1. Can we recover the pair (L, ϕ) from the *section ring* $R = R(X, L) \coloneqq \bigoplus_{m\geq 0} H^0(X, mL)$?

Here, the section ring $R(X, L)$ is just thought of as a graded $\mathbf{C}$-algebra. In this setting, the curvature form $dd^c\phi$ of ϕ is a Kähler form on X, so the top wedge power $(dd^c\phi)^n$ is a volume form on X.

Fact 26.2. The de Rham cohomology class of $dd^c\phi$ coincides with the first Chern class $c_1(L)$ of L; in particular,

$$\int_X (dd^c\phi)^n = (L^n) \coloneqq \langle c_1(L)^n, [X]\rangle.$$

On the other hand, the asymptotic Riemann–Roch theorem says that

$$\dim H^0(X, mL) = (L^n) \cdot \frac{m^n}{n!} + O(m^{n-1}).$$

Thus, we can recover $\int_X (dd^c\phi)^n$ from $R(X, L)$. Using Bergman kernels, we can essentially recover ϕ, as well!

26.2. **Bergman Kernels: Classical Situation.** Consider a measure space (X, μ) and a closed subspace $\mathcal{H} \subseteq L^2(X, \mu)$. Moreover, assume that the following two conditions are satisfied:

(1) for any $h \in \mathcal{H}$, the value $h(x)$ is well-defined for all $x \in X$;
(2) the linear functional $\mathcal{H} \to \mathbf{C}$, given by $h \mapsto h(x)$, is *bounded* for all $x \in X$, i.e. there is a constant $C_x > 0$ such that

$$|h(x)| \le C_x \|h\|$$

for all $h \in \mathcal{H}$.

Example 26.3. Let $X = \mathbf{D}$ be the unit disc and let μ be the Lebesgue measure. Consider

$$\mathcal{H} := \left\{ \mathbf{D} \xrightarrow{h} \mathbf{C} \text{ holomorphic such that } \int_{\mathbf{D}} |h(x)|^2 d\mu(x) < +\infty \right\}.$$

Under these assumptions, the Riesz representation theorem implies that for each $x \in X$, there exists $k_x \in \mathcal{H}$ such that

$$h(x) = \langle h, k_x \rangle = \int_X h(y)\overline{k_x(y)} d\mu(y) \tag{26.1}$$

for all $h \in \mathcal{H}$. The function k_x is called the *Bergman kernel* for the point x; alternatively, we write $k_x(y) = K(x, y)$. Taking $h = k_y$ in (26.1) gives

$$k_y(x) = \int_X k_y(z)\overline{k_x(z)} d\mu(z) \Longrightarrow k_y(x) = \overline{k_x(y)}.$$

Finally, setting $y = x$, we find that

$$k_x(x) = \int_X |k_x(z)|^2 d\mu(z).$$

Definition 26.4. The function $K(x) := k_x(x)$ is called the *Bergman kernel on the diagonal.*

Note that $K(x) = \int_X |k_x(z)|^2 d\mu(z) = \|k_x\|^2$ is the square of the norm of the "evaluation at x" functional.

There are 2 points of view on Bergman kernels, the first of which is in terms of an orthonormal basis. From now on, fix an orthonormal basis $(h_j)_1^\infty$ for $\mathcal{H}$.

Claim 26.5.

(1) For any $x, y \in X$, we have $k_x(y) = \sum_j h_j(y)\overline{h_j(x)}$.
(2) For any $x \in X$, we have $K(x) = \sum_j |h_j(x)|^2$.

This claim requires justification.

Lemma 26.6. *For any $N > 0$, we have $\sum_{j=1}^N |h_j(x)|^2 \le K(x) < +\infty$ for all $x \in X$.*

Proof. Pick $a_1, \dots, a_N \in \mathbf{C}$ with $\sum_1^N |a_j|^2 \le 1$, and set $h = \sum_{j=1}^N a_j b_j \in \mathcal{H}$. Then, $\|h\| \le 1$, so

$$|h(x)|^2 = |\langle h, k_x \rangle|^2 \le \|h\|^2 \|k_x\|^2 \le K(x).$$

Since the coefficients $(a_j)_1^N$ arbitrary, we may conclude. □

implies that the partial sum $\sum_{j=1}^N |h_j(x)|^2$ converges pointwise in x, so $\sum_j h_j(y)\overline{h_j(x)}$ converges pointwise and hence $\sum_j h_j(\cdot)\overline{h_j(x)}$ converges in L^2 (i.e. in $\mathcal{H}$) to some function $k'_x \in \mathcal{H}$. Thus, Claim 26.5 follows if we can show that $k'_x = k_x$. However, for all ℓ, we have $\langle h_\ell, k'_x \rangle = h_\ell(x) = \langle h_\ell, k_x \rangle$, so $k'_x = k_x$ as required.

Example 26.7. If $X = \mathbf{D}$, then an orthonormal basis consists of $h_j(x) = \sqrt{\frac{j+1}{\pi}}x^j$; orthonormality is clear, since one can check in polar coordinates that $\int_{\mathbf{D}} x^j \overline{x^\ell} = 0$ unless $j = \ell$. Thus,

$$K_x(y) = \sum_j h_j(y)\overline{h_j(x)} = \frac{1}{\pi}\frac{1}{(1-\overline{x}y)^2},$$

and hence

$$K(x) = \frac{1}{\pi}\frac{1}{(1-|x|^2)^2}.$$

Remark 26.8. For any $g \in L^2(X,\mu)$ (i.e. g does not necessarily belong to $\mathcal{H}$), set

$$\widehat{g}(x) := \langle g, k_x\rangle$$

The second point of view on Bergman kernels is an extremal characterization, exemplified by the proposition below.

Proposition 26.9. *For any $x \in X$, we have*

$$K(x) = \sup_{\|h\|\leq 1} |h(x)|^2 = \sup_{h\neq 0} \frac{|h(x)|^2}{\|h\|^2}.$$

In other words, the function $\sqrt{K(x)}$ is the norm of the 'evaluation at x' map.

Proof. Observe that

$$\sup_{h\neq 0}\frac{|h(x)|^2}{\|h\|^2} = \sup_{h\neq 0}\frac{|\langle h, k_x\rangle|^2}{\|h\|^2} = \|k_x\|^2 = K(x).$$

□

27. March 19th

There will be no class on Wednesday, March 21st and Friday, March 23rd.

27.1. **Bergman Kernels: Classical Situation (Continued).** Recall the setup from last time: let (X,μ) be a measure space and let $\mathcal{H} \subseteq L^2(X,\mu)$ be a closed subspace (which is then also a Hilbert space) and assume that:

(1) $h(x)$ is well-defined for all $x \in X$;
(2) the linear functional $h \mapsto h(x)$ on $\mathcal{H}$ is bounded for all $x \in X$.

By the Riesz representation theorem, there exists $k_x \in \mathcal{H}$ such that

$$h(x) = \langle h, k_x\rangle = \int_X h(y)\overline{k_x(y)}d\mu(y)$$

for all $h \in \mathcal{H}$; the function k_x is the *Bergman kernel* at the point x. The function $K(x) := k_x(x)$ is the *Bergman kernel on the diagonal.*

Properties 27.1.

(1) If $(h_j)_j$ is an orthonormal basis for $\mathcal{H}$, then
$$k_x(y) = \sum_j h_j(y)\overline{h_j(x)} \qquad \text{and} \qquad K(x) = \sum_j |h_j(x)|^2.$$
(2) For $g \in L^2(X,\mu)$, set $\widehat{g}(x) := \langle g, k_x\rangle$ for $x \in X$. Then, $\widehat{g} \in \mathcal{H}$ is the orthogonal projection of g onto $\mathcal{H}$.

(3) For each $x \in X$, $K(x)$ is the square of the norm of the evaluation map $h \mapsto h(x)$; that is,

$$K(x) = \sup_{h \neq 0} \frac{|h(x)|^2}{\|h\|^2} = \sup_{\|h\| \leq 1} |h(x)|^2,$$

and the supremum is attained for $h = k_x$.

Example 27.2. If $X = \mathbf{B}$ is the unit ball in $\mathbf{C}^n$ and $\mu = dV$ is the Lebesgue measure, consider the space $\mathcal{H} \coloneqq L^2(X, \mu) \cap \mathcal{O}(X)$ of holomorphic L^2-functions. In this case, one can show that

$$K(z) = \frac{1}{\operatorname{vol}(\mathbf{B})} \frac{1}{(1 - |z|^2)^{n+1}}.$$

Typically, the Bergman kernel on the diagonal blows up as one approaches the boundary, as is the case here.

Example 27.3. If $X = \mathbf{D}^n$ is the unit polydisc and $\mu = dV$, consider the space $\mathcal{H} \coloneqq L^2(X, \mu) \cap \mathcal{O}(X)$ of holomorphic L^2-functions. As the Bergman kernel on a product is the product of the Bergman kernels, it follows from Example 27.2 that

$$K(z) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - |z_j|^2)^2}.$$

Example 27.4. If $X = \mathbf{C}^n$ and $\mu = dV$, then the space $\mathcal{H} \coloneqq L^2(X, \mu) \cap \mathcal{O}(X)$ of holomorphic L^2-functions is only the constant functions $\mathbf{C}$, so $K(z) \equiv 1$.

Example 27.5. If $X = \mathbf{C}^n$ and $\mu = e^{-|z|^2} dV$, then the space $\mathcal{H} \coloneqq L^2(X, \mu) \cap \mathcal{O}(X)$ of holomorphic L^2-functions is known as the *Bargmann–Foch space*. In this case, the Bergman kernel on the diagonal is

$$K(z) = e^{|z|^2}.$$

Example 27.6. If $X = \mathbf{C}$ and $\mu = \frac{1}{(1+|z|^2)^{N+2}} dV$, then the space $\mathcal{H} \coloneqq L^2(X, \mu) \cap \mathcal{O}(X)$ of holomorphic L^2-functions consists of polynomials of degree $\leq N$. (There is an analogous description when $X = \mathbf{C}^n$.) In this case, we have

$$K(z) = \frac{N+1}{\pi} \left(1 + |z|^2\right)^N.$$

27.2. **Bergman Kernel Asymptotics.** Consider an open subset $\Omega \subseteq \mathbf{C}^n$, a C^2 "weight" function $\phi \colon \Omega \to \mathbf{R}$, and the measure $\mu_m = e^{-2m\phi} dV$, where dV is the Lebesgue measure on Ω. Set

$$\mathcal{H}_m \coloneqq L^2(\Omega, \mu_m) \cap \mathcal{O}(\Omega),$$

and write K_m for the Bergman kernel on the diagonal, so K_m is a function $\Omega \to \mathbf{R}_{>0}$. We are interested in the asymptotics of K_m as $m \to +\infty$.

Proposition 27.7. *For $z \in \Omega$, there is a pointwise inequality*

$$\limsup_{m \to +\infty} m^{-n} K_m(z) e^{-2m\phi(z)} dV(z) \leq \mathbf{1}_{\Omega_0}(z) \frac{(dd^c\phi)^n}{n!}(z), \tag{27.1}$$

where $\Omega_0 \coloneqq \{z \in \Omega \colon (dd^c\phi)(z) > 0\}$ is the locus in Ω where $dd^c\phi$ is a positive $(1,1)$-form.

Proof. Without loss of generality, assume $z = 0$. Use coordinates $(\xi_1, \dots, \xi_n)$ on $\mathbf{C}^n$. To prove (27.1), we use the extremal characterization of $K_m(z)$; that is, write

$$K_m(0) = \sup_{h \neq 0} \frac{|h(0)|^2}{\int_\Omega |h|^2 e^{-2m\phi} dV}$$

Pick $h \in \mathcal{O}(\Omega)$ such that $\int_\Omega |h|^2 e^{-2m\phi} dV \leq 1$. We must estimate the value $|h(0)|$ from above.

First, assume that the Taylor expansion of ϕ near 0 is of the form

$$\phi = \phi(0) + \frac{1}{2}\sum_{j=1}^{n} \lambda_j |\xi_j|^2 + o(|\xi|^2),$$

where $\lambda_j > 0$. In particular, $(dd^c\phi)(0) > 0$. Pick $0 < \lambda_j < \lambda_j'$ for all $1 \le j \le n$ so that $\phi \le \phi(0) + \sum_{j=1}^{n} \lambda_j' |\xi_j|^2$ for $|\xi_j| \le \delta$; here, $\delta > 0$ is some number (depending on the λ_j''s!) that we will fix later. Then, using the polar coordinates $\xi_j = r_j e^{i\theta_j}$, we have the inequalities

$$\begin{aligned}
1 &\ge \int_{|\xi_j|\le\delta} |h|^2 e^{-2m\phi} dV \\
&\ge e^{-2m\phi(0)} \int_{|\xi_j|\le\delta} |h|^2 e^{-m\sum_{j=1}^{n}\lambda_j'|\xi_j|^2} dV \\
&= e^{-2m\phi(0)} \int_{0\le r_j\le\delta} r_1\cdots r_n e^{-m\sum_{j=1}^{n}\lambda_j' r_j^2} dr_1\cdots dr_n \underbrace{\int_{0\le\theta_j\le 2\pi} |h(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})|^2 d\theta_1\cdots d\theta_n}_{\ge(2\pi)^n|h(0)|^2} \\
&\ge e^{-2m\phi(0)} (2\pi)^n |h(0)|^2 \prod_{j=1}^{n} \frac{1}{2m\lambda_j'}\left(1 - e^{-m\lambda_j'\delta^2}\right)
\end{aligned}$$

where the upper bound on $(2\pi)^n|h(0)|^2$ follows from the Cauchy estimates. Now, taking the supremum over all such h, we have

$$m^{-n} K_n(0) e^{-2m\phi(0)} \le \pi^{-n} \prod_{j=1}^{n} \lambda_j' (1 - e^{-m\lambda_j'\delta^2})^{-1}. \tag{27.2}$$

Now,

$$(dd^c\phi)(0) = \frac{i}{\pi}\partial\overline{\partial}\phi(0) = \frac{1}{\pi}\sum_{j=1}^{n} \lambda_j i dz_j \wedge d\overline{z}_j,$$

and so

$$\frac{(dd^c\phi)^n}{n!} = \pi^{-n}\prod_{j=1}^{n} \lambda_j dV \tag{27.3}$$

at 0. Combining (27.2) and (27.3), first let $m \to +\infty$ and then let $\lambda_j' \searrow \lambda_j$ to recover (27.1).

