# ARITHMETIC GEOMETRY LEARNING SEMINAR: ADIC SPACES

Description. These notes are from the arithmetic geometry learning seminar on adic spaces at the University of Michigan during the Winter 2017 semester, organized by Bhargav Bhatt. The goal of the seminar is to understand Huber's paper [Hub93], supplemented by [Con14, Wed12]. See [Dat17, Mur17] for other notes from the seminar.

Contents




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Notes were taken by Matt Stevenson, who is responsible for any and all errors.

## 1. January 12th – Motivation (Bhargav Bhatt)

Let $X/k$ be an algebraic variety and let $Z \hookrightarrow X$ be a subvariety. The *formal completion* of $X$ along $Z$, denoted $\widehat{X}_Z$, is the direct limit of the $n$-th infinitesimal neighborhoods of $Z$ in $X$. This is an algebraic version of a tubular neighborhood of $Z$, e.g. the étale fundamental group $\widehat{X}_Z$ coincides with the étale fundamental group of $Z$.

The goal is to study the object "$\widehat{X}_Z - Z$", a punctured tubular neighborhood of $Z$. The desideratum: $\widehat{X}_Z - Z$ should not change if we modify $X$ along $Z$. More precisely, if $g\colon Y \to X$ is proper and $g^{-1}(X\backslash Z) \xrightarrow{\simeq} X\backslash Z$, then $\widehat{Y}_{g^{-1}(Z)} - g^{-1}(Z) \xrightarrow{\simeq} \widehat{X}_Z - Z$. Call this condition $(*)$.

Consider

$$\left(\widehat{X}_Z\right)^{\mathrm{ad}}_{\eta} := \left\{ \mathrm{Spec}(R) \xrightarrow{\pi} X : \begin{array}{r} R \text{ valuation ring}, \\ \pi(\text{generic points}) \in X\backslash Z, \\ \pi(\text{any other point}) \in Z. \end{array} \right\} \Big/ \begin{array}{c}\text{equivalence of} \\ \text{valuations}\end{array}$$

This depends only on $\widehat{X}_Z$, and it satisfies $(*)$ by the valuative criterion for properness.

Huber's theory makes $\left(\widehat{X}_Z\right)^{\mathrm{ad}}_{\eta}$ into a spectral space, it endows $\left(\widehat{X}_Z\right)^{\mathrm{ad}}_{\eta}$ with a structure sheaf $\mathcal{O}^+$, and we have an isomorphism

$$\left(\left(\widehat{X}_Z\right)^{\mathrm{ad}}_{\eta}, \mathcal{O}^+\right) \xrightarrow{\simeq} \varprojlim_{g\colon Y \to X} \widehat{Y}_{g^{-1}(Z)},$$

where the inverse limit is taken in the category of locally ringed spaces. This isomorphism is due to Raynaud originally. Finally, the assignment $(X, Z) \mapsto \left(\left(\widehat{X}_Z\right)^{\mathrm{ad}}_{\eta}, \mathcal{O}^+\right)$ depends only on the formal completion $\widehat{X}_Z$.

The main references for the seminar are Huber's papers [Hub93] and [Hub94], supplemented by Conrad's seminar notes [Con14] and Wedhorn's notes [Wed12].

## 2. January 26th – Huber rings (Rankeya Datta)

This talk is based on Wedhorn's notes [Wed12] and the first quarter of Huber's paper [Hub93]. Rankeya has posted notes [Dat17], which are much more detailed than the lecture and contain much supplementary material.

2.1. **Notation.** Let $A$ be a topological commutative ring. Given $a \in A$, denote by $\ell_a\colon A \to A$ the continuous map which is left multiplication by $a$ (in particular, $\ell_a$ is a homeomorphism iff $a$ is a unit). Given subsets $U, V \subseteq A$, denote by $U \cdot V$ or $UV$ the subgroup of $A$ generated by $\{uv\colon u \in U, v \in V\}$. A local ring is this talk is not necessarily noetherian. Valuations are written multiplicatively. Given a field $K$ with a valuation $|\cdot|$ on it, denote the valuation ring by $R_{|\cdot|}$; it is not assumed to be complete and we allow the trivial valuation, but our valuations our injective (i.e. we do not allow semivaluations).

### 2.2. Adic rings.

**Definition 2.1.** A topological ring $A$ is *adic* if there is an ideal $I \subseteq A$ such that $\{I^n\colon n > 0\}$ is a neighborhood basis of 0. The ideal $I$ is called an *ideal of definition* of $A$.

**Example 2.2.** (1) Any commutative ring $R$ (with the discrete topology) is adic with the zero ideal being the ideal of definition.

(2) Given any commutative ring $R$ and an ideal $I$, topologize $R$ by declaring a neighborhood basis of 0 to be powers of $I$, then $R$ is adic with ideal of definition of $I$.

(3) The prototype for today's talk: $K$ is a perfectoid field and $K^\circ$ is its valuation ring, then $K^\circ$ is adic with ideal of definition $aK^{\circ\circ}$ for any nonzero $a \in K^{\circ\circ}$ (i.e. $a$ is a pseudouniformizer).

**Exercise 2.3.** If $A$ is adic and $I, J$ are ideal of definition, then $\sqrt{I} = \sqrt{J}$. The converse holds if $I, J$ are finitely-generated ideals, but it fails in general.

2.3. **Huber rings.** The following is not that appears in [Hub93], but rather it is the definition from [Wed12].

**Definition 2.4.** A topological ring $A$ is called a *Huber ring* (or an *f-adic ring*[1]) if there exists an open subring $A_0 \subseteq A$ such that the induced topology on $A_0$ is adic for some finitely generated ideal $I$ of $A_0$. The ring $A_0$ is called a *ring of definition* (rod), and $I$ is called an *ideal of definition* (iod).

This reconciles Wedhorn's definition with Huber's definition.

**Proposition 2.5.** *Let $A$ be a topological ring, then the following are equivalent:*

*(1) $A$ is Huber;*
*(2) there exists a subset $U \subseteq A$ and a finite subset $T \subseteq U$ such that $\{U^n : n > 0\}$ is a neighborhood basis of* $0$ *in $A$, and $T \cdot U = U^2 \subseteq U$.*

*Proof.* From our definition, it is clear what $U$ and $T$ must be: take $U = I$ and $T$ to be a generating set of $I$. To go back, take $A_0$ to be the smallest subring of $A$ containing $U$. A complete proof can be found in [Wed12, Proposition 6.1]. □

**Example 2.6.** (1) Any adic ring with a finitely generated ideal of definition is Huber (so this includes any commutative ring, equipped with the discrete topology).
(2) If $K$ is perfectoid, then $K$ is Huber with ring of definition $K^\circ$ and ideal of definition generated by any pseudouniformizer.
(3) If $K$ is perfectoid and $A$ is any non-Archimedean normed $K$-algebra, then $A$ is Huber with ring of definition $\{x \in A\colon |x| \leq 1\}$ and ideal of definition $pA_0$, for any $p \in K^{\circ\circ}$. (This works more generally for any non-Archimedean valued field $K$, which is not necessarily complete). This holds in particular for the Tate algebra

$$K\langle x_1, \ldots, x_n\rangle = \left\{ f \in K[[x_1, \ldots, x_n]]\colon f = \sum_\nu a_\nu \underline{x}^\nu \text{ and } |a_\nu| \to 0 \text{ as } |\nu| \to +\infty \right\}.$$

One can ask: which valued fields are Huber? If $(K, |\cdot|)$ is a valued field with value group $\Gamma$, then we have the *valuation topology* whose neighborhood basis of 0 is given by sets of the form

$$K_{\leq\gamma} = \{x \in K\colon |x| < \gamma\}$$

for $\gamma \in \Gamma$. As any two elements of $\Gamma$ are comparable, given any two sets of this form, one must be contained in the other. The valuation ring $R_{|\cdot|}$ is a natural open subring, so we now ask: when is $K$ Huber with ring of definition $R_{|\cdot|}$?

Observe that the topology on $K$ is Hausdorff because $\bigcap_{\gamma\in\Gamma} K_{<\gamma} = (0)$. In particular, the induced topology on $R_{|\cdot|}$ is Hausdorff. Then, $R_{|\cdot|}$ is a ring of definition iff there exists $a \in R_{|\cdot|}\setminus\{0\}$ such that $\bigcap_{n>0} a^n R_{|\cdot|} = (0)$.

**Proposition 2.7.** *Assume $R_{|\cdot|} \neq K$ (so $K$ does not have the trivial valuation). The following are equivalent:*

*(1) the valuation topology on $K$ is Huber with ring of definition $R_{|\cdot|}$;*
*(2) there exists $a \in R_{|\cdot|}\setminus\{0\}$ such that $\bigcap_{n>0} a^n R_{|\cdot|} = (0)$;*
*(3) $R_{|\cdot|}$ has a height-1 prime;*
*(4) $\bigcap_{\substack{\mathfrak{p}\colon prime \\ \mathfrak{p}\neq(0)}} \mathfrak{p} \neq (0)$.*

[1] The 'f' in f-adic stands for finite.

We already discussed the equivalence of (1) and (2), and it is not hard to see the equivalence of (3) and (4) using the fact that the ideals of a valuation ring are totally ordered, but the equivalence of (2) and (3) is tricky. Once an use the following nice fact (which is somewhat tricky to see).

**Lemma 2.8.** *If $R$ is a valuation domain and $I$ is a proper ideal of $R$, then $\bigcap_{n>0} I^n$ is a prime ideal.*

It appears we have only classified the valued field which are Huber with ring of definition equal to its valuation ring, but we will see later that any Huber valued field has its valuation ring as a ring of definition.

### 2.4. Tate rings.

**Definition 2.9.** An element $a$ of a topological ring $A$ is *topologically nilpotent* if $a^n \to 0$ as $n \to +\infty$. A Huber ring $A$ is *Tate* if it contains a topologically nilpotent unit.

For example, any nontrivially-valued field is a Tate ring, and any non-Archimedean normed algebra over a non-trivially valued field is a Tate ring. In fact, every ring that will come up in Bhargav's class is a Tate ring.

**Exercise 2.10.** If $A$ is adic with ideal of definition $I$, then $a \in A$ is topologically nilpotent iff $a \in \sqrt{I}$.

**Lemma 2.11.** *If $A$ is a topological ring with topologically nilpotent element $a$, then for any $b \in A$, $ba^n \to 0$ as $n \to +\infty$.*

The lemma is easy to see if you use the continuity of the left multiplication by $b$.

**Proposition 2.12.** *Let $A$ be a Tate ring with ring of definition $B$, and topologically nilpotent unit $g$. Then,*

*(1) $g^n \in B$ for all $n \gg 0$;*
*(2) if $g^n \in B$, then $B$ is $g^nB$-adic (i.e. $B$ is an adic ring with ideal of definition $g^nB$);*
*(3) if $g^n \in B$, then $A = B[1/g^n]$.*

*Proof.* For (1), use that $B$ is open. For (2), since $B$ is a ring of definition, its topology is $I$-adic for some (finitely generated) ideal $I$ of $B$; in particular, $I$ is open in $B$, hence open in $A$. So, for all $m \gg 0$, $g^{nm} \in I$, and hence $(g^nB)^m \subseteq I$. Since $g$ is a unit of $A$, $\ell_g$ is a homeomorphism. This implies that $g^nB$ is also open (because it is the image of the open set $B$), so some power of $I$ is contained in $g^nB$. For (3), $g^n$ is a unit of $A$, so by the universal property of localization, we have obvious injection $i \colon B[1/g^n] \hookrightarrow A$. To see that $i$ is surjective, it suffices to show that for all $a \in A$, $g^n a \in B$ for $n \gg 0$. This follows from Lemma 2.11. $\square$

### 2.5. Boundedness.

Fix a topological ring $A$.

**Definition 2.13.**
(1) A subset $S \subseteq A$ is *bounded* if for all open neighborhoods $U$ of 0, there exists an open neighborhood $V$ of 0 such that $\{vs \colon v \in V, s \in S\} \subseteq U$. Put differently, $S$ is bounded if $\bigcap_{s\in S} \ell_s^{-1}(U)$ contains an open neighborhood of 0.
(2) An element $a \in A$ is *power-bounded* if $\{a^n \colon n > 0\}$ is a bounded set. The set of power-bounded elements is denoted by $A^\circ$, and the set of topologically nilpotent elements is denoted by $A^{\circ\circ}$.

For example, if $K$ is a rank-1 valued field, then $K^\circ$ is the valuation ring and $K^{\circ\circ}$ is the maximal ideal.

**Proposition 2.14.** [Properties of boundedness] *Let $A$ be a topological ring.*

*(1) Any finite subset of $A$ is bounded.*
*(2) Any subset of a bounded set is bounded.*
*(3) Finite unions of bounded sets are bounded.*
*(4) If $A$ is adic, then $A$ is bounded.*
*(5) If $A$ is Huber, then any ring of definition of $A$ is bounded.*
*(6) Any topologically nilpotent element of $A$ is power bounded.*
*(7) If $A$ is a Tate ring with ring of definition $A_0$ and topologically nilpotent unit $g \in A_0$, then $S$ is bounded iff $S \subseteq g^n A_0$ for some $n \in \mathbf{Z}$.*

(8) *If $A$ is Huber, $S \subseteq A$ is bounded iff the subgroup generated by $S$ is bounded.*
(9) *If $A_1, A_2$ are bounded subrings of a Huber ring $A$, so is $A_1 \cdot A_2$.*
(10) *The collection of bounded open subrings of a Huber ring $A$ is a directed set under inclusions.*
(11) *If $B$ is a bounded subring of $A$ and $x \in A^\circ$, then $B[x]$ is bounded.*
(12) *If $A$ is Huber, $a \in A^\circ$, and $t \in A^{\circ\circ}$, then $at \in A^{\circ\circ}$.*
(13) *If $A$ is Huber and $a_1, a_2 \in A^{\circ\circ}$, then $a_1 + a_2 \in A^{\circ\circ}$.*

*Proof.* For (4), take $U = A$ and $V = I$. For (5), use that the ring of definition of a Huber ring is adic and (4). For (6), if $a \in A$ is topologically nilpotent and $I$ is an ideal of definition (or if we're in a topological ring, an open subset), then there is $N_0$ such that for all $n \geq N_0$, $a^n \in I$. Since $\{a, a^2, \ldots, a^{N_0-1}\}$ is bounded (it is a finite set), so is the union with $\{a^n : n \geq N_0\}$. For (9), there exists some open subgroup $V$ such that $V \cdot A_2 \subseteq I$, where $I$ is an ideal of definition, and there exists some open neighborhood $V'$ of 0 such that $V' \cdot A_1 \subseteq V$. Then, $V' \cdot (A_1 \cdot A_2) = (V' \cdot A_1) \cdot A_2 \subseteq V \cdot A_2 \subseteq I$. For (11), for any $n > 0$, there exists $n_0 > 0$ such that $I^{n_0}\{a_1^m : m \gg 0\}$, $I^{n_0}\{a_2^m : m \gg 0\} \subseteq I^n$. Pick $n_m > n_0$ such that for all $i \geq n_0$, $a_1^i, a_2^i \in I^{n_0}$. Then, for all $i \geq 2n_m$, $(a_1 + a_2)^i \in I^n$ by binomial expansion. □

**Proposition 2.15.** *Let $A$ be a Huber ring.*

(1) *$A^\circ$ is the filtered direct limit (i.e. the union) of bounded open subrings of $A$; in particular, $A^\circ$ a ring.*
(2) *$A^{\circ\circ}$ is a radical ideal of $A^\circ$.*
(3) *$A^\circ$ is integrally closed in $A$.*

*Proof.* For (3), take $x \in A$ integral over $A^\circ$. There exists a bounded open subring $B \subseteq A^\circ$ such that $x$ is integral over $B$. Then, $B[x] = B + Bx + \ldots + Bx^{n-1}$ if $x$ satisfies a monic polynomial equation of degree $n$ with coefficients in $B$. Since each $Bx^i$ is bounded, $B[x]$ is bounded, and hence $x \in A^\circ$. □

**Example 2.16.** If $K$ is a valuation field, its valuation ring is bounded (irrespective of the rank).

From now on, $A$ will always denote a Huber ring.

**Proposition 2.17.** *Let $A_0$ be a subring of $A$. The following are equivalent:*

(1) *$A_0$ is a ring of definition;*
(2) *$A_0$ is open and bounded.*

*Thus, $A^\circ$ is the union of all of its rings of definition.*

*Proof.* Let $B$ be a ring of definition of $A$, with $I$ an ideal of definition (these exist since $A$ is Huber). Since $A_0$ is open, there exists $m > 0$ such that $I^m \subseteq A_0$ (warning: this does not imply that $I^m$ is an ideal of $A_0$!). Note that $\{I^{m+i} : i \geq 0\}$ remains a neighborhood basis of 0 in $A_0$. Since $A_0$ is bounded, there exists $n > m$ such that $I^n A_0 \subseteq I^m$. Let $J$ be the ideal of $A_0$ generated by a generating set $\{x_1, \ldots, x_r\}$ of $I^n$ (as an ideal of $B$). Thus, $J \subseteq I^m$ and we claim that $I^{m+n} \subseteq J$. This will prove that $J$ is an ideal of definition. It suffices to show that $ik$, with $i \in I^m$ and $k \in I^n$, are in $J$. Write $k = b_1x_1 + \ldots + b_rx_r$ for $b_i \in B$, then $ik = (ib_1)x_1 + \ldots + (ib_r)x_r$ and $ib_j \in I^m \subseteq A_0$, so $ik \in J$. □

Let's isolate a key part of the above proof.

**Lemma 2.18.** *If $A$ is Huber, $A_0$ is an open bounded subring of $A$, $B$ is a ring of definition with ideal of definition $I$, then there exists $n > 0$ such that $I^n \subseteq A_0$ and if $\{x_1, \ldots, x_r\}$ is a generating set of $I^nB$, then $A_0x_1 + \ldots + A_0x_r$ is an ideal of definition of $A_0$.*

**Corollary 2.19.** *If $A_1 \subseteq A_2$ are rings of definition of $A$ and $I$ is a finitely generated ideal of $A_1$, then $I$ is an ideal of definition of $A_1$ iff $IA_2$ is an ideal of definition of $A_2$.*

The hard direction of the corollary, which requires the lemma, is the only if direction (i.e. assuming $IA_2$ is an ideal of definition of $A_2$).

**Proposition 2.20.** [Consequences of Proposition 2.17]

*(1) If $A_1, A_2$ are rings of definition, then $A_1 \cdot A_2$ is a ring of definition.*
*(2) If $A$ is a topological ring, then $A$ is Huber iff any open subring of $A$ is Huber.*
*(3) Any bounded subring of $A$ is contained in a ring of definition.*
*(4) If $A$ is Huber, then $A$ is bounded iff $A$ is adic.*

*Proof.* For the forward direction of (2), suppose $B \subseteq A$ is an open subring. If $A_0$ is a ring of definition of $A$, $A_0 \cap B$ is open and bounded, and it is contained in $B$, hence it is a ring of definition of $B$.

