ON THE DE RHAM COHOMOLOGY OF ALGEBRAIC VARIETIES

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Abstract. This is the final project for Prof. Bhargav Bhatt's Math 613 in the Winter of 2015 at the University of Michigan. The purpose of this paper is to prove a result of Grothendieck [Gro66] showing, for a smooth scheme locally of finite type over $\mathbb{C}$, the algebraic de Rham cohomology is isomorphic to the singular cohomology (with $\mathbb{C}$-coefficients) of its analytification. Our proof follows the exposition in [EC14, Chapter 2], and it is heavily-influenced by [Gro66] and [Gil].

1. Introduction

On a smooth manifold X of real dimension n, the sheaves $\Omega^k_{C^\infty,X}$ of smooth k-forms on X give rise to the smooth de Rham complex $\Omega^\bullet_{C^\infty,X}$, where the differential of the complex is the exterior derivative d. Taking the global sections of the smooth de Rham complex gives the complex of abelian groups $\Gamma(X, \Omega^\bullet_{C^\infty,X})$, and the k-th cohomology group $H^k_{dR}(X/\mathbb{R}) := H^k(\Gamma(X, \Omega^\bullet_{C^\infty,X}))$ is called the de Rham cohomology of X. The "classical" de Rham theorem provides an isomorphism between the de Rham cohomology groups of X and the singular cohomology groups of X.

Theorem 1. *Let X be a smooth manifold, then $H^*_{dR}(X/\mathbb{R}) \xrightarrow{\simeq} H^*_{sing}(X, \mathbb{R})$.*

This isomorphism associates a cohomology class $[\omega] \in H^k_{dR}(X/\mathbb{R})$ with the functional $H_k(X, \mathbb{R}) \to \mathbb{R}$ given by

$$[\gamma] \mapsto \int_\gamma \omega.$$

Moreover, the complex of sheaves $\Omega^\bullet_{C^\infty,X}$ is a resolution of the sheaf $\mathbb{R}_X$ of locally constant $\mathbb{R}$-valued functions on X:

$$0 \to \mathbb{R}_X \to \Omega^0_{C^\infty,X} \xrightarrow{d} \Omega^1_{C^\infty,X} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{C^\infty,X} \to 0.$$

The Poincaré lemma implies that the $\Omega^p_{C^\infty,X}$ are acyclic (the key here is that, in this context, there are partitions of unity). Therefore, the sheaf cohomology of X with coefficients in the constant sheaf $\mathbb{R}_X$ can be computed from the cohomologies of the complex $\Gamma(X, \Omega^\bullet_{C^\infty,X})$. This result is known as the de Rham theorem.

Theorem 2. (Classical de Rham Theorem) *Let X be a smooth manifold, then $H^*(X, \mathbb{R}_X) \simeq H^*_{dR}(X/\mathbb{R})$.*

When one considers instead a complex manifold X of (complex) dimension n, we still have an isomorphism between sheaf cohomology $H^*(X, \mathbb{C}_X)$ of the constant sheaf $\mathbb{C}_X$ and the singular cohomology $H^*(X, \mathbb{C})$ with $\mathbb{C}$-coefficients. However, the rest of the story becomes more complicated. As before, the sheaves $\Omega^p_{holo,X}$ of holomorphic p-forms on X give rise to the holomorphic de Rham complex $\Omega^\bullet_{holo,X}$, where the differential of the complex is the ∂-operator. The holomorphic Poincaré lemma implies that the holomorphic de Rham complex is a resolution of the sheaf $\mathbb{C}_X$ of locally constant $\mathbb{C}$-valued functions by coherent analytic sheaves:

$$0 \to \mathbb{C}_X \to \Omega^0_{holo,X} \xrightarrow{\partial} \Omega^1_{holo,X} \xrightarrow{\partial} \dots \xrightarrow{\partial} \Omega^n_{holo,X} \to 0.$$

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In general[1], the sheaves $\Omega^p_{holo,X}$ are not acyclic (this corresponds to the fact that there are no holomorphic partitions of unity on a complex manifold). Therefore, we cannot necessarily conclude that the sheaf cohomology groups $H^*(X, \mathbb{C}_X)$ coincide with the cohomology groups of the complex $\Gamma(X, \Omega^\bullet_{holo,X})$ of global holomorphic forms. This problem disappears if we consider instead the hypercohomology $\mathbb{H}^*(X, \Omega^\bullet_{holo,X})$ of the holomorphic de Rham complex, which does indeed calculate $H^*(X, \mathbb{C}_X)$; this in turn is isomorphic to the singular cohomology group $H^*(X, \mathbb{C})$ with $\mathbb{C}$-coefficients.

Theorem 3. (Analytic de Rham Theorem) *Let X be a complex manifold, then*

$$\mathbb{H}^*(X, \Omega^\bullet_{holo,X}) \simeq H^*(X, \mathbb{C}).$$

In [Ser56], Serre provided a dictionary linking the world of complex manifolds with that of smooth projective schemes over $\mathbb{C}$. One may ask: what is the right notion of de Rham cohomology in the latter context, and do we get analogues of the de Rham theorems? As before, the "right" de Rham cohomology for a smooth scheme over $\mathbb{C}$ is the hypercohomology of the complex of sheaves of differentials, where now the differentials are algebraic instead of holomorphic. In fact, [Ser56] provides a natural map between this algebraic de Rham cohomology and the hypercohomology of the holomorphic de Rham complex on the associated analytic space. For a projective scheme, we show that this is an isomorphism (this is our Theorem 7).

The questions with which we are concerned in this paper is, when the smooth scheme over $\mathbb{C}$ is no longer assumed to be projective, is there still an isomorphism between the algebraic de Rham cohomology and the singular cohomology of the analytification? The goal of this paper is to describe the relationship between these two hypercohomologies, and ultimately to explain Grothendieck's proof from [Gro66] of this isomorphism. This will be an "algebraic" de Rham theorem, which we call Grothendieck's theorem.