In general, one can write the Taylor series of ϕ at 0 as

$$\phi = \phi(0) + 2\mathrm{Re}\,(p(\xi)) + q(\xi,\overline{\xi}) + o(|\xi|^2),$$

where $p(\xi)$ is a holomorphic polynomial and q is a Hermitian form. After performing the substitution $h \mapsto he^{-mp}$, one can without loss of generality set $p = 0$. Now, perform a unitary change of variables to diagonalize q, so that it can be written as $q = \sum_j \lambda_j |\xi_j|^2$ for some $\lambda_j \in \mathbf{R}$. If $\lambda_j > 0$ for all j, then we are done by the previous case. If $\lambda_j \le 0$ for some j, the same argument goes through (and in fact, the right-hand side of (27.2) only becomes larger!). □

Next, we will have a version of the Bergman kernels for line bundles as well as a version of Proposition 27.7. Moreover, we will have a criterion for equality to occur in (27.1) using the $\overline{\partial}$-equation.

28. March 26th

There will be no class on Wednesday, March 28th. Today we begin discussing the global version of the Bergman kernels that were introduced last time. We follow the presentation of [Ber10, §4.2].

28.1. **Bergman Kernels for Line Bundles.** Let X be a compact complex manifold of dimension n, L a holomorphic line bundle on X, ϕ a smooth[8] metric on L, and μ a smooth measure (or volume form) on X (which corresponds to a metric on K_X). Consider the inner product on the space $H^0(X, L)$ of global holomorphic sections of L:

$$\langle u, v\rangle := \int_X u\overline{v}e^{-2\phi}d\mu$$

for $u, v \in H^0(X, L)$. Let $(u_j)_j$ be an orthonormal basis for $H^0(X, L)$.

Definition 28.1. The *Bergman function* of (X, L, ϕ, μ) (on the diagonal) is

$$B := \sum_j |u_j|^2 e^{-2\phi},$$

which is a function on X. The *Bergman kernel* (again on the diagonal) is $K := Be^{2\phi}$, so

$$\frac{1}{2}\log K = \frac{1}{2}\log B + \phi$$

is a metric on L (in the usual additive terminology for metrics).

As in the classical case, there is an extremal characterization of the Bergman function: for $x \in X$, we have

$$B(x) = \sup_{u\neq 0} \frac{|u(x)|^2 e^{-2\phi(x)}}{\|u\|^2} = \sup_{\|u\|\leq 1} |u(x)|e^{-2\phi(x)}.$$

Consider now the Bergman kernel B_m for $(mL, m\phi, \mu)$ and we will look at what happens as $m \to +\infty$. The same argument as last time (using the sub-mean-value principle) yields the following proposition:

Proposition 28.2. *On X, there is a pointwise inequality*

$$\limsup_{m\to+\infty} m^{-n} B_m d\mu \leq \mathbf{1}_{\{dd^c\phi>0\}}(x)\frac{(dd^c\phi)^n}{n!}, \tag{28.1}$$

where the term $B_m d\mu$ is called the Bergman measure *on X.*

Integrating (28.1) over all of X yields another estimate:

Corollary 28.3. *If $X_0 := \{dd^c\phi > 0\} \subseteq X$, then*

$$\limsup_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) \leq \int_{X_0} (dd^c\phi)^n. \tag{28.2}$$

Proof. By construction, we have $h^0(X, mL) = \int_X B_m(x)d\mu(x)$. Now, we have

$$\limsup_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) = \limsup_{m\to+\infty} \frac{n!}{m^n}\int_X B_m(x)d\mu(x) \leq \int_X \limsup_{m\to+\infty} \frac{n!}{m^n} B_m(x)d\mu(x) \leq \int_{X_0} (dd^c\phi)^n$$

where the second inequality follows from (28.1), and we would like to use Fatou's lemma to get the first inequality. To use Fatou's lemma, one needs an *upper bound* on $m^{-n}B_m$. Fix $x \in X$. Pick local coordinates z at x such that $z(x) = 0$ and a local trivialization s of L at x such that

$$\phi \circ s = q(z, \overline{z}) + o(|z|^2),$$

where q is a Hermitian form. Consider $u \in H^0(X, mL)$ with $\|u\|^2 = \int_X |u|^2 e^{-2m\phi}d\mu \leq 1$; in particular, we have

$$1 \geq \int_{|z|^2\leq \frac{1}{m}} |u \circ s|^2 e^{-2m(\phi\circ s)}d\mu.$$

[8]Unlike in certain past situations, we will really use that ϕ is smooth, or at least C^2.

Now, if $|z|^2 \le \frac{1}{m}$, then $|e^{2m\phi}| \le C < +\infty$ for some constant $C > 0$ independent of m. By approximating the smooth measure $d\mu$ by the Lebesgue measure, we find that

$$1 \ge C^{-1} \int_{|z|^2 \le \frac{1}{m}} |u \circ s|^2 d\mu \ge C' m^{-n} |u(s(x))|^2,$$

where the last inequality follows from the sub-mean-value inequality, and $C' > 0$ is again a constant independent of m. Thus, $m^{-n}|u(s(x))|^2 e^{-2m\phi(s(x))} \le C''$ and taking the supremum over such u gives that $m^{-n}B_m(x) \le C''$ (and one can make C'' uniform in x, as well). For this reason, we can indeed apply Fatou's lemma, which completes the proof. □

The more interesting estimates are from the opposite direction, for which one must actually construct sections.

Theorem 28.4. *Suppose that $dd^c\phi > 0$, i.e. ϕ is a positive metric, and $\mu = \frac{(dd^c\phi)^n}{n!}$. Then,*

$$\lim_{m\to+\infty} m^{-n} B_m = 1$$

pointwise on X.

Remark 28.5. There is a long story of more precise asymptotics for B_m starting with Bouche [Bou90] and Tian [Tia90]; other examples include [BBS08, Cat99, Zel98].

Corollary 28.6. [Asymptotic Riemann–Roch] *If $dd^c\phi > 0$, then*

$$\lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) = (L^n). \tag{28.3}$$

From the more precise asymptotics discussed in Remark 28.5, one can show e.g. that the m^{n-1}-term in the asymptotic expansion of the function $m \mapsto \frac{n!}{m^n} h^0(X, mL)$ is the intersection number $(K_X \cdot L^{n-1})$.

Proof. Theorem 28.4 and the dominated convergence theorem shows that the left-hand side of (28.3) is given by

$$\int_X \lim_{m\to+\infty} \frac{n!}{m^n} B_m (dd^c\phi)^n = \int_X (dd^c\phi)^n = (L^n),$$

where the final equality holds because $dd^c\phi$ is a smooth form in the first Chern class $c_1(L)$ of L. □

To prove Theorem 28.4, one needs an estimate on B_m *from below*. This amounts to constructing sections u of mL with $\|u\| = 1$ and $|u(x)|e^{-\phi(x)}$ not too small.

Proof of Theorem 28.4. Fix $x \in X$. Pick local coordinates z at x such that $z(x) = 0$ and a local trivialization s of L at x such that

$$\phi \circ s = \frac{1}{2}|z|^2 + o(|z|^2),$$

Pick a smooth function $\chi\colon \mathbf{C}^n \to \mathbf{R}$ such that $\chi \equiv 1$ on the ball of radius 1 centered at 0, and $\chi \equiv 0$ outside the ball of radius 2 centered at 0. Define a local section h_m of mL at x by the formula

$$h_m = s^m \cdot \chi(m^{1/2}\delta_m z),$$

where $\delta_m \to 0$ but $m^{1/2}\delta_m \to +\infty$ as $m \to +\infty$. In fact, one can view h_m as a *global* (smooth) section of mL, which is supported near x. On the support, one has $\phi \sim \frac{1}{2}|z|^2$.

The idea is now to use Hörmander's theorem to solve the equation $\overline{\partial}\, u_m = f_m := \overline{\partial}\, h_m$ with the correct L^2-estimates. To do so, we need to compute the L^2-norm of f_m. In the end, we will set $s_m := h_m - u_m$. This is holomorphic, and one must estimate $\|s_m\|^2$ and $s_m(x)e^{-m\phi(x)}$ to get the correct bound from below on $B_m(x)$.

First, we estimate $\|h_m\|$: by a change of variables, we find that

$$\int_X |h_m|^2 e^{-2m\phi} d\mu \sim \int_{\{|\xi| \le \frac{2}{m^{1/2}\delta_m}\} \subseteq \mathbf{C}^n} \chi(m^{1/2}\delta_m \xi) e^{-m|\xi|^2} dV(\xi) \pi^{-n},$$

where $dV(\xi)$ denotes the Lebesgue measure on $\mathbf{C}^n$. Taking $\eta = m^{1/2}\xi$, this last integral becomes

$$\frac{1}{(\pi m)^n}\int_{|\eta|\le\frac{2}{\delta_m}}\chi(\delta_m\eta)e^{-|\eta|^2}dV(\eta).$$

As $\delta_m \to 0$, we have $\chi(\delta_m\cdot)\to 1$ and $\frac{2}{\delta_m}\to+\infty$, and so

$$\frac{1}{(\pi m)^n}\int_{|\eta|\le\frac{2}{\delta_m}}\chi(\delta_m\eta)e^{-|\eta|^2}dV(\eta)\sim\frac{1}{(\pi m)^n}\int_{\mathbf{C}^n}e^{-|\eta|^2}dV(\eta)=m^{-n}.$$

The conclusion is that $\|h_m\|^2\sim m^{-n}$. We will come back next class and estimate the norm of f_m, which will nearly complete the proof. ☐

29. March 30th

29.1. **Bergman Kernels for Line Bundles (Continued).** The goal is to complete the proof of the Bergman function asymptotics that we began last time. Let us recall the setup: X is a compact complex manifold of dimension n, L is a holomorphic line bundle on X, ϕ is a smooth metric on L, and μ is a smooth measure on X. There is an inner product on the global (not necessarily holomorphic, but only measurable) sections of L given by

$$\langle u,v\rangle := \int_X u\overline{v}e^{-2\phi}.$$

If $(u_j)_j$ is an orthonormal basis of $H^0(X,L)$, then the *Bergman function* is

$$B := \sum_j |u_j|^2e^{-2\phi}.$$

This, in practice, is a finite sum since X is assumed to be compact. There is an alternate 'extremal' characterization of B, namely we can write

$$B(x)=\sup_{u\neq0}\frac{|u(x)|^2e^{-2\phi(x)}}{\|u\|^2},$$

which shows that the definition of B is independent of the choice of orthonormal basis.

We are interested in the asymptotics of the Bergman functions of mL as $m\to+\infty$. Let B_m denote the Bergman function of mL.

Theorem 29.1. *Suppose $dd^c\phi>0$ and $\mu:=\frac{(dd^c\phi)^n}{n!}$. Then,*

$$\lim_{m\to+\infty}m^{-n}B_m=1$$

pointwise on X.

As a corollary of Theorem 29.1, we showed a version of the asymptotic Riemann–Roch theorem.

Corollary 29.2. *If $dd^c\phi>0$, then*

$$\lim_{m\to+\infty}\frac{n!}{m^n}h^0(X,mL)=(L^n).$$

Last time, we began discussing the proof of Theorem 29.1.

Proof of Theorem 29.1. We only need to prove that $\liminf_{m\to+\infty}m^{-n}B_m\ge1$. Pick $x\in X$, local coordinates z at x with $z(x)=0$, and a local trivializing section s of L such that

$$\phi\circ s=\frac{1}{2}|z|^2+o(|z|^2),$$

where $\frac{1}{2}|z|^2=\frac{1}{2}\left(|z_1|^2+\ldots+|z_n|^2\right)$. Pick a smooth function $\chi\colon\mathbf{C}^n\to\mathbf{R}$ such that $\chi(\zeta)=1$ if $|\zeta|\le1$, and $\chi(\zeta)=0$ if $|\zeta|\ge2$. Pick a sequence $\delta_m>0$ such that $\delta_m\to0$ but $\delta_m m^{1/2}\to+\infty$, e.g. $\delta_m=m^{-1/3}$. Set

$h_m := s^m \cdot \chi(m^{1/2}\delta_m z)$, where here we think of z as a map from a small open neighbourhood of x to a small open ball around the origin in $\mathbf{C}^n$. Thus, h_m is a global (smooth) section of mL with support near x. We will now "correct" h_m (by solving a $\overline{\partial}$-equation) to get something holomorphic with similar estimates.