For (3), suppose $B \subseteq A$ is a bounded subring. Take a ring of definition $A_0$, then $B \cdot A_0$ is open, bounded, and it contains $B$. $\square$

**Example 2.21.** If $A = \mathbf{Q}_p[t]/(t^2)$, then $A^\circ = \mathbf{Z}_p \oplus \mathbf{Q}_p t$ and $A^\circ$ is **not** bounded, even though $A$ is Huber. Note that the ring of definition of $A$ is $\mathbf{Z}_p \oplus \mathbf{Z}_p t$ with the $p$-adic topology.

The discussion of Huber rings will continue next week.

## 3. February 2nd – Huber rings continued (Rankeya Datta) & Valuation spectra (Emanuel Reinecke)

We recall briefly some facts established last time, which will be needed today.

**Lemma 3.1.** *Let $A$ be a Huber ring, and let $A_0$ be an open bounded subring. If $B$ is a ring of definition of $A$ with ideal of definition $I$, then there exists $n > 0$ such that $I^n \subseteq A_0$ and such that if $\{x_1, \dots, x_r\}$ generates $I$ as an ideal of $B$, then the ideal of $A_0$ generated by $\{x_1, \dots, x_r\}$ is an ideal of definition of $A_0$.*

**Corollary 3.2.** *Let $A$ be a Huber ring with subring $A_0$, then $A_0$ is a ring of definition iff $A_0$ is open and bounded.*

**Corollary 3.3.** *Let $A$ be a Huber ring. Let $A_0 \subseteq A_1$ be rings of definition. If $I$ is a finitely generated ideal of $A_0$, then $I$ is an ideal of definition of $A_0$ iff $IA_1$ is an ideal of definition of $A_1$.*

**Proposition 3.4.** *Let $A$ be Tate with ring of definition $B$ and topologically nilpotent unit $g \in A$. If $g^n \in B$, then $B$ is $g^n B$-adic.*

### 3.1. Norm on Hausdorff Tate rings.

Let $A$ be a Tate ring with ring of definition $A_0$ and a topologically nilpotent unit $g \in A_0$. Suppose $A$ is Hausdorff (or equivalently, $\bigcap_{n \geq 1} g^n A_0 = (0)$), then we can define a norm $|\cdot| \colon A \to \mathbf{R}_{\geq 0}$ as follows: for $a \in A$, set

$$|a| \coloneqq \inf\{2^{-n} \colon a \in g^n A_0, n \in \mathbf{Z}\}.$$

Here, the use of 2 is an arbitrary choice; we could have used any number bigger than 1. Note that $|\cdot|$ is a norm precisely because $|a| = 0$ iff $a \in g^n A_0$ for all $n \gg 0$ iff $a \in \bigcap_{n \geq 1} g^n A_0 = (0)$, i.e. $A$ is Hausdorff.

The norm is submultiplicative in general. In particular, if $A$ is complete in the $g^n A_0$-adic topology, then $A$ is a Banach ring. Note that $|a| \leq 1$ iff there exists $n \geq 0$ such that $a \in g^n A_0$, which is equivalent to saying that $a \in A_0$ (because we chose $g \in A_0$). Thus, $A_{\leq 1} = A_0$.

The metric topology from the norm will coincide with the topology on the Tate ring $A$, but the norm itself does heavily depend on the choice of the ring of definition $A_0$.

### 3.2. Valued fields.

Recall that a valued field is a field equipped with a non-trivial valuation. Last time, we answered the following question: which valued fields $(K, |\cdot|)$ are Huber with ring of definition $R_{|\cdot|}$? We characterized these valued fields as those whose valuation ring $R_{|\cdot|}$ has a unique height-1 prime (this need not happen if e.g. $R_{|\cdot|}$ does not have finite Krull dimension).

What if we weaken the requirement slightly and ask the following: which valued fields $(K, |\cdot|)$ are Huber? The second question is in fact not more general, because the valuation ring is open in the valuation topology and the valuation ring is also bounded in the valuation topology, hence it is a ring of definition by Corollary 3.2.

**Exercise 3.5.** Let $K$ be a valued field which is Huber. Show the following:

(1) if $\mathfrak{q}$ is the unique height-1 prime of $R_{|\cdot|}$ and if we identify the localization $\left(R_{|\cdot|}\right)_{\mathfrak{q}}$ with a subring of $K$, then $\mathfrak{q}\left(R_{|\cdot|}\right)_{\mathfrak{q}} = \mathfrak{q}$ as sets.
(2) $\left(R_{|\cdot|}\right)_{\mathfrak{q}}$ is bounded in $K$ and $K^\circ = \left(R_{|\cdot|}\right)_{\mathfrak{q}}$.[2]
(3) $K^{\circ\circ} = \mathfrak{q}$.

3.3. **Adic homomorphisms.** The adic homomorphisms will be the natural structure preserving ring maps between Huber rings. Huber calls them f-adic homomorphism, but we follow Wedhorn's convention.

**Definition 3.6.** Let $A, B$ be Huber rings. An *adic homomorphism* $f\colon A \to B$ is a continuous ring map such that there are rings of definition $A_0$ of $A$, $B_0$ of $B$, and an ideal of definition $I$ of $A_0$ such that $f(A_0) \subseteq B_0$ and $f(I)B_0$ is an ideal of definition of $B_0$.

Adic homomorphisms are very special in the class of continuous ring homomorphisms, unless you're dealing with Tate rings, as we will see later.

**Example 3.7.** (1) Consider $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ noetherian local rings equipped with the $\mathfrak{m}_R$-adic and $\mathfrak{m}_S$-adic topologies. If $\phi\colon R \to S$ is any integral ring map, then $\phi$ is adic.
(2) For any Huber ring $A$ of characteristic $p > 0$, the Frobenius endomorphism $\mathrm{Frob}_A\colon A \to A$ is adic.
(3) [Non-example] Let $R$ be a ring with the discrete topology and let $S$ be any Huber ring with a non-discrete topology, then any ring homomorphism $R \to S$ is **not** adic, even though it must be continuous. (This is because any ideal of definition must be nilpotent, the expansion of a nilpotent ideal its still nilpotent, and hence would define the discrete topology.)

**Proposition 3.8.** [Properties of adic homomorphisms] *Let $f\colon A \to B$ be an adic homomorphism of Huber rings.*
*(1) $f$ is bounded, i.e. $f$ maps bounded sets to bounded sets. In particular, $f(A^\circ) \subseteq B^\circ$.*
*(2) $f(A^{\circ\circ}) \subseteq B^{\circ\circ}$ (in fact, one only needs $f$ to be continuous for this to hold).*
*(3) If $A_0, B_0$ are rings of definition of $A, B$ respectively such that $f(A_0) \subseteq B_0$, then for any ideal of definition $I$ of $A_0$, the expansion $f(I)B_0$ is an ideal of definition of $B_0$.*
*(4) If $A_0$ is a ring of definition of $A$, then there exists a ring of definition $B_0$ of $B$ such that $f(A_0) \subseteq B_0$.*
*(5) If $f$ is only a continuous ring homomorphism, then for any ring of definition $B_0$ of $B$, there exists a ring of definition $A_0$ of $A$ such that $f(A_0) \subseteq B_0$.*
*(6) If $I$ is an open ideal of $A$, then its expansion $f(I)B_0$ is open.*
*(7) Let $A', B'$ be open subrings of $A, B$ respectively such that $f(A') \subseteq B'$, then $f\colon A \to B$ is adic iff the restriction $f|_{A'}\colon A' \to B'$ is adic.*

*Proof.* See [Dat17, Proposition 4.3.3]. □

**Example 3.9.** The following is a non-trivial example of a continuous ring map between Huber rings that is not adic. Let $K$ be a perfectoid field. Consider the inclusion $i\colon K^\circ \hookrightarrow K^\circ[[x]]$, where $K^\circ[[x]]$ is given the $(t, x)$-adic topology, for any nonzero $t \in K^{\circ\circ}$. Then, $i$ is continuous but it is not adic. One can see this by using Proposition 3.8(3): the ideal generated by $(t)$ in $K^\circ$ is an ideal of definition, but its expansion in $K^\circ[[x]]$ does not have radical equal to that of $(t, x)$, and hence they cannot generate the same topology.

More generally, if $R$ has prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ with $\mathrm{ht}(\mathfrak{p}) < \mathrm{ht}(\mathfrak{q})$, consider the identity map $R \to R$, where the domain is equipped with the $\mathfrak{p}$-adic topology and the codomain is equipped with the $\mathfrak{q}$-adic topology. Then, this continuous ring map is not adic.

**Corollary 3.10.** *Let $A, B, C$ be Huber rings.*
*(1) If $f\colon A \to B$ and $g\colon B \to C$ are adic, then $g \circ f\colon A \to C$ is adic.*
*(2) If $f\colon A \to B$ and $g\colon B \to C$ are continuous ring homomorphisms such that $g \circ f$ is adic, then $g$ is adic.*

[2] Hint: use that there are no proper subrings in between $\left(R_{|\cdot|}\right)_{\mathfrak{q}}$ and $K$.

**Proposition 3.11.** *Let $f\colon A \to B$ be a continuous ring homomorphism between Huber rings. If $A$ is Tate, then $B$ is Tate and $f$ is adic.*

*Proof.* The ring $B$ is Tate because any continuous ring homomorphism maps a topologically nilpotent unit of $A$ to a topologically nilpotent unit of $B$. To see that $f$ is adic, choose a ring of definition $B_0$ of $B$ and a ring of definition $A_0 \subseteq f^{-1}(B_0)$; such objects exist by Proposition 3.8(5). Let $g \in A_0$ be a topologically nilpotent unit. By Corollary 3.3, $A_0$ is $gA_0$-adic and hence $B_0$ is $f(g)B_0$-adic. $\square$

3.4. **Completion of Huber rings.** Fix the following setup: let $A$ be a Huber ring with ring of definition $A_0$, and ideal of definition $I \subseteq A_0$. As the powers $I^n$ are subgroups of $A$, we can form the inverse limit $\widehat{A} \coloneqq \varprojlim_n A/I^n$ as an abelian group. It carries a natural ring structure, which we will describe later (the naive guess of viewing $\widehat{A}$ as a subset of the infinite product $\prod_n A/I^n$ and using coordinate-wise multiplication does not work, because $I$ is not an ideal of $A$, only of $A_0$).

Once the ring structure on $\widehat{A}$ has been constructed, it will have the following properties.

**Proposition 3.12.**
1. *$\widehat{A}$ is Hausdorff in the natural topology[3].*
2. *The natural map $i\colon A \to \widehat{A}$ is a continuous ring map, and $i(A)$ is dense in $\widehat{A}$.*
3. *As $I$ is an ideal of $A_0$, we can form the inverse limit $\widehat{A_0} \coloneqq \varprojlim_n A/I^n$ as a ring. This is a ring now coordinate-wise multiplication. Then, the inclusion $j\colon \widehat{A_0} \hookrightarrow \widehat{A}$ is ring homomorphism.*
4. *An element $x = (x_n + I^n)_{n\geq 1} \in \widehat{A}$ belongs to $\widehat{A_0}$ iff $x_1 \in A_0$.*
5. *The subring $\widehat{A_0}$ is open in $\widehat{A}$.*
6. *For any $x \in \widehat{A}$, there exists $y \in \widehat{A_0}$ and $a \in A$ such that $x = y + i(a)$.*

*Proof.* For (4), use that $x_n - x_1 \in I \subseteq A_0$ for all $n \geq 1$. For (5), the subset $U = (A_0/I \times A/I^2 \times A/I^3 \times \ldots) \cap \widehat{A} \subset \widehat{A_0}$ is open (it is the preimage of the open $A_0/I$ in $A$ under the first projection $\widehat{A} \to A/I$) and if a subring contains an open set, then it must be open. For (6), if $x = (x_1 + I, x_2 + I^2, \ldots)$, take $y = (0, (x_2 - x_1) + I^2, (x_3 - x_1) + I^3, \ldots)$ and $a = x_1$. $\square$

**Remark 3.13.** The ring structure on $\widehat{A}$ can be described as the set of Cauchy sequences in $A$ modulo the null sequences, but we will give a more explicit construction below.

The multiplication on $\widehat{A}$ can be described as follows: first we define $i(a) \cdot x$, for $x \in \widehat{A}$ and $a \in A$. Since $\ell_a\colon A \to A$ is continuous, there exists $c > 0$ such that $I^c \subseteq \ell_a^{-1}(A_0)$; thus, for all $n \geq 1$, $I^{n+c}a \subseteq I^n$. If $x = (x_1 + I, x_2 + I^2, \ldots)$, then $i(a) \cdot x \coloneqq (ax_{c+1} + I, ax_{c+2} + I^2, ax_{c+3} + I^3, \ldots)$. This gives an element of the inverse limit because

$$ax_{c+n+1} - ax_{c+n} = a(x_{c+n+1} - x_{c+n}) \in aI^{n+c} \subseteq I^n.$$

Now, in general: if $x_1, x_2 \in \widehat{A}$, choose $y_1, y_2 \in \widehat{A_0}$ and $a_1, a_2 \in A$ such that $x_1 = y_1 + i(a_1)$ and $x_2 = y_2 + i(a_2)$, then we may define the product $x_1 x_2$ as

$$x_1 x_2 = y_1 y_2 + i(a_1) y_2 + i(a_2) y_1 + i(a_1 a_2).$$

Therefore, $\widehat{A}$ is a ring with open subring $\widehat{A_0}$.

**Proposition 3.14.**
1. *The ring $\widehat{A}$ is Huber with ring of definition $\widehat{A_0}$, and ideal of definition $I\widehat{A_0}$. Moreover, for all $n \geq 1$, there is a natural projection $\widehat{A_0} \twoheadrightarrow A_0/I^n$, with kernel $I^n\widehat{A_0}$.*
2. *There is an isomorphism $\widehat{A} \simeq \widehat{A_0} \otimes_{A_0} A$, i.e. the diagram*

---

[3] The natural topology is the inverse limit topology: the infinite product $\prod_n A/I^n$ has the product topology (and hence is Hausdorff, because each factor $A/I^n$ is discrete, hence Hausdorff) and $\widehat{A}$ is naturally a subset of $\prod_n A/I^n$, so it is equipped with the subspace topology.

$$\begin{tikzcd} A_0 \arrow[r, hook] \arrow[d, hook] & A \arrow[d] \\ \widehat{A_0} \arrow[r, "j"] & \widehat{A} \end{tikzcd}$$

*is a pushout square.*

*Proof.* See [Sta17, Tag 05GG]. □

**Corollary 3.15.** *Suppose $A_0$ is noetherian.*

*(1) The natural map $A \to \widehat{A}$ is flat.*
*(2) If $A$ is a finitely generated $A_0$-algebra, then $\widehat{A}$ is a finitely generated $\widehat{A_0}$-algebra.*

The flatness conclusion in Corollary 3.15 will not be true for the rings appearing in the perfectoid theory, because there we are very far from the noetherian world.

3.5. **Valuation spectrum.** This talk is based on Conrad's notes [Con14], and the second section of Huber's paper [Hub93]. A brief description of where we are heading: the goal is to associate, to any Huber ring $A$, a spectral space $\mathrm{Cont}(A)$, whose points are the continuous valuations on $A$. We will then show that, if $A$ is a Tate algebra over a complete non-Archimedean field, then

$$\mathrm{Spa}(A, A^\circ) \coloneqq \{v \in \mathrm{Cont}(A) \colon v(A^\circ) \leq 1\}$$

naturally contains the rigid-analytic space $\mathrm{Sp}(A)$, and there is a canonical equivalence of categories

$$\mathrm{Shv}(\mathrm{Spa}(A, A^\circ)) \xrightarrow{\simeq} \mathrm{Shv}(\mathrm{Sp}(A)).$$

Today, we describe this construction when $A$ is discrete.

To motivate the definitions to come, let's think about how to define the basic spaces of algebraic geometry: if $A$ is a ring, then

$$\mathrm{Spec}(A) = \left\{\text{ring maps } A \xrightarrow{x} K \colon K \text{ field}\right\} / \sim,$$

where two maps $x \colon A \to K$ and $x' \colon A \to K'$ are equivalent, written $x \sim x'$, if there is an injection of fields $K \hookrightarrow K'$ and a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "x"'] \arrow[dr, "x'"] & \\ K \arrow[rr, hook] & & K' \end{tikzcd}$$

To define the valuation spectrum, we use a similar definition, but consider instead certain maps (namely, valutions) from a ring to a totally ordered abelian group. One such notion is that of a Riemann–Zariski space.

**Definition 3.16.** Let $A$ be an integral domain inside of a field $K$. The *Riemann–Zariski space* is

$$\mathrm{RZ}(K, A) = \left\{\text{valuations } K \xrightarrow{v} \Gamma \cup \{0\} \colon \begin{array}{c} \Gamma \text{ is a totally ordered abelian group,} \\ v(x) \leq 1 \text{ for all } x \in A. \end{array}\right\} / \sim$$

where two valuations $v \colon K \to \Gamma \cup \{0\}$ and $v' \colon K \to \Gamma' \cup \{0\}$ are equivalent, written $v \sim v'$, if there is a, map $\Gamma \longrightarrow \Gamma'$ of ordered abelian groups such that the following diagram commutes:

$$\begin{tikzcd} & K \arrow[dl, "v"'] \arrow[dr, "v'"] & \\ \Gamma \cup \{0\} \arrow[rr, hook] & & \Gamma' \cup \{0\} \end{tikzcd}$$

Equivalently, $\mathrm{RZ}(K, A)$ consists of all valuations on $K$ that are centered on $A$.

If $x \in K$, set

$$U(x) := \mathrm{RZ}(K, A[x]) = \{v \in \mathrm{RZ}(K, A) \colon v(x) \leq 1\}.$$

We define the topology on $\mathrm{RZ}(K, A)$ by declaring that sets of the form $U(x)$ form a subbasis for the topology. If $K = \mathrm{Frac}A$, then these sets can be alternatively described as follows: write $x = \frac{a}{b}$ for some $a, b \in A$ with $b \neq 0$. then,

$$U(x) = U(a, b) := \{v \in RZ(K, A) \colon v(a) \leq v(b)\}.$$

Set $\mathrm{RZ}(K) := \mathrm{RZ}(K, \mathbf{Z})$, i.e. the set of all valuations on $K$.

**Example 3.17.**
(1) $\mathrm{RZ}(\mathbf{F}_q)$ consists of just the trivial valuation $v_0$, because $\mathbf{F}_q$ is torsion and every totally ordered abelian group is torsion-free, so the group homomorphism between them is trivial.
(2) $\mathrm{RZ}(\mathbf{Q}) = \{v_0, v_2, v_3, \ldots, v_p, \ldots\}$, where $v_0$ denotes the trivial valuation and $v_p$ denotes the $p$-adic valuations. In fact, $\mathrm{RZ}(\mathbf{Q})$ is homeomorphic to $\mathrm{Spec}(\mathbf{Z})$.
(3) If $C$ is a smooth projective curve over an algebraically closed field $k$ and $K = k(C)$ denotes the function field of $C$, then $\mathrm{RZ}(K, k) \simeq C$ as topological spaces; see [Har77, Exercise II.4.12(a)]. Briefly, to a point $p \in C$, the local ring $\mathcal{O}_{C,p}$ is a valuation ring of $K$, and hence the associated valuation (or equivalence class thereof) is an element of $\mathrm{RZ}(K, k)$.