2. Preliminaries

2.1. **Complex Analytic Spaces.** Let $U \subset \mathbb{C}^n$ be an open subset, and let $\mathcal{O}_U$ be the sheaf of holomorphic functions on U. Take $f_1, \ldots, f_m \in \mathcal{O}_U(U)$, then the zero locus $Z := \{z \in U : f_1(z) = \ldots = f_m(z) = 0\}$ carries a natural structure sheaf $\mathcal{O}_Z := \mathcal{O}_U/\mathcal{I}_Z$, where $\mathcal{I}_Z$ is the corresponding sheaf of ideals of Z. We say that $(Z, \mathcal{O}_Z)$ is a *complex analytic variety*.

Just as a scheme over $\mathbb{C}$ can be viewed as a locally ringed space that locally looks like a complex algebraic variety, a complex analytic space is a locally ringed space that locally looks like a complex analytic variety. These will be one of our main objects of study. The relationships between schemes over $\mathbb{C}$ and complex analytic spaces are detailed in e.g. [Ser56] or [sga71], but we will discuss briefly some results that are needed for our purposes.

Definition 1. A *complex analytic space* $(X, \mathcal{O}_X)$ is a locally ringed space in which every point has an open neighbourhood that is isomorphic (as a locally ringed space) to a complex analytic variety. That is, for any $x \in X$, there exists an open neighbourhood U of X so that $(U, \mathcal{O}_U)$ is isomorphic to a complex analytic variety as locally ringed spaces. This isomorphism is called a *local model* at the point x.

Often, the complex analytic space $(X, \mathcal{O}_X)$ is denoted just by X, omitting the structure sheaf. When referring to the sheaves of holomorphic differential forms on a complex analytic space X, we will denote it by Ω^*_X instead of $\Omega^*_{holo,X}$ when it is clear from context that X is analytic.

[1] If X is a Stein manifold, then Cartan's theorem B [EC14, Theorem 2.3.10] says that all coherent analytic sheaves on X are acyclic. In particular, the sheaves of holomorphic differentials $\Omega^*_{holo,X}$ are acyclic, so the analytic de Rham cohomology $\mathbb{H}^*(X, \Omega^\bullet_{holo,X})$ coincides with $H^*(\Gamma(X, \Omega^\bullet_{holo,X}))$, which is analogous to the de Rham cohomology of the real case.

In the sequel, a scheme will refer to a scheme locally of finite type. Given a scheme X over $\mathbb{C}$, there is a natural way to construct an associated analytic space, denoted by X^{an}, which is called the analytification of X. We first construct the analytification when X is an affine scheme, and then proceed to the general case.

Consider the affine scheme $X = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. The points of the analytification X^{an} are the $\mathbb{C}$-points of X, which is to say $X^{an} = X(\mathbb{C}) = \{z \in \mathbb{C}^n : f_1(z) = \ldots = f_m(z) = 0\}$. The structure sheaf $\mathcal{O}_{X^{an}}$ of X^{an} is precisely the sheaf of holomorphic functions on $X(\mathbb{C})$, when viewed as an analytic subset of $\mathbb{C}^n$. Now, for an arbitrary scheme X, take any cover $\{U_i\}$ of X by open affine subsets and glue the locally ringed spaces $\{U_i^{an}\}$ to produce the analytification X^{an} of X.

Example 1. Let X be an elliptic curve over $\mathbb{C}$, i.e. the projective closure of an affine scheme of the form $\operatorname{Spec} \mathbb{C}[x, y]/(y^2 - f(x))$ for some cubic polynomial $f(x) \in \mathbb{C}[x]$. The analytification of the elliptic curve

$$X^{an} = \left\{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : y^2 z = F(x, z)\right\},$$

where $F(x, z)$ is the homogenization of $f(x)$. This is a closed Riemann surface of genus 1, hence diffeomorphic to a torus.

Furthermore, there is a natural morphism $\phi : X^{an} \to X$ of locally ringed spaces, which induces a pullback functor on sheaves. For a sheaf $\mathcal{F}$ on X, we will denote the pull-back by

$$\mathcal{F}^{an} := \phi^*\mathcal{F} = \phi^{-1}\mathcal{F} \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{O}_{X^{an}}.$$

Results relating a scheme and its analytification, as well those discussing how sheaves behave under this analytification procedure, are generally known as Serre's GAGA theorems, referring to [Ser56]. Two such results will be useful for our purposes.

Theorem 4. *Let X be a smooth projective scheme over $\mathbb{C}$. The functor ϕ^* is an equivalence between the category of coherent sheaves of $\mathcal{O}_X$-modules on X and the category of coherent sheaves of $\mathcal{O}_{X^{an}}$-modules on X_{an}.*

Proof. See [Ser56, Theorem 2, Theorem 3] □

Even better, there is a relationship between the cohomology of the coherent sheaf and the cohomology of its pullback by the analytification functor.

Theorem 5. *Let X be a smooth projective scheme over $\mathbb{C}$ and let $\mathcal{F}$ be a coherent sheaf on X, then for any $q \geq 0$, there is a natural isomorphism $\epsilon^q : H^q(X, \mathcal{F}) \longrightarrow H^q(X^{an}, \mathcal{F}^{an})$.*

Proof. See [Ser56, Theorem 1]. □

When X is a smooth projective scheme over $\mathbb{C}$, the above theorem, along with the Hodge-to-de Rham degeneration , will suffice to show that the algebraic and the analytic de Rham cohomologies coincide. This is our Theorem 7.

2.2. **Sheaves of differentials.**

Definition 2. Let A be a finitely-generated reduced $\mathbb{C}$-algebra. The *module of Kähler differentials* $\Omega^1_{A/\mathbb{C}}$ is the A-module generated by the symbols $\{da : a \in A\}$ module and the following relations:

(1) $dc = 0$ for $c \in \mathbb{C}$.
(2) $d(ab) = a \cdot db + b \cdot da$ for $a, b \in A$.