As $\phi \circ s \sim \frac{1}{2}|z|^2$ on the support of h_m (in the sense that the ratio tends to 1 as $m \to +\infty$), it follows that

$$dd^c\phi \overset{\text{loc}}{=} \frac{i}{\pi}\,\partial\,\overline{\partial}(\phi \circ s) \sim \frac{i}{2\pi}(dz_1 \wedge d\,\overline{z}_1 + \ldots + dz_n \wedge d\,\overline{z}_n),$$

and so

$$\mu = \frac{(dd^c\phi)^n}{n!} \sim \pi^{-n} \bigwedge_{i=1}^{n} \frac{i}{2} dz_j \wedge d\,\overline{z}_j, \tag{29.1}$$

and the right-hand side of (29.1) is the Lebesgue measure on $\mathbf{C}^n$. Now, we get that

$$\|h_m\|^2 = \int_X |h_m|^2 e^{-2m\phi} d\mu \sim \int_{\{|\zeta| \le \frac{2}{m^{1/2}\delta_m}\} \subseteq \mathbf{C}^n} \chi(m^{1/2}\delta_m\zeta) e^{-m|\zeta|^2} \pi^{-n} dV(\zeta), \tag{29.2}$$

where $dV(\zeta)$ denotes the Lebesgue measure on $\mathbf{C}^n$. If one makes the change of variables $\eta = m^{1/2}\zeta$, then the right-hand side of (29.2) becomes

$$(m\pi)^{-n} \int_{\{|\eta| < \frac{2}{\delta_m}\}} \chi(\delta_m \eta) e^{-|\eta|^2} dV(\eta) \sim (m\pi)^{-n} \int_{\mathbf{C}^n} e^{-|\eta|^2} dV(\eta) = m^{-n}, \tag{29.3}$$

where the comparison above follows because the region $\{|\eta| < \frac{2}{\delta_m}\}$ exhausts $\mathbf{C}^n$ as $m \to +\infty$.

Next (in order to solve a $\overline{\partial}$-equation), we must estimate $\|\,\overline{\partial}\,h_m\|^2$ from above. We know that $\overline{\partial}\,h_m$ is supported where $1 \le m^{1/2}\delta_m|z| \le 2$, so there is an estimate of the form $|\,\overline{\partial}\,h_m|^2 e^{-2m\phi} \le Cm\delta_m^2 e^{-2m(\phi\circ s)}$ on this region. and hence

$$\begin{aligned}
\|\,\overline{\partial}\,h_m\|^2 &= \int_X |\,\overline{\partial}\,h_m|^2 e^{-2m\phi} d\mu \\
&\le (m\pi)^{-n} Cm\delta_m^2 \int_{\{\frac{1}{\delta_m} \le |\eta| \le \frac{2}{\delta_m}\}} e^{-|\eta|^2} dV(\eta) \\
&\le C' m^{1-n} \delta_m^2 e^{-1/\delta_m^2},
\end{aligned}$$

The key factor in this last estimate is the e^{-1/δ_m^2} term, which goes to zero very fast as $m \to +\infty$.

Now, we use Hörmander's theorem to solve $\overline{\partial}\,u_m = f_m := \overline{\partial}\,h_m$ with L^2-estimates. (The problem is that f_m is a $(0,1)$-form, not an $(n,1)$-form - we will come back to this). This is done with the weight $m\phi$, which is ok since $dd^c(m\phi) = mdd^c\phi = m\omega$, where $\omega := dd^c\phi$ is a Kähler form on X.

The upshot of Hörmander's theorem is that there exists u_m such that $\overline{\partial}\,u_m = f_m$ and satisfying

$$\|u_m\|^2 = \int_X |u_m|^2 e^{-2m\phi} d\mu \le \frac{1}{m} \int_X |f_m|^2 e^{-2m\phi} d\mu. \tag{29.4}$$

By the previous estimate, the right-hand side of (29.4) is bounded above by

$$Cm^{-n}\delta_m^2 e^{-1/\delta_m^2},$$

which tends to zero very rapidly as $m \to +\infty$ (it does so faster than m^{-n}, for example). Set $s_m := h_m - u_m$, then $s_m \in H^0(X, mL)$ (i.e. s_m is a *holomorphic* section) and $\|s_m\|^2 \sim \|h_m\|^2 \sim m^{-n}$. Now, we must estimate $|s_m(x)|e^{-m\phi(x)}$ from below. On the one hand, we know that $|h_m(x)|e^{-m\phi(x)} = 1$ for all $m \ge 1$ by construction. The "correction term" u_m is holomorphic on $\Omega_m = \{|z| \le \frac{1}{m}\}$ (because when $|z| < \frac{1}{m}$, we have $|m^{1/2}\delta_m z| < 1$, so $\chi(m^{1/2}\delta_m z) \equiv 1$ on this region), and $m\phi \circ s \approx 0$ on Ω_m (because $\phi \circ s \sim \frac{1}{2}|z|^2$). Thus, the submean-value principle implies that

$$|u_m(x)|e^{-2m\phi(x)} \le Cm^{2n} \int_{\Omega_m} |u_m|^2 e^{-2m\phi} d\mu \le Cm^{2n}\|u_m\|^2 \le Cm^{2n}m^{-n}\delta_m^2 e^{-1/\delta_m^2} \to 0 \text{ as } m \to +\infty.$$

In the above, we have used that $\mathrm{vol}(\Omega_m)$ is (up to a constant factor) equal to m^{2n}. It follows that

$$|s_m(x)|e^{-2m\phi(x)} \to 1$$

as $m \to +\infty$ (and, in fact, the convergence is very fast!). Finally, we find that

$$B_m(x) \geq \frac{|s_m(x)|e^{-2m\phi(x)}}{\|s_m\|^2} \sim \frac{1}{m^{-n}},$$

and hence $\liminf_{m\to+\infty} m^{-n}B_m(x) \geq 1$, as required.

The last problem to be dealt with is that we use the "wrong version" of Hörmander's theorem. One way to get around this is roughly as follows: write $mL = K_X + (mL - K_X)$, and view f_m as an $(n, 1)$-form with values in $mL - K_X$. Now, $mL - K_X \gg 0$ for $m \gg 0$, and use Hörmander's theorem for the line bundle $mL - K_X$ for a suitable choice of positive metric. □

Next time, we will begin discussing variations of Bergman kernels (that is, how they behave in a family), which will lead into Berndtsson's theorem [Ber09] on the positivity of direct images.

30. April 2nd

30.1. **Positivity of Direct Images.** There are various versions of Berndtsson's theorem [Ber09] on the positivity of direct images, but we will concern ourselves with the more geometric one.

Let X and Y be complex manifolds, and let $p\colon X \to Y$ be a proper holomorphic submersion. (For practical purposes, one can think of Y as being 1-dimensional.)

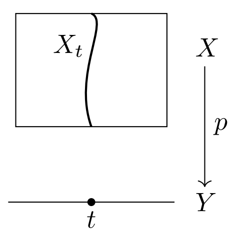


Figure 7. The fibre of p above a point $t \in Y$ is denoted by X_t.

For a line bundle L on X, consider the *direct image* $E := p_*(L + K_X)$, which is a coherent sheaf on Y. The idea is that, under suitable hypothesis, E is a vector bundle with certain positivity properties. In any case, set

$$E_t := H^0(X_t, (L + K_X)|_{X_t}) \simeq H^0(X_t, L|_{X_t} + K_{X_t})$$

for $t \in Y$. Basically, if L is semipositive (in the sense that it admits a smooth metric ϕ such that the curvature is semipositive, i.e. $dd^c\phi \geq 0$), then the Ohsawa–Takegoshi theorem implies that E is a vector bundle.

Fix a smooth metric ϕ on L. Then, one can define an L^2-norm (i.e. a Hermitian metric) on E_t for every $t \in Y$: if $u \in E_t$, then

$$\|u\|^2 := \int_{X_t} |u|^2 e^{-2\phi},$$

where we think of $u \in E_t$ as a holomorphic $(n, 0)$-form on X_t with values in $L|_{X_t}$, so the expression $|u|^2e^{-2\phi}$ is a well-defined volume form on X_t.

Theorem 30.1. [Ber09] *If X is Kähler and (L, ϕ) is (semi)positive, then:*
1. *E is a vector bundle on Y;*
2. *$\|\cdot\|$ is a smooth Hermitian metric on E;*
3. *$(E, \|\cdot\|)$ is (semi)positive in the sense of Nakano.*

There are various notions of positivity for vector bundles that we will introduce; Nakano positivity is the strongest such condition.

There is an interesting special case of Theorem 30.1: consider $X = Y \times Z$, so the fibres $X_t = Z$ are independent of $t \in Y$. Further, assume that L is the pullback of a line bundle M on Z, so that $L|_{X_t} = M$. Consider a smooth metric ϕ on L such that $\phi|_{(L|_{X_t})}$ may depend on t; that is, each copy of $L|_{X_t}$ is isomorphic to M, but the metric varies from fibre to fibre. This is still a nontrivial situation.

30.2. **Positivity Notions for Line Bundles.** There are algebraic and analytic notions of positivity for line bundles, and we begin with the former.

30.2.1. *Algebraic Notions of Positivity.* Let X be a complex projective manifold, and let L be an algebraic line bundle on X (in fact, since X is projective, every holomorphic line bundle on X is algebraic).

Definition 30.2. Consider the following algebraic positivity notions defined by sections of the line bundle:

- L is *very ample* if there exists an embedding $\phi\colon X \hookrightarrow \mathbf{P}^N$ such that $L = \phi^*\mathcal{O}(1)$; equivalently, the rational map into the projective space $\mathbf{P}(H^0(X,L)^\vee)$ defined by $H^0(X,L)$ is an embedding.
- L is *ample* if mL is very ample for some $m \geq 1$.
- L is *globally generated* (or *base point free*) if for any $x \in X$, there exists $s \in H^0(X,L)$ such that $s(x) \neq 0$; equivalently, the rational map into the projective space $\mathbf{P}(H^0(X,L)^\vee)$ defined by $H^0(X,L)$ is a morphism.
- L is *semiample*[9] if mL is globally generated for some $m \geq 1$.

There are obvious implications

$$\begin{array}{ccc} \text{very ample} & \Longrightarrow & \text{ample} \\ \Downarrow & & \Downarrow \\ \text{globally generated} & \Longrightarrow & \text{semiample} \end{array}$$

but there are no further implications in general.

30.2.2. *Numerical Notions of Positivity.* Numerical conditions refer to ones that are characterized by intersection-theoretic properties. A classical example is ampleness:

Theorem 30.3. [Nakai–Moishezon Criterion] *A line bundle L is ample iff $\int_V c_1(L)^{\dim V} > 0$ for every irreducible subvariety $V \subseteq X$ with $\dim V > 0$.*

Definition 30.4. A line bundle L is *numerically trivial*, written $L \equiv 0$, if $\deg(L|_C) = 0$ for any curve $C \subseteq X$.

Definition 30.5. The *Néron–Severi group* of X is $N^1(X) \coloneqq \operatorname{Pic}(X)/\equiv$; equivalently, $N^1(X)$ can be constructed as the quotient of the group $\operatorname{Div}(X)$ of Cartier divisors on X modulo numerical equivalence.

As a corollary of the Nakai–Moishezon criterion, one can show that:

Corollary 30.6. *If $L_1 \equiv L_2$, then L_1 is ample iff L_2 is ample.*

There are inclusions

$$N^1(X) \hookrightarrow N^1(X)_{\mathbf{Q}} \coloneqq N^1(X) \otimes_{\mathbf{Z}} \mathbf{Q} \hookrightarrow N^1(X)_{\mathbf{R}} \coloneqq N^1(X) \otimes_{\mathbf{Z}} \mathbf{R} \hookrightarrow H^{1,1}(X,\mathbf{R}) \coloneqq H^{1,1}(X,\mathbf{C}) \cap H^2(X,\mathbf{R}),$$

where the embedding $N^1(X)_{\mathbf{R}} \hookrightarrow H^{1,1}(X,\mathbf{R})$ is as a linear subspace. Note that $H^{1,1}(X,\mathbf{R})$ (and hence $N^1(X)_{\mathbf{R}}$) is a finite-dimensional real vector space. Furthermore, ampleness can be characterized by the image of the line bundle in $N^1(X)_{\mathbf{R}}$, from which one deduces that ampleness is an "open" condition.

Fact 30.7. There is an open convex cone $\operatorname{Amp}(X) \subseteq N^1(X)_{\mathbf{R}}$ such that $L \in \operatorname{Pic}(X)$ is ample iff its image in $N^1(X)_{\mathbf{R}}$ lies in $\operatorname{Amp}(X)$.

[9] Another term for 'semiample' in the literature is 'eventually free'.

Definition 30.8. A line bundle L is *nef*[10] if its image in $N^1(X)_{\mathbf{R}}$ lies in the closure $\mathrm{Nef}(X)$ of $\mathrm{Amp}(X)$.

Theorem 30.9. [Kleiman's Theorem] *A line bundle L is nef iff $\deg(L|_C) \geq 0$ for any curve $C \subseteq X$.*

From Kleiman's theorem, one deduces the implications:

$$\text{ample} \Longrightarrow \text{semiample} \Longrightarrow \text{nef}.$$

30.2.3. *Analytic Notions of Positivity.* The advantage of working analytically is that one does not have to worry about first working over $\mathbf{Q}$, and then passing to $\mathbf{R}$. Instead, everything is already defined over $\mathbf{R}$.

Definition 30.10. A cohomology class $\alpha \in H^{1,1}(X, \mathbf{R})$ is a *Kähler class* if it contains a Kähler form.

Fact 30.11. The set $\mathcal{K} \subseteq H^{1,1}(X, \mathbf{R})$ consisting of Kähler classes is an open convex cone, called the *Kähler cone.*

The above fact is always holds, but it is possible that $\mathcal{K}$ is empty for a general complex manifold (indeed, this occurs iff X is not Kähler).

Definition 30.12. A line bundle L is *(semi)positive* if it admits a smooth metric ϕ such that

- (positive) $dd^c\phi > 0$;
- (semipositive) $dd^c\phi \geq 0$.