These Riemann–Zariski spaces have been used to some success, e.g. by Zariski to prove resolution of singularities in dimensions 2 and 3 in characteristic 0.

3.6. **Valuations.** Throughout, let $A$ be a ring.

**Definition 3.18.** A *valuation* on $A$ is a map $v \colon A \to \Gamma \cup \{0\}$, where $\Gamma$ is a totally ordered abelian group (written multiplicatively, so that 1 denotes the unit of $\Gamma$) such that
(1) for all $\gamma \in \Gamma$, $\gamma \cdot 0 = 0 = 0 \cdot \gamma$ and $0 < \gamma$;
(2) $v(0) = 0$, and $v(1) > 1$;
(3) for all $x, y \in A$, $v(xy) = v(x)v(y)$, i.e. $v$ is a group homomorphism;
(4) for all $x, y \in A$ ,$v(x + y) \leq \max\{v(x), v(y)\}$, i.e. $v$ is non-Archimedean.

The *value group* $\Gamma_v$ of $v$ is the subgroup of $\Gamma$ generated by $\mathrm{im}(v)\backslash\{0\}$.

A quick warning: $v(A) \not\subseteq \Gamma_{\leq 1} \cup \{0\}$ in general (e.g. if $A$ is a field and $v \colon A \to \mathbf{R}$ is a nontrivial rank-1 valuation on $A$, then there must exist $a \in A$ with $v(a) > 1$).

**Remark 3.19.** We use the multiplicative terminology for valuations (which is different from what is usually done in commutative algebra). In particular, if $\Gamma = \mathbf{R}_{>0}$, then a valuation (in the sense of Definition 3.18) is often called a (semi)norm.

There is a direct generalization of the Riemann–Zariski space of a field to any ring, called the valuation spectrum.

**Definition 3.20.** The *valuation spectrum* of $A$ is the set

$$\mathrm{Spv}(A) := \left\{\text{valuations } A \xrightarrow{v} \Gamma \cup \{0\}\right\} / \sim,$$

where the equivalence relation $\sim$ is as in Definition 3.16.

For $f, g \in A$, define

$$\mathrm{Spv}(A)\left(\frac{f}{g}\right) := \{v \in \mathrm{Spv}(A) \colon v(f) \leq v(g) \neq 0\}.$$

One should think of $\mathrm{Spv}(A)\left(\frac{f}{g}\right)$ as the locus in $\mathrm{Spv}(A)$ where $v\left(\frac{f}{g}\right) \leq 1$, though this does not necessarily make sense, as $g$ need not be invertible (we even allow $g$ to be zero!). We will often write $U\left(\frac{f}{g}\right) := \mathrm{Spv}(A)\left(\frac{f}{g}\right)$, analogous to how one writes $D(f)$ in scheme theory. We define the topology on $\mathrm{Spv}(A)$ by declaring that sets of the form $U\left(\frac{f}{g}\right)$ form a subbasis for the topology.

When using this notation, one must be careful: if $f, g, h \in A$, then in general one has the inclusion $\mathrm{Spv}(A)\left(\frac{hf}{hg}\right) \subseteq \mathrm{Spv}(A)\left(\frac{f}{g}\right)$, but you don't necessarily have equality! For example, $\mathrm{Spv}(A)\left(\frac{1}{1}\right) = \mathrm{Spv}(A)$, but $\mathrm{Spv}(A)\left(\frac{h}{h}\right) = \{v \in \mathrm{Spv}(A) \colon v(h) \neq 0\}$, which is smaller than $\mathrm{Spv}(A)$ in general.

**Exercise 3.21.** If $f_1, f_2, g \in A$, then

$$\mathrm{Spv}(A)\left(\frac{f_1}{g}\right) \cap \mathrm{Spv}(A)\left(\frac{f_2}{g}\right) = \mathrm{Spv}(A)\left(\frac{f_1, f_2}{g}\right) := \{v \in \mathrm{Spv}(A) \colon v(f_1) \leq v(g) \text{ and } v(f_2) \leq v(g)\}.$$

**Example 3.22.** If $A$ is a field, then $\mathrm{Spv}(A) = \mathrm{RZ}(A)$.

**Remark 3.23.** The assignment $A \mapsto \mathrm{Spv}(A)$ gives a contravariant functor from the category of commutative rings to the category of topological spaces. Indeed, to a ring map $\varphi \colon A \to B$, one associates the continuous map $\varphi^* \colon \mathrm{Spv}(B) \to \mathrm{Spv}(A)$ given by $v \mapsto v \circ \varphi$.

**Definition 3.24.** If $v \in \mathrm{Spv}(A)$, the *support* of $v$ is the ideal $\mathfrak{p}_v = \mathrm{supp}(v) := v^{-1}(0)$ of $A$.

**Exercise 3.25.** For any $v \in \mathrm{Spv}(A)$, $\mathfrak{p}_v$ is a prime ideal of $A$.

Given a valuation $v \colon A \to \Gamma \cup \{0\}$, it factors as

$$A \to A/\mathfrak{p}_v \hookrightarrow \kappa(\mathfrak{p}_v) := \mathrm{Frac}(A/\mathfrak{p}_v) \xrightarrow{\tilde{v}} \Gamma \cup \{0\},$$

where $\tilde{v}$ denotes the unique extension of the valuation to the residue field $\kappa(\mathfrak{p}_v)$. Let $R_v \subseteq \kappa(\mathfrak{p}_v)$ denote the valuation ring of $\tilde{v}$. Observe that $\Gamma_v = \kappa(\mathfrak{p}_v)^*/R_v^*$. It follows that $v \sim w$ iff $\mathfrak{p}_v = \mathfrak{p}_w$ and $R_v = R_w$, which is perhaps a more concrete description of the equivalence of valuations.

We can define the "kernel/support" map $\varphi \colon \mathrm{Spv}(A) \to \mathrm{Spec}(A)$, given by $v \mapsto \mathfrak{p}_v$. The map $\varphi$ helps us to better understand $\mathrm{Spv}(A)$, by realizing it as a "fibration" over the affine scheme $\mathrm{Spec}(A)$.

**Lemma 3.26.** *$\varphi \colon \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ is continuous.*

*Proof.* It suffices to show that for any $b \in A$, $\varphi^{-1}(D(b))$ is open: observe that

$$\varphi^{-1}(D(b)) = \{v \in \mathrm{Spv}(A) \colon v(b) \neq 0\} = \mathrm{Spv}(A)\,(0/b)\,,$$

which is open in $\mathrm{Spv}(A)$. □

The fibres of the support map $\varphi$ are in fact Riemann–Zariski spaces, as the following exercise makes precise.

**Exercise 3.27.** [Functoriality properties of Spv] Let $S \subseteq A$ be a multiplicatively-closed subset, and let $\mathfrak{a} \subset A$ be an ideal. If $B$ denotes either $S^{-1}A$ or $A/\mathfrak{a}$ and $\phi \colon A \to B$ denotes the canonical morphism, then

$$\begin{tikzcd} \mathrm{Spv}(B) \arrow[r, "\phi^*"] \arrow[d, "\varphi_B"'] & \mathrm{Spv}(A) \arrow[d, "\varphi_A"] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

is a Cartesian diagram of topological spaces. More concretely,

$$\mathrm{Spv}(S^{-1}A) = \{v \in \mathrm{Spv}(A) \colon \varphi(v) \notin S\}$$

and

$$\mathrm{Spv}(A/\mathfrak{a}) = \{v \in \mathrm{Spv}(A) \colon \varphi(v) \subseteq \mathfrak{a}\}.$$

In particular, for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, $\varphi^{-1}(\mathfrak{p}) = \mathrm{Spv}(\kappa(\mathfrak{p})) = RZ(\kappa(\mathfrak{p}))$. Therefore, as sets, one can think of $\mathrm{Spv}(A)$ as fibered over $\mathrm{Spec}(A)$, where the fibre above a prime ideal is the Riemann–Zariski space of its residue field.

**Example 3.28.** (1) The space $\mathrm{Spv}(\mathbf{Z})$ is a fibration over $\mathrm{Spec}(\mathbf{Z})$, where the fibre over $(0)$ is a copy of $\mathrm{RZ}(\mathbf{Q}) = \mathrm{Spec}(\mathbf{Z})$, and the fibre over the prime $(p)$ is the point $\mathrm{RZ}(\mathbf{F}_p) = \{v_{p,0}\}$. One can check that the point $v_{p,0}$ is closed in $\mathrm{Spv}(\mathbf{Z})$, but the closure of $v_0 \in \mathrm{RZ}(\mathbf{Q})$ in $\mathrm{Spv}(\mathbf{Z})$ is all of $\mathrm{Spv}(\mathbf{Z})$, and the closure of $v_p \in \mathrm{RZ}(\mathbf{Q})$ is $\{v_p, v_{p,0}\} \subset \mathrm{Spv}(\mathbf{Z})$.

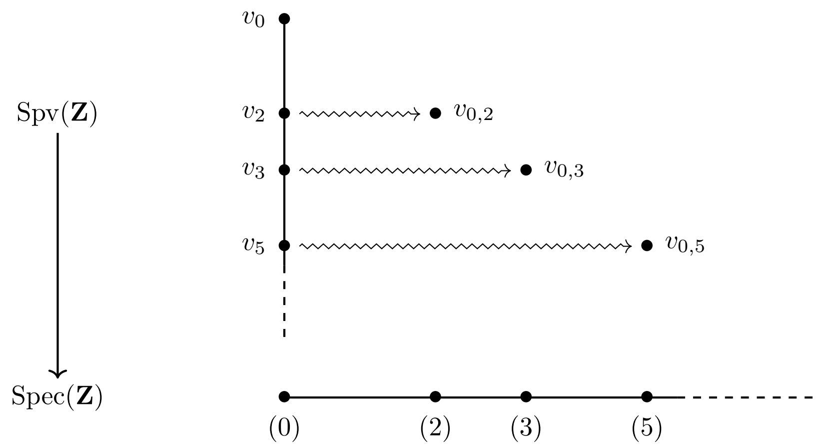


FIGURE 1. $\mathrm{Spv}(\mathbf{Z})$ is a fibration over $\mathrm{Spec}(\mathbf{Z})$. The arrows $\rightsquigarrow$ denote specializations (though not all specializations are shown).

(2) The space $\mathrm{Spv}\, k((y))[[x]]$ has many points, but let us describe a few of them. It is fibered over $\mathrm{Spec}\, k((y))[[x]]$, which consists of the generic point $(0)$ and the closed point $(x)$, because $k((y))[[x]]$ is a dvr. Over the closed point $(x)$ lies $\mathrm{RZ}\,(k((y)))$; this fibre contains e.g. the $y$-adic valuation $\overline{v'}$ on $k((y))$ and the trivial valuation $\overline{\eta}$. Over the generic point $(0)$ lies $\mathrm{RZ}\,(k((y))((x)))$; this fibre contains e.g. the trivial valuation $\eta$, and the $x$-adic valuation $v$.

Given a point in the fibre of the support map, there is a recipe to construct more points in the fibre: given a valuation ring $R$ inside a field $K$, let $k = R/\mathfrak{m}_R$ denote the residue field of $R$. Suppose there is a valuation on $k$ whose valuation $\overline{R'}$ is not all of $k$, and consider the pullback diagram

$$\begin{tikzcd} R' \arrow[r, hook] \arrow[d] & R \arrow[d] \\ \overline{R'} \arrow[r, hook] & k \end{tikzcd}$$

Note that the ideal $\mathfrak{m}$ is contained in $R'$ (but it is not necessarily maximal in $R'$), and the map $R' \to \overline{R'}$ is the quotient by $\mathfrak{m}$.

**Lemma 3.29.** *The subring $R'$, defined by the above pullback diagram, is a valuation ring of $K$, called the* composite *of $R$ and $\overline{R'}$.*

*Proof.* It is clear that $R'$ is a ring. If $x \in K \backslash R'$, then we would like to show that $x^{-1} \in R'$. If $x \notin R$, then $x^{-1} \in R \backslash R^* = \mathfrak{m}$, but $\mathfrak{m} \subset R'$, so we're done. If $x \in R \backslash R' \subset R^*$, then $x^{-1} \in R^*$. However, $x \bmod \mathfrak{m} \notin \overline{R'}$, so $x^{-1} \bmod \mathfrak{m} \in \overline{R'}$, so $x^{-1} \in R'$. $\square$

**Example 3.30.** With notation as in Example 3.28(2), if $K = k((y))((x))$ and $R = k((y))[[x]]$, then the residue field is $k((y))$ and an example of a valuation ring in $k((y))$ is $\overline{R'} = k[[y]]$. Then, $R' = k[[y]] + xk((y))[[x]]$.

**Exercise 3.31.** With notation as above, $R = R'_{\mathfrak{m}}$.

The recipe described above is the only way to get new valuations on $K$, in the following sense.

**Fact 3.32.** [Mat89, Theorem 10.1]

(1) If $R \subseteq K = \mathrm{Frac}(R)$ is a valuation ring, then there is a bijection
$$\{\text{valuation rings } R' \subseteq R \subseteq K\} \xrightarrow{\simeq} \mathrm{Spv}\,(R/\mathfrak{m}_R)\,,$$
given by $R' \mapsto \overline{R'} \subseteq R/\mathfrak{m}_R$.

(2) If $R' \subseteq K$ is a valuation ring, then there is a bijection
$$\{\text{valuation rings } R' \subseteq R \subseteq K\} \xrightarrow{\simeq} \mathrm{Spec}(R'),$$
given by $R \mapsto \mathfrak{m}_R \cap R'$.

Moreover, the value group of the new valuation ring $R'$ can be described in terms of the value groups of $R$ and $\overline{R'}$ as follows: there is a short exact sequence of abelian groups
$$\begin{tikzcd} 1 \arrow[r] & R^*/(R')^* \arrow[r] \arrow[d, "\simeq"] & K^*/(R')^* \arrow[r] \arrow[d, "\simeq"] & K^*/R* \arrow[r] \arrow[d, "\simeq"] & 1 \\ & k^*/(\overline{R'})^* \simeq \Gamma_{\overline{v'}} & \Gamma_{v'} & \Gamma_v & \end{tikzcd}$$
Thus, we can describe $\Gamma_{v'}$ (at least as a group) as an extension of $\Gamma_v$ by $\Gamma_{\overline{v'}}$.

**Exercise 3.33.** If $v, v', \overline{v'}$ are as in Example 3.30, then $\Gamma_{v'} \simeq \Gamma_v \times \Gamma_{\overline{v'}} \simeq x^{\mathbf{Z}} \times y^{\mathbf{Z}} \simeq \mathbf{Z}^2$, with the lexicographic order.

## 4. February 9th – Valuation spectra continued (Emanuel Reinecke)

Last time, we associated, to a discrete ring $A$, its valuation spectrum $\mathrm{Spv}(A)$. Today, we will discuss the topology on $\mathrm{Spv}(A)$: our first goal is to show that $\mathrm{Spv}(A)$ is a spectral space, which is a tameness criterion, defined below.

**Definition 4.1.** A topological space $X$ is called *spectral* if it satisfies the following:

(1) $X$ is sober, i.e. every irreducible (closed) subset of $X$ has a unique generic point;
(2) $X$ is quasi-compact and quasi-separated[4];
(3) $X$ admits a base of quasi-compact open subsets (that is closed under taking finite intersections).

The condition that $\mathrm{Spv}(A)$ is sober will allow us to talk about specializations in a meaningful way, and it will reduce the problem of understanding irreducible closed subsets of $\mathrm{Spv}(A)$ to understanding specializations. The properties (2) and (3) will allow us to discuss the constructible topology on $\mathrm{Spv}(A)$, which we will define shortly.

**Example 4.2.** [Non-example] A non-quasi-separated scheme is an example of a non-spectral space, though such objects are tricky to construct. While the affine line with the doubled origin is quasi-separated (indeed, any noetherian scheme is quasi-separated), this idea can nonetheless be used to construct an example: let $U_i = \mathbf{A}^\infty_k = \mathrm{Spec}\,(k[x_1, x_2, \ldots])$, and let $X$ be the scheme obtained by gluing $U_1$ and $U_2$ along the complement of the origin; that is, $X$ is the infinite-affine space with a double origin. Then, $U_1$ and $U_2$ are quasi-compact open subsets of $X$, but $U_1 \cap U_2 = \bigcup_{i=1}^\infty D(x_i)$ is not quasi-compact.

**Example 4.3.** If $A$ is a ring, then $\mathrm{Spec}(A)$ is spectral. More generally, any qcqs scheme is spectral.

In fact, affine schemes are the only examples of spectral spaces, in the sense below.

**Theorem 4.4.** [Hoc69] *If $X$ is a spectral space, then there exists a ring $A$ and a homeomorphism* $X \simeq \mathrm{Spec}(A)$.

[4]Recall that a topological space $X$ is *quasi-separated* if the intersection of any two quasi-compact open subsets is again quasi-compact.

4.1. **Constructible topology.** While we may be familiar with the definition of constructible sets in noetherian topological spaces, we will recall the general definition.

**Definition 4.5.** Let $X$ be a qcqs topological space. A subset $C \subseteq X$ is *constructible* if we can write

$$C = \bigcup_{i=1}^{r} (U_i \cap (X \backslash V_i))$$

for some quasi-compact open sets $U_i, V_i \subset X$.

**Remark 4.6.** A qcqs topological space $X$ is noetherian iff all subspaces are quasi-compact. Thus, if $X$ is a qcqs noetherian space, we cover the usual definition of a constructible set, namely a finite union of locally closed subsets.

**Exercise 4.7.** [Con14, Proposition 3.2.1] If $U \subseteq \mathrm{Spec}(A)$ is open, then $U$ is quasi-compact iff $\mathrm{Spec}(A) \backslash U = \mathrm{Spec}(A/(f_1, \ldots, f_n))$, i.e. the complement of $U$ is cut out by finitely-many equations.

**Definition 4.8.** The *constructible topology* on a qcqs topological space $X$ is the topology on $X$ whose base consists of all constructible subsets; denote it by $X_{\mathrm{cons}}$.

**Remark 4.9.** If $X$ has a base of quasi-compact open subsets, then the identity map $X_{\mathrm{cons}} \to X$ is continuous.

**Example 4.10.** If $A$ is a noetherian domain of dimension 1 and $X = \mathrm{Spec}(A)$, then $X = \{\eta\} \cup X^0$, where $\eta$ is the generic point of $X$, and $X^0$ denotes the set of closed points. As closed points are constructible, they are clopen in $X_{\mathrm{cons}}$ and so $X^0$ is a discrete subset of $X_{\mathrm{cons}}$.

For any $x \in X^0$, can write $X = \{x\} \cup (X \backslash \{x\})$ two open subsets in the constructible topology, each of which contains either $x$ or the generic point $\eta$. Thus, $X_{\mathrm{cons}}$ is Hausdorff.

If there is a constructible subset $C \subset X$ with $\eta \in C$, there exists $U \subset C$ open (if it was locally closed but still contained the generic point, then it would contain everything). In particular, $X \backslash C$ is finite.