There is a $\mathbb{C}$-linear derivation $d : A \to \Omega^1_{A/\mathbb{C}}$ with the universal property that any other $\mathbb{C}$-linear derivation $A \to M$, where M is some A-module, factors through $d : A \to \Omega^1_{A/\mathbb{C}}$ in a unique way. Let $\Omega^i_{A/\mathbb{C}} := \bigwedge^i \Omega^1_{A/\mathbb{C}}$, then the universal property shows that there are maps $d : \Omega^i_{A/\mathbb{C}} \to \Omega^{i+1}_{A/\mathbb{C}}$ so that they form a complex

$$A \xrightarrow{d} \Omega^1_{A/\mathbb{C}} \xrightarrow{d} \Omega^2_{A/\mathbb{C}} \xrightarrow{d} \Omega^3_{A/\mathbb{C}} \xrightarrow{d} \ldots \quad (1)$$

Definition 3. On the affine scheme $X = \text{Spec } A$, the *sheaf of Kähler differentials* is $\Omega_X := \left(\Omega^1_{A/\mathbb{C}}\right)^{\sim}$. For a general scheme X, take any open affine cover and glue the sheaves of Kähler differentials on the open affines to produce the sheaf Ω^1_X. (See [Har77, Section 2.8] for a more intrinsic description of Ω^1_X on a general scheme.) Moreover, the complex of modules Eq. (1) induces a complex of sheaves (of abelian groups, whose terms are coherent $\mathcal{O}_X$-modules[2])

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \Omega^3_X \xrightarrow{d} \dots, \tag{2}$$

which is called the *algebraic de Rham complex* of X.

There are some important properties of the sheaf of Kähler differentials on a smooth scheme X: first, Ω^1_X is locally free of finite rank, hence coherent. Consequently, we can consider its analytification, along with the analytification of its exterior powers. In fact, the analytification of the sheaf $\Omega^q_{X/\mathbb{C}} = \bigwedge^q \Omega^1_{X/\mathbb{C}}$ of Kähler q-forms is the sheaf $\Omega^q_{X^{an}}$ of holomorphic q-forms on the analytic space X^{an}. See [Har77, Section 2.8] for a careful discussion.

In Section 3, we will define the algebraic de Rham cohomology $H^*_{dR}(X/\mathbb{C})$ of a smooth scheme X over $\mathbb{C}$ to be the hypercohomology of the algebraic de Rham complex. When X is an affine scheme, this definition reduces to the cohomology of the complex Eq. (1). In the following example, we see that the algebraic de Rham cohomology of an affine hyperelliptic curve coincides with the singular cohomology with $\mathbb{C}$-coefficients of its analytification.

Example 2. Let $q(x) \in \mathbb{C}[x]$ be any monic polynomial of degree e with no multiple roots, and let $A = \mathbb{C}[x, y]/(y^2 - q(x))$. Then, $X = \text{Spec } A$ is called an affine hyperelliptic curve of degree e. The sheaf of differentials $\Omega^1_{X/\mathbb{C}}$ is a free A-module of rank 1 and the algebraic de Rham cohomology is $H^0_{dR}(X/\mathbb{C}) \simeq \mathbb{C}$ and $H^1_{dR}(X/\mathbb{C}) \simeq \mathbb{C}^{e-1}$.

A priori, the module of Kähler differentials is

$$\Omega^1_{A/\mathbb{C}} = \frac{A \cdot dx + A \cdot dy}{(2ydy - q'(x)dx)}.$$

As $q(x)$ has no double roots, $q'(x)$ and $q(x)$ are coprime, so there are polynomials $a(x), b(x) \in \mathbb{C}[x]$ so that $1 = a(x)q(x) + b(x)q'(x)$. Let $\omega := a(x)ydx + 2b(x)dy$, then we claim that $\Omega^1_{A/\mathbb{C}} = A \cdot \omega$. To see this, notice that

$$dx = a(x)q(x)dx + b(x)q'(x)dx = a(x)y^2dx + 2b(x)ydy = y\omega$$

and

$$dy = a(x)q(x)dy + b(x)q'(x)dy = a(x)y^2dx + b(x)q'(x)dy = \frac{1}{2}a(x)yq'(x)dx + b(x)q'(x)dy = \frac{q'(x)}{2}\omega.$$

In this case, the algebraic de Rham complex is just $A \xrightarrow{d} \Omega^1_{A/\mathbb{C}} = A\omega$. The kernel of d is precisely the constants $\mathbb{C} \subset A$ and the cokernel is the $\mathbb{C}$-span of $\{\omega, x\omega, \dots, x^{e-2}\omega\}$. Therefore, the algebraic de Rham cohomology of X is

$$H^*_{dR}(X/\mathbb{C}) = \begin{cases} \mathbb{C}, & * = 0 \\ \mathbb{C}^{e-1}, & * = 1. \end{cases}$$

If we are instead concerned with smooth projective schemes over $\mathbb{C}$, we could compactify X to its projective closure in $\mathbb{P}^2(\mathbb{C})$ (by adding 1 or 2 points depending on the degree e) and then use excision to compute the algebraic de Rham cohomology of the projective closure. (Alternatively, one could cover the projective closure by open affines compute the algebraic de Rham cohomology in a Čech-style.) Either way, we see that the algebraic de Rham cohomology of the projective closure also coincides with the singular cohomology of its analytification, which is a closed Riemann surface of genus $\frac{e-1}{2}$ or $\frac{e-2}{2}$, depending on the parity of the degree e.

[2]The algebraic de Rham complex is a complex of sheaves of abelian groups, where the morphisms are induced by the derivation $d\colon A \to \Omega^1_{A/\mathbb{C}}$. Even though all of the terms of the complex are themselves coherent $\mathcal{O}_X$-modules, this is not necessarily a complex of coherent $\mathcal{O}_X$-modules. See [Gil, Example 1] for a counterexample.