In this language, the Kodaira embedding theorem asserts that L is ample iff L is positive. Thus, under the embedding $N^1(X) \hookrightarrow H^{1,1}(X, \mathbf{R})$, we have $\mathrm{Amp}(X) = \mathcal{K} \cap N^1(X)_{\mathbf{R}}$.

As a consequence, one can also describe the nef cone in analytic terms: L is nef iff for any Kähler form ω and any $\epsilon > 0$, there exists a smooth metric ϕ_ϵ on L such that $dd^c\phi_\epsilon \geq -\epsilon\omega$. There are examples where L is nef but there does not exist a smooth semipositive metric on L; see [DPS94, Example 1.7].

30.3. **Positivity Notions for Vector Bundles.** Unlike the case of line bundles, the positivity properties of vector bundles are messier. To be precise, a *holomorphic vector bundle of rank r on X* refers to a holomorphic map $\pi\colon E \to X$ such that there is a covering $(U_i)_{i\in I}$ of X and for all i, there are homeomorphisms

$$\pi^{-1}(U_i) = E|_{U_i} \xrightarrow{\varphi} U_i \times \mathbf{C}^r$$

such that

$$(U_i \cap U_j) \times \mathbf{C}^r \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (U_i \cap U_j) \times \mathbf{C}^r$$

is of the form $(x, v) \mapsto (x, g_{ij}(x)v)$, where the transition function $g_{ij}\colon U_i \cap U_j \to \mathrm{GL}_r(\mathbf{C})$ is holomorphic. The g_{ij}'s are cocycles, in the sense that $g_{ij}g_{ji} = g_{ij}g_{jk}g_{ki} = \mathrm{id}$.

An algebraic vector bundle on X is defined similarly, where the transition functions are demanded to be algebraic. One could also think of a vector bundle as a locally free sheaf, but we instead take the above geometric approach.

Next time, we will begin discussing the algebraic and analytic notions of positivity for vector bundles.

31. April 4th

Today we will spend more time on the various positivity notions for vector bundles.

[10] Other terms for 'nefness' in the literature are 'numerically eventually free' and 'numerically effective'.

31.1. **Ample and Positive Vector Bundles.** Given a finite dimensional vector space V (over $\mathbf{C}$), let

$$\mathbf{P}(V) := \{\text{1-dimensional subspaces of } V\},$$

and

$$\mathbf{P}(V^\vee) := \{\text{1-dimensional quotients of } V\} = \{\text{hyperplanes of } V\}.$$

These are our conventions for the projective space of a vector space and of its dual.

Let X be a complex projective manifold, and let E be a (holomorphic) vector bundle on X of rank $r \geq 1$. Write $E^\vee$ for the dual vector bundle. The *projectivization* $\mathbf{P}(E^\vee)$ is locally given as follows: if $E^\vee \simeq U \times \mathbf{C}^r$ locally, then $\mathbf{P}(E^\vee) \simeq U \times \mathbf{P}^{r-1}$. Thus, one has a map $\pi\colon \mathbf{P}(E^\vee) \to X$ with $\pi^{-1}(x) \simeq \mathbf{P}^{r-1}$. The projectivization $\mathbf{P}(E^\vee)$ comes with the Serre line bundle $\mathcal{O}_{\mathbf{P}(E^\vee)}(1)$ such that $\mathcal{O}_{\mathbf{P}(E^\vee)}(1)|_{\pi^{-1}(x)} \simeq \mathcal{O}(1)$ on $\pi^{-1}(x) \simeq \mathbf{P}^{r-1}$.

In algebraic geometry, the standard definition of ampleness of a vector bundle first appeared in [Har66].

Definition 31.1. [Hartshorne] The vector bundle E is *ample/nef* if the line bundle $\mathcal{O}_{\mathbf{P}(E^\vee)}(1)$ on $\mathbf{P}(E^\vee)$ is ample/nef.

As is the case for line bundles, there are cohomological characterizations of ampleness and it has good functorial properties. The best reference for this material is [Laz04].

Example 31.2. If $L_1, \ldots, L_r$ are line bundles on X, then $L_1 \oplus \ldots \oplus L_r$ is ample iff each L_i is ample.

Example 31.3. If E is a vector bundle on X, then E is ample iff the symmetric power $S^m E$ is ample for some (equivalently, any) $m \geq 1$.

Remark 31.4. If $r = \operatorname{rank}(E) > 1$, then $\operatorname{rank}(S^m E) > r$ for $m > 1$. For this reason, the condition in Example 31.3 is often difficult to check.

31.2. **Differential Geometry Notions.** In order to define various differential-geometric notions of positivity of vector bundles, we require some further language from differential geometry. We follow the presentation of [Dem12].

Let X be a complex manifold, and let $E \xrightarrow{\pi} X$ be a holomorphic vector bundle of rank r on X. Locally, for $V \subseteq X$ open, there is a local trivialization $\theta\colon E|_V \xrightarrow{\simeq} V \times \mathbf{C}^r$, which is a biholomorphism over V, such that

$$(V_\alpha \cap V_\beta) \times \mathbf{C}^r \xrightarrow{\theta_\alpha \circ \theta_\beta^{-1}} (V_\alpha \cap V_\beta) \times \mathbf{C}^r$$

is of the form $(x, \xi) \mapsto (x, g_{\alpha\beta}(x))$, and the transition functions $g_{\alpha\beta}\colon V_\alpha \cap V_\beta \to \mathrm{GL}_r(\mathbf{C})$ are holomorphic (and satisfy the usual cocycle conditions).

Any trivialization $\theta\colon E|_V \xrightarrow{\simeq} V \times \mathbf{C}^r$ defines a frame $(e_1, \ldots, e_r)$, where $e_j \in C^\infty(V, E|_V)$ are local smooth sections of E. It is given by

$$\theta(x, e_j(x)) = (x, (0, \ldots, 1, \ldots, 0)),$$

where 1 is placed in the j-th position. Any other local smooth section $s \in C^\infty(V, E|_V)$ can be written as

$$s = \sum_{\lambda=1}^{r} \sigma_\lambda e_\lambda,$$

where $\sigma_\lambda \in C^\infty(V, \mathbf{C})$ are smooth complex-valued functions satisfying the transformation rule

$$\begin{pmatrix} \sigma_1^\alpha \\ \vdots \\ \sigma_r^\alpha \end{pmatrix} = \sigma^\alpha = g_{\alpha\beta}\sigma^\beth$$

on $V_\alpha \cap V_\beta$.

Definition 31.5. A *Hermitian metric* on E is a function $|\cdot|\colon E \to \mathbf{R}_{\geq 0}$ on the total space such that

(1) the restriction of $|\cdot|^2$ to each fibre $E_x = \pi^{-1}(x) \simeq \mathbf{C}^r$ is a positive Hermitian metric/form;

(2) the function $|\cdot|^2\colon E \to \mathbf{R}_{\geq 0}$ is smooth.

Write $\langle \cdot, \cdot \rangle$ for the corresponding inner product.

Given a trivialization $\theta\colon E|_V \overset{\simeq}{\to} V \times \mathbf{C}^r$, one gets a matrix $H := (h_{\lambda\mu})_{1\leq\lambda,\mu\leq r}$ with coefficients in $C^\infty(V, \mathbf{C})$ given by

$$h_{\lambda\mu}(x) = \langle e_\lambda(x), e_\mu(x) \rangle$$

for $x \in X$. One also gets an induced map

$$C^\infty_{\bullet,\bullet}(X, E) \times C^\infty_{\bullet,\bullet}(X, E) \longrightarrow C^\infty_{\bullet,\bullet}(X, \mathbf{C})$$

given by $(s, t) \mapsto \{s, t\}$. The form $\{s, t\}$ is defined as follows: write $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda$ and $t = \sum_\mu \tau_\mu \otimes e_\mu$, and set

$$\{s, t\} = \sum_{\lambda,\mu} \underbrace{\langle e_\lambda, e_\mu \rangle}_{h_{\lambda\mu}} \sigma_\lambda \wedge \overline{\tau_\mu},$$

where $h_{\lambda\mu}$ is a function and $\sigma_\lambda \wedge \overline{\tau_\mu}$. Here, $C^\infty_{\bullet,\bullet}(X, E)$ denotes the smooth forms on X with values in E, and $C^\infty_{\bullet,\bullet}(X, \mathbf{C})$ denotes the smooth forms on X with values in $\mathbf{C}$.

To the above data, we can associated a canonical connection on E, called the Chern connection. If E is a Hermitian holomorphic vector bundle (i.e. a holomorphic vector bundle equipped with a Hermitian metric), we want to define the curvature of E, which involves second derivatives. Thus, we first need to define a first derivative, which is what a connection does.

There is always a $\overline{\partial}$-operator[11] $\overline{\partial}\colon C^\infty_{p,q}(X, E) \to C^\infty_{p,q+1}(X, D)$ defined as follows: if $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda$ is a local section of E on V with respect to the trivialization $\theta\colon E|_V \simeq V \times \mathbf{C}^r$, then set

$$\overline{\partial}\, s := \sum_\lambda \overline{\partial}\, \sigma_\lambda \otimes e_\lambda.$$

One must show that this is independent of the choice of trivialization θ: on $V_\alpha \cap V_\beta$, we have $\sigma^\alpha = g_{\alpha\beta}\sigma^\beta$, so $\overline{\partial}\, \sigma^\alpha = g_{\alpha\beta}\, \overline{\partial}\, \sigma^\beta$ since $\overline{\partial}\, g_{\alpha\beta} = 0$.

Fact 31.6. There exists a unique connection D on E compatible with the complex structure and with the Hermitian metric on E. It satisfies (and is characterized by) the following properties:

(1) $D\colon C^\infty_m(X, E) \to C^\infty_{m+1}(X, E)$ is a linear differential operator of degree 1;
(2) $D(f \wedge s) = df \wedge s + (-1)^{\deg(f)} f \wedge Ds$ for $f \in C^\infty_\bullet(X, \mathbf{C})$ and $s \in C^\infty_\bullet(X, E)$;
(3) $d\{s, t\} = \{Ds, t\} + (-1)^{\deg(s)}\{s, Dt\}$ for $s, t \in C^\infty_\bullet(X, E)$;
(4) if we write $D = D' + D''$, where D' increases bidegree by $(1, 0)$ and D'' increases bidegree by $(0, 1)$, then $D'' = \overline{\partial}$.

The operator D is called the *Chern connection* of the Hermitian metric.

One perform local calculations to show that D is uniquely defined locally, and for that reason it is globally defined (and uniquely so).

Let us perform a computation in a holomorphic trivialization: given the biholomorphism $\theta\colon E|_V \overset{\simeq}{\to} V \times \mathbf{C}^r$ and forms $s, t \in C^\infty_\bullet(X, E)$, which we write as $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda$ and $t = \sum_\mu \tau_\mu \otimes e_\mu$, then

$$\{s, t\} = \sum_{\lambda,\mu} h_{\lambda\mu}\sigma_\lambda \wedge \overline{\tau_\mu} = \sigma^t \wedge H\overline{\tau},$$

where σ^t denotes the transpose.

Claim 31.7. We have $D' \simeq_\theta \overline{H}^{-1}\, \partial\, \overline{H}$.

[11]In [Dem12], $\overline{\partial}$ is written as d''.

A special case of Claim 31.7 is when E is a line bundle: $H > 0$ is a positive function on $V \subseteq X$, so we can write $H = e^{-2\varphi}$ for some $\varphi \in C^\infty(V, \mathbf{R})$. Then,

$$D' \simeq_\theta e^{2\varphi}\,\partial\, e^{-2\varphi},$$

which is an expression that arose earlier.

The proof of Claim 31.7 is a direct computation using the properties (1-4) in Fact 31.6.

We can now define the curvature. The Chern connection can be viewed as a linear differential operator $D\colon C^\infty_m(X,E) \to C^\infty_{m+1}(X,E)$, and, in fact, a calculation with the Leibnitz rule shows that the composite $D^2\colon C^\infty_m(X,E) \to C^\infty_{m+2}(X,E)$ is of the form

$$D^2 s = \Theta(D) \wedge s,$$

where $\Theta(D) \in C^\infty_2(X, \mathrm{End}(E))$ is called the *curvature tensor* of the connection D.

32. April 6th

32.1. **Positivity of Hermitian Vector Bundles.** Consider a holomorphic vector bundle $E \to X$ equipped with a Hermitian metric $\|\cdot\|\colon E \to \mathbf{R}_{\geq 0}$, i.e. a function on the total space of E that defines a Hermitian inner product on each fibre. Given a trivialization $\theta\colon E|_V \overset{\simeq}{\to} V \times \mathbf{C}^r$ on some open set $V \subseteq X$, we get a local frame $(e_1, \dots, e_r)$ with $e_j \in C^\infty(V, E)$. Any section $s \in C^\infty(V, E)$ can be written as

$$s = \sum_\lambda \sigma_\lambda \otimes e_\lambda,$$

where $\sigma = (\sigma_\lambda)_\lambda \in C^\infty(V, \mathbf{C})^r$. Write

$$h_{\lambda\mu}(x) := \langle e_\lambda(x), e_\mu(x) \rangle$$

for $1 \leq \lambda, \mu \leq r$ and $x \in X$, and set $H := (h_{\lambda\mu})^r_{\lambda,\mu=1}$, which is a positive definite Hermitian matrix for each fixed $x \in X$.

Furthermore, consider the sesquilinear mapping $C^\infty_\bullet(X,E) \times C^\infty_\bullet(X,E) \to C^\infty_\bullet(X,E)$, given by $(s,t) \mapsto \{s,t\}$, where the form $\{s,t\}$ is defined as follows: write $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda$ and $t = \sum_\mu \tau_\mu \otimes e_\mu$, and set

$$\{s,t\} := \sum_{\lambda,\mu} h_{\lambda\mu} \sigma_\lambda \wedge \overline{\tau_\mu} = \sigma^t \wedge H \overline{\tau},$$

where one thinks of σ and τ as column vectors.

