THE ALGEBRAIC TORUS THEOREM

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Abstract. This are notes for a talk given in Dani Wise's class on cube complexes at McGill University. This is a brief survey on the algebraic torus theorem, proved by Dunwoody & Swenson in [2], which generalizes Stalling's theorem on the ends of groups to give a structure theorem for finitely-generated groups with a polycyclic-by-finite codimension-1 subgroup.

1. Preliminaries

1.1. **Polycyclic-by-finite groups.** Let G be a group. Recall that a *subnormal series* of G is a sequence of subgroups of the form

$$1 = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G,$$

Notice that we require only that $H_i \lhd H_{i+1}$ for each i, not that $H_i \lhd G$ necessarily. Furthermore, recall that G is said to be *solvable* if it admits a subnormal series such that H_{i+1}/H_i is abelian for all i.

Now, we say that G is *polycyclic* if it admits a subnormal series such that H_{i+1}/H_i is cyclic for all i. Equivalently, polycyclic groups are solvable groups that satisfy a maximal condition on subgroups, namely that each subgroup is finitely-generated. Examples of polycyclic groups are given below:

(1) Finitely-generated abelian groups: this is obvious, just throw in one generator at a time to make the subnormal series.
(2) Finitely-generated nilpotent groups: to see this, use the lower central series as a subnormal series and induct on the nilpotency class.
(3) Finite solvable groups (and hence any odd order finite group, by the Felt-Thompson theorem): obvious, as they satisfy the maximal condition on subgroups.
(4) Maltsev and Auslander & Swan showed that polycyclic groups correspond exactly to solvable subgroups of $GL(2, \mathbb{Z})$.

In fact, we have the following inclusions in the class of finitely-generated groups:

$$\text{cyclic} \subset \text{abelian} \subset \text{nilpotent} \subset \text{polycyclic} \subset \text{solvable} \subset \text{finitely-generated}.$$

Note that the inclusion of polycyclic groups in solvable groups is strict: indeed, consider the Baumslag-Solitar group $BS(2,1) = \langle a, b | aba^{-1} = b^2 \rangle$, which is solvable but not polycyclic. To see this, note that $\langle\langle b \rangle\rangle$ is isomorphic to the group of dyadic rationals, and hence is not finitely-generated. Similarly, the inclusion of nilpotent groups in polycyclic groups is strict: S_3 is a finite solvable group and hence is polycyclic, but is not nilpotent.

A *virtually polycyclic group* is a group with a finite-index polycyclic subgroup. In fact, one can show that a virtually polycyclic group will have a normal finite-index polycyclic subroup, and so it makes sense to rename a virtually polycyclic group as a *polycyclic-by-finite group* (because the group G is an extension of a polycyclic group H by a finite group G/H). These two terms will be used interchangeably.

The *Hirsch length* $h(G)$ of a polycyclic group G is the number of infinite factors in a subnormal series. If G is only polycyclic-by-finite, the Hirsch length of G is defined to be the Hirsch length of a finite-index normal polycyclic subgroup $H \lhd G$, i.e. $h(G) := h(H)$. There is an obvious question as to whether the Hirsch length

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is well-defined, but recall that the Schreier refinement theorem stipulates that any two subnormal series have equivalent refinements (that is, the factor groups are isomorphic but possibly reordered); thus, the number of infinite factor groups is the same for both refinements. However, the number of infinite factors is constant under taking refinements (use 3rd isomorphism theorem), so the Hirsch length is indeed well-defined.

Remark that $h(G) = 0$ iff G is finite.