3. Hypercohomology and Algebraic de Rham Cohomology

Let $Shv(X)$ be the Grothendieck abelian category of sheaves of abelian groups on a locally ringed space X. Denote by $Shv(X)^\bullet$ the category of cochain complexes over $Shv(X)$.

Definition 4. For $\mathcal{F}^\bullet \in Shv(X)^\bullet$, the *k-th cohomology sheaf of the complex* $\mathcal{F}^\bullet$ is

$$\mathscr{H}^k(\mathcal{F}^\bullet) := \mathfrak{ker}(\mathcal{F}^k \to \mathcal{F}^{k+1})/\mathfrak{im}(\mathcal{F}^{k-1} \to \mathcal{F}^k),$$

viewed as quotient of sheaves. That is, $\mathscr{H}^k(\mathcal{F}^\bullet)$ is the sheafification of the presheaf

$$U \mapsto \frac{\ker(\mathcal{F}^k(U) \to \mathcal{F}^{k+1}(U))}{\operatorname{im}(\mathcal{F}^{k-1}(U) \to \mathcal{F}^k(U))}.$$

Moreover, a map of complexes $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a *quasi-isomorphism* if the induced map on cohomology sheaves $\mathscr{H}^k(\mathcal{F}^\bullet) \to \mathscr{H}^k(\mathcal{G}^\bullet)$ is an isomorphsim for all k.

Below we define the hypercohomology of a complex in a rather non-constructive manner and refer the reader to [Wei94, Section 5.7] for its usual description as a collection of derived functors.

Definition 5. The *k-th hypercohomology* is a functor $\mathbb{H}^k(X, -)\colon Shv(X)^\bullet \to \mathrm{Ab}$ satisfying the following two conditions:

(1) If $f^\bullet\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism of complexes, then $\mathbb{H}^k(X, f^\bullet)$ is an isomorphism.
(2) If $\mathcal{I}^\bullet$ is a complex of injectives, then $\mathbb{H}^k(X, \mathcal{I}^\bullet) = H^k(\Gamma(X, \mathcal{I}^\bullet))$.

As every complex of sheaves admits a quasi-isomorphism to a complex of injective sheaves, it follows from the definition that the hypercohomology of any complex can be computed in terms of these injectives. Indeed, if $\mathcal{A}^\bullet \in Shv(X)^\bullet$ and $A^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism into some complex of injectives $\mathcal{I}^\bullet$, then

$$\mathbb{H}^k(\mathcal{A}^\bullet) \simeq \mathbb{H}^k(\mathcal{I}^\bullet) = H^k(\Gamma(X, \mathcal{I}^\bullet)).$$

To compute hypercohomology, we can of course use the long exact sequence in cohomology associated to a short exact sequence of complexes. However, it will often be more helpful to build a spectral sequence abutting to the desired cohomology group. There are 2 natural such spectral sequences, and at the moment we are concerned with just one of them.

Proposition 1. *For any $\mathcal{F}^\bullet \in Shv(X)^\bullet$, there are convergent spectral sequences*

$$E_1^{p,q} = H^p(X, \mathcal{F}^q) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet)$$

and

$$E_2^{p,q} = H^p(X, \mathscr{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet).$$

Proof. This is explained for a general abelian category in [Wei94, Section 5.7]. □

Definition 6. Let $\Omega_X^\bullet$ be the algebraic de Rham complex on a smooth scheme X over $\mathbb{C}$. The *algebraic de Rham cohomology* $H^*_{dR}(X/\mathbb{C})$ is defined to be the hypercohomology of the algebraic de Rham complex. That is,

$$H^k_{dR}(X/\mathbb{C}) := \mathbb{H}^k(X, \Omega_X^\bullet).$$

In this context, the E_1-spectral sequence of Proposition 1 is called the *Hodge-to-de Rham spectral sequence*, and it takes the form

$$E^1_{p,q} = H^p(X, \Omega_X^q) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet).$$

When the scheme X is sufficiently nice, this spectral sequence will degenerate at the E_1-page (i.e. the differentials d_r on the E_r-page are zero for all $r \geq 1$).

Theorem 6. (Hodge-to-de Rham degeneration) *Let X be a smooth, proper scheme over $\mathbb{C}$. Then, the Hodge-to-de Rham spectral sequence degenerates at the E_1-page.*

Proof. See [Ill96, Theorem 6.9]. □

In particular, when X is affine, the degeneration of the Hodge-to-de Rham spectral sequence implies that $H^k\Gamma(X, \Omega_X^\bullet) = \mathbb{H}^k(X, \Omega_X^\bullet)$. Indeed, the bottom row (i.e. $p = 0$ row) on the E_1-page consists of the groups $H^0(X, \Omega^k)$ and the differentials are horizontal. The degeneration at the E_1-page means that the only terms on the E_2-page are the groups $H^k(\Gamma(X, \Omega^\bullet))$ and the differentials are zero. It follows that these terms are isomorphic to those of the E_∞-page, i.e.

$$H^k(\Gamma(X, \Omega^\bullet)) \simeq \mathbb{H}^k(X, \Omega^\bullet).$$

Therefore, in the affine case, the algebraic de Rham cohomology $H^*_{dR}(X/\mathbb{C})$ can be thought of as very much the direct analogue of the usual de Rham cohomology, which is to say global closed forms module global exact forms.

The degeneration of the Hodge-to-de Rham spectral sequence, combined with Theorem 5, is enough to show that the algebraic and analytic de Rham cohomologies coincide when X is also assumed to be projective (in fact, the projective assumption can be weakened to proper; see [sga71] for details).