Finally, we had the Chern connection, which was a map $D\colon C^\infty_m(X,E) \to C^\infty_{m+1}(X,E)$ for each $0 \leq m \leq n$ characterized by the following properties:

1. [Leibnitz Rule] $D(f \wedge s) = df \wedge s + (-1)^{\deg(f)} f \wedge Ds$ for $f \in C^\infty_\bullet(X, \mathbf{C})$ and $s \in C^\infty_\bullet(X,E)$;
2. [Compatibility with Hermitian Structure] $d\{s,t\} = \{Ds, t\} + (-1)^{\deg(s)}\{s, Dt\}$ for $s,t \in C^\infty_\bullet(X,E)$;
3. [Compatibility with Holomorphic Structure] if we write $D = D' + D''$ where D' is the $(1,0)$-part of the operator and D'' is the $(0,1)$-part, then $D'' = \overline{\partial}$.

Given a *holomorphic* trivialization $E|_V \overset{\simeq}{\to} V \times \mathbf{C}^r$, then one can view D' as a map

$$D'\colon C^\infty_{p,q}(X, \mathbf{C})^r \to C^\infty_{p+1,q}(X,\mathbf{C})^r.$$

Then,

$$D' = \overline{H}^{-1}\,\partial\,\overline{H}. \tag{32.1}$$

That is, $D's = \overline{H}^{-1}\,\partial\, H\sigma$, where σ is the representation of s in this trivialization as a vector of forms on V. The proof of (32.1) is left as an exercise (or see [Dem12, p.269]).

The operator $D^2\colon C^\infty_m(X,E) \to C^\infty_{m+2}(X,E)$ is of the form $D^s = \Theta(E) \wedge s$ for all $s \in C^\infty_m(X,E)$, where $\Theta(E) \in C^\infty_2(X, \mathrm{End}(E))$ is the *curvature form* of the Chern connection D.

Claim 32.1. We have $i\Theta(E) \in C^\infty_{1,1}(X, \mathrm{Herm}(E))$.

Proof. For forms $s, t \in C^\infty_\bullet(X, E)$, applying the compatibility of D with the Hermitian structure twice, we find that

$$0 = d^2\{s,t\} = \{D^s, t\} + \{s, D^2 t\} = \{\Theta(E) \wedge s, t\} + \{s, \Theta(E) \wedge t\}.$$

If one unwinds the definitions, this implies $i\Theta(E) \in C^\infty_2(X, \mathrm{Herm}(E))$. Now, to see that $i\Theta(E)$ is of type-(1,1), consider a holomorphic trivialization $E|_V \overset{\simeq}{\to} V \times \mathbf{C}^r$, and write $D = D' + \overline{\partial}$ and $D' = \overline{H}^{-1}\,\partial\,\overline{H}$. Then,

$$D^2 = (D' + \overline{\partial})^2 = (D')^2 + (D'\,\overline{\partial} + \overline{\partial}\,D') + \underbrace{\overline{\partial}^2}_{=0}.$$

Further,

$$(D')^2 = \overline{H}^{-1}\,\partial\,\overline{H}\overline{H}^{-1}\,\partial\,\overline{H} = \overline{H}^{-1}\,\partial^2\,\overline{H} = 0.$$

Thus, $D^2 = D'\,\overline{\partial} + \overline{\partial}\,D'$, which is an operator of type-(1,1), and hence $\Theta(E) \in C^\infty_{1,1}(X, \mathrm{End}(E))$. □

One can also compute $i\Theta(E)$ locally with respect to the above holomorphic trivialization: write

$$\begin{aligned}
D^2 s &= (D'\,\overline{\partial} + \overline{\partial}\,D')s \\
&= \overline{H}^{-1}\,\partial\,\overline{H}\,\overline{\partial}\,\sigma + \overline{\partial}\,\overline{H}^{-1}\,\partial\,\overline{H}\sigma \\
&= \overline{H}^{-1}\,\partial\,\overline{H} \wedge \overline{\partial}\,\sigma + \partial\,\overline{\partial}\,\sigma + \overline{\partial}(\overline{H}^{-1}\,\partial\,H \wedge \sigma) + \overline{\partial}\,\partial\,\sigma \\
&= \overline{H}^{-1}\,\partial\,\overline{H} \wedge \overline{\partial}\,\sigma + \overline{\partial}(\overline{H}^{-1}\,\partial\,\overline{H}) \wedge \sigma - \overline{H}^{-1}\,\partial\,\overline{H} \wedge \overline{\partial}\,\sigma \\
&= \overline{\partial}(\overline{H}^{-1}\,\partial\,\overline{H}).
\end{aligned}$$

Thus, we find that $i\Theta(E)$ is locally given by $i\,\overline{\partial}(\overline{H}^{-1}\,\partial\,\overline{H})$.

For example, if E is a line bundle, then $H = \overline{H} = e^{-2\varphi}$ for some $\varphi\colon V \to \mathbf{R}$, so the above formula shows that

$$i\Theta(E) = i\,\overline{\partial}(e^{2\varphi}\,\partial\,e^{-2\varphi}) = -2i\,\overline{\partial}\,\partial\,\varphi = 2i\,\partial\,\overline{\partial}\,\varphi.$$

Thus, $i\Theta(E)$ is (up to a constant) the same curvature form that we associated to a metric on a line bundle.

Now, there is an induced Hermitian form on $TX \otimes E$ that arises from the curvature form. Recall that $i\Theta(E) \in C^\infty_{1,1}(X, \mathrm{End}(E))$ is a $(1,1)$-form with values in $\mathrm{Herm}(E)$, and this induces a smooth Hermitian form θ_E on $TX \otimes E$: use local coordinates $(z_1, \ldots, z_n)$ on $V \subseteq X$ and an *orthonormal frame* $(e_1, \ldots, e_r)$ of $E|_V$, and then we can write

$$i\Theta(E) = \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le r}} c_{jk\lambda\mu} i dz_j \wedge d\,\overline{z}_k \otimes e^*_\lambda \otimes \overline{e_\mu},$$

where e^*_λ denotes the dual basis vector and we think of $e^*_\lambda \otimes e_\mu$ as a local section of $\mathrm{End}(E)$. The coefficients satisfy $\overline{c_{jk\lambda\mu}} = c_{kj\mu\lambda}$. Then, set:

$$\theta_E \coloneqq \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(dz_j \otimes e^*_\lambda) \otimes \overline{(dz_k \otimes e^*_\mu)}.$$

This expression θ_E is thought of as a Hermitian form on $TX \otimes E$; that is, if $u = \sum_{j,\lambda} u_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_x X \otimes E_x$, then

$$\theta_E(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(x) u_{j\lambda} \overline{u_{k\mu}}.$$

If E is a line bundle, then the rank of $TX \otimes E$ is just the dimension n of X, but when E has rank $r > 1$, the rank of $TX \otimes E$ is nr, and for this reason the notions of positivity of E are much hairier than in the line bundle case.

Definition 32.2. [Nakano, 1955] The Hermitian vector bundle E is *Nakano positive* if θ_E is a positive Hermitian form on $TX \otimes E$, i.e.

$$\theta_E(u, u) > 0$$

for all nonzero $u \in TX \otimes E$.

There is a weaker notion of positivity that only demands positivity of θ_E on 'indecomposables'.

Definition 32.3. [Griffiths, 1969] The Hermitian vector bundle E is *Griffiths positive* if

$$\theta_E(\xi \otimes s, \xi \otimes s) > 0$$

for all $\xi \in T_xX\backslash\{0\}$ and all $s \in E_x\backslash\{0\}$. Equivalently, for all $\xi \in T_xX\backslash\{0\}$, the Hermitian form

$$s \mapsto \theta_E(\xi \otimes s, \xi \otimes s)$$

on E_x is positive definite.

We can define Nakano/Griffiths negative, semipositive, seminegative in the same way. Following [Dem12], we write $E >_{\text{Nak}} 0$, $E >_{\text{Grif}} 0$ and so on.

Remark 32.4. For line bundles, Nakano and Griffiths positivity coincide (and they are simply the usual notion of a positive line bundle).

We list various general properties of Nakano and Griffiths positivity without proof (see [Dem12, Ch. VII, Proposition 6.10]):

- $E >_{\text{Nak}} 0 \Longrightarrow E >_{\text{Grif}} 0$;
- $E_{\text{Grif}} > 0 \iff E^\vee <_{\text{Grif}} 0$ (but this fails for Nakano positivity!);
- if $0 \to S \to E \to Q \to 0$ is an exact sequence of Hermitian vector bundles, then
  - (a) $E \geq_{\text{Grif}} 0 \Longrightarrow Q \geq_{\text{Grif}} 0$;
  - (b) $E \leq_{\text{Grif}} 0 \Longrightarrow S \leq_{\text{Grif}} 0$;
  - (b') $E \leq_{\text{Nak}} 0 \Longrightarrow S \leq_{\text{Nak}} 0$.

  The analogous statements are true for '$<$' and '$>$', but the assertion (a') is false.

The third point asserts that 'positivity increases in quotients' (for an algebraic example, the quotient of an ample vector bundle is again ample); on the other hand, 'negativity increases in subbundles'.

One nice aspect of Nakano positive vector bundles is that they satisfy an analogue of the Kodaira vanishing theorem.

Theorem 32.5. [Nakano Vanishing Theorem] *If X is compact Kähler and $E >_{\text{Nak}} 0$, then $H^{n,q}(X, E) = 0$ for all $q > 0$.*

See [Dem12, Ch. VII, Theorem 7.3] for a proof of the Nakano vanishing theorem.

Example 32.6. [Dem12, Ch. VII, Example 8.4] If $X = \mathbf{P}^n$ and $E = T\mathbf{P}^n$, then $E >_{\text{Grif}} 0$ but $E \not>_{\text{Nak}} 0$. That E is not Nakano positive can be deduced from the Nakano vanishing theorem.

There is a comparison between the algebraic and analytic definitions of positivity.

Fact 32.7. [Dem12, Ch. VII, Corollary 11.13] If X and E are algebraic and $E >_{\text{Grif}} 0$, then E is ample.

It is a longstanding conjecture of Griffiths that the converse holds.

Conjecture 32.8. [Griffith's Conjecture] If X and E are algebraic and E is ample, then $E >_{\text{Grif}} 0$.

The conjecture is known if $\dim(X) = 1$ (and even then it is not trivial!). If E is ample, then one gets a metric on some symmetric power of E, and the difficulty is to construct from this a metric on E.

33. April 9th

33.1. Positivity of Hermitian Vector Bundles (Continued). Consider a Hermitian holomorphic vector bundle $(E, \|\cdot\|)$, on which we have the Chern connection $D = D^{1,0} + \overline{\partial}$. For a local section s, we have $D^2 s = \Theta(E) \wedge s$, where $i\Theta(E) \in C^\infty_{(1,1)}(X, \mathrm{Herm}(E))$ is the (Chern) curvature form.

If $(z_1, \ldots, z_n)$ are local coordinates on X and $(e_1, \ldots, e_r)$ is a local orthonormal (hence, not holomorphic) frame of E, then we can write

$$i\Theta(E) = i \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le r}} c_{jk\lambda\mu} dz_j \wedge d\,\overline{z}_k \otimes e^*_\lambda \otimes e_\mu,$$

where $\overline{c_{jk\lambda\mu}} = c_{kj\mu\lambda}$. From this data, we can define a Hermitian form θ_E on $TX \otimes E$ given by

$$\theta_E = \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le r}} c_{jk\lambda\mu} (dz_j \otimes e^*_\lambda) \otimes \overline{(dz_k \otimes e^*_\mu)}.$$

Using θ_E, we can define two notions of positivity for $(E, \|\cdot\|)$, which we recall from last time:

- E is *Nakano positive* if $\theta_E > 0$ on $TX \otimes E$;
- E is *Griffiths positive* if $\theta_E(\xi \otimes s, \xi \otimes s) > 0$ for all $\xi \in T_x X \backslash \{0\}$, $s \in E_x \backslash \{0\}$, and $x \in X$.

33.2. Computations with Hermitian Vector Bundles. The presentation here follows [Dem12, Ch. V, §15].

Computation 33.1. Given a Hermitian vector space V of dimension $n+1$, set $X = \mathbf{P}(V)$ and let $\underline{V} \coloneqq X \times V$ be the trivial vector bundle on X equipped with the given Hermitian metric. There is a tautological line subbundle $\mathcal{O}(-1) \subseteq \underline{V}$ given by

$$\mathcal{O}(-1) = \{([x], v) \in \mathbf{P}(V) \times V : v \in \mathbf{C}x\}.$$

Define $H \coloneqq \underline{V}/\mathcal{O}(-1)$ to be the quotient bundle, which is a rank n vector bundle on X. One can check that $TX \simeq H \otimes \mathcal{O}(1)$, where $\mathcal{O}(1) = \mathcal{O}(-1)^*$ is the dual of the tautological line bundle.

Question 33.2. Is the Hermitian vector bundle H Nakano/Griffiths positive?

By the definition of H, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{V} \longrightarrow H \longrightarrow 0,$$

and so one expects at least that $H \ge_{\mathrm{Grif}} 0$, since $\underline{V} \ge_{\mathrm{Grif}} 0$ (in fact, $\underline{V} =_{\mathrm{Grif}} 0$, in the sense that the trivial bundle is both Griffiths semipositive and seminegative).