Using this fact, you can show:

**Exercise 4.11.** [Con14, Example 3.2.12] With notation as in Example 4.10, $X$ is the 1-point compactification of $X^0$.

**Exercise 4.12.** [Con14, Example 3.2.13] Describe the constructible topology on $\mathbf{A}^2_k$.

The open subsets in the constructible topology are (possibly infinite) unions of constructible sets, and we call them the *ind-constructible* sets. The closed subsets are called *pro-constructible* sets, i.e. they are (possibly infinite) intersections of constructible sets (and hence may no longer be constructible).

**Definition 4.13.** A map $f\colon X \to Y$ between spectral spaces is called *spectral* if $f$ continuous and $f_{\mathrm{cons}}\colon X_{\mathrm{cons}} \to Y_{\mathrm{cons}}$ is continuous.

In light of the above definition, we can define the category of spectral spaces, with spectral maps as morphisms. Certain basic facts about spectral spaces and spectral maps are summarized below.

**Proposition 4.14.** [Con14, 3.3] *Let $X$ be a spectral space.*

*(1) $X_{\mathrm{cons}}$ is compact (i.e. quasi-compact and Hausdorff).*
*(2) A subset $Z \subseteq X$ is constructible iff $Z$ is clopen in $X_{\mathrm{cons}}$.*
*(3) If $Z \subseteq X$ proconstructible, then $Z$ is constructible iff $X \backslash Z$ is proconstructible.*
*(4) If $Z \subseteq X$ is proconstructible, then the closure $\overline{Z}$ in the usual topology is $\overline{Z} = \bigcup_{z \in Z} \overline{\{z\}}$.*
*(5) If $Z \subseteq X$ is proconstructible, then $Z \subseteq X$ is spectral with the subspace topology. Furthermore, every constructible set $D \subseteq Z$ is of the form $Z \cap C$ for some $C \subseteq X$ constructible.*
*(6) If $f\colon X \to Y$ is a continuous map between spectral spaces, then $f$ is spectral iff the $f$-preimages of proconstructible sets are proconstructible iff the $f$-preimages of constructible sets are constructible iff $f$ is qc.*

*(7) If $\{X_i\}$ is an inverse system of spectral spaces with spectral transition maps, then $\varprojlim X_i$ is spectral and the projection maps are spectral; in particular, this is the inverse limit in the category of spectral spaces.*

**Theorem 4.15.** [Hoc69] *Let $X'$ be a quasi-compact topological space, and let $\Sigma \subset P(X')$ be a collection of clopen subsets, which is stable under finite intersections. Let $X$ be the set $X'$, equipped with the topology generated by $\Sigma$ as a base. If $X$ is $T_0$, then it is spectral with $\Sigma$ as a base of quasi-compact open sets, and $X_{\mathrm{cons}} = X'$.*

One should think of Hochster's theorem as saying that the original topology can be reconstructed from the constructible topology.

Our first main theorem, namely the spectrality of $\mathrm{Spv}(A)$, is an application of Theorem 4.15.

**Theorem 4.16.** *If $A$ is a ring, then $X = \mathrm{Spv}(A)$ is spectral, and all open sets of the form*

$$X\left(\frac{f}{g}\right) = \{v \in X : v(f) \le v(g) \ne 0\}$$

*are quasi-compact. Furthermore,* Spv *is a contravariant functor from the category of rings to the category of spectral spaces.*

Last time, we saw one instance of this theorem: there is a homeomorphism $\mathrm{Spv}(\mathbf{Q}) = \mathrm{RZ}(\mathbf{Q}) \simeq \mathrm{Spec}(\mathbf{Z})$.

**Exercise 4.17.** Find a ring $A$ such that $\mathrm{Spv}(\mathbf{Z}) \simeq \mathrm{Spec}(A)$.

**Remark 4.18.** The fact that the open set $X(f/g)$ is quasi-compact is not obvious, because $X(f/g) \ne \mathrm{Spv}(A[f/g])$ (e.g. take $f = g$, then $A[f/g] = A$, but $X(f/g)$ consists of all valuations that are nonzero on $f$).

*Proof of Theorem 4.16.* Recall that two valuations $v$ and $w$ are equivalent iff $\mathfrak{p}_v = \mathfrak{p}_w$ and $R_v = R_w$, or equivalently if $v$ and $w$ have the same order relations, i.e. $v(a) \le v(b)$ iff $w(a) \le w(b)$ for all $a, b \in A$ .

**Step 1**. $X$ is $T_0$.

If $v, w \in X$ are distinct, then we want to show that there is an open set containing one but not the other. If $\mathfrak{p}_v \ne \mathfrak{p}_w$, then WLOG there is $a \in A$ such that $a \in \mathfrak{p}_v$, but $a \notin \mathfrak{p}_w$. Then, $w \in X(a/a)$, but $v \notin X(a/a)$. If $\mathfrak{p}_v = \mathfrak{p}_w$, then there exists $a, b \in A \backslash \{0\}$ such that $v(a) \le v(b)$, but $w(a) > w(b)$. Then, $X(a/b)$ is the desired open set.

**Step 2**. Embed $X \subset \mathcal{P}(A \times A)$, where $\mathcal{P}(A \times A)$ denotes the power set of $A \times A$.

There is a map $X \to \mathcal{P}(A \times A)$ given by sending $v \in X$ to the binary relation $|_v$ on $A$ defined by $a|_v b$ iff $v(a) \ge v(b)$ (recall that a binary relation $|$ on $A$ determines the subset $\{(a, b) \in A \times A : a|b\}$ of $A$, and hence an element of $\mathcal{P}(A \times A)$). This map is injective, because two valuations are equivalent iff they have the same order relations.

In fact, the image of $\varphi$ consists of all binary relations $|$ on $A$ satisfying the following 6 conditions: for all $a, b, c \in A$,

(1) $a|b$ or $b|a$,
(2) $a|b$ and $b|c$ implies $a|c$,
(3) $a|b$ and $a|c$ implies $a|b + c$,
(4) $a|b$ implies $ac|bc$,
(5) $ac|bc$ and $0 \not| c$ implies $a|b$,
(6) $0 \not| 1$.

It is clear that every binary relation of the form $|_v$ satisfies (1-6). For the converse, the idea is that the conditions (1-6) encode the set $v(A \backslash \mathfrak{p}_v)$, which generates the value group $\Gamma_v$. More precisely, consider the quotient $M := A/\sim$, where we declare $a \sim b$ iff $a|b$ and $b|a$. On $M$, define multiplication by multiplication on representatives (this is well-defined by (4)), and $M \backslash \{\overline{0}\}$ is closed under multiplication (by (5)); thus, we get a commutative monoid with identity 1 (using (6) in order to say that $\overline{1}$ and $\overline{0}$ are not equivalent). Let $\Gamma_v$ be the Grothendieck group of $M$ and let $M \backslash \{\overline{0}\} \to \Gamma_v$ be the canonical map, which is injective by (5). We can define a total ordering on $\Gamma_v$ via the rule $\overline{a} \ge \overline{b}$ iff $a|b$. One can then show that the canonical map $A \to \Gamma_v \cup \{0\}$ is a valuation.

**Step 3**. Endow $\mathcal{P}(A \times A) = \{0,1\}^{A\times A}$ with the product topology, where $\{0,1\}$ has the discrete topology. As $\{0,1\}$ is compact in the discrete topology, $\mathcal{P}(A \times A)$ is compact by Tychonoff's theorem. Moreover, one can check that conditions (1-6) are closed. For example, let us show that the condition (3) is closed. This is equivalent to the following logical statement: for any $a, b, c$, either $a|b + c$ or $a \not| b$ or $a \not| c$. If $\pi_{f,g} \colon \mathcal{P}(A \times A) \to \{0, 1\}$ denotes the projection onto the factor corresponding to $(f, g) \in A \times A$, then

$$\mathcal{P}(A \times A) \supseteq \{\text{binary relations} \mid \text{satisfying (3)}\} = \bigcap_{a,b,c\in A} \left(\pi^{-1}_{a,b+c}(1) \cup \pi^{-1}_{a,b}(0) \cup \pi^{-1}_{a,c}(0)\right).$$

As the projections are continuous, it follows that (3) is a closed conditon.

**Step 4**. Apply Hochster's Theorem 4.15. Define $X'$ to be the set $X$, equipped with the subspace topology of $\mathcal{P}(A \times A)$. As $X'$ is defined by closed conditions, it is quasi-compact by Tychonoff's theorem. Furthermore, we can write

$$\begin{aligned} X\left(\frac{f}{s}\right) &= \{v \in X \colon v(f) \le v(s) \ne 0\} \\ &= \{v \in X \colon v(f) \le v(s)\} \cap \Big(X \backslash \{v \in X \colon v(s) \le v(0)\}\Big) \\ &= X \cap \pi^{-1}_{s,f}(1) \cap \pi^{-1}_{0,s}(0), \end{aligned}$$

so the set $X(f/s)$ is clopen in $X$. Said differently, the sets that define the usual topology on $X$ are clopen in $X'$, and they are also stable under intersections. Hochster's criterion now shows that $X$ is spectral, where sets of the form $X(f/g)$ form the subbase of quasi-compact open sets.

It remains to prove the functoriality of Spv. Iff $\varphi \colon A \to B$ is a ring map, it induces a continuous map $\mathrm{Spv}(\varphi) \colon \mathrm{Spv}(B) \to \mathrm{Spv}(A)$ with the property that

$$\mathrm{Spv}(\varphi)^{-1}\left(\mathrm{Spv}(A)\left(\frac{f}{g}\right)\right) = \mathrm{Spv}(B)\left(\frac{\varphi(f)}{\varphi(g)}\right).$$

In particular, $\mathrm{Spv}(\varphi)$ is quasi-compact, and hence spectral by Proposition 4.14(6). □

**Remark 4.19.** A similar analysis shows that the support map $\mathrm{Spv}(A) \to \mathrm{Spec}(A)$ is spectral (by showing that the preimages of the distinguished opens $D(f) \subseteq \mathrm{Spec}(A)$ are quasi-compact). Moreover, a special case of Theorem 4.16 is that the Riemann–Zariski spaces are quasi-compact, which historically was one of Zariski's major contributions to the study of valuations.

4.2. **Specializations.** Our goal now is to understand specializations in $\mathrm{Spv}(A)$. They occur in 2 types: the "vertical ones" which occur in the fibres of the support map, and the "horizontal ones" which change the support.

**Definition 4.20.** A *vertical* (or *secondary*) specialization/generization of a valuation $v \in \mathrm{Spv}(A)$ is a specialization/generization $w$ of $v$ such that $\mathrm{supp}(v) = \mathrm{supp}(w)$.

Recall that valuations on the residue field $\kappa(\mathfrak{p}_v)$ are in bijection with valuation rings $R \subseteq \kappa(\mathfrak{p}_v)$.

**Lemma 4.21.** *If $v, w$ are valuations on a field $K$, then $v$ is a specialization of $w$ iff $R_v \subseteq R_w$.*

**Example 4.22.** With notation as in Example 3.28(1), Lemma 4.21 implies that the trivial valuation $v_0 \in \mathrm{supp}^{-1}(0) \subset \mathrm{Spv}(\mathbf{Z})$ specializes to every other point of $\mathrm{Spv}(\mathbf{Z})$.

*Proof of Lemma 4.21.* We have $R_v \subseteq R_w$ iff for all $g \in K$, $v(g) \le 1$ implies $w(g) \le 1$. This is equivalent to the assertion that for all $f \in K$, $s \in K^*$, $v(f) \le v(s)$ implies $w(f) \le w(s)$; said differently, for all $f, s \in K$, $v \in \mathrm{Spv}\, K(f/s)$ implies $w \in \mathrm{Spv}\, K(f/s)$, because valuations on fields are injective. In particular, every open neighborhood of $v$ contains $w$, so $v \in \overline{\{w\}}$. □

Let us examine what specialization affects at the level of value groups. Recall from last time that, if a valuation $v$ specializes to $w$, there is a short exact sequence of abelian groups

$$1 \to \underbrace{R_v^*/R_w^*}_{=w(R_v^*)} \to \underbrace{K^*/R_v^*}_{=\Gamma_w} \xrightarrow{\pi} \underbrace{K^*/R_w^*}_{=\Gamma_v} \to 1$$

In fact, the map $\pi$ is order-preserving. To show this, it suffices to show that $\pi((\Gamma_w)_{\leq 1}) \subseteq (\Gamma_v)_{\leq 1}$. As $\pi\left((\Gamma_w)_{\leq 1}\right) = \pi(R_w/R_w^*) = R_w/R_w^*$ and $(\Gamma_v)_{\leq 1} = R_v/R_v^*$, this is clear.

A subgroup $H \subseteq \Gamma_v$ is said to satisfy $(*)$ if $\Gamma_v/H$ is totally ordered and such that the natural projection map $\Gamma_v \twoheadrightarrow \Gamma_v/H$ is order-preserving. There is a bijection

$$\{\text{valuation ring } R_w \subseteq R_v \subseteq K\} \longleftrightarrow \{\text{subgroups } H \subseteq \Gamma_w \text{ satisfying } (*)\}$$

where a valuation ring $R_w \subseteq R_v$ is sent to the subgroup $w(R_v^*)$, and the subgroup $H \subseteq \Gamma_w$ is sent to the valuation ring $v^{-1}\left((\Gamma_w/H)_{\leq 1}\right) \subseteq R_v$.

**Definition 4.23.** If $\Gamma$ is a totally ordered abelian group, a subgroup $H \subseteq \Gamma$ is *convex* if for any $h, h' \in H$ and $\gamma \in \Gamma$, $h \leq \gamma \leq h'$ implies $\gamma \in H$.

**Exercise 4.24.** [Con14, 4.2] If $v$ is a valuation and $H \subseteq \Gamma_v$ is a subgroup, then $H$ satisfies $(*)$ iff $H$ is convex.

**Definition 4.25.** If $v\colon A \to \Gamma_v \cup \{0\}$ is a valuation and $H \subseteq \Gamma_v$ is a convex subgroup, then the valuation $v_{/H}\colon \Gamma_v/H \cup \{0\}$ is defined by

$$v_{/H}(a) := \begin{cases} v(a) \bmod H & v(a) \neq 0, \\ 0 & v(a) = 0. \end{cases}$$

The valuations of the form $v_{/H}$ allow us to construct a bijection between vertical generizations of a fixed valuation and convex subgroups of its value group. More precisely, let $v\colon A \to \Gamma_v \cup \{0\}$ be a valuation, then there is a bijection

$$\{\text{vertical generizations of } v\} \longleftrightarrow \{\text{convex subgroups of } H \subseteq \Gamma_v\},$$

where a convex subgroup $H \subseteq \Gamma_v$ is sent to the valuation $v_{/H}$. See [Con14, Proposition 4.2.4] for further details.

## 5. February 10th – Valuation spectra continued (Emanuel Reinecke)

Last time, we first showed that the valuation spectrum $\mathrm{Spv}(A)$ of a ring $A$ is spectral. Furthermore, we showed that vertical generizations of a fixed valuation $v \in \mathrm{Spv}(A)$ correspond bijectively to both valuation subrings of $R_v \subseteq \kappa(\mathfrak{p}_v)$, and convex subgroups of $\Gamma_v$. To a convex subgroup $H \subseteq \Gamma_v$, the associated vertical generization of $v$ was built by using the quotient $\Gamma_v/H$.

### 5.1. **Horizontal specializations.**

**Definition 5.1.** If $v \in \mathrm{Spv}(A)$ and $H \subseteq \Gamma_v$ is a convex subgroup, then define a map $v_{|H}\colon A \to H \cup \{0\}$ by

$$v_{|H}(a) := \begin{cases} v(a) & v(a) \in H, \\ 0 & v(a) \notin H. \end{cases}$$

**Example 5.2.** With notation as in Example 3.28(1), the $p$-adic valuation $v_p \in \mathrm{RZ}(\mathbf{Q}) \subset \mathrm{Spv}(\mathbf{Z})$ has value group equal to $\mathbf{Z}_{(p)}$, which has two convex subgroups: $\mathbf{Z}_{(p)}$ itself and the trivial subgroup 1. Clearly, $(v_p)_{|\mathbf{Z}_{(p)}} = v_p$, but $(v_p)\,|_1 = v_{p,0} \in \mathrm{RZ}(\mathbf{F}_p) \subset \mathrm{Spv}(\mathbf{Z})$.

**Definition 5.3.** If $v \in \mathrm{Spv}(A)$, define the *characteristic group* $c\Gamma_v$ of $v$ to be the minimal convex subgroup of the value group $\Gamma_v$ that contains $(\Gamma_v)_{\geq 1} \cap v(A)$.

**Example 5.4.** Let $v \in \mathrm{Spv}(A)$.

(1) If $A$ is a field , then $v(A^*) = \Gamma_v$ and hence $c\Gamma_v = \Gamma_v$. In particular, the characteristic group can be as large as possible.
(2) If $A$ is the valuation ring for $v$, i.e. $v(A) = (\Gamma_v)_{\leq 1}$, then $c\Gamma_v$ is generated by $(\Gamma_v)_{=1} = 1$, so $c\Gamma_V = 1$. In particular, the characteristic group can be as small as possible.

**Lemma 5.5.** *If $H$ is a convex subgroup, then $v_{|_H}$ is a valuation iff $c\Gamma_v \subseteq H$.*

*Proof.* If $v_{|_H}$ is a valuation, it suffices to show that $(\Gamma_v)_{\leq 1} \cap v(A) \subseteq H$, because $H$ is convex. Suppose this is not the case: then, there exists $a \in A$ such that $1 < v(a) \notin H$; in particular, $v_{|_H}(a) = 0$. On the other hand, $v_{|_H}(1+a) = \max\{v_{|_H}(1) = 1, v_{|_H}(a)\} = v_{|_H}(a) = 0$, hence $1 \in \mathfrak{p}_{v_{|_H}}$, a contradiction. The converse is left as an exercise; see [Con14, Proposition 4.3.6]. ☐

**Lemma 5.6.** *If $H \subseteq \Gamma_v$ is a convex subgroup and $c\Gamma_v \subseteq H$, then $v_{|_H}$ is a specialization of $v$.*

*Proof.* If $v_{|_H} \in \mathrm{Spv}(A)(f/g)$, then $v_{|_H}(f) \leq v_{|_H}(g) \neq 0$, so $v(g) \in H$. Suppose that $v(f) > v(g)$, then this implies that $v(f) \notin H$, so $v(f) < 1$; in particular, $H \ni v(g) < v(f) < 1$, and, by convexity, $v(f) \in H$. This is a contradiction, because $1 \in H$. Thus, $v \in \mathrm{Spv}(A)(f/g)$. It follows that every open neighborhood of $v_{|_H}$ also contains $v$. ☐

One can ask: when passing from $v$ to $v_{|_H}$, what happens on the level of valuations rings? If $H \subseteq \Gamma_v$ is a convex subgroup such that $c\Gamma_v \subseteq H$, consider the diagram

$$\begin{tikzcd}
 & R_v \arrow[r, "v"] \arrow[d, hook] & (\Gamma_v)_{\leq 1} \cup \{0\} \arrow[d, hook] & \\
A \arrow[r, "j"] & \kappa(\mathfrak{p}_v) \arrow[r, "v"] & \Gamma_v \cup \{0\} \arrow[r] & \Gamma_v/H \cup \{0\} \\
 & (R_v)_{\mathfrak{p}} \arrow[u, hook] \arrow[rr, "v_{/H}"] & & (\Gamma_v/H)_{\leq 1} \cup \{0\} \arrow[u, hook]
\end{tikzcd}$$

Here, $j\colon A \to \kappa(\mathfrak{p}_v)$ is the canonical map, and $\mathfrak{p}$ is the maximal ideal of $R_v$ corresponding to the valuation $v_{/H}\colon \kappa(\mathfrak{p}_v) \to \Gamma_v/H \cup \{0\}$. That is,

$$\mathfrak{p} = \Big(\{x \in R_v \colon v(x) < \delta \text{for all } \delta \in H\}\Big)_{\leq 1}.$$

From the diagram, we can observe that $c\Gamma_v \subseteq H$ iff $v_{/H}(A) \subseteq (\Gamma_v/H)_{\leq 1}$ iff $j(A) \subseteq (R_v)_{\mathfrak{p}}$.