Theorem 7. *Let X be a smooth projective scheme over $\mathbb{C}$, then $\mathbb{H}^*(X, \Omega_X^\bullet) \simeq \mathbb{H}^*(X^{an}, \Omega_{X^{an}}^\bullet)$. In particular, the algebraic de Rham cohomology $H^*_{dR}(X/\mathbb{C})$ is isomorphic to the singular cohomology $H^*(X^{an}, \mathbb{C})$ with $\mathbb{C}$-coefficients.*

Proof. As in Proposition 1, there are the canonical E_1-spectral sequences $E_1^{p,q} = H^p(X, \Omega_X^q)$ and $E_{1,an}^{p,q} = H^p(X^{an}, \Omega_{X^{an}}^q)$ abutting to $\mathbb{H}^{p+q}(X, \Omega_X^\bullet)$ and $\mathbb{H}^{p+q}(X^{an}, \Omega_{X^{an}}^\bullet)$, respectively. Theorem 5 asserts that there are natural termwise-isomorphisms $H^p(X, \Omega_X^q) \simeq H^p(X^{an}, \Omega_{X^{an}}^q)$ for all p, q, as X is assumed to be projective. This collection of isomorphisms yields an isomorphism of spectral sequences $\{E_1^{p,q}\} \xrightarrow{\simeq} \{E_{1,an}^{p,q}\}$. In particular, there are isomorphisms of the abutments

$$\mathbb{H}^*(X, \Omega_X^\bullet) \simeq \mathbb{H}^*(X^{an}, \Omega_{X^{an}}^\bullet). \tag{3}$$

In Section 1, we remarked that the holomorphic Poincaré lemma implies that the augmentation map $\mathbb{C}_{X^{an}}^\bullet \to \Omega_{X^{an}}^\bullet$ is a quasi-isomorphism, where $\mathbb{C}_{X^{an}}^\bullet$ denotes the trivial complex $0 \to \mathbb{C}_{X^{an}} \to 0$. In particular, there are isomorphisms $H^*(X^{an}, \mathbb{C}) \simeq \mathbb{H}^*(X^{an}, \mathbb{C}_{X^{an}}^\bullet) \simeq \mathbb{H}^*(X^{an}, \Omega_{X^{an}}^\bullet)$. Composing these with Eq. (3) produces the isomorphism $H^*(X^{an}, \mathbb{C}) \simeq H^*_{dR}(X/\mathbb{C})$. □

4. Grothendieck's Theorem

Let X be a smooth scheme over $\mathbb{C}$, then the analytification functor of Section 2 defines a map $\phi \colon X^{an} \to X$, which by Theorem 5 gives maps on the cohomologies $H^p(X, \Omega_X^q) \to H^p(X^{an}, \Omega_{X^{an}}^q)$. Hence we get an induced map on hypercohomology

$$\mathbb{H}^*(X, \Omega_X^\bullet) \to \mathbb{H}^*(X^{an}, \Omega_{X^{an}}^\bullet). \tag{4}$$

In [Gro66], Grothendieck asserted moreover that the morphism of Eq. (4) is an isomorphism, which shows that the algebraic de Rham cohomology and the analytic de Rham cohomology both capture the same information about the scheme X and its analytification X^{an}. This theorem is our goal in this section; we follow the proof in [EC14, Chapter 2].

Theorem 8. (Grothendieck) *Let X be a smooth scheme over $\mathbb{C}$. The natural map $\mathbb{H}^*(X, \Omega_X^\bullet) \to \mathbb{H}^*(X^{an}, \Omega_{X^{an}}^\bullet)$ is an isomorphism.*

Recall that the holomorphic Poincaré lemma says that the augmentation map $\mathbb{C}_{X^{an}} \to \Omega_{X^{an}}^\bullet$ gives a quasi-isomorphism between the trivial complex $\mathbb{C}_{X^{an}}^\bullet \colon 0 \to \mathbb{C}_{X^{an}} \to 0$ and the holomorphic de Rham complex $\Omega_{X^{an}}^\bullet$. Consequently, we get a chain of isomorphisms

$$H^*(X^{an}, \mathbb{C}_{X^{an}}) \simeq \mathbb{H}^*(X^{an}, \mathbb{C}_{X^{an}}^\bullet) \simeq \mathbb{H}^*(X^{an}, \Omega_{X^{an}}^\bullet).$$

Therefore, an immediate corollary of Theorem 8 is that the algebraic de Rham cohomology $H^*_{dR}(X/\mathbb{C})$ coincides with the singular cohomology $H^*(X^{an}, \mathbb{C})$ with $\mathbb{C}$-coefficients. This is the "algebraic" de Rham theorem from Section 1.

4.1. **Reduction to the affine case.** Let us assume that Theorem 8 holds for affine schemes, and use this to prove the general case. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine cover of X, and for any list of indices $i_0, \ldots, i_N \in I$, let $U_{i_0,\ldots,i_N} := U_{i_0} \cap \ldots \cap U_{i_N}$. For a sheaf $\mathcal{F}$ of abelian groups on X and for each $p \geq 0$, let

$$\check{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{\substack{i_0,\ldots,i_p \in I \\ i_0 < \ldots < i_p}} \mathcal{F}(U_{i_0,\ldots,i_p}).$$

There are maps $\delta \colon \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$ so that $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ forms a complex of abelian groups called the Čech complex of $\mathcal{F}$ (these are the usual maps of the Čech complex; see [Har77, Chapter III.4] for the conventions).

For each $q \geq 0$, the above gives the Čech complex $\check{C}(\mathcal{U}, \Omega_X^q)$ associated to Ω_X^q; these can be assembled into a double complex so that the differentials $d \colon \Omega_X^q \to \Omega_X^{q+1}$ induce maps $\check{C}^p(\mathcal{U}, \Omega_X^q) \to \check{C}^p(\mathcal{U}, \Omega_X^{q+1})$ satisfying $\delta d = d\delta$. If $K^{p,q} := \check{C}^p(\mathcal{U}, \Omega_X^p)$, then the double complex $K^{\bullet,\bullet}$ is of the form

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\check{C}^2(\mathcal{U}, \mathcal{O}_X) & \xrightarrow{d} & \check{C}^2(\mathcal{U}, \Omega_X^1) & \xrightarrow{d} & \check{C}^2(\mathcal{U}, \Omega_X^2) & \xrightarrow{d} & \ldots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\check{C}^1(\mathcal{U}, \mathcal{O}_X) & \xrightarrow{d} & \check{C}^1(\mathcal{U}, \Omega_X^1) & \xrightarrow{d} & \check{C}^1(\mathcal{U}, \Omega_X^2) & \xrightarrow{d} & \ldots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\check{C}^0(\mathcal{U}, \mathcal{O}_X) & \xrightarrow{d} & \check{C}^0(\mathcal{U}, \Omega_X^1) & \xrightarrow{d} & \check{C}^0(\mathcal{U}, \Omega_X^2) & \xrightarrow{d} & \ldots
\end{array}$$

where all of the squares commute.