Indeed, fix $a \in X = \mathbf{P}(V)$ and pick an orthonormal basis $e_0, \ldots, e_n$ of V such that $[e_0] = a$. Define an embedding $\mathbf{C}^n \hookrightarrow \mathbf{P}(V)$ given by

$$z = (z_1, \ldots, z_n) \mapsto [e_0 + z_1 e_1 + \ldots + z_n e_n].$$

In particular, $0 \mapsto a$ under this embedding. One can check that

$$\Theta(\mathcal{O}(\pm 1))|_{\mathbf{C}^n} = \pm \sum_{j,k=1}^{n} dz_j \wedge d\,\overline{z}_k\,.$$

It follows that $\mathcal{O}(1)$ is both Nakano and Griffiths positive (indeed, this form is positive definite; it can be written e.g. as $\partial\overline{\partial} \log(1 + |z|^2)$). One also has a formula for the curvature of H at $a \in X$:

$$\Theta(H)_a = \sum_{1 \le j,k \le n} dz_j \wedge d\,\overline{z}_k \otimes \widetilde{e_k}^* \otimes \widetilde{e_j},$$

where $(\widetilde{e_1}, \dots, \widetilde{e_n})$ denotes the image of $(e_1, \dots, e_n)$ in the quotient $H|_{\mathbf{C}^n}$. The corresponding Hermitian form on $TX \otimes H$ can be written (near a) as

$$\theta_H = \sum_{1 \le j,k \le n} (dz_j \otimes \widetilde{e_k}^*) \otimes \overline{(dz_k \otimes \widetilde{e_j}^*)}$$

For a local section $u = \sum_{1 \le j,\lambda \le n} u_{j\lambda} \frac{\partial}{\partial z_j} \otimes \widetilde{e}_\lambda \in (TX \otimes H)_a$, we have

$$\theta_H(u,u) = \sum_{1 \le j,\lambda \le n} u_{j\lambda} \overline{u_{\lambda j}}.$$

It follows that $H \not\geq_{\mathrm{Nak}} 0$ if $n \ge 2$: for example, pick $u = \frac{\partial}{\partial z_1} \otimes \widetilde{e_n} - \frac{\partial}{\partial z_n} \otimes \widetilde{e_1}$, so $u_{1n} = 1$ and $u_{n1} = -1$, and hence $\theta_H(u,u) = 0$. However, $H \geq_{\mathrm{Grif}} 0$. Indeed, it suffices to check that $\theta_H(u,u) \ge 0$ for a simple tensor u of the form $u = \xi \otimes \widetilde{e_\lambda}$ for some tangent vector $\xi \in TX$. In this case, write $\xi = \sum_{j=1}^n b_j \frac{\partial}{\partial z_j}$, then

$$\theta_H(u,u) = |b_\lambda|^2 \ge 0.$$

Thus, we have shown that $H \geq_{\mathrm{Grif}} 0$.

Computation 33.3. Let E be a Hermitian holomorphic vector bundle on X of rank r, and let E^* be the dual vector bundle. Write $\pi \colon \mathbf{P}(E^*) \to X$ for the projectivization, and it contains a tautological bundle $S \subseteq \pi^* E$ of corank 1 given as follows: for $x \in X$ and $\xi \in E_x^*$, set

$$S_{[\xi]} = \xi^{-1}(0) \subseteq E_x.$$

The quotient $\mathcal{O}_E(1) \coloneqq \pi^* E / S$ is the tautological (Serre) line bundle on $\mathbf{P}(E^*)$. There is a short exact sequence

$$0 \longrightarrow S \longrightarrow \pi^* E \longrightarrow \mathcal{O}_E(1) \longrightarrow 0,$$

which we would like to use to compute the curvature of $\mathcal{O}_E(1)$ in terms of the curvature of E.

Consider $a \in \mathbf{P}(E^*)$, and set $x = \pi(a) \in X$. Pick local coordinates $z_1, \dots, z_n$ on Z centered at x. One cannot pick a frame that is both holomorphic and orthonormal in general, but one can construct a holomorphic frame that is "almost orthonormal", in the sense that there is a small correction term controlled by the curvature:

Fact 33.4. There exists a local holomorphic frame $e_1, \dots, e_r$ of E such that

$$\langle e_\lambda, e_\mu \rangle = \delta_{\lambda\mu} - \sum_{1 \le j,k \le n} c_{jk\lambda\mu} z_j \overline{z_k} + O(|z|^3),$$

for $1 \le \lambda, \mu \le r$, where the $c_{jk\lambda\mu}$ are the coefficients of the curvature; that is,

$$\Theta(E) = \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le r}} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu.$$

Such a frame is called a *normal frame.*

Now, represent $a \in \mathbf{P}(E^*)$ by a vector $\sum_{\lambda=1}^r a_\lambda e_\lambda^*$ of length 1. Then, one can extend $(z_1, \dots, z_n)$ to local coordinates $(z_1, \dots, z_n, \xi_1, \dots, \xi_{r-1})$ on $\mathbf{P}(E^*)$ at a such that

$$\Theta(\mathcal{O}_{\mathbf{P}(E^*)}(1))_a = \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le r}} c_{jk\mu\lambda} a_\lambda \overline{a_\mu} dz_j \wedge d\overline{z}_k + \sum_{1 \le \lambda \le r-1} d\xi_\lambda \wedge d\overline{\xi_\lambda}.$$

Furthermore,

$$\Theta(E^*) = - \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le r}} c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes e_\lambda \otimes e_\mu^*.$$

Therefore, if $E \geq_{\mathrm{Grif}} 0$, then $E^* \leq_{\mathrm{Grif}} 0$, so $\mathcal{O}_{\mathbf{P}(E^*)}(1) \ge 0$ (here, $\mathcal{O}_{\mathbf{P}(E^*)}(1)$ is a line bundle, so there is no distinction between Nakano and Griffiths positivity). Similarly, the same calculation goes through for strict inequalities, which leads to the following corollary:

Corollary 33.5. *If $E >_{\text{Grif}} 0$, then E is ample (in the sense of Hartshorne: $\mathcal{O}_{\mathbf{P}(E^*)}(1)$ is ample on $\mathbf{P}(E^*)$).*

Proof. The above calculation shows that $\mathcal{O}_{\mathbf{P}(E^*)}(1) > 0$, and hence $\mathcal{O}_{\mathbf{P}(E^*)}(1)$ is ample by the Kodaira embedding theorem. □

33.3. **Criteria for Positivity/Negativity.** In order to prove Berndtsson's theorem, we require some more direct criteria for Nakano and Griffiths positivity and negativity. One such criterion is provided below:

Proposition 33.6. *Consider a Hermitian holomorphic vector bundle $(E, \|\cdot\|)$ on X.*

(a) $E \leq_{\text{Grif}} 0$ iff the function $\log\|\cdot\|\colon E \to \mathbf{R} \cup \{-\infty\}$ is plurisubharmonic.
(b) $E <_{\text{Grif}} 0$ iff the function $\log\|\cdot\|$ is strictly plurisubharmonic on $E\backslash\{$zero section$\}$.

Remark 33.7. The corresponding statement of Proposition 33.6 for $\geq_{\text{Grif}}$ is false (or at least not the equivalence), even for trivial bundles!.

The proof of Proposition 33.6 will be discussed next time, in addition to a criterion for Nakano positivity.

34. April 11th

The plan for today is to explain various positivity and negativity criteria, one of which was stated last time.

34.1. **Positivity of Vector Bundles (Continued).** Let $(E, \|\cdot\|)$ be a holomorphic Hermitian vector bundle, with Chern connection $D = D^{1,0} + \overline{\partial}$. For a local section s, we have $D^2 s = \Theta(E) \wedge s$, where $i\Theta(E) \in C^\infty_{(1,1)}(X, \text{Herm}(E))$ is the (Chern) curvature form. Furthermore, there is an associated Hermitian form θ_E on $TX \otimes E$.

Recall that E is *Nakano positive* if θ_E is positive definite, and E is *Griffiths positive* if $\theta_E(\xi \otimes s, \xi \otimes s) > 0$ for $\xi \in T_x X\backslash\{0\}$, $s \in E_x\backslash\{0\}$, and $x \in X$ (i.e. θ_E is positive on simple tensors). Similarly, we can define negativity, seminegativity, and semipositivity. This can be reformulated using the sesquilinear map

$$C^\infty_p(X, E) \times C^\infty_q(X, E) \longrightarrow C^\infty_{p+q}(X, \mathbf{C})$$

that we denoted (following [Dem12]) by $(s, t) \mapsto \{s, t\}$; more precisely,

- $E \geq_{\text{Grif}} 0$ iff
$$\{u, i\Theta(E) \wedge u\} = \{i\Theta(E) \wedge u, u\} \geq 0 \tag{34.1}$$
(as a (1,1)-form) for every local section $u \in C^\infty(V, E)$ defined on an open set $V \subseteq X$ (equivalently, it suffices to verify this condition for local holomorphic sections of E).
- $E \geq_{\text{Nak}} 0$ iff
$$\sum_{1 \leq j,k \leq n} \{u_j, i\Theta(E) \wedge u_k\} \geq 0 \tag{34.2}$$
(as a (1,1)-form) for any local (holomorphic) sections $u_1, \ldots, u_n$ of E.

Similarly, one can characterize (strict) positivity and negativity and seminegativity in this manner.

34.2. **Criterion for Griffiths Negativity.** The criterion for Griffiths negativity that was stated last time is the following:

Proposition 34.1.

(a) $E \leq_{\text{Grif}} 0$ iff the function $\log\|\cdot\|$ is plurisubharmonic on E.
(b) $E <_{\text{Grif}} 0$ iff the function $\log\|\cdot\|$ is strictly plurisubharmonic on $E\backslash\{$zero section$\}$.

Remark 34.2. If $\text{rank}(E) \geq 2$, then it is *not* true that $E \geq_{\text{Grif}} 0$ iff $-\log\|\cdot\|$ is psh on E; indeed, see Example 34.3 below.

Example 34.3. For a Hermitian **C**-vector space V with $r = \dim V \geq 2$, set $E = X \times V$. If $e_1, \ldots, e_r$ is an orthonormal basis for V, then

$$\|\sigma_1 e_1 + \ldots + \sigma_n e_r\|^2 = \sum_{j=1}^{r} |\sigma_j|^2.$$

In this case, $\theta_E = 0$, so $E \geq_{\mathrm{Grif}} 0$; however, the function

$$-\log \|(x, v)\| = -\frac{1}{2} \log \sum_{j=1}^{r} |\sigma_j|^2,$$

where $v = \sum_{j=1}^{r} \sigma_j e_j$, is not psh if $r \geq 2$.

Proof of Proposition 34.1. For (a), assuming $E \leq_{\mathrm{Grif}} 0$, it suffices to prove the following: if $u \colon V \to E$ is a local holomorphic section of E defined on an open subset $V \subseteq X$, then the function $\log \|u\|$ is psh on V (because the function $\log \|\cdot\|$ is always psh on fibres). Said differently, we must show that $i\,\partial\,\overline{\partial} \log \|u\|^2 \geq 0$, and this will follow from the definitions. Indeed,

$$\overline{\partial} \log \|u\|^2 = \frac{1}{\|u\|^2}\,\overline{\partial}\,\|u\|^2 = \frac{1}{\|u\|^2}\,\overline{\partial}\{u, u\} \tag{34.3}$$

Now, by the definition of the Chern connection, we have $d\{u, u\} = \{Du, u\} + \{u, Du\}$; in addition, $Du = D^{1,0}u + \overline{\partial}\,u = D^{1,0}u$ since u is holomorphic. Thus, $d\{u, u\} = \{D^{1,0}u, u\} + \{u, D^{1,0}u\}$, where the first term has bidegree $(1, 0)$ and the second term has bidegree $(0, 1)$. It follows that

$$\begin{cases} \partial\{u, u\} = \{D^{1,0}u, u\}, \\ \overline{\partial}\{u, u\} = \{u, D^{1,0}u\}. \end{cases}$$

Combining this observation with (34.3) gives

$$\overline{\partial} \log \|u\|^2 = \frac{1}{\|u\|^2}\{u, D^{1,0}u\}. \tag{34.4}$$

Similarly,

$$\partial\{u, D^{1,0}u\} = \{D^{1,0}u, D^{1,0}u\} + \{u, \overline{\partial}\,D^{1,0}u\}, \tag{34.5}$$

where we use that $(D^{1,0})^2 = 0$. Using the Leibniz rule as well as the equations (34.4) and (34.5), we find that

$$\begin{aligned} i\,\partial\,\overline{\partial} \log \|u\|^2 &= i\,\partial \left(\frac{1}{\|u\|^2}\{u, D^{1,0}u\} \right) \\ &= \frac{i}{\|u\|^2} \left(\{D^{1,0}u, D^{1,0}u\} + \{u, \overline{\partial}\,D^{1,0}u\} \right) - \frac{\bar{i}}{\|u\|^4}\{D^{1,0}u, u\} \wedge \{u, D^{1,0}u\} \\ &= \frac{i}{\|u\|^2}\{u, \partial\,D^{1,0}u\} + \frac{i}{\|u\|^4} \left(\{u, u\} \cdot \{D^{1,0}u, D^{1,0}u\} - \{D^{1,0}u, u\} \wedge \{u, D^{1,0}u\} \right). \end{aligned}$$

The first term of the above can be rewritten as

$$\frac{i}{\|u\|^2}\{u, \partial\,D^{1,0}u\} = -\frac{1}{\|u\|^2}\{u, i\,\overline{\partial}\,D^{1,0}u\} = -\frac{1}{\|u\|^2}\{u, iD^2u\} = -\frac{1}{\|u\|^2}\{u, i\Theta(E) \wedge u\} \geq 0,$$

by the assumption that $E \leq_{\mathrm{Grif}} 0$. The second term is non-negative by a Cauchy–Schwarz argument, which we leave as an exercise. This completes one direction of (a). The converse is shown by picking the sections u in a clever way, which we will not explain here. The assertion (b) follows in the same manner. ☐

34.3. **Criterion for Nakano Positivity.** Let $E \to X$ be a holomorphic Hermitian vector bundle and pick $x \in X$. Set $n = \dim X$, and let $z_1, \dots, z_n$ be local holomorphic coordinates near x. Given local holomorphic sections $u_1, \dots, u_n$ of E at x, define an $(n-1, n-1)$-form T_u on X near x by the formula

$$T_u \coloneqq \sum_{1 \le j,k \le n} \epsilon_{jk} \{u_j, u_k\} \alpha_{jk}, \tag{34.6}$$

where

$$\alpha_{jk} = dz_1 \wedge \dots \wedge \widehat{dz_j} \wedge \dots dz_n \wedge d\,\overline{z}_1 \wedge \dots \wedge \widehat{d\,\overline{z}_k} \wedge \dots \wedge d\,\overline{z}_n,$$

and the ϵ_{jk}'s are suitable unimodular constants (i.e. complex numbers of absolute value 1) such that $T_u \ge 0$ as an $(n-1, n-1)$-form. In fact, $\epsilon_{jk} = (-1)^{j+k} i^{(n-1)^2}$ works.