Now, $\mathrm{supp}(v_{|_H}) = j^{-1}(\mathfrak{p}(R_v)_{\mathfrak{p}})$. Thus, there exists a factorization $v_{|_H}\colon A \to (R_v)_{\mathfrak{p}}/\mathfrak{p}(R_v)_{\mathfrak{p}} \to H \cup \{0\}$, and one can check that the corresponding valuation ring is $R_v/\mathfrak{p}$. Therefore, there is an extension of valued fields

$$\kappa\left(\mathfrak{p}_{v_{|_H}}, R_{v_{|_H}}\right) \hookrightarrow \left((R_v/\mathfrak{p})_{\mathfrak{p}}, R_v/\mathfrak{p}\right).$$

**Definition 5.7.** If $v \in \mathrm{Spv}(A)$, a subset $\Sigma \subset A$ is *$v$-convex* if for all $s, t \in \Sigma$ and $a \in A$, $v(s) \leq v(a) \leq v(t)$ implies $a \in \Sigma$.

**Example 5.8.** The prime ideal $\mathrm{supp}(v_{|_H})$ of $A$ is $v$-convex.

For any $v \in \mathrm{Spv}(A)$, the $v$-convex prime ideals of $A$ classify the horizontal specializations of $v$, in the following sense.

**Theorem 5.9.** [Wed12, Proposition 4.18] *If $v \in \mathrm{Spv}(A)$, then there exists an order-preserving bijection*

$$\{\textit{horizontal specializations of } v\} \longleftrightarrow \{v\textit{-convex prime ideals of } A\},$$

*where a horizontal specialization $w$ of $v$ is sent to the prime $\mathfrak{p}_w$; conversely, given a $v$-convex prime ideal $\mathfrak{q}$ of $A$, the valuation $w_{\mathfrak{q}} \in \mathrm{Spv}(A)$ defined by*

$$w_{\mathfrak{q}}(a) := \begin{cases} v(a) & a \notin \mathfrak{q}, \\ 0 & a \in \mathfrak{q}. \end{cases}$$

*is a horizontal specialization of $v$. In fact, if $H_{\mathfrak{q}}$ denotes the minmal convex subgroup of $\Gamma_v$ containing $v(A \backslash \mathfrak{q})$, then $w_{\mathfrak{q}} = v_{|H_{\mathfrak{q}}}$.*

**Remark 5.10.** As ideals of a valuation ring are totally ordered by inclusion, Theorem 5.9 implies that the same holds for horizontal specializations of $v$, where $w \leq w'$ iff $w$ specializes to $w'$. The maximal horizontal specialization is $v_{|c\Gamma_v}$.

Moreover, if $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$, then the point $\mathfrak{q}_1$ of $\mathrm{Spec}(A)$ specializes to $\mathfrak{q}_2$. In particular, horizontal specializations are determined by their support.

If $v \in \mathrm{Spv}(A)$, consider the diagram

$$\begin{tikzcd}
\{\text{horizontal specializations of } v\} \arrow[r, hook] \arrow[d, "\simeq"] & \overline{\{v\}} \arrow[r, hook] \arrow[d] & \mathrm{Spv}(A) \arrow[d] \\
\{v\text{-convex prime ideals of } A\} \arrow[r, hook] & \mathrm{Spec}(A/\mathfrak{p}_v) \arrow[r, hook] & \mathrm{Spec}(A)
\end{tikzcd}$$

To relate specializations on the top row to specializations on the bottom row, we have the "going-down theorem" (also called the "going-up theorem" in [Con14]).

**Corollary 5.11.** [Wed12, Proposition 4.20] *If $v \in \mathrm{Spv}(A)$ and if $\mathfrak{p} \in \mathrm{Spec}(A)$ specializes to $\mathfrak{p}_v$ in $\mathrm{Spec}(A)$, then there exists $w \in \mathrm{Spv}(A)$ such that $\mathfrak{p}_w = \mathfrak{p}$, and $w$ specializes to $v$.*

The following is the main theorem of this section. It details how any specialization in the valuation spectrum can be decomposed into a horizontal specialization followed by a vertical specialization, or vice-versa. It originally appears in [HK94].

**Theorem 5.12.** [Con14, Theorem 4.5.2] *If $v, w \in \mathrm{Spv}(A)$ are such that $v$ specializes to $w$, then exactly one of the following occurs:*

*(1) $w$ is a horizontal specialization of a vertical specialization; that is, there exists $v' \in \mathrm{Spv}(A)$ and a diagram of specializations*

$$\begin{tikzcd}
v \arrow[d, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\
v' \arrow[r, rightsquigarrow] & w
\end{tikzcd}$$

*(2) $w$ is a vertical specialization of a horizontal specialization, except possibly when $v(A) \subseteq \Gamma_{v,\leq 1}$ and $v(A \backslash \mathfrak{p}_v) = 1$; that is, there exists $w' \in \mathrm{Spv}(A)$ and a diagram of specializations*

$$\begin{tikzcd}
v \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] & w' \arrow[d, rightsquigarrow] \\
& w
\end{tikzcd}$$

The key ingredient in the proof of Theorem 5.12 is the "Exchange Lemma", below.

**Lemma 5.13.** [Con14, Proposition 4.5.1] *Let $v, v', w, w' \in \mathrm{Spv}(A)$.*

*(1) If $v \rightsquigarrow w$ is a horizontal specialization and $w \rightsquigarrow w'$ is a vertical specialization, then there exists a vertical specialization $v \rightsquigarrow v'$ and a commutative diagram*

$$\begin{tikzcd} v \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow, "\exists"'] & w \arrow[d, rightsquigarrow] \\ v' \arrow[r, rightsquigarrow] & w' \end{tikzcd}$$

(2) *If $v \rightsquigarrow w$ is a horizontal specialization and $v \rightsquigarrow v'$ is a vertical specialization, then there exists a unique vertical specialization $w \rightsquigarrow w'$ such that*

$$\begin{tikzcd} v \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & w \arrow[d, rightsquigarrow, "\exists!"] \\ v' \arrow[r, rightsquigarrow] & w' \end{tikzcd}$$

**Remark 5.14.** The caveats in Theorem 5.12(2) are necessary (ask Emanuel for an example).

5.2. **The space** $\mathrm{Spv}(A, I)$**.** For the rest of the talk, fix a ring $A$ and an ideal $I \subseteq A$ such that $\mathrm{Spec}(A)\backslash V(I)$ is quasi-compact, i.e. there exists a finitely-generated ideal $J \subseteq A$ such that $V(I) = V(J)$; this occurs iff $\sqrt{I}$ is the radical of some finitely-generated ideal of $A$.

In all of our applications, we will have the setup in the example below.

**Example 5.15.** Let $A$ be a Huber ring and let $I = A^{\circ\circ}$ be the ideal of topologically nilpotent elements. This data satisfies the above property: if $(A_0, J)$ is a pair of definition for $A$, then $\sqrt{I} = \sqrt{J \cdot A}$.

Consider the fibre product diagram

$$\begin{tikzcd} Y := \{v \in \mathrm{Spv}(A) \colon v(I) = 0\} \arrow[r, hook] \arrow[d] & \mathrm{Spv}(A) \arrow[d] \\ V(I) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd}$$

The goal of this section is to construct a "tubular neighborhood" $\mathrm{Spv}(A, I)$ of $Y$ in $\mathrm{Spv}(A)$. Roughly, $\mathrm{Spv}(A, I)$ will be defined by the following condition: if $v \in \mathrm{Spv}(A)$, then $v \in \mathrm{Spv}(A, I)$ iff all proper horizontal specializations lie over $V(I)$. After defining $\mathrm{Spv}(A, I)$, we will prove its 2 main properties: $\mathrm{Spv}(A, I)$ is a spectral space, and there is a spectral retraction $r\colon \mathrm{Spv}(A) \twoheadrightarrow \mathrm{Spv}(A, I)$.

If $v \in \mathrm{Spv}(A)$ and $H \subseteq \Gamma_v$ convex subgroup with $c\Gamma_v \subseteq H$, then $v_{|H}$ lies over $V(I)$ iff $v(I) \cap H = \emptyset$. Note that $v(I) \cap H = v(\sqrt{I}) \cap H$. There are 2 cases to consider:

**Case 1**. If $v(I) \cap c\Gamma_v \neq \emptyset$, then no horizontal specialization lives over $V(I)$; thus, the retraction ought to be just the constant map to the "most special" specialization $v_{|c\Gamma_v}$.

**Case 2**. If $v(I) \cap c\Gamma_v = \emptyset$, then after some horizontal specializations, we will lie over $V(I)$; we would like to describe the last point before this occurs.

**Definition 5.16.** If $\Gamma$ is a totally ordered abelian group and $H \subset \Gamma$ is a subgroup, then an element $\gamma \in \Gamma \cup \{0\}$ is said to be *cofinal for $H$* if for all $h \in H$, there exists $n \geq 0$ such that $\gamma^n < h$.

**Example 5.17.**
(1) A cofinal element of $\Gamma_v \cup \{0\}$ is zero.
(2) Any element $\gamma \in \Gamma_{\geq 1}$ is not cofinal, for any $H$. In particular, if $v(I) \cap c\Gamma_v \neq \emptyset$, there exists $\gamma \in v(I)$ that is not cofinal.

The last observation in Example 5.17 follows immediately from the following exercise.

**Exercise 5.18.** If $v \in \mathrm{Spv}(A)$, then $v(I) \cap c\Gamma_v \neq \emptyset$ iff $v(I) \cap v(A)_{\geq 1} = \emptyset$.

**Remark 5.19.** When $\Gamma = \Gamma_v$ for some $v \in \mathrm{Spv}(A)$ and $H = \Gamma_v$, then cofinal elements for $H$ are also called *pseudouniformizers*.

**Lemma 5.20.** *If $v \in \mathrm{Spv}(A)$ and if $H \subseteq \Gamma_v$ is a convex subgroup with $c\Gamma_v \subsetneq H$, then*

$$\mathfrak{a} = \mathfrak{a}_H := \{a \in A \colon v(a) \text{ is cofinal for } H\}$$

*is a radical ideal of $A$.*

*Proof.* If $a, b \in \mathfrak{a}$, then $v(a+b) \leq \max\{v(a), v(b)\}$, so $a+b$ is clearly cofinal. Let $b \in A$ and $a \in \mathfrak{a}$. If $v(b) \leq 1$, it is also clear that $ba$ is cofinal, because $v(ba) \leq v(a)$. If $v(b) > 1$, since $c\Gamma_v \subsetneq H$, we can find an element $h_0 \in H \backslash c\Gamma_v$ and (by passing to the inverse, if necessary) we may assume that $h_0 < c\Gamma_v$. Thus, for all $h \in H$,

$$v(ab)^n = v(a)^n v(b)^n < v(a)^n h_0^{-1},$$

because $v(b) \in c\Gamma_v$. Then, $v(a)^n h_0^{-1} < h$ for $n \gg 0$, because $v(a)$ is cofinal and $h_0 \in H$. Therefore, $\mathfrak{a}_H$ is an ideal. It is clearly radical. $\square$

**Proposition 5.21.** *If $v \in \mathrm{Spv}(A)$ and $v(I) \cap c\Gamma_v = \emptyset$, then:*

*(1) there exists a greatest convex subgroup $H_I \subseteq \Gamma_v$ containing $c\Gamma_v$ and such that $v(a)$ is cofinal for $H_I$, for all $a \in I$;*
*(2) if, in addition, $v(I) \neq 0$, then $H_I \supsetneq c\Gamma_v$ and $v(I) \cap H_I \neq \emptyset$.*

The conclusions of Proposition 5.21 need not hold when $v(I) \cap c\Gamma_v \neq \emptyset$, because in this case there was always an element which was not cofinal.

**Exercise 5.22.** The subgroup $H_I$ is contained in any convex subgroup $H$ of $\Gamma_v$ satisfying $v(I) \cap H \neq \emptyset$.

The geometric interpretation of Proposition 5.21 is captured by the above exercise. Indeed, having non-empty intersection with a convex subgroup $H$ means that $\mathfrak{p}_v$ does not contain $I$; thus, whenever a specialization does not lie over $V(I)$, it must be more generic than the one given by $H_I$. That is, the convex subgroup $H_I$ gives the "last" specialization prior to lying over $V(I)$.

**Definition 5.23.** Define the subgroup $c\Gamma_v(I)$ of $c\Gamma_v$ as

$$c\Gamma_v(I) = \begin{cases} c\Gamma_v & v(I) \cap c\Gamma_v \neq \emptyset, \\ H_I & v(I) \cap c\Gamma_v = \emptyset. \end{cases}$$

The subgroup $c\Gamma_v(I)$ gives, by construction, the "last" horizontal specialization prior to lying over $V(I)$.

**Lemma 5.24.** *If $v \in \mathrm{Spv}(A)$ and $\sqrt{I} = \sqrt{(a_1, \ldots, a_n)}$ for some $a_1, \ldots, a_n \in A$, then the following are equivalent:*

*(1) $c\Gamma_v(I) = \Gamma_v$;*
*(2) $\Gamma_v = c\Gamma_v$, or $v(a)$ is cofinal in $\Gamma_v$ for all $a \in I$*
*(3) $\Gamma_v = c\Gamma_v$, or $v(A_i)$ cofinal in $\Gamma_v$ for all $i = 1, \ldots, n$.*

*Proof.* The equivalent of (1) and (2) follows immediately from Proposition 5.21, and the equivalence of (2) and (3) is straightforward. $\square$

**Definition 5.25.** The subset $\mathrm{Spv}(A, I)$ of $\mathrm{Spv}(A)$ is given by

$$\mathrm{Spv}(A, I) = \{v \in \mathrm{Spv}(A) \colon c\Gamma_v(I) = \Gamma_v\},$$

and it is equipped with the subspace topology.

By construction, $\mathrm{Spv}(A, I)$ only depends on $\sqrt{I}$, and it consists of all valuations such that every proper specialization lies over $V(I)$. Moreover, one can recover $\mathrm{Spv}(A)$ by setting $I = 0$.

**Example 5.26.** If $A = I$, then $\mathrm{Spv}(A, A) = \{v \in \mathrm{Spv}(A) \colon c\Gamma_v = \Gamma_v\}$. Said differently, $\mathrm{Spv}(A, A)$ consists of all valuations without horizontal specializations.

There is a retraction of sets $r \colon \mathrm{Spv}(A) \to \mathrm{Spv}(A, I)$ given by $v \mapsto v_{|c\Gamma_v(I)}$. We will see that both $\mathrm{Spv}(A, I)$ and $r$ are, in fact, spectral.

**Proposition 5.27.** *(1) $X = \mathrm{Spv}(A, I)$ is a spectral space.*

*(2) A basis $\mathcal{R}$ of quasi-compact open sets for the topology on $X$ is given by those of the form*

$$X(T/s) = \{v \in X \colon v(f_i) \leq v(s) \neq 0 \text{ for } i = 1, \ldots, n\},$$

*where $T = \{f_1, \ldots, f_n\} \subset A$ is a finite subset such that $I \subseteq \sqrt{T \cdot A}$.*

*(3) The retraction $r$ is spectral; in particular, $r$ is continuous.*

*(4) If $v \in \operatorname{Spv}(A)$ lies over $\operatorname{Spec}(A) \backslash V(I)$, so too does the retraction $r(v)$.*

**Remark 5.28.** While the retraction $r \colon \operatorname{Spv}(A) \to \operatorname{Spv}(A, I)$ is continuous and spectral, the inclusion map $\operatorname{Spv}(A, I) \hookrightarrow \operatorname{Spv}(A)$ is continuous (because $\operatorname{Spv}(A, I)$ is equipped with the subspace topology), but it is not spectral.

*Proof of Proposition 5.27.* Assume WLOG that $I = (a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A$. The collection $\mathcal{R}$ consists of open sets: indeed, if $X(T/s) \in \mathcal{R}$, then $X(T/s) = X \cap \operatorname{Spv}(T/s)$. Observe that $X(T/s) = X(T \cup \{s\}/s)$; it will be useful to assume that, whenever we write $X(T/s)$, $s \in T$.

To see that $\mathcal{R}$ is closed under finite intersections, note that iff $X(T_1/s_1), X(T_2/s_2) \in \mathcal{R}$ then $T \coloneqq T_1T_2 = \{f_1f_2 \colon f_1 \in T_1, f_2 \in T_2\}$ and $X(T_1/s_1) \cap X(T_2/s_2) = X(T/s_1s_2)$. Moreover, $I^2 \subseteq \sqrt{T \cdot A}$, so taking radicals shows that $T$ is of the desired form.

Let us now check that $\mathcal{R}$ is a base: if $v \in \operatorname{Spv}(A, I)$ and if $U$ is an open neighborhood of $v$ in $\operatorname{Spv}(A)$, choose$g_1, \ldots, g_{n_1} \in A$ such that $v \in W \coloneqq \operatorname{Spv}(A)(\frac{g_1, \ldots, g_{n_1}}{s}) \subseteq U$. We must construct a neighborhood $W'$ of $v$ that intersect $W$. Consider two cases:

**Case 1**. If $\Gamma_v = c\Gamma_v$, then there exists $d \in A$ such that $v(ds) \geq 1$. Then, $W' \coloneqq X(gd, 1/sd)$ is such a neighborhood.

**Case 2**. If $c\Gamma_v \neq \Gamma_v$, then Lemma 5.24 implies that $v(a_i)$ is cofinal in $\Gamma_v$ for all $i = 1, \ldots, n$. It follows that $v(a_i)^k \leq v(s)$ for $k \gg 0$, hence we can set $W' \coloneqq X(\frac{g, a_1^k, \ldots, a_m^k}{s}) \subseteq W \cap X$.