Definition 7. The k-th *Čech cohomology* $\check{H}^k(\mathcal{U}, \Omega_X^\bullet)$ (with respect to the open cover $\mathcal{U}$) of the de Rham complex $\Omega_X^\bullet$ is the k-th cohomology of the totalization $\mathrm{Tot}(K^{\bullet,\bullet})$. Recall that the totalization of $K^{\bullet,\bullet}$ is the complex of abelian groups whose terms are

$$\mathrm{Tot}(K^{\bullet,\bullet})^k := \bigoplus_{p+q=k} \check{C}^p(\mathcal{U}, \Omega_X^q)$$

and the differentials are given by $\delta + (-1)^p d$. We can also define the Čech cohomology for any complex of sheaves of abelian groups in the same manner. See [EC14, Section 2.8] for further details.

Similar to how Čech cohomology of a quasi-coherent sheaf $\mathcal{F}$ computes the sheaf cohomology of $\mathcal{F}$ precisely because the cohomology of $\mathcal{F}$ on any piece of the affine cover is trivial, the Čech cohomology of the algebraic de Rham complex $\Omega_X^\bullet$ will compute the hypercohomology because the sheaf cohomology of Ω_X^q is trivial on any piece of the affine cover, for all $q > 0$.

Theorem 9. *For all $k \geq 0$, there is an isomorphism $\check{H}^k(\mathcal{U}, \Omega_X^\bullet) \simeq \mathbb{H}^k(X, \Omega_X^\bullet)$.*

Proof. See [EC14, Theorem 2.8.1]. □

Now, since $\check{H}^*(\mathcal{U}, \Omega_X^\bullet)$ is realized as the cohomology of the totalization of a double complex, there is a natural E_1-spectral sequence converging to $\check{H}^*(\mathcal{U}, \Omega_X^\bullet)$, whose terms are given by

$$E_1^{p,q} = H^q\left(\check{C}^p(\mathcal{U}, \Omega_X^\bullet)\right) = \prod_{\substack{i_0,\ldots,i_p \in I \\ i_0 < \ldots < i_p}} H^q\left(\Gamma(U_{i_0,\ldots,i_p}, \Omega_X^\bullet)\right).$$

However, since $U_{i_0,\ldots,i_p}$ is affine, there are by assumption isomorphisms

$$H^q\left(\Gamma(U_{i_0,\ldots,i_p}, \Omega_X^\bullet)\right) \simeq H^q\left(\Gamma\left((U_{i_0,\ldots,i_p})^{an}, \Omega_{X^{an}}^\bullet\right)\right). \tag{5}$$

Why is this helpful? Well, the same argument can be repeated in the analytic case! The analytifications $\mathcal{U}^{an} := \{(U_i)^{an}\}_{i\in I}$ form an open cover of the analytic space X^{an} by analytic varieties, and there are isomorphisms $\check{H}^*(\mathcal{U}^{an}, \Omega^\bullet_{X^{an}}) \simeq \mathbb{H}^*(X^{an}, \Omega^\bullet_{X^{an}})$ between the Čech cohomology and the analytic de Rham cohomology. The E_1-spectral sequence in this setting converges to $\check{H}^*(\mathcal{U}^{an}, \Omega^\bullet_{X^{an}})$ and its terms are given by

$$E_{1,an}^{p,q} = H^q\left(\check{C}^p(\mathcal{U}^{an}, \Omega^\bullet_{X^{an}})\right) = \prod_{\substack{i_0,\dots,i_p\in I \\ i_0<\dots<i_p}} H^q\left(\Gamma\left((U_{i_0,\dots,i_p})^{an}, \Omega^\bullet_{X^{an}}\right)\right).$$

By our previous considerations, the two spectral sequences $\{E_1^{p,q}\}$ and $\{E_{1,an}^{p,q}\}$ are isomorphic because of the termwise isomorphisms of Eq. (5) (and the differentials clearly commute). In particular, their E_∞-pages are isomorphic, so for any $k \geq 0$,

$$\mathbb{H}^k(X, \Omega^\bullet_X) \simeq \check{H}^k(\mathcal{U}, \Omega^\bullet_X) \simeq \check{H}^k(\mathcal{U}^{an}, \Omega^\bullet_{X^{an}}) \simeq \mathbb{H}^k(X^{an}, \Omega_{X^{an}})$$

There is a slight problem with this reduction as stated, namely that the intersection of the open affines $U_{i_0,\dots,i_N}$'s are not necessarily themselves affine because we have not made the assumption that X is separated. To get around this, we should first repeat the above argument for separated schemes X, and then the general case follows by repeating the argument once more since the $U_{i_0,\dots,i_N}$'s will certainly be separated, if not affine.

Therefore, we have shown that it suffices to prove Grothendieck's theorem (our Theorem 8) for a smooth affine scheme over $\mathbb{C}$. This is done in the subsequent subsection.

4.2. **The affine case.** In the previous subsection, we reduced the proof of Theorem 8 to the case of an affine scheme. In this setting, the analytic de Rham theorem allows us to reformulate Theorem 8 to the folllowing statement.