1.2. **Stalling's Theorem on ends of groups.** Let G be a finitely-generated group, and let $e(G)$ denote the number of ends of G (remove any finite collection of edges from the Cayley graph of G, then $e(G) \leq n$ if the resulting space has $\leq n$ infinite connected components; the number of ends is the minimal such n)[1] The classical formulation of Stalling's theorem on ends of groups says the following:

Theorem 1. (Stallings) *Let G be finitely-generated and $e(G) > 1$, then one of the following holds:*

(1) *G is an amalgam over a finite group, i.e. $G = H *_C K$, where C is a finite group and $H, K \neq C$.*
(2) *G is an HNN extension relative to an isomorphism of finite groups, i.e. $G = H*_C$ where C is finite.*

More succinctly, Stalling's theorem says that a finitely-generated group with more than one end splits over a finite subgroup. Said differently, suppose $J < G$ is a finite subgroup with codimension-1[2], then G splits over a subgroup commensurable with J.

To see that these two hypotheses are the same, notice that $e(G) > 1$ means that the Cayley graph Γ has more than one infinite connected component if we remove a finite collection of edges. The quotient $J\backslash\Gamma$ just kills some finite bits of the Cayley graph, and so it also has more than 1 end. The converse follows similarly.

Recall that two subgroups $J, H < G$ are commensurable if the intersection $J \cap H$ has finite index in both J and H, i.e. $[J\colon J \cap H] < \infty$ and $[H\colon J \cap H] < \infty$. Since J is finite, $J \cap H$ is finite; since H has a finite subgroup of finite-index, it must be finite. Thus, the above conclusion does say that G splits over a finite group.

Example 2. Let $G = \langle a, b \mid b^n, [a, b]\rangle \simeq \mathbb{Z} \times \mathbb{Z}/n$. This is a virtually infinite cyclic group, so $e(G) = 2$. The Cayley graph Γ of G with respect to the generating set $\{a, b\}$ looks like an infinite line with a cycle of length n attached at each vertex of the line. The quotient of Γ by $\mathbb{Z}/n$ is an infinite line, hence $\mathbb{Z}/n < G$ is a finite subgroup of codimension-1. Let $id\colon \mathbb{Z}/n \to \mathbb{Z}/n$ be the identity morphism, then $G = (\mathbb{Z}/n)*_{id}$ is an HNN extension.

This formulation of Stalling's theorem is what we will generalize, namely we relax the condition that J be finite with codimension-1 to let J be polycyclic-by-finite with codimension-1.

2. The algebraic torus theorem

Recall from hyperbolic geometry that a Fuchsian group is a discrete subgroup of isometries of $\mathbb{H}^2$ (that is, a discrete subgroup of $\langle PSL(2, \mathbb{R}), z \mapsto -\overline{z}\rangle$ that acts properly discontinuously on $\mathbb{H}^2$). A co-Fuchsian group is a Fuchsian group whose action on $\mathbb{H}^2$ is co-compact. Examples of Fuchsian groups include $SL(2, \mathbb{Z})$ and its congruence subgroups; examples of co-Fuchsian groups are harder to come by, but include e.g. the $(2, 3, 7)$-triangle group.[3]

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[1]Recall that if G is a finitely-generated group, $e(G) \in \{0, 1, 2, \infty\}$ and we can classify those groups with certain number of ends: $e(G) = 0$ iff G is finite, $e(G) = 2$ iff G is virtually infinite cyclic. Thus, in some sense, generic groups are either one-ended or have infinitely-many ends. Some special cases are $e(F_X) = \infty$ for $|X| \in (1, \infty)$ and $e(\mathbb{Z}^2) = 1$.

[2]Let G be a finitely-generated group with Cayley graph Γ, then $H < G$ is *codimension-1* if for some $r > 0$, there are two or more H-orbits of deep components $K_i \subset \Gamma$, in the sense that $K_i \not\subset N_s(H)$ for any $s > 0$. In class, we proved that $H < G$ is codimension-1 iff $H\backslash\Gamma$ has more than one end; the latter is the characterization that we will use here.

[3]The (p, q, r)-triangle group is a Coxeter group with 3 generators given by the presentation

$$\Delta(p, q, r) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^r = (ca)^q = 1\rangle.$$

This group can be realized as the group of hyperbolic isometries generated by the reflections in the sides of a hyperbolic triangle with interior angles π/p, π/q, and π/r.