We have shown all of (2), except for the quasi-compactness of the sets $X(T/s)$. For (3), we claim that $r^{-1}(X(T/s)) = W \coloneqq \operatorname{Spv}(A)(T/s)$. This immediately implies that $r$ is continuous, and it will also imply that $r$ is spectral, once we know that the constituents of $\mathcal{R}$ are quasi-compact. To see the inclusion $r^{-1}(X(T/s)) \subseteq W$, observe that $X(T/s) \subseteq W$ and everything in $r^{-1}(X(T/s))$ is a generization of some element of $X(T/s)$; however, $W$ is open, so it is closed under taking generizations, hence $r^{-1}(X(T/s)) \subseteq W$. The other inclusion is left as an exercise.

For (1), set $X'$ to be the set $X$ with the topology generated by boolean combinations of $\mathcal{R}$. Then, (3) implies that $r \colon (\operatorname{Spv} A)_{\mathrm{cons}} \twoheadrightarrow X'$ is continuous. As $\operatorname{Spv} A$ is spectral, it follows that $(\operatorname{Spv} A)_{\mathrm{cons}}$ is quasi-compact and, by surjectivity, $X'$ is quasi-compact. Let $\Sigma = \mathcal{R}$ be the collection of clopen subsets of $X'$. Note that $X$ is $T_0$, because $X \subseteq \operatorname{Spv}(A)$ is equipped with the subspace topology, and $\operatorname{Spv}(A)$ is $T_0$ Then, Theorem 4.15 implies that $\operatorname{Spv}(A, I)$ is spectral, with $\Sigma = \mathcal{R}$ as base of quasi-compact open subsets. This proves (1), and the quasi-compactness in (2). $\square$

## 6. February 16th – Continuous valuations and adic spectra (Takumi Murayama)

This talk is based on the third section of Huber's paper [Hub93]. Takumi has posted notes [Mur17], which are much more detailed than the lecture and contain much supplementary material. All rings are commutative and unital. We briefly recall some definitions from the previous lectures.

**Definition 6.1.** If $A$ is a ring, the *valuation spectrum* of $A$ is the set

$$\operatorname{Spv}(A) \coloneqq \Big\{\text{valuations } v \colon A \to \Gamma \cup \{0\}\Big\} \Big/ \begin{tikzcd} & \Gamma_v \cup \{0\} \arrow[dd, "\simeq"] \\ A \arrow[ur, "v"] \arrow[dr, "w"'] & \\ & \Gamma_w \cup \{0\} \end{tikzcd}$$

where $\Gamma_v := \langle \mathrm{im}(v)\backslash 0\rangle \subseteq \Gamma$ is the *value group* of $v$. The topology on $\mathrm{Spv}(A)$ is generated by rational domains, i.e. subsets of the form

$$R(f/g) = \{v \in \mathrm{Spv}(A) \colon v(f) \leq v(g) \neq 0\},$$

for $f, g \in A$.

The goal is to associate, to a Huber ring $A$, spectral spaces $\mathrm{Spa}(A, A^+) \subseteq \mathrm{Cont}(A) \subseteq \mathrm{Spv}(A)$ such that $\mathrm{Cont}(A)$ remembers the topology on $A$, and $\mathrm{Spa}(A, A^+)$ also remembers the subring $A^+ \subseteq A$. The relationships are summarized in the diagram below:

$$\begin{tikzcd}
\mathrm{Ring}^{\mathrm{op}} \arrow[r, "\mathrm{discrete}"] \arrow[drr, "\mathrm{Spv}"'] & \mathrm{HubRing}^{\mathrm{op}} \arrow[r, "\mathrm{Cont}"] \arrow[dr, dashed, "\exists"] & \mathrm{Top} \\
& & \mathrm{SpecSp} \arrow[u, hook]
\end{tikzcd}$$

Here, HubRing denotes the category of Huber rings (with adic morphisms) and SpecSp denotes the category of spectral spaces (with spectral maps).

6.1. $\mathrm{Cont}(A)$**.** From now on, all rings $A$ are Huber rings.

**Definition 6.2.** A valuation $v \in \mathrm{Spv}(A)$ is *continuous* if any of the following equivalent conditions holds:

(1) $v \colon A \to \Gamma_v \cup \{0\}$ is continuous, where $\Gamma_v \cup \{0\}$ has the order topology;
(2) for all $\gamma \in \Gamma_v$, $\{f \in A \colon v(f) < \gamma\}$ is open;
(3) the topology on $A$ is finer than that induced by $v$.

Define $\mathrm{Cont}(A) := \{v \in \mathrm{Spv}(A) \colon v \text{ continuous}\}$, equipped with the subspace topology.

**Example 6.3.** If $A$ is discrete, then $\mathrm{Cont}(A) = \mathrm{Spv}(A)$.

**Example 6.4.** If $v \in \mathrm{Spv}(A)$ has $\Gamma_v = 1$, then $v \in \mathrm{Cont}(A)$ iff $\mathrm{supp}(v)$ is open.

**Example 6.5.** With notation as in Example 3.28(2), the valuations $v, w, \overline{\eta}, \overline{w} \in \mathrm{Spv}\,(k((y))[[x]])$ are continuous, but the trivial valuation $\eta \in \mathrm{Spv}\,(k((y))[[x]])$ is not continuous, because $(0)$ is not open in $k((y))[[x]]$.

**Remark 6.6.** The assignment $A \mapsto \mathrm{Cont}(A)$ is a functor: if $f \colon A \to B$ continuous ring homomorphism, then $\mathrm{Cont}(f) \colon \mathrm{Cont}(B) \to \mathrm{Cont}(A)$ given by precomposition, i.e. $v \mapsto v \circ f$.

6.2. **Spectrality of** $\mathrm{Cont}(A)$**.** Last time, we showed that the space $\mathrm{Spv}(A, I)$ is spectral, and we will use this to show that $\mathrm{Cont}(A)$ is also spectral. Let us first recall some definitions from last time.

**Definition 6.7.** If $v \in \mathrm{Spv}(A)$, the *characteristic subgroup* $c\Gamma_v$ is the convex subgroup of $\Gamma$ generated by the subset $\{v(a) \colon v(a) \geq 1\}$. An element $\gamma \in \Gamma \cup \{0\}$ is said to be *cofinal in a subgroup* $H \subseteq \Gamma$ if for all $h \in H$, there exists $n \in \mathbf{N}$ such that $\gamma^n < h$.

Last time, we proved the following:

**Proposition 6.8.** *If $I \subseteq A$ is an ideal, the space*

$$\mathrm{Spv}(A, I) := \{v \in \mathrm{Spv}(A) \colon \textit{either } \Gamma_v = c\Gamma_v, \textit{ or } v(a) \textit{ is cofinal in } \Gamma_v \textit{ for all } a \in I\}$$

*is spectral, provided $\sqrt{I}$ is the radical of some finitely-generated ideal of $A$;* $\mathrm{Spv}(A, I)$ *has a quasi-compact basis of open, constructible subsets of the form*

$$R(T/s) = \{v \in \mathrm{Spv}(A, I) \colon v(f_i) \leq v(s) \neq 0 \textit{ for all } i\},$$

*where $T = \{f_1, \dots, f_n\} \subset A$ is a nonempty finite subset such that $I \subseteq \sqrt{T \cdot A}$, and $s \in A$.*

Recall that $A^{\circ\circ} := \{a \in A \colon a^n \to 0 \text{ as } n \to \infty\}$ is the ideal of topologically nilpotent elements.

**Lemma 6.9.** *If $T = \{f_1, \dots, f_n\} \subset A$ is a nonempty finite subset, then $T \cdot A$ is open in $A$ iff $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$.*

*Proof.* If $I$ is an ideal of definition of $A$, then $T \cdot A$ is open iff $I^n \subseteq T \cdot A$ for some $n > 0$, but by properties of radical ideals, this occurs iff $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$ (here, we are using that $\sqrt{A^{\circ\circ} \cdot A} = \sqrt{I \cdot A}$). ☐

**Theorem 6.10.** $\operatorname{Cont}(A) = \{v \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A) \colon v(a) < 1$ *for all* $a \in A^{\circ\circ}\}$.

**Corollary 6.11.** $\operatorname{Cont}(A)$ *is closed in* $\operatorname{Spv}(A, A^{\circ\circ} \cdot A)$, *hence it is spectral and closed under specialization.*

*Proof.* Observe that $\operatorname{Spv}(A, A^{\circ\circ} \cdot A) \setminus \operatorname{Cont}(A) = \bigcup_{a \in A^{\circ\circ}} \operatorname{Spv}(A, A^{\circ\circ} \cdot A)(1/a)$, hence $\operatorname{Cont}(A)$ is closed. Moreover, it is a general fact that a closed subspace of a spectral space is again spectral. ☐

*Proof of Theorem 6.10.* The inclusion $\subseteq$ is straightforward. If$w \in \operatorname{Cont}(A)$ and $a \in A^{\circ\circ}$, then $w(a^n) = w(a)^n < \gamma$ for any $\gamma \in \Gamma_w$ and $n = n(\gamma) \gg 0$, by continuity. Thus, $w(a) < 1$ by choosing $\gamma = 1$. It also follows that $w(a)$ is cofinal in $\Gamma_w$. For the opposite inclusion $\supseteq$, let $v \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A)$ be such that $v(a) < 1$ for all $a \in A^{\circ\circ}$. We proceed in 2 steps.

**Step 1**. Show that $v(a)$ is cofinal in $\Gamma_v$ for all $a \in A^{\circ\circ}$.

If $\Gamma_v \neq c\Gamma_v$, then this is obvious. If $\Gamma_v = c\Gamma_v$, take $\gamma \in \Gamma_v$. If $\gamma \geq 1$, then we are done since $v(a) < 1$ by hypothesis. If $\gamma < 1$, then $\frac{v(t')}{v(t)} < \gamma$ for some $t, t' \in A$, hence $v(t)^{-1} < \gamma$. Let $n \in \mathbf{N}$ be such that $ta^n \in A^{\circ\circ}$, then $v(ta^n) < 1$, $v(a)^n < \gamma$, as required.

**Step 2**. Show that $v \in \operatorname{Cont}(A)$.

Pick a set $S = \{t_1, \dots, t_r\}$ of generators of $I$, where $I$ is an ideal of definition of $A$. Set $\delta = \max\{v(t_1), \dots, v(t_r)\}$, then $\delta < 1$ because $t_i \in A^{\circ\circ}$ for $i = 1, \dots, r$. By Step 1, there exists $n \in \mathbf{N}$ such that $\delta^n < \gamma$, so $v(S^n \cdot I) = v(I^{n+1}) < \gamma$, hence $I^{n+1} \subseteq \{f \in A \colon v(f) < \gamma\}$. That is, $v$ is continuous. ☐

**Proposition 6.12.** *The functor* $\operatorname{Cont} \colon \mathrm{HubRing}^{\mathrm{op}} \to \mathrm{Top}$ *sends adic morphisms to spectral maps.*

*Proof.* If $f \colon A \to B$ is adic, then $\operatorname{Cont}(f)^{-1}(R(T/s)) = R(f(T)/f(s))$ and this set is still a rational domain precisely because $f$ is adic. ☐

### 6.3. **Analytic points.**

**Definition 6.13.** A valuation $v \in \operatorname{Cont}(A)$ is *analytic* if $\operatorname{supp}(v)$ is not open in $A$. Let $\operatorname{Cont}(A)_a$ denote the subset of analytic points of $\operatorname{Cont}(A)$, and let $\operatorname{Cont}(A)_{na} \coloneqq \operatorname{Cont}(A) \setminus \operatorname{Cont}(A)_a$.

**Example 6.14.** If $A$ is discrete, then there are no analytic points.

**Proposition 6.15.** *If* $T \subset A^{\circ\circ}$ *is a finite subset such that* $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$, *then* $v \in \operatorname{Cont}(A)$ *is analytic iff* $v(t) \neq 0$ *for some* $t \in T$.

*Proof.* Observe that $\operatorname{supp}(v)$ is open iff $(T \cdot A)^n \subseteq \operatorname{supp}(v)$ for some $n > 0$, which occurs iff $T \subseteq \operatorname{supp}(v)$, by taking radicals. ☐

**Corollary 6.16.** *If* $A$ *is Tate, then* $\operatorname{Cont}(A) = \operatorname{Cont}(A)_a$; *that is, every point is analytic.*

*Proof.* If $u \in A^{\circ\circ} \cap A^*$ is a topologically nilpotent unit, then take $T = \{u\}$ in Proposition 6.15. ☐

**Example 6.17.** With notation as in Example 3.28(2), suppose $k((y))[[x]]$ is equipped with the $x$-adic topology. Then, the ideal $(x)$ is open, so the points $v, w \in \operatorname{Spv}(k((y))[[x]])$ are analytic, but $\overline{\eta}, \overline{w}$ are not.

The above example is one instance of a more general phenomenon in [Hub94], where the "closed fibre" of the valuation spectrum of a dvr will consist of analytic points (it corresponds to a formal scheme), and the "generic fibre" consists of non-analytic points (it corresponds to a rigid-analytic space).

**Remark 6.18.** A valuation $v \in \operatorname{Cont}(A)$ is analytic iff $v$ has a "Tate neighborhood", in the sense of [Hub94].

**Proposition 6.19.** *Let* $f \colon A \to B$ *be a continuous ring homomorphism, and let* $g \coloneqq \operatorname{Cont}(f) \colon \operatorname{Cont}(B) \to \operatorname{Cont}(A)$. *Then,*

*(1) g preserves non-analytic points;*

*(2) if $f$ is adic, then $g$ preserves analytic points.*

*Proof.* If $v \in \mathrm{Cont}(B)$ is not analytic, then $\mathrm{supp}(v)$ is open in $B$, so the preimage by the continuous map $f$ is also open, hence $v \circ f$ is not analytic. □

6.4. **The adic spectrum.** The space $\mathrm{Cont}(A)$ is not exactly the space that we would like to use in applications; the preferred definition is hinted at by the lemma below.

**Lemma 6.20.** ["Adic nullstellensatz"]

*(1) There is an inclusion-reversing bijection*

$$\mathcal{G}_a := \left\{ \begin{matrix} \textit{open, integrally closed} \\ \textit{subrings of } A \end{matrix} \right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{ \begin{matrix} \textit{pro-constructible subsets of } \mathrm{Cont}(A) \\ \textit{which are intersections of subsets of the} \\ \textit{form } \{v \in \mathrm{Cont}(A) \colon v(a) \le 1\} \end{matrix} \right\},$$

*where*

$$\mathcal{G}_a \ni G \overset{\sigma}{\mapsto} \{v \in \mathrm{Cont}(A) \colon v(g) \le 1 \textit{ for all } g \in G\},$$

*and the inverse functor is given by*

$$F \mapsto \{a \in A \colon v(a) \le 1 \textit{ for all } v \in F\}.$$

*(2) If $G \in \mathcal{G}_A$ is such that $G \subseteq A^\circ$, then $\sigma(G)$ is dense in* $\mathrm{Cont}(A)$.

**Example 6.21.** The subring $A^\circ$ of power-bounded elements belongs to $\mathcal{G}_A$.

**Definition 6.22.** (1) A subring $A^+ \subseteq A$ is called a *ring of integral elements* if it is open, integrally closed, and it is contained in $A^\circ$.

(2) A *Huber pair* $(A, A^+)$ is a Huber ring $A$ together with a ring of integral elements $A^+ \subseteq A$. A morphism $\Phi \colon (A, A^+) \to (B, B^+)$ of Huber pairs is a ring homomorphism $f \colon A \to B$ such that $f(A^+) \subseteq B^+$. We say $\Phi$ is adic or continuous if $f$ is.

(3) The *adic spectrum* of the Huber pair $(A, A^+)$ is

$$\mathrm{Spa}(A, A^+) := \sigma(A^+) = \{v \in \mathrm{Cont}(A) \colon v(f) \le 1 \text{ for all } f \in A^+\}.$$

The terminology of Huber pairs is found in [Con14], but [Hub93, Wed12] call them affinoid rings.

**Remark 6.23.** If $f \in A$ is such that $v(f) \le 1$ for all $v \in \mathrm{Spa}(A, A^+)$, then $f \in A^+$.

**Remark 6.24.** The assignment $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ is a functor $\mathrm{HubPair}_{\mathrm{cont}}^{\mathrm{op}} \to \mathrm{Top}$, where $\mathrm{HubPair}_{\mathrm{cont}}$ denotes the category of Huber pairs with continuous morphisms.

**Remark 6.25.** The adic spectrum $\mathrm{Spa}(A, A^+)$ has a structure presheaf, which is described in [Hub94].

**Example 6.26.** If $B$ is the integral closure of $\mathbf{Z} \cdot 1 + A^{\circ\circ}$ (i.e. the smallest subring of $A$ containing $A^{\circ\circ}$), then $\mathrm{Cont}(A) = \mathrm{Spa}(A, B)$.

*Proof of Lemma 6.20.* For (1), it is easy to check that $\sigma \circ \tau = id$. The opposite direction is harder, i.e. to show $\tau \circ \sigma = id$. If $G \in \mathcal{G}_A$, then $G \subseteq \tau(\sigma(G))$ is easy. Suppose there is $a \in \tau(\sigma(G)) \backslash G$, and we will construct a valuation $v \in \sigma(G)$ such that $v(a) > 1$, which yields a contradiction. For the construction, see [Mur17, Lemma 3.1] for a nice "proof by picture".

For (2), we will show the stronger statement that every point $v \in \mathrm{Cont}(A)$ is a vertical specialization of a point in $\sigma(G)$. There are two cases to consider:

(1) If $\mathrm{supp}(v)$ is open, then the vertical generization $v_{/\Gamma_v} \in \sigma(G)$.

(2) If $\mathrm{supp}(v)$ is not open, then $\mathrm{supp}(v) \not\supseteq A^{\circ\circ}$, so there exists $a \in A^{\circ\circ}$ such that $v(a) > 0$. Let $H$ be the largest convex subgroup of $\Gamma_v$ with $v(a) \notin H$. Consider the composition $w \colon A \xrightarrow{v} \Gamma_v \cup \{0\} \twoheadrightarrow \Gamma_v / H \cup \{0\}$, then one can check that $w(g) \le 1$.

□

## 6.5. Spectrality of $\mathrm{Spa}(A, A^+)$.

**Definition 6.27.** If $(A, A^+)$ is a Huber pair, a *rational domain* in $\mathrm{Spa}(A, A^+)$ is a subset of the form

$$R(T/s) \coloneqq \{v \in \mathrm{Spa}(A, A^+) \colon v(t) \leq v(s) \neq 0 \text{ for all } t \in T\},$$

where $s \in A$, and $T \subset A$ is a nonempty finite subset such that $T \cdot A$ is open in $A$.

**Theorem 6.28.** *If $X = \mathrm{Spa}(A, A^+)$ is an adic spectrum, then:*

*(1) $X$ is spectral; that is, the functor $\mathrm{HubPair}_{\mathrm{ad}}^{\mathrm{op}} \to \mathrm{Top}$ factors through $\mathrm{SpecSp}$.*
*(2) Rational domains form a quasi-compact basis of open, constructible subsets of $X$, which are closed under finite intersection.*

*Proof.* In Lemma 6.20, we showed that $X$ is a proconstructible subset of $\mathrm{Cont}(A)$, and proconstructible subsets of spectral spaces are again spectral. The factorization is immediate.