By construction, if H is the hyperplane defined by $z_n = \sum_{\ell=1}^{n-1} c_\ell z_\ell$, then $T_u|_H$ is some multiple of

$$dz_1 \wedge \dots \wedge dz_{n-1} \wedge d\,\overline{z}_1 \wedge d\,\overline{z}_n,$$

and so $i^{(n-1)^2}$ times this form is ≥ 0 on H. Said differently, the restriction of T_u to any hyperplane near x is non-negative.

Proposition 34.4. *We have $E \ge_{\text{Nak}} 0$ iff $i\,\partial\,\overline{\partial}\,T_u \le 0$ (as an (n,n)-form) for any local holomorphic sections $u_1, \dots, u_n$ of E at any point x such that $Du_j = 0$ at x for $1 \le j \le n$*

One should think of the sections $u_1, \dots, u_n$ appearing in the statement of Proposition 34.4 to be "constant to order 1" near the given point.

Proof. If $E \ge_{\text{Nak}} 0$, then computing as in the proof of Proposition 34.1 we find that

$$\overline{\partial}\,T_u = \sum_{1 \le j,k \le n} \epsilon_{jk}\,\overline{\partial}\{u_j, u_k\} \wedge \alpha_{j,k} = \sum_{1 \le j,k \le n} \epsilon_{jk} \{u_j, D^{1,0} u_k\} \wedge \alpha_{jk},$$

and hence

$$\partial\,\overline{\partial}\,T_u = \sum_{1 \le j,k \le n} \epsilon_{jk} \left(\{D^{1,0} u_j, D^{1,0} u_k\} + \{u_j, \overline{\partial}\,D^{1,0} u_j\} \right) \wedge \alpha_{jk},$$

where the term $\{D^{1,0} u_j, D^{1,0} u_k\}$ vanishes at x by assumption. Thus, we get that

$$i\,\partial\,\overline{\partial}\,T_u = \sum_{1 \le j,k \le n} \epsilon_{jk} (-1) \{u_j, i\Theta(E) \wedge u_k\} \wedge \alpha_{jk} \le 0,$$

where the pointwise inequality holds by the assumption that $E \ge_{\text{Nak}} 0$.

The converse holds by the same calculation and the following fact: given any $e \in E_x$, there exists a holomorphic section u of E at x such that $u(x) = e$ and $Du = 0$ at x. Indeed, given any u with $u(x) = e$, we can "correct" u as

$$u + \sum_{j=1}^{n} z_j v_j,$$

where $z_1, \dots, z_n$ are local coordinates centered at x, and $v_1, \dots, v_n$ are suitable local holomorphic sections of E near x. Then, at x, we have

$$D^{1,0}\left(u + \sum_{j=1}^{n} z_j v_j\right) = D^{1,0} u + \sum_{j=1}^{n} v_j(x) \otimes dz_j,$$

and we can choose the v_j's so that this expression is zero. ∎

35. April 13th

35.1. **Positivity of Direct Images.** Let X be a Kähler manifold of dimension $n+m$, Y a complex manifold of dimension n, and $p\colon X \to Y$ a proper holomorphic submersion. The fibre $X_t \coloneqq p^{-1}(t)$ is a compact Kähler manifold of dimension n, for $t \in Y$. Let L be a holomorphic line bundle on X, ϕ a smooth metric on L, and write K_X for the canonical bundle on X. Consider the direct image sheaf $E \coloneqq p_*(K_X + L)$; it is perhaps more natural to think of this as $E = p_*(K_{Y/X} + L)$ instead.

Berndtsson's theorem [Ber09] on the positivity of direct images is the following:

Theorem 35.1. [Berndtsson, 2009]

(1) If L is semipositive (i.e. there exists a semipositive metric ϕ on L), then E is a holomorphic vector bundle on Y with fibres $E_t = H^0(X_t, K_{X_t} + L|_{X_t})$ for $t \in Y$.

(2) Equip the fibre E_t with the Hermitian inner product

$$\langle u, v\rangle \coloneqq \int_{X_t} c_n u \wedge \overline{v} e^{-2\phi_t},$$

where $L_t = L|_{X_t}$, $\phi_t = \phi|_{L_t}$, and $c_n u \wedge \overline{v} e^{-2\phi_t}$ is an (n,n)-form on X_t. Then, E is a holomorphic Hermitian vector bundle on Y. Further, if ϕ is (semi)positive, then $(E, \|\cdot\|)$ is Nakano (semi)positive.

The statement in (1) follows from the (geometric version of the) Ohsawa–Takegoshi theorem (use any semipositive metric on L). To prove (2), we may assume that Y is a ball in $\mathbf{C}^m$ (because the statements are local on the base). For simplicity, we assume $m = 1$, so $Y = \mathbf{D}$, and use the coordinate t on $\mathbf{D}$. (The main applications of the positivity of direct images are when the base is 1-dimensional, so this is not a serious reduction.)

We will use the criterion for Nakano positivity that was proved last time. (Note that when Y is 1-dimensional (as we assumed here) then Griffiths and Nakano positivity coincide.) As Nakano positivity is a local property, we can work at $t = 0$. What must be proved is the following: for any holomorphic section $u \in H^0(Y, E)$ such that $D^{1,0}u = 0$ at $t = 0$, then there is an inequality of currents

$$i\,\partial\overline{\partial}\,T_u \le 0,$$

where $D^{1,0}$ is the $(1,0)$-part of the Chern connection. In this case, the setting is slightly simpler because T_u is a function: indeed, we have $T_u = \|u\|^2$, where

$$\|u\|^2(t) = \int_{X_t} c_n u(t) \wedge \overline{u(t)} e^{-2\phi_t}.$$

In order to prove this, we should carefully interpret the sections of E. We think of a (local or global) section $u \in C^\infty(Y, E)$ as an equivalence class $[U]$, where $U \in C^\infty_{n,0}(X, L)$, and:

- we need that $\overline{\partial}\,U$ vanishes on fibres, so we can write $\overline{\partial}\,U = \eta \wedge dt + \nu \wedge d\overline{t}$ in $C^\infty_{n,1}(X, L)$;
- $U_1 \sim U_2$ if $U_1 - U_2|_{X_t} = 0$ for all $t \in \mathbf{D}$, which occurs iff $U_1 - U_2 = \gamma \wedge dt$ for $\gamma \in C^\infty_{n-1,0}(X, L)$.

In addition, we can write the Hermitian structure as follows: for $u_t, v_t \in E_t = H^0(X_t, K_{X_t} + L|_{X_t})$, define a volume form

$$[u_t, v_t] \coloneqq c_t u_t \wedge \overline{v_t} e^{-2\phi_t}$$

on X_t, and its integral

$$\langle u_t, v_t\rangle \coloneqq \int_{X_t} [u_t, v_t].$$

Also write $\|u_t\|^2 \coloneqq \langle u_t, u_t\rangle$. Now, if $u, v \in C^\infty(V, E)$, we can view $\langle u, v\rangle$ as a function of t: write $u = [U]$ and $v = [V]$ for $U, V \in C^\infty_{n,0}(X, L)$, then

$$\langle u, v\rangle \coloneqq p_*\left([U, V]\right),$$

where $[U, V] = c_n U \wedge \overline{V} \in C^\infty_{n,n}(X)$ and p_* means that one is integrating over fibres. It is easy to check that this is well-defined (that is, independent of the choice of U and V).

To compute $i\,\partial\,\overline{\partial}\,T_u = i\,\partial\,\overline{\partial}\,\|u\|^2$, we need to use the Leibnitz rule, and to do so we must understand the Chern connection $D = D^{1,0} + \overline{\partial}$ on E. As above, any $u \in C^\infty_{p,q}(Y,E)$ (for $0 \le p,q \le 1$) can be represented as $u = [U]$, where $U \in C^\infty_{p+n,q}(X,L)$ such that $\overline{\partial}\,U = \cdot \wedge dt + \cdot \wedge d\bar{t}$, and $U_1 \sim U_2$ if $U_1 - U_2 = \cdot \wedge dt$. Then, the $(0,1)$-part $\overline{\partial}$ of the Chern connection is given by what one expects, i.e.

$$\overline{\partial}[U] = [\overline{\partial}\,U].$$

This is well-defined since $\overline{\partial}(\gamma \wedge dt) = \overline{\partial}\,\gamma \wedge dt$. If we write $\overline{\partial}\,U = \eta \wedge dt + \nu \wedge d\bar{t}$ where $\nu \in C^\infty_{n,0}(X,L)$, then

$$\overline{\partial}[U] = [\nu] \wedge dt.$$

The $(1,0)$-part of the Chern connection on L (induced by ϕ) is $\partial^\phi \colon C^\infty_{p,q}(X,L) \to C^\infty_{p+1,q}(X,L)$, and it is given by

$$\partial^\phi\,U = e^{2\phi}\,\partial\,e^{-2\phi}U.$$

(This is an abuse of notation: it works like this in a given frame of L.) Now, consider $u \in C^\infty_{0,q}(Y,E)$ (for $0 \le q \le 1$), and write $u \in [U]$ for $U \in C^\infty_{n,q}(X,L)$ such that $\overline{\partial}\,U = \cdot \wedge dt + \cdot \wedge d\bar{t}$. Then, $\partial^\phi\,U \in C^\infty_{n+1,q}(X,L)$, so

$$\partial^\phi\,U = \mu \wedge dt,$$

where $\mu \in C^\infty_{n,q}(X,L)$.

Lemma 35.2. *For u as above, $D^{1,0}u = [\mu] \wedge dt$.*[12]

Proof. One uses the definition of the Chern connection to write

$$d\langle u,v\rangle = \{D^{1,0}u, v\} + \{u, \overline{\partial}\,v\},$$

where $u,v \in C^\infty(Y,E)$ and where $\{\cdot,\cdot\} \colon C^\infty_\bullet(Y,E) \times C^\infty_\bullet(Y,E) \to C^\infty_\bullet(Y,\mathbf{C})$ is the usual sesquilinear pairing. Now, one wants to work on X: write $u = [U]$ and $v = [V]$ for $U,V \in C^\infty_{n,0}(X,L)$, then

$$\begin{aligned}\partial\langle u,v\rangle &= \partial\,p_*\left([U,V]\right)\\ &= c_n p_*\left(\partial(U \wedge \overline{V}e^{-2\phi})\right)\\ &= c_n p_*(\partial^\phi\,U \wedge \overline{V}e^{-2\phi}) + (-1)^n c_n p_*(U \wedge \overline{\overline{\partial}\,V}e^{-2\phi}),\end{aligned}$$

where the third equality follows from the Leibnitz rule. Writing $\overline{\partial}^\phi\,U = \mu \wedge dt$ and $\overline{\partial}\,V = \eta \wedge dt + \nu \wedge d\bar{t}$, we get that

$$\partial\langle u,v\rangle = \{[\mu] \wedge dt, v\} + \{u, [\nu]d\bar{t}\}$$

where one must be careful with signs to get this final equality. □

Now, granted Lemma 35.2, we can compute $i\,\partial\,\overline{\partial}\,T_u$ after picking a good representative U of u. This will be discussed next class.

36. April 16th

This is the last class, and we will aim to complete the proof of Berndtsson's theorem on the positivity of direct images.

[12] This is not quite correct, one instead has $D^{1,0}u = [P(\mu)] \wedge dt$, where P is an orthogonal projection onto the holomorphic forms on fibres. This is explained next class, as well as in [Ber09]. (Briefly, this is necessary so that this expression is well-defined, i.e. independent of the choice of U.)

36.1. **Positivity of Direct Images (Continued).** Recall the setup from last time: X is a Kähler manifold of dimension $n+1$ and we fix a Kähler form ω, $Y = \mathbf{D}$, $p\colon X \to Y$ is a proper[13] holomorphic submersion, L is a holomorphic line bundle on X, ϕ is a smooth (semi)positive metric on L, and $E = p_*(K_X + L)$ is the direct image sheaf. The fibres of E are $E_y = H^0(E_y, K_{X_y} + L|_{X_y})$ for $y \in Y$. We use the coordinate τ on $Y = \mathbf{D}$ and set $t := p^*\tau$.

The goal is to prove Berndtsson's theorem on the positivity of direct images:

Theorem 36.1. [Berndtsson, 2009]

(1) E is a holomorphic Hermitian vector bundle on Y;
(2) E is Nakano (semi)positive.