Theorem 10. *Let X be a smooth affine scheme over $\mathbb{C}$, then for any $k \geq 0$, there are isomorphisms*

$$\mathbb{H}^k(X, \Omega^\bullet_X) \simeq H^k(X_{an}, \mathbb{C}). \tag{6}$$

It will be helpful to realize X as living inside some smooth *projective* scheme; to do so, we will need to apply (a form of) Hironaka's theorem on resolution of singularities. Hironaka's theorem asserts that there exists a smooth projective scheme Y over $\mathbb{C}$ and an open embedding $i\colon X \hookrightarrow Y$ so that the complement $U := Y - X$, a closed subset of Y, is a normal crossing divisor in Y. See [Hir64] for the proof of this result, and see [Har77, Chapter V.3, Remark 3.8.1] for a discussion of normal crossing divisors.

Analytifying all of the above spaces, there is an embedding $i\colon X^{an} \hookrightarrow Y^{an}$. Originally, we were concerned with holomorphic forms on X^{an}, but it is advantageous to "push forward the problem to Y^{an}" by considering a sheaf of forms on Y^{an} instead. With this in mind, we define $\Omega^k_{Y^{an}}(*U^{an})$ to be the subsheaf of meromorphic k-forms on the complex analytic space Y^{an} which restrict to holomorphic k-forms on X^{an}. Said differently, $\Omega^k_{Y^{an}}(*U^{an})$ is the subsheaf of meromorphic k-forms on Y^{an} that can only have poles on the complement $U^{an} = Y^{an} - X^{an}$ of X^{an}. In order to say something intelligent about the hypercohomology of the complex $\Omega^\bullet_{Y^{an}}(*U^{an})$, we can realize the terms $\Omega^k_{Y^{an}}(*U^{an})$ as subsheaves of ostensibly simpler sheaves in such a way that the inclusion of complexes is a quasi-isomorphism.

Let $\Omega^k_{C^\infty, X^{an}}$ be the sheaf of smooth $\mathbb{C}$-valued k-forms on the analytic space X^{an}, then $\Omega^k_{Y^{an}}(*U^{an})$ is clearly a subsheaf of $i_*\left(\Omega^k_{C^\infty, X^{an}}\right)$, since sections of $\Omega^k_{Y^{an}}(*U^{an})$ are holomorphic (in particular, smooth) k-forms when restricted to X^{an}. The inclusions $\Omega^k_{Y^{an}}(*U^{an}) \hookrightarrow i_*\left(\Omega^k_{C^\infty, X^{an}}\right)$ of sheaves on Y^{an} induce an inclusion of complexes $\Omega^\bullet_{Y^{an}}(*U^{an}) \hookrightarrow i_*\left(\Omega^\bullet_{C^\infty, X^{an}}\right)$ of sheaves on Y^{an}.

Theorem 11. (Atiyah-Hodge) *The inclusion $\Omega^\bullet_{Y^{an}}(*U^{an}) \hookrightarrow i_*\left(\Omega^\bullet_{C^\infty, X^{an}}\right)$ is a quasi-isomorphism.*

Proof. See [HA55, Section 3]. □

Theorem 11 is sometimes called the Fundamental Lemma of Atiyah & Hodge, or the (analytic) Atiyah-Hodge Lemma. The upshot of Theorem 11 is that there is an isomorphism in hypercohomology

$$\mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(*U^{an})) \simeq \mathbb{H}^*(Y^{an}, i_* \left(\Omega^\bullet_{C^\infty, X^{an}}\right)). \tag{7}$$

Our goal is to first construct an isomorphism $\mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(*U^{an})) \simeq H^*(\Gamma(X, \Omega^\bullet_X))$ and then build an isomorphism $\mathbb{H}^*(Y^{an}, i_* \left(\Omega^\bullet_{C^\infty, X^{an}}\right)) \simeq H^*(X^{an}, \mathbb{C})$. These combine with Eq. (7) to prove Theorem 10.

For a non-negative integer n, let $\Omega^k_Y(nU)$ and $\Omega^k_{Y^{an}}(nU^{an})$ be the subsheaves of $\Omega^k_Y(*U)$ and $\Omega^k_{Y^{an}}(*U^{an})$ with poles of order less than or equal to n on U and U^{an}, respectively. It follows that we can write $\Omega^k_Y(*U) = \varinjlim_n \Omega^k_Y(nU)$ and $\Omega^k_{Y^{an}}(*U^{an}) = \varinjlim_n \Omega^k_{Y^{an}}(nU^{an})$. This descriptions of $\Omega^*_Y(*U)$ and $\Omega^*_{Y^{an}}(*U^{an})$ as filtered colimits of coherent sheaves is a crucial ingredient in the proof of the next two propositions.

Proposition 2. *There are isomorphisms* $\mathbb{H}^*(Y, \Omega^\bullet_Y(*U)) \simeq \mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(*U^{an}))$.

Proof. As Y is smooth and projective, Theorem 5 says that $H^p(Y, \Omega^q_Y(nU)) \simeq H^p(Y^{an}, \Omega^q_{Y^{an}}(nU^{an}))$ for any $n \geq 0$. Moreover, cohomology commutes with filtered colimits (at least in this case, since Y^{an} is compact and Y is a noetherian space), there are isomorphisms

$$\begin{aligned} H^p(Y, \Omega^q_Y(*U)) = H^p(Y, \varinjlim_n \Omega^q_Y(nU)) &= \varinjlim_n H^p(Y, \Omega^q_Y(nU)) \\ &\simeq \varinjlim_n H^p(Y^{an}, \Omega^q_{Y^{an}}(nU^{an})) \\ &= H^p(Y^{an}, \varinjlim_n \Omega^q_{Y^{an}}(nU^{an})) = H^p(Y^{an}, \Omega^q_{Y^{an}}(*U^{an})). \end{aligned}$$

However, $H^p(Y, \Omega^q_Y(*U))$ and $H^p(Y^{an}, \Omega^q_{Y^{an}}(*U^{an}))$ are the (p, q)-terms of the E_1-page of the first quadrant spectral sequences associated to (and converging to) the hypercohomologies $\mathbb{H}^{p+q}(Y, \Omega^\bullet_Y(*U))$ and $\mathbb{H}^{p+q}(Y^{an}, \Omega^\bullet_{Y^{an}}(*U^{an}))$, respectively. As before, the termwise isomorphism of the spectral sequences gives an isomorphism on the E_∞-pages, i.e we have the desired isomorphisms

$$\mathbb{H}^*(Y, \Omega^\bullet_Y(*U)) \simeq \mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(*U^{an})).$$

□

Proposition 3. *There are isomorphisms* $\mathbb{H}^*(Y, \Omega^\bullet_Y(*U)) \simeq \mathbb{H}^*(X, \Omega^\bullet_X)$.