For (2), recall that sets of the form $R(T/s)$, where $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$, form a basis for $\mathrm{Spv}(A, A^{\circ\circ} \cdot A)$. Thus, the only thing to check is that the conditions on $T$ are equivalent, but this was Lemma 6.9. $\square$

## 6.6. Nonemptiness criterion.

Recall that if $B$ is a ring, then $\mathrm{Spec}(B) = \emptyset$ iff $B = 0$. We would like to have a similar criterion for when the adic spectrum is empty.

**Proposition 6.29.** *If $(A, A^+)$ is a Huber pair, then*

*(1) $\mathrm{Spa}(A, A^+) = \emptyset$ iff $A/\overline{\{0\}} = 0$;*
*(2) $\mathrm{Spa}(A, A^+)_a = \emptyset$ iff $A/\overline{\{0\}}$ has the discrete topology.*

*Proof.* Note that $\mathrm{Spa}(A/\overline{\{0\}}, A^+/\overline{\{0\}}) \to \mathrm{Spa}(A, A^+)$ is a bijection, since $v(\overline{\{0\}}) = 0$ for all $v \in \mathrm{Cont}(A)$. Consider first the implications $\Leftarrow$. For (1), if $A/\overline{\{0\}} = 0$, then the zero ring has no valuations with our definitions. For (2), we already saw that a discrete ring has no analytic points.

Next, consider the impliciations $\Rightarrow$. For (1) (assuming (2) holds): if $\mathrm{Spa}(A, A^+) = \emptyset$, then $\mathrm{Spa}(A, A^+)_a = \emptyset$, so $A/\overline{\{0\}}$ is discrete. If $A/\overline{\{0\}} \neq 0$, then there exist valuations on $A$, contradicting $\mathrm{Spa}(A, A^+) = \emptyset$. For (2), we proceed in three steps.

**Step 1**. If $B \subseteq A$ is an open subring, $f \colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is the corresponding morphism on spectra, and $T = \{\mathfrak{p} \in \mathrm{Spec}(B) \colon \mathfrak{p} \subseteq B \text{ open}\}$, then $f^{-1}(T) = \{\mathfrak{p} \in \mathrm{Spec}(A) \colon \mathfrak{p} \subseteq A \text{ open}\}$, and moreover the restriction $\mathrm{Spec}(A) \backslash f^{-1}(T) \xrightarrow{\simeq} \mathrm{Spec}(B) \backslash T$ is an isomorphism.

If $\mathfrak{p} \in \mathrm{Spec}(B) \backslash T$, take $s \in B^{\circ\circ}$ such that $s \notin \mathfrak{p}$. Thus, for all $a \in A$, there exists $n \in \mathbf{N}$ such that $s^n a \in B$, since $B$ is open. This implies that $B_s \to A_s$ is an isomorphism. The description of $f^{-1}(T)$ follows by Proposition 6.19(2).

**Step 2**. Let $B$ be a ring of definition of $A$ with ideal of definition $I$ and let $\mathfrak{p} \subseteq \mathfrak{q}$ be two primes in $B$. If $I \subseteq \mathfrak{q}$, then $I \subseteq \mathfrak{p}$. Geometrically, $V(I)$ contains every irreducible component of $\mathrm{Spec}(B)$ that it touches.

Suppose that $I \not\subseteq \mathfrak{p}$. Let $u \in \mathrm{Spv}(B)$ with $\mathfrak{p} = \mathrm{supp}(u)$ and such that the valuation ring for $u$ dominates $(B/\mathfrak{p})_{\mathfrak{q}/\mathfrak{p}}$. First, use the retraction map $r \colon \mathrm{Spv}(B) \to \mathrm{Spv}(B, I)$, then $r(u) \in \mathrm{Cont}(B)$ and $I \not\supseteq \mathrm{supp}(r(u))$, hence $\mathrm{supp}(r(u))$ is not open. By Step 1, there exists $v \in \mathrm{Cont}(A)$ such that $r(u) = v|_B$. Since every point in $\mathrm{Cont}(A)$ generalizes to a point in $\mathrm{Spa}(A, A^+)$, there exists $w \rightsquigarrow v$; then, $w \in (\mathrm{Spa}(A, A^+))_a$.

**Step 3**. $A/\overline{\{0\}}$ is discrete.

Consider $\varphi \colon B \to (1 + I)^{-1} B =: C$, then $\varphi(I) \cdot C \subseteq \mathcal{R}(C)$, where $\mathcal{R}(C)$ denotes the Jacobson radical of $C$. Thus, every maximal ideal in $C$ contains $I$. By Step 2, the same is true for prime ideals, i.e. $\varphi(I) \cdot C \subseteq \mathcal{N}(C)$, where $\mathcal{N}(C)$ denotes the nilradical. As $I$ is finitely-generated, this implies that $\varphi(I^n) \cdot C = \{0\}$. By the definition of localization, there exists $i \in I$ such that $(1 + i) I^n = \{0\}$ in $B$, so $I^n \subseteq I^{n+1}$, hence $I^n = I^{n+1} = I^k$ for all $k \geq n$. Thus, $A/\overline{\{0\}}$ is discrete. $\square$

6.7. **Invariance under completion.** If we think of $\mathrm{Spa}(A, A^+)$ as a "punctured tubular neighborhood", then we would expect it not to change under completion; this is the aim of this section.

If $(A, A^+)$ is a Huber pair, then the completion $(\widehat{A}, \widehat{A^+})$ is a Huber pair, where $\widehat{A^+} \coloneqq \overline{(A^+)^\wedge}$ is the integral closure of the completion of $A^+$.

**Proposition 6.30.** *The canonical map $g\colon \mathrm{Spa}(\widehat{A}, \widehat{A^+}) \to \mathrm{Spa}(A, A^+)$ is a homeomorphism, which identifies rational domains in the source and target.*

**Lemma 6.31.** *Let $X$ be a quasi-compact subset of $\mathrm{Spa}(A, A^+)$, and let $s \in A$ be such that $v(s) \neq 0$ for all $v \in X$. Then, there exists a neighborhood $U$ of $0$ in $A$ such that $v(u) < v(s)$ for all $v \in X$ and all $u \in U$.*

*Proof.* Let $T \subset A^{\circ\circ}$ be a nonempty finite subset such that $T \cdot A$ is open, and consider the rational domain $X_n \coloneqq R(T^n/s) \subseteq \mathrm{Spa}(A, A^+)$. Then, $X \subseteq \bigcup_{n\in\mathbf{N}} X_n$. As $X$ is quasi-compact, $X \subseteq X_m$ for some $m > 0$, and one can show that $U = T^m \cdot A^{\circ\circ}$ works. $\square$

**Lemma 6.32.** [Mur17, Lemma 3.14] *Let $A$ be complete, and let $s, t_1, \ldots, t_n \in A$ be such that $I = (t_1, \ldots, t_n)A$ is an open ideal. Then, there exists a neighborhood $U$ of $0$ in $A$ such that*

$$R\left(\frac{t_1, \ldots, t_n}{s}\right) = R\left(\frac{t_1', \ldots, t_n'}{s'}\right)$$

*for any $s' \in s + U$, $t_i' \in t_i \in U$, and $I' = (t_1', \ldots, t_n')A$ open.*

*Proof of Proposition 6.30.* First, continuous valuations extend continuously to $\widehat{A}$, in a unique manner. Thus, there is a bijection $g\colon \mathrm{Spa}(\widehat{A}, \widehat{A^+}) \to \mathrm{Spa}(A, A^+)$. Second, we have already shown that rational domains pull back to rational domains. It remains to show that if $U$ is a rational domain, then $g(U)$ is a rational domain. Let $i\colon A \to \widehat{A}$ be the completion map. By Lemma 6.32 and the fact that $i(A)$ is dense in $\widehat{A}$, we can write $U = R(i(T)/i(s))$ for $s \in A$ and $T \subset A$ satisfying the usual conditions. As $U$ is quasi-compact, $v(i(s)) \neq 0$ for all $v \in U$, so Lemma 6.31 implies there is a open neighborhood $G$ of $0$ such that $v(i(g)) \leq v(i(s))$ for all $v \in U$ and all $g \in G$. Finally, let $D$ be a finite subset of $G$ such that $D \cdot A$ is open, then $g(U) = R(T \cup D/s)$. $\square$

## 7. February 23rd – Tate algebras of finite type (Matt Stevenson)

7.1. **Motivation.** If $k$ is a complete non-Archimedean field and $A$ is a Tate algebra over $k$, then to this data we can associate a rigid-analytic space $\mathrm{Sp}(A)$ with its Grothendieck topology $\mathcal{T}_A$ (to be defined later). It follows from Deligne's completeness theorem and [GV71, IV.4.2.4] that there is a unique sober[5] topological space $X_A$ that admits an equivalence of topoi

$$\mathrm{Shv}(\mathcal{T}_A) \simeq \mathrm{Shv}(X_A).$$

Our goal for today is to show $X_A \simeq \mathrm{Spa}(A, A^\circ)$ and moreover that there is a *canonical* equivalence of topoi.

As an analogy, consider the following classical result: let $A$ be a $k$-algebra of finite type, let $X = \mathrm{Spec}(A)$ (or more generally, any scheme locally of finite type over a field), and let $X_0 = \mathrm{MaxSpec}(A)$ (i.e. the subspace of closed points of $X$). Then, there is an inclusion-preserving bijection

$$\{\text{constructible subsets of } X\} \xrightarrow{\simeq} \{\text{constructible subsets of } X_0\},$$

given by $C \mapsto C_0 \coloneqq C \cap X_0$, such that $C \subseteq X$ is open/closed iff $C_0 \subseteq X_0$ is open/closed. Therefore, there is an equivalence of topoi

$$\mathrm{Shv}(X) \xrightarrow{\simeq} \mathrm{Shv}(X_0),$$

given by $\mathcal{F} \mapsto \mathcal{F}|_{X_0}$.

[5]A topological space is *sober* if every irreducible closed subset has a unique generic point. Easy examples include locally Hausdorff topological spaces or schemes.

In our setting, the adic spectrum $\mathrm{Spa}(A, A^\circ)$ will play the role of the affine scheme $\mathrm{Spec}(A)$, and the rigid space $\mathrm{Spa}(A)$ will play the role of the maximal spectrum $\mathrm{MaxSpec}(A)$; however, the proof is more involved as certain intermediate steps/spaces are required.

Why is such a result important? One reason is that a sober topological space is more-or-less determined by its sheaf theory, in the following sense.

**Theorem 7.1.** [MLM94, Proposition IX.5.2] *If $X, Y$ are sober topological spaces, then there is a bijection*

$$\mathrm{Hom}_{\mathrm{Top}}(X, Y) \xrightarrow{\simeq} \mathrm{Mor}_{\mathrm{Topoi}}(\mathrm{Shv}(X), \mathrm{Shv}(Y)) := \{\textit{adjoint pairs } (F, G) \textit{ such that } G \textit{ is exact}\} / \sim,$$

*given by $f \mapsto (f_*, f^{-1})$.*

In particular, if $X = \{*\}$ is a point, then the theorem recovers $|Y| = \mathrm{Mor}_{\mathrm{Topoi}}(\mathrm{Sets}, \mathrm{Shv}(Y))$, where $|Y|$ denotes the underlying set of $Y$.

7.2. **Tate algebras and rigid-analytic spaces.** Let $k$ be a field, complete with respect to a non-trivial rank-1 valuation $|\cdot|$. Consider the $k$-algebra

$$k\langle T_1, \dots, T_n\rangle := \left\{ \sum_{\nu \in (\mathbf{Z}_{\geq 0})^n} a_\nu T^\nu \colon a_\nu \in k, a_\nu \to 0 \text{ as } |\nu| \to +\infty \right\}$$

The $k$-algebra $k\langle T_1, \dots, T_n\rangle$ is equipped topology induced by the Gauss norm

$$\left\| \sum_{\nu \in (\mathbf{Z}_{\geq 0})^n} a_\nu T^\nu \right\| := \max_\nu |a_\nu|.$$

**Exercise 7.2.** $k\langle T_1, \dots, T_n\rangle$ is a complete Tate ring.

**Definition 7.3.** A complete topological $k$-algebra $A$ is a *Tate algebra over $k$* if there is a continuous, open, surjective $k$-algebra homomorphism $k\langle T_1, \dots, T_n\rangle \twoheadrightarrow A$.

**Remark 7.4.** In Huber's works, Tate algebras are also called Tate rings topologically of finite type over $k$. In rigid-analytic geometry, these are called $k$-affinoid algebras. In Berkovich geometry, these are called strictly $k$-affinoid algebras (however, Berkovich's $k$-affinoid algebras need not be Tate algebras in our sense, because there we allow $k$ to be trivially valued).

**Fact 7.5.**
1. Any Tate algebra over $k$ is noetherian and all ideals are closed.
2. [Noether normalization] For any Tate algebra $A$ over $k$, there is a finite monomorphism (i.e. an injective continuous $k$-algebra homomorphism) $k\langle T_1, \dots, T_d\rangle \hookrightarrow A$ for some $d \geq 0$.
3. Every $k$-algebra homomorphism between Tate algebras over $k$ is continuous, and induces a map on the subrings of power-bounded elements.

Fix a Tate algebra $A$ over $k$ and let $\mathrm{Max}\, A$ denote the set of maximal ideals of $A$. A point of $\mathrm{Max}\, A$ is denoted by $x$, and the corresponding maximal ideal of $A$ is denoted $\mathfrak{m}_x$.

**Fact 7.6.** For any $x \in \mathrm{Max}\, A$, the residue field $A/\mathfrak{m}_x$ is a finite extension of $k$. In particular, there is a unique valuation $|\cdot|_x$ on $A$ extending $|\cdot|$ on $k$ such that $\mathrm{supp}(|\cdot|_x) = \mathfrak{m}_x$.

*Proof.* The residue field $A/\mathfrak{m}_x$ is a Tate algebra over $k$, so by Noether normalization, there exists a finite monomorphism $k\langle T_1, \dots, T_d\rangle \hookrightarrow A/\mathfrak{m}_x$. As $A/\mathfrak{m}_x$ is a field, the Krull dimension of $k\langle T_1, \dots, T_d\rangle$ must be zero, so $d = 0$. Thus, there is an finite injective $k$-algebra homomorphism $k \hookrightarrow A/\mathfrak{m}_x$, as required.

As $A/\mathfrak{m}_x$ is a finite extension of $k$, there is a unique extension of the valuation $|\cdot|$ on $k$ to a valuation on $A/\mathfrak{m}_x$. Thus, there is a valuation $|\cdot|_x \colon A \to A/\mathfrak{m}_x \to \mathbf{R}_{\geq 0}$ on $A$ that extends the valuation on $k$ and with support equal to $\mathfrak{m}_x$ and Furthermore, as any valuation with support equal to $\mathfrak{m}_x$ must factor through $A/\mathfrak{m}_x$, $|\cdot|_x$ is the unique such valuation. $\square$

**Lemma 7.7.** *If $A \to B$ is a $k$-algebra homomorphism between Tate algebras $A, B$ over $k$, then it induces a map*
$$\operatorname{Max} B \to \operatorname{Max} A.$$

*Proof.* If $f\colon A \to B$ is a $k$-algebra homomorphism and $\mathfrak{m} \in \operatorname{Max} B$, then there is a chain of inclusions
$$k \hookrightarrow A/f^{-1}(\mathfrak{m}) \hookrightarrow B/\mathfrak{m},$$
and $B/\mathfrak{m}$ is a finite field extension of $k$, so $A/f^{-1}(\mathfrak{m})$ is a domain of finite dimension over a field, hence it is itself a field. Therefore, $f^{-1}(\mathfrak{m})$ is maximal. □

Recall that $\operatorname{Spa}(A, A^\circ)$ consists of all continuous valuations $v$ on $A$ such that $v(A^\circ) \leq 1$.

**Exercise 7.8.** There is an injection $\operatorname{Max} A \hookrightarrow \operatorname{Spa}(A, A^\circ)$ given by $x \mapsto |\cdot|_x$.

**Definition 7.9.** A subset $U \subseteq \operatorname{Max} A$ is called *rational* if there are $f_1, \dots, f_n, g \in A$ such that $(f_1, \dots, f_n) = A$ and
$$U = R\left(\frac{f_1, \dots, f_n}{g}\right) \coloneqq \{x \in \operatorname{Max} A\colon |f_i(x)| \leq |g(x)| \text{ for } i = 1, \dots, n\}.$$
To $U$, we associate the $A$-algebra $A_U \coloneqq \frac{A\langle T_1, \dots, T_n\rangle}{(gT_i - f_i)}$; when equipped with the residue norm, $A_U$ becomes a complete Tate ring, and it is often written as $A\langle \frac{f}{g}\rangle$.

**Remark 7.10.** In rigid geometry, one often imposes the condition that $(g, f_1, \dots, f_n) = A$, instead of $(f_1, \dots, f_n) = A$; however, these are equivalent.

**Remark 7.11.** While $k\langle T_1, \dots, T_n\rangle^\circ = k^\circ\langle T_1, \dots, T_n\rangle$, in general $A^\circ\langle \frac{f}{g}\rangle \not\simeq A\langle \frac{f}{g}\rangle^\circ$ (though there is always an integral map $A^\circ\langle \frac{f}{g}\rangle \to A\langle \frac{f}{g}\rangle$. For an example where this fails, see [Con14, Example 12.2.8]. We briefly explain the construction here: if $k$ is a discretely-valued field with uniformizer $\pi$, set $A = \frac{k\langle x, y\rangle}{(x^2 + \pi^4 + y^5)}$, then there is an injection
$$A^0\langle \frac{\pi}{y}, \frac{x}{y}\rangle \hookrightarrow A\langle \frac{\pi}{y}, \frac{x}{y}\rangle^\circ,$$
but $\frac{x^2}{y}$ is not in the image of $A^0\langle \frac{\pi}{y}, \frac{x}{y}\rangle$.