We know (1) and it remains to show (2).

Recall that a smooth section $u \in C^\infty(Y, E)$ of E is given by an equivalence class $u = [U]$, where $U \in C^\infty_{n,0}(X, L)$ satisfies $\bar\partial U = \nu \wedge dt + \eta \wedge d\bar t$ and $[U_1] = [U_2]$ iff $U_1 - U_2 = \gamma \wedge dt$.

With this notation, the Hermitian metric on E is given as follows:

$$\langle [U], [V] \rangle := c_n p_*(U \wedge \overline{V} e^{-2\phi}),$$

where $c_n = i^{n^2}$ is the usual unimodular constant, $U \wedge \overline{V} e^{-2\phi}$ is viewed as an (n,n)-form on X, and the pushforward p_* is the integration along fibres; that is, if α is a form of bidegree $(n+1-r, n+1-s)$, then $p_*\alpha$ is of bidegree $(1-r, 1-s)$, and this operation satisfies a projection formula

$$p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta$$

for forms α, β.

The Chern connection on E can be decomposed as $D = D^{1,0} + \bar\partial$ and the components act as follows: for $u = [U]$ such that $\bar\partial U = \eta \wedge dt + \nu \wedge d\bar t$, then[14]

$$\bar\partial u = (-1)^n [\nu] \otimes d\tau.$$

Now, if the $(1,0)$-part of the Chern connection on L is denoted by ∂^ϕ, then write $\partial^\phi U = \mu \wedge dt$, and we have

$$D^{1,0} u = [P(\mu)] \otimes d\bar\tau,$$

where $P(\mu)$ is the orthogonal projection of μ onto the space of L-valued $(n,0)$-forms on X that are holomorphic on the fibres of p. This was omitted last class, but one really must use $P(\mu)$ instead of μ in order for this to be well-defined (i.e. independent of the choice of U).

With this data, we have

$$\begin{cases} \partial\langle u, v\rangle = \{D^{1,0}u, v\} + \{u, \bar\partial v\}, \\ \bar\partial\langle u, v\rangle = \{u, D^{1,0}v\} + \{\bar\partial u, v\}, \end{cases}$$

for E-valued forms u and v, where $\{\cdot,\cdot\}$ denotes the usual pairing on E-valued forms. We won't verify these formulas, but they do require proof.

To prove (2), we use the following criterion for Nakano positivity (which coincides with Griffiths positivity in this setting): for any $y \in Y$, we have

$$i\,\partial\bar\partial\,\|u\|^2 \le 0$$

for all $u \in H^0(Y, E)$ such that $(D^{1,0}u)(u) = 0$. Below, we do this only for $y = 0 \in \mathbf{D}$.

Write such a global section u as $u = [U]$, where $U \in C^\infty_{n,0}(X, L)$, and the condition that $\bar\partial u = 0$ translates to $\bar\partial U = \eta \wedge dt$ for some $\eta \in C^\infty_{n-1,1}(X, L)$. We have

$$\|u\|^2 = c_n p_*(U \wedge \overline{U} e^{-2\phi}),$$

[13] It is possible that we really require p to be projective in order to apply the relevant Ohsawa–Takegoshi extension theorem.

[14] In [Ber09], there is no $(-1)^n$ term in $\bar\partial u$, but this is because Berndtsson write $\bar\partial U$ as $dt \wedge \eta + d\bar t \wedge \nu$ instead.

and we will use this to compute $i\,\partial\,\overline{\partial}\,\|u\|^2$. (We will later need to make a clever choice of representative U in order to verify the positivity criterion.)

Observe that

$$\begin{aligned}\overline{\partial}\,\|u\|^2 &= c_n\,\overline{\partial}\,p_*(U\wedge\overline{U}e^{-2\phi})\\ &= c_n p_*(\overline{\partial}(U\wedge\overline{U}e^{-2\phi}))\\ &= c_n p_*(\overline{\partial}\,U\wedge\overline{U}e^{-2\phi})+(-1)^n c_n p_*(U\wedge\overline{\partial^\phi\,U}e^{-2\phi}).\end{aligned}$$

The first term vanishes for bidegree reasons: indeed, the projection formula gives that

$$\begin{aligned}p_*(\overline{\partial}\,U\wedge\overline{U}e^{-2\phi}) &= p_*(\eta\wedge dt\wedge\overline{U}e^{-2\phi})\\ &= (-1)^n p_*(\eta\wedge\overline{U}e^{-2\phi}\wedge p^*d\tau)\\ &= (-1)^n p_*(\eta\wedge\overline{U}e^{-2\phi})\wedge d\tau,\end{aligned}$$

and this is a form of bidegree $(1,0)$. However, $\overline{\partial}\,U\wedge\overline{U}e^{-2\phi}$ has bidegree $(n,n+1)$, and hence $p_*(\overline{\partial}\,U\wedge\overline{U}e^{-2\phi})$ has bidegree $(0,1)$. It follows that $p_*(\overline{\partial}\,U\wedge\overline{U}e^{-2\phi})$ must be zero.

Differentiating the formula for $\overline{\partial}\,\|u\|^2$ once more gives the expression

$$\partial\,\overline{\partial}\,\|u\|^2=(-1)^n c_n p_*(\partial^\phi\,U\wedge\overline{\partial^\phi\,U}e^{-2\phi})+c_n p_*(U\wedge\overline{\overline{\partial}\,\partial^\phi\,U}e^{-2\phi}. \tag{36.1}$$

We will keep the first term of (36.1), but rewrite the second term using the formula

$$\partial^\phi\,\overline{\partial}+\overline{\partial}\,\partial^\phi=2\,\partial\,\overline{\partial}\,\phi\wedge\cdot \tag{36.2}$$

as operators. This seems promising because $\partial\,\overline{\partial}\,\phi$ is (up to a constant factor) the curvature of ϕ, and this is something we control. To that end, we have

$$p_*(U\wedge\overline{\overline{\partial}\,U}e^{-2\phi})=p_*(U\wedge\overline{\eta}\wedge d\overline{t}e^{-2\phi})=p_*(U\wedge\overline{\eta}e^{-2\phi}\wedge d\overline{\tau} \tag{36.3}$$

is zero again for bidegree reasons; thus, differentiating (36.3), we get

$$0=\overline{\partial}\,p_*(U\wedge\overline{\overline{\partial}\,U}e^{-2\phi})=p_*(\overline{\partial}\,U\wedge\overline{\overline{\partial}\,U}e^{-2\phi})+(-1)^n p_*(U\wedge\overline{\partial^\phi\,\overline{\partial}\,U}e^{-2\phi}). \tag{36.4}$$

Now, combining (36.1), (36.2), and (36.4) we get that

$$i\,\partial\,\overline{\partial}\,\|u\|^2=(-1)^n ic_n p_*(\partial^\phi\,U\wedge\overline{\partial^\phi\,U}e^{-2\phi})-2c_n p_*(U\wedge\overline{U}\wedge i\,\partial\,\overline{\partial}\,\phi e^{-2\phi})+(-1)^n ic_n p_*(\overline{\partial}\,U\wedge\overline{\overline{\partial}\,U}e^{-2\phi}). \tag{36.5}$$

The goal is to show that (36.5) is non-positive. The second term is ≤ 0 since $i\,\partial\,\overline{\partial}\,\phi\ge 0$. In order to deal with the first and third terms, we must choose U cleverly, using the hypothesis that $(D^{1,0}u)(0)=0$.

Proposition 36.2. [Ber09, Proposition 4.2] *Given u as above, we can choose U such that*

(1) $\overline{\partial}\,U=\eta\wedge dt$, where $\eta\wedge\omega|_{X_0}=0$ (i.e. $\eta|_{X_0}$ is primitive);
(2) the $(n+1,0)$-form $\partial^\phi\,U$ is zero at every point on X_0.

We will not prove Proposition 36.2, but Berndtsson's proof uses ideas as before, in that one plays with the operators $\overline{\partial},\overline{\partial}^*$ and the Lefschetz decomposition (some Hodge-theoretic input).

Granted Proposition 36.2, look at (36.5) when $\tau=0$, and we want it to be ≤ 0. The first term vanishes (at $\tau=0$) by Proposition 36.2(2). Therefore, at $\tau=0$, we have

$$i\,\partial\,\overline{\partial}\,\|u\|^2\le(-1)^n ic_n p_*(\overline{\partial}\,U\wedge\overline{\overline{\partial}\,U}e^{-2\phi}). \tag{36.6}$$

While it is true that $c_p\alpha\wedge\overline{\alpha}\ge 0$ whenever α is a $(p,0)$-form, we do not quite have that setup in (36.6). Nonetheless, Proposition 36.2(1) allows us to write

$$\begin{aligned}(-1)^n ic_n p_*(\overline{\partial}\,U\wedge\overline{\overline{\partial}\,U}e^{-2\phi}) &= ic_n p_*(\eta\wedge\overline{\eta}e^{-2\phi}\wedge dt\wedge d\overline{t})\\ &= p_*(c_n\eta\wedge\overline{\eta}e^{-2\phi})\wedge id\tau\wedge d\overline{\tau}.\end{aligned}$$

As $id\tau \wedge d\overline{\tau} \geq 0$, we just need to show that $p_*(c_n\eta \wedge \overline{\eta}e^{-2\phi}) \leq 0$ at $\tau = 0$. However, $\eta|_{X_0}$ is an $(n-1,1)$-form so (by a calculation that we did a long time ago) we have

$$c_n\eta \wedge \overline{\eta}e^{-2\phi} = \left(|\eta \wedge \omega|^2 e^{-2\phi} - |\eta|^2 e^{-2\phi}\right) dV_\omega$$

on X_0 and by assumption $\eta \wedge \omega$ vanishes on X_0, so

$$c_n\eta \wedge \overline{\eta}e^{-2\phi} = -|\eta|^2 e^{-2\phi} dV_\omega \leq 0.$$

This is a similar calculation to what one does when proving the Hodge index theorem analytically. This completes the verification that $i\,\partial\,\overline{\partial}\,\|u\|^2 \leq 0$, and hence it completes the proof of the positivity of direct images.

References

[BBS08] Robert Berman, Bo Berndtsson, and Johannes Sjöstrand. A direct approach to Bergman kernel asymptotics for positive line bundles. *Ark. Mat.*, 46(2):197–217, 2008.

[Ber95] Bo Berndtsson. L^2-methods for the $\overline{\partial}$-equation. http://www.math.chalmers.se/~bob/not3.pdf, 1995.

[Ber09] Bo Berndtsson. Curvature of vector bundles associated to holomorphic fibrations. *Ann. of Math. (2)*, 169(2):531–560, 2009.

[Ber10] Bo Berndtsson. An introduction to things $\overline{\partial}$. In *Analytic and algebraic geometry*, volume 17 of *IAS/Park City Math. Ser.*, pages 7–76. Amer. Math. Soc., Providence, RI, 2010.

[Bo13] Zbigniew Bł ocki. Suita conjecture and the Ohsawa-Takegoshi extension theorem. *Invent. Math.*, 193(1):149–158, 2013.

[Bou90] Thierry Bouche. Convergence de la métrique de Fubini-Study d'un fibré linéaire positif. *Ann. Inst. Fourier (Grenoble)*, 40(1):117–130, 1990.

[Bou17] S. Boucksom. Singularities of plurisubharmonic functions and multiplier ideals. http://sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf, 2017.

[Cat99] David Catlin. The Bergman kernel and a theorem of Tian. In *Analysis and geometry in several complex variables (Katata, 1997)*, Trends Math., pages 1–23. Birkhäuser Boston, Boston, MA, 1999.

[Che11] Bo-Yong Chen. A simple proof of the Ohsawa-Takegoshi extension theorem. *arXiv:1105.2430v1*, 2011.

[Con] Brian Conrad. Differential Geometry Handouts. http://math.stanford.edu/~conrad/diffgeomPage/handouts.html.

[Dem12] Jean-Pierre Demailly. Complex-Analytic and Differential Geometry. https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf, 2012.

[DPS94] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.*, 3(2):295–345, 1994.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[Hö65] Lars Hörmander. L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113:89–152, 1965.

[Hö90] Lars Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[Har66] Robin Hartshorne. Ample vector bundles. *Inst. Hautes Études Sci. Publ. Math.*, (29):63–94, 1966.

[Jon18] Mattias Jonsson. Some functional analysis. http://www.math.lsa.umich.edu/~mattiasj/710/refs/fcnan.pdf, 2018.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Man93] Laurent Manivel. Un théorème de prolongement L^2 de sections holomorphes d'un fibré hermitien. *Mathematische Zeitschrift*, 212(1):107–122, 1993.

[Nad90] Alan Michael Nadel. Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. *Ann. of Math. (2)*, 132(3):549–596, 1990.

[OT87] Takeo Ohsawa and Kensho Takegoshi. On the extension of L^2 holomorphic functions. *Mathematische Zeitschrift*, 195(2):197–204, 1987.

[Pă07] Mihai Păun. Siu's invariance of plurigenera: a one-tower proof. *J. Differential Geom.*, 76(3):485–493, 2007.

[Siu96] Yum-Tong Siu. The Fujita conjecture and the extension theorem of Ohsawa-Takegoshi. *Geometric complex analysis (Hayama, 1995)*, 577592, 1996.

[Siu98] Yum-Tong Siu. Invariance of plurigenera. *Inventiones mathematicae*, 134(3):661–673, 1998.

[Tia90] Gang Tian. On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geom.*, 32(1):99–130, 1990.

[Voi02] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.

[Zel98] Steve Zelditch. Szegö kernels and a theorem of Tian. *Internat. Math. Res. Notices*, (6):317–331, 1998.