Finally, we state the algebraic torus theorem.[4]

Theorem 3. *Let G be a finitely-generated group and let $J < G$ be a virtually polycyclic subgroup of Hirsch length n. If J has codimension-1, but there is no infinite-index subgroup $I < J$ of codimension-1 in G, then one of the following holds:*

1. *G is virtually polycyclic of Hirsch length $n+1$ and J is normalized by a finite-index subgroup of G (i.e. there is $H < G$ of finite-index such that $H < N_G(J)$, the normalizer of J).*
2. *G splits (as a free product with amalgamation or as an HNN extension) over a subgroup commensurable with J.*
3. *G is an extension of a virtually polycyclic group of Hirsch length $n-1$ by a co-Fuchsian group, i.e*
$$\text{co-Fuchsian} \hookrightarrow G \twoheadrightarrow \text{virtually polycyclic.}$$
4. *There is a finite graph of groups decomposition of G (i.e. G is the π_1 of a graph of groups), where the underlying graph has a vertex incident to every edge for which the corresponding vertex group H has a virtually polycyclic subgroup N of Hirsch length $n-1$ such that $N < J < H$.*

   In this case, H/N is a Fuchsian group and the edge groups of the decomposition each contain N and correspond to the conjugacy classes consisting of peripheral subgroups[5] of H/N.

Remark that if $n = 0$, then (2) says that G splits over a virtually polycyclic subgroup which is finite; in particular, G splits over a finite group. Obviously, (3) cannot occur, as a group of Hirsch length -1 is nonsense.

The "converse" of this statement is not in general true; that is, we have examples of a group G that splits over a subgroup J, but no subgroup commensurable to J has codimension-1. For example, take $G = D_\infty$ the infinite dihedral group. It has a presentation $\langle x, y | x^2 = y^2 = 1\rangle$ and so $D_\infty = \langle x\rangle *_{\{1\}} \langle y\rangle$, but $D_\infty/\langle x\rangle \simeq \mathbb{Z}/2\mathbb{Z}$. In particular, $[D_\infty \colon \langle x\rangle] = 2$, so $\langle x\rangle$ cannot have codimension-1 (recall we proved that $[G\colon H] < \infty$ implies that H is not codimension-1 in G).

Finally, we mention an application of this result to cube complexes. In 2005, Sageev and Wise proved the following characterization of certain groups acting properly on a CAT(0) cube complex.

Theorem 4. *(Sageev-Wise 2005, [3]) Suppose that G is a group for which there is a bound on the order of its finite subgroups. Suppose that G acts properly on a CAT(0) cubical complex. Then for $H < G$, either*

1. *H contains a rank-2 free subgroup.*
2. *H is virtually a finitely-generated abelian group.*

3. Appendix

Lemma 5. *A group is polycyclic iff it is solvable and every subgroup is finitely-generated.*

Proof. Let G be polycyclic, then G has a subnormal series with cyclic factors and in particular abelian factors, hence G is solvable. If $H < G$, then H is also polycyclic (take intersection of the subnormal series with H) and polycyclic plus solvable implies that H is finitely-generated.

Conversely, if G is solvable and such that every subgroup is finitely-generated, consider the solvable series $1 = G_{r+1} \lhd G_r \lhd \ldots \lhd G_1 = G$. The abelian factors G_i/G_{i+1} are abelian and finitely-generated, hence polycyclic. Thus, the solvable series can be refined to a polycyclic series, by adding isomorphic copies of polycyclic series for each factor. □

[4]The name of the theorem likely derives from the torus theorem of Peter Scott, which is a special case of the algebraic torus theorem: take $n = 2$ and restrict G to be a 3-manifold group.

[5]A *peripheral subgroup* of a Fuchsian group is the stabilizer of a parabolic limit point or a component of the domain of discontinuity in $\partial\mathbb{H}^2 \simeq S^1$.

References

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