Proof. The plan is to build the Čech complexes for $\mathbb{H}^*(Y, \Omega^\bullet_Y(*U))$ and $\mathbb{H}^*(X, \Omega^\bullet_X)$, respectively, as in Definition 7 and show that all of the terms of the two complexes are isomorphic. To ensure that the cohomology of the totalization of the Čech double complex does yield the correct hypercohomology, we need to first check that the sheaf $\Omega^*_Y(*U)$ is acyclic on an affine open subset of Y (we already checked this for Ω^*_X on X). Indeed, let $V \subset Y$ be an affine open, then applying the fact that cohomology commutes with filtered colimits once again we find that for all n, q

$$H^p(V, \Omega^q_Y(*U)) = H^p(V, \varinjlim_n \Omega^q_Y(nU)) \simeq \varinjlim_n H^p(V, \Omega^q_Y(nU)) = 0,$$

where $H^p(V, \Omega^q_Y(nU)) = 0$ because $\Omega^q_Y(nU)$ is coherent. Therefore, we can take any open affine cover $\mathcal{U} = \{U_i\}_{i \in I}$ of Y and the Čech cohomology with respect to $\mathcal{U}$ computes the hypercohomology, i.e.

$$\mathbb{H}^*(Y, \Omega^\bullet_Y(*U)) \simeq \check{H}^*(\mathcal{U}, \Omega^\bullet_Y(*U)).$$

As the pushforward $i_*\Omega^\bullet_X$ coincides with $\Omega^\bullet_Y(*D)$, the terms of the Čech double complex associated to $\Omega^\bullet_Y(*U)$ with respect to $\mathcal{U}$ can be written as

$$\check{C}^p(\mathcal{U}, \Omega^q_Y(*U)) = \check{C}^p(\mathcal{U}, i_*\Omega^q_X) = \prod_{\substack{i_0, \ldots, i_p \in I \\ i_0 < \ldots < i_p}} \Gamma(U_{i_0, \ldots, i_p} \cap X, \Omega^q_X). \tag{8}$$

Furthermore, let $\mathcal{U}|_X := \{U_i \cap X\}_{i\in I}$ be the restriction of the cover $\mathcal{U}$ to X, then the terms of the Čech double complex associated to $\Omega_X^\bullet$ with respect to $\mathcal{U}|_X$ can be written as

$$\check{C}^p(\mathcal{U}|_X, \Omega_X^q) = \prod_{\substack{i_0,\dots,i_p\in I \\ i_0<\dots<i_p}} \Gamma(U_{i_0,\dots,i_p} \cap X, \Omega_X^q). \tag{9}$$

However, the terms of the Čech double complexes Eq. (8) and Eq. (9) are identical, so there is an equality between the Čech cohomologies, i.e. $\check{H}^*(\mathcal{U}, \Omega_Y^\bullet(*U)) = \check{H}^*(\mathcal{U}|_X, \Omega_X^\bullet)$. In particular, there is the desired isomorphism

$$\mathbb{H}^*(X, \Omega_X^\bullet) \simeq \check{H}^*(\mathcal{U}|_X, \Omega_X^\bullet) = \check{H}^*(\mathcal{U}, \Omega_Y^*(*U)) \simeq \mathbb{H}^*(Y, \Omega_Y^\bullet(*U)).$$

□

Theorem 11, Proposition 3, and Proposition 2 combine to give the string of isomorphisms

$$\mathbb{H}^*(X, \Omega_X^\bullet) \simeq \mathbb{H}^*(Y, \Omega_Y^\bullet(*U)) \simeq \mathbb{H}^*(Y^{an}, \Omega_{Y^{an}}^\bullet) \simeq \mathbb{H}^*(Y^{an}, i_*\left(\Omega_{C^\infty, X^{an}}^\bullet\right)).$$

Therefore, to prove Theorem 10, it suffices to show that $\mathbb{H}^*(Y^{an}, i_*\Omega_{C^\infty, X^{an}}^\bullet) \simeq H^*(X^{an}, \mathbb{C})$, where the latter term is the singular cohomology of the analytic space X^{an} with $\mathbb{C}$-coefficients. This is precisely the assertion of our final proposition.

Proposition 4. *There are isomorphisms* $\mathbb{H}^*(Y^{an}, i_*\left(\Omega_{C^\infty, X^{an}}^\bullet\right)) \simeq H^*(X^{an}, \mathbb{C})$.

Proof. Pointwise multiplication gives a natural $\mathcal{O}_{C^\infty, Y^{an}}$-module structure to the sheaf $i_*\left(\Omega_{C^\infty, X^{an}}^\bullet\right)$, which indicates that $i_*\left(\Omega_{C^\infty, X^{an}}^\bullet\right)$ is acyclic (indeed, the $\mathcal{O}_{C^\infty, Y^{an}}$-module structure allows us to use partitions of unity). Consequently, the hypercohomology is nothing more than

$$\mathbb{H}^k(Y^{an}, i_*\Omega_{C^\infty, X^{an}}^\bullet) \simeq H^k(\Gamma(Y^{an}, i_*\left(\Omega_{C^\infty, X^{an}}^\bullet\right))) \simeq H^k(\Gamma(X^{an}, \Omega_{C^\infty, X^{an}}^\bullet)) \simeq H^k(X^{an}, \mathbb{C}),$$

where the last isomorphism is the classical de Rham theorem. □

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