**Proposition 7.12.** *(1) If $U$ is a rational subset of* $\operatorname{Max} A$*, $A_U$ is a Tate algebra over $k$.*
*(2)* [Universal property] *For any morphism $A \to B$ of Tate algebras over $k$ such that the induced map* $\operatorname{Max} B \to \operatorname{Max} A$ *has image in $U$, there is a unique factorization*
$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A_U \arrow[d, dashed, "\exists!"] \\ & B \end{tikzcd}$$
*(3)* $\operatorname{Max} A_U \simeq U$*.*

*Proof.* For (1), let $\varphi\colon k\langle S_1, \dots, S_m\rangle \twoheadrightarrow A$ be a continuous, open, surjective $k$-algebra homomorphism, then it induces
$$\varphi \hat{\otimes}_k 1\colon k\langle S_1, \dots, S_m, T_1, \dots, T_n\rangle \twoheadrightarrow A\langle T_1, \dots, T_n\rangle,$$
which in turn admits a continuous, open, surjective $k$-algebra homomorphism onto $A_U$, so $A_U$ is a Tate algebra over $k$. For (2) and (3), see [BGR84, 6.1.4/1]. □

**Proposition 7.13.** *(1) The intersection of rational subsets in* $\operatorname{Max}(A)$ *is a rational subsets.*
*(2) If $U \subseteq \operatorname{Spa}(A, A^\circ)$ is a rational subset of* $\operatorname{Spa}(A, A^\circ)$*, then $U \cap \operatorname{Max}(A)$ is a rational subset of* $\operatorname{Max}(A)$*.*

*Proof.* For (1), let $U = R\left(\frac{f_1,\dots,f_n}{f_0}\right)$ and $V = R\left(\frac{g_1,\dots,g_m}{g_0}\right)$ be two rational subsets of $\mathrm{Max}(A)$, where $f_0,\dots,f_n \in A$ and $g_0,\dots,g_m \in A$ are such that $(f_0,\dots,f_n) = A = (g_0,\dots,g_m)$. Set

$$W = R\left(\frac{f_i g_j}{f_0 g_0} : i = 0,\dots,n\ ,\ j = 0,\dots,m\right).$$

then $W$ is a rational domain, because $\mathcal{J} \coloneqq (f_i g_j : i = 0,\dots,n\ ,\ j = 0,\dots,m) = A$; indeed, there are $h_0,\dots,h_n \in A$ such that $h_0 f_0 + \ldots + h_n f_n = 1$, so $h_0(f_0 g_j) + \ldots + h_n(f_n g_j) = g_j$ for all $j = 0,\dots,m$. It follows that $A = (g_0,\dots,g_m) \subseteq \mathcal{J}$, hence $\mathcal{J} = A$.

We claim that $W = U \cap V$. The inclusion $U \cap V \subseteq W$ is clear. Conversely, if $x \in W$, then $|f_i g_0(x)| \le |f_0 g_0(x)|$ for all $i = 1,\dots,n$. The element $g_0$ must be a unit of $A$; otherwise, there exists $y \in \mathrm{Max}\,A$ such that $g_0(y) = 0$, hence $g_j(y) = 0$ for all $j = 1,\dots,m$ and $(g_0,\dots,g_m) \subseteq \mathfrak{m}_y$, a contradiction. Thus, $|g_0(x)| \neq 0$, so $|f_i(x)| \le |f_0(x)|$, i.e. $x \in U$. Similarly, $x \in V$.

For (2), if $U = R(T/s)$ is a rational subset of $\mathrm{Spa}(A, A^\circ)$, then $T = \{f_1,\dots,f_n\} \subset A$ is a finite subset such that $T \cdot A$ is open, or equivalently $A = A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$, i.e. $A = \sqrt{T \cdot A}$ (here, $A = A^{\circ\circ} \cdot A$ because $A$ is a Tate ring). □

**Definition 7.14.** If $A$ is a Tate algebra over $k$, the associated *rigid-analytic space* $X = \mathrm{Sp}(A)$ consists of the following data:

(1) the underlying set is $\mathrm{Max}\,A$;
(2) the Grothendieck topology on $\mathrm{Max}\,A$:
  (a) the category of rational subsets of $\mathrm{Max}\,A$ (where morphisms are inclusions);
  (b) an admissible cover of a rational subset $U \subseteq \mathrm{Max}\,A$ is a collection $(U_i)_{i\in I}$ of rational subsets of $\mathrm{Max}\,A$ such that there is a finite subset $J \subseteq I$ such that
$$U = \bigcup_{i\in I} U_i = \bigcup_{j\in J} U_j.$$
(3) the structure sheaf $\mathcal{O}_X$ on the Grothendieck topology: if $U \subseteq \mathrm{Max}\,A$ is a rational subset, then
$$\mathcal{O}_X(U) \coloneqq A_U.$$

**Remark 7.15.** It is a highly nontrivial theorem (called Tate's acyclicity theorem) that $\mathcal{O}_X$ is indeed a sheaf. Moreover, one generally defines the category underlying Grothendieck topology on $X$ to be the category of affinoid domains in $\mathrm{Max}\,A$, but this is equivalent to our definition by the Gerritzen–Grauert theorem.

The goal of today can now be made precise: we will prove the following.

**Theorem 7.16.** *If $A$ is a Tate algebra over $k$ and $X = \mathrm{Sp}(A)$, then:*

*(1) $\mathrm{Spa}(A, A^\circ)$ is spectral and the rational subsets form a basis of quasi-compact, constructible subsets;*
*(2) $\mathrm{Max}\,A$ is dense in $\mathrm{Spa}(A, A^\circ)_{\mathrm{cons}}$;*
*(3) there is a canonical equivalence of topoi $\mathrm{Shv}(X) \simeq \mathrm{Shv}(\mathrm{Spa}(A, A^\circ))$.*

Of course, Takumi proved (1) in a more general setting, but it falls out of the method of proof.

7.3. **Extensions of valuations to rational subsets.** Let $X = \mathrm{Sp}(A)$ be the rigid-analytic space associated to a Tate algebra $A$ over $k$.

**Proposition 7.17.** [Hub93, 4.7.1] *Let $g_i, f_{i,1},\dots,f_{i,n(i)} \in A$ be such that $(f_{i,1},\dots,f_{i,n(i)}) = A$ for $i = 1,\dots,k$, and consider the rational subset*
$$U = \bigcap_{i=1}^{k} R_X\left(\frac{f_{i,1},\dots,f_{i,n(i)}}{g_i}\right)$$
*of $X$. If $v \in \mathrm{Spa}(A, A^\circ)$, then the following are equivalent:*

*(1) there is an extension $v_U$ of $v$ to $\mathcal{O}_X(U)$ such that $v_U\left(\mathcal{O}_X(U)^\circ\right) \le 1$;*

*(2)* $v(f_{i,j}) \leq v(g_i)$ *for all* $j = 1, \ldots, n(i)$ *and* $i = 1, \ldots, k$.

**Remark 7.18.** If $v \in \mathrm{Spa}(A, A^\circ)$ and $U \subseteq X$ is a rational subset, to say that there exists an extension $v_U$ of $v$ to $\mathcal{O}_X(U)$ such that $v_U(\mathcal{O}_X(U)^\circ) \leq 1$ is equivalent to saying $v$ lies in the image of the map

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X(U)^\circ) \to \mathrm{Spa}(A, A^\circ)$$

induced by the canonical map $\pi_U \colon A \to \mathcal{O}_X(U)$.

*Proof.* Assuming (1), observe that each $g_i$ is a unit in $A$: indeed, if $g_i$ was not a unit, then there exists $x \in \mathrm{Max}\, A$ such that $g_i(x) = 0$, and hence $f_{ij}(x) = 0$ for all $j = 1, \ldots, n(i)$; in particular, $(g_i, f_{i,1}, \ldots, f_{i,n(i)}) \subseteq \mathfrak{m}_x$, and hence cannot generate the unit ideal, a contradiction. As $g_i$ is a unit, $f_{ij}/g_i \in A$ and, if $A \overset{\pi_U}{\to} \mathcal{O}_X(U)$ is the canonical map, then we claim that $\pi_U(f_{ij}/g_i)$ is power-bounded. As $\mathcal{O}_X(U)$ is a Tate algebra, $\pi_U(f_{ij}/g_i)$ is power-bounded iff $\rho(f_{ij}/g_i) \leq 1$. But, $\rho(f_{ij}/g_i) = \max_{x \in \mathrm{Max}\, \mathcal{O}_X(U)} \frac{|f_{ij}(x)|}{|g_i(x)|} \leq 1$, because $\mathrm{Max}\, \mathcal{O}_X(U) \simeq U$. Thus,

$$v\left(f_{ij}/g_i\right) = v_U\left(\pi_U\left(f_{ij}/g_i\right)\right) \leq 1 \Longrightarrow v(f_{ij}) \leq v(g_i).$$

Assuming (2), let $L = \mathrm{Frac}(A/\mathrm{supp}(v))$ and let $B \subseteq L$ be the valuation ring corresponding to the unique rank-1 generization of $v$. so $A(v) \subseteq B \subseteq L$. As $v$ extends the valuation on $k$ by continuity, $A(v) \cap k = k^\circ$, hence $B \cap k \supseteq A(v) \cap k = k^\circ$. However, a valuation on a field has rank-1 iff its valuation ring is maximal among proper subrings of the field, so $B \cap k = k^\circ$.

Let $|\cdot|$ denote the valuation on $L$ extending $|\cdot|$ on $k$ such that $B = \{x \in L \colon |x| \leq 1\}$ (this exists because $B$ is rank-1 valuation ring in $L$ and $B \cap k = k^\circ$). Let $(K, |\cdot|)$ denote the completion of $(L, |\cdot|)$. If $\varphi$ denotes the composition $A \to A/\mathrm{supp}(v) \to L \to K$, then $\varphi$ is continuous because each map is.

**Claim**. $\varphi(g_i)$ is a unit, and $\varphi(f_{ij}/g_i)$ is power-bounded.

If $\varphi(g_i) = 0$, then the image of $g_i$ in $L$ is zero, so $v(g_i) = 0$, hence $v(f_{ij}) = 0$ for all $j = 1, \ldots, n(i)$, hence $A = (f_{i,1}, \ldots, f_{i,n(i)}) \subseteq \mathrm{supp}(v)$, a contradiction; thus, $\varphi(g_i)$ is a unit in $K$. Furthermore, as $A(v) \subseteq B$, $v(a) \leq 1$ implies that $|a| \leq 1$, so $|\frac{f_{ij}}{g_i}| \leq 1$ in $L$, hence $\varphi(f_{ij})/\varphi(g_i)$ has norm $\leq 1$ in $K$, so it is power-bounded.

By the universal property Proposition 7.12(2), there is a continuous $A$-algebra homomorphism $\varphi_U \colon \mathcal{O}_X(U) \to K$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_U"] \arrow[dr, "\varphi"'] & \mathcal{O}_X(U) \arrow[d, dashed, "\exists! \varphi_U"] \\ & K \end{tikzcd}$$

Thus, we get a continuous valuation $v_U$ on $\mathcal{O}_X(U)$ extending $v$ on $A$. Moreover, $v_U(\pi_U(f_{ij}/g_i)) = v(f_{ij}/g_i) \leq 1$, so $v_U \leq 1$ on $A^\circ[f_{ij}/g_i \colon i, j]$, and hence on its integral closure $C$ in $\mathcal{O}_X(U)$. As $v_U$ is continuous, it also holds that $v_U \leq 1$ on the closure $\overline{C}$ of $C$ in $\mathcal{O}_X(U)$. Note that $\overline{C}$ is a ring of integral elements, so we appeal to the following result of Huber:

**Lemma 7.19.** [Hub94, Lemma 4.4] *If $A$ is a Tate algebra over $k$, then $A^\circ$ is the unique ring of integral elements of $A$ such that $k^\circ \subseteq A^\circ$ and $(k, k^\circ) \to (A, A^+)$ is topologically of finite type.*

The proof of the lemma is not difficult, but requires machinery and terminology developed in [Hub94]. It follows immediately that $\overline{C} = \mathcal{O}_X(U)^\circ$, so $v_U(\mathcal{O}_X(U)^\circ) \leq 1$, as required. $\square$

**Remark 7.20.** For any rational subset $U \subseteq X$, $\mathcal{O}(U)$ is the completion of a localization of $A$, hence any continuous valuation on $A$ extends to at most one continuous valuation on $\mathcal{O}(U)$.

7.4. **The space of prime filters.** Let $X = \mathrm{Sp}(A)$ be the rigid-analytic space associated to a Tate algebra $A$ over $k$. The strategy is to construct an auxiliary topological space $\mathcal{P}(X)$ such that there is a bijection between the rational subsets of $X$ and a basis of the topology on $\mathcal{P}(X)$ (from which we can deduce that the categories of sheaves on the two spaces coincide), and then show that $\mathcal{P}(X)$ is homeomorphic to the adic spectrum $\mathrm{Spa}(A, A^\circ)$.

**Definition 7.21.** . A set $\mathcal{F}$ of rational subsets of $X$ is a *prime filter* if:

(1) $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;
(2) if $U_1, U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$;
(3) if $U_1 \in \mathcal{F}$ and $U_2$ is a rational subset of $X$ with $U_1 \subseteq U_2$, then $U_2 \in \mathcal{F}$;
(4) if $U_1, \dots, U_n \subseteq X$ are rational subsets of $X$ such that $U_1 \cup \dots \cup U_n \in \mathcal{F}$, then there is $i \in \{1, \dots, n\}$ such that $U_i \in \mathcal{F}$.

**Remark 7.22.** A *filter* is a collection of rational subsets satisfying (1-3). Condition (4) is equivalent to the following: if $U_1 \cup U_2 \in \mathcal{F}$, then either $U_1 \in \mathcal{F}$ or $U_2 \in \mathcal{F}$, hence the word prime.

**Definition 7.23.** Let $\mathcal{P}(X)$ denote the set of prime filters on $X$. Given a rational subset $U \subseteq X$, set

$$U^\sim := \{\mathcal{F} \in \mathcal{P}(X) \colon U \in \mathcal{F}\}.$$

The set $\mathcal{P}(X)$ is equipped with the topology generated by subsets of the form $U^\sim$, which we call the *rational* subsets of $\mathcal{P}(X)$.

The goal is to associate, to each prime filter, a valuation in the adic spectrum. The support of these valuations are described below.

**Lemma 7.24.** *For all $\mathcal{F} \in \mathcal{P}(X)$ and $a \in A$, the following are equivalent:*

*(1) for all $F \in \mathcal{F}$, there exists $x \in F$ such that $a(x) = 0$;*
*(2) for all $c \in k^*$, there exists $F \in \mathcal{F}$ such that for all $x \in F$, $|a(x)| \le |c|$.*

*For $\mathcal{F} \in \mathcal{P}(X)$, let $\mathfrak{p}_\mathcal{F}$ be the subset of $A$ satisfying the above conditions. Then, $\mathfrak{p}_\mathcal{F}$ is a prime ideal of $A$.*

*Proof.* Given $a \in A$ satisfying (2), let us show that $a$ satisfies (1). Suppose, for sake of contradiction, that there is $F' \in \mathcal{F}$ such that for all $x \in F'$, $a(x) \neq 0$. Take $c \in k^*$ such that $|a(x)| > |c| > 0$. By (2), there exists $F'' \in \mathcal{F}$ such that for all $y \in F''$, $|a(y)| \le |c|$. Set $F = F' \cap F''$, then $F \in \mathcal{F}$ and for any $x \in F$, $|c| \le |a(x)| > |c|$, a contradiction.

It remains to check that $\mathfrak{p}_\mathcal{F}$ is a prime ideal of $A$. Given $a, b \in \mathfrak{p}_\mathcal{F}$ and $c \in k^*$, there are $F_a, F_b \in \mathcal{F}$ such that $|a(x)| \le |c|$ for all $x \in F_a$, and $|b(x)| \le |c|$ for all $x \in F_b$. Then, $F = F_a \cap F_b \in \mathcal{F}$ and $|(a+b)(x)| \le \max\{|a(x)|, |b(x)|\} \le |e|$ for all $x \in F$, so $a + b \in \mathfrak{p}_\mathcal{F}$. Given $a \in \mathfrak{p}_\mathcal{F}$, $b \in A$, and $F \in \mathcal{F}$, there exists $x \in F$ such that $a(x) = 0$, hence $(ab)(x) = 0$ and $ab \in \mathfrak{p}_\mathcal{F}$. Thus, $\mathfrak{p}_\mathcal{F}$ is a prime ideal of $A$.

The opposite implication is left as an exercise. □

**Proposition 7.25.** [Hub93, 4.7.2] *Given $\mathcal{F} \in \mathcal{P}(X)$, there exists a unique $v_\mathcal{F} \in \mathrm{Spv}(A)$ such that for all $a, b \in A$, $v_\mathcal{F}(a) \le v_\mathcal{F}(b)$ iff one of the following occurs:*

*(1) $a \in \mathfrak{p}_\mathcal{F}$;*
*(2) there exists $F \in \mathcal{F}$ such that for all $x \in F$, $|a(x)| \le |b(x)|$.*

*Moreover, $v_\mathcal{F} \in \mathrm{Spa}(A, A^\circ)$ and $\mathrm{supp}(v_\mathcal{F}) = \mathfrak{p}_\mathcal{F}$.*

*Proof.* Define a binary relation $|_\mathcal{F}$ on $A$ by declaring that $b|_\mathcal{F} a$ iff either $a \in \mathfrak{p}_\mathcal{F}$ or there exists $F \in \mathcal{F}$ such that for all $x \in F$, $|a(x)| \le |b(x)|$. This binary relation satisfies the 6 properties found in the proof of [Hub93, Proposition 2.2]: for any $a, b, c \in A$,

(1) Either $a|_\mathcal{F} b$ or $b|_\mathcal{F} a$;
(2) If $a|_\mathcal{F} b$ and $b|_\mathcal{F} c$, then $a|_\mathcal{F} c$;
(3) If $a|_\mathcal{F} b$ and $a|_\mathcal{F} c$, then $a|_\mathcal{F} b + c$;
(4) If $a|_\mathcal{F} b$, then $ac|_\mathcal{F} bc$;

(5) If $ac|_{\mathcal{F}}bc$ and $0 \nmid_{\mathcal{F}}$, then $a|_{\mathcal{F}}b$;
(6) $0 \nmid_{\mathcal{F}} 1$.

As an example, let's prove (4): If $b \in \mathfrak{p}_{\mathcal{F}}$, then $bc \in \mathfrak{p}_{\mathcal{F}}$, because $\mathfrak{p}_{\mathcal{F}}$ is an ideal. If $b \notin \mathfrak{p}_{\mathcal{F}}$, then there exists $F \in \mathcal{F}$ such that for all $x \in F$, $|a(x)| \leq |b(x)|$; in particular, for all $x \in F$, $|(ac)(x)| \leq |(bc)(x)|$, and so $ac|_{\mathcal{F}}bc$.

Thus, there exists a valuation $v_{\mathcal{F}}$ on $A$ with the property that $v_{\mathcal{F}}(a) \leq v_{\mathcal{F}}(b)$ iff $b|_{\mathcal{F}}a$.

If $a \in A^\circ$, then $|a(x)| \leq 1$ for all $x \in X$, hence $1|_{\mathcal{F}}a$ and so $v_{\mathcal{F}}(a) \leq v_{\mathcal{F}}(1) = 1$.

If $v_{\mathcal{F}}(a) = 0$, then $0|_{\mathcal{F}}a$, so either $a \in \mathfrak{p}_{\mathcal{F}}$ or there exists $F \in \mathcal{F}$ such that $|a(x)| \leq 0$ for all $x \in F$. If $a \in \mathfrak{p}_{\mathcal{F}}$, we're done. Otherwise, for any $F' \in \mathcal{F}$, $F'' = F \cap F' \neq \emptyset$ and for $x \in F''$, $a(x) = 0$; in particular, there is $x \in F'$ such that $a(x) = 0$, so $a \in \mathfrak{p}_{\mathcal{F}}$. Thus, $\mathrm{supp}(v_{\mathcal{F}}) = \mathfrak{p}_{\mathcal{F}}$.

If $a \in A \backslash \mathfrak{p}_{\mathcal{F}}$, then $v_{\mathcal{F}}(a) > 0$. Take $e_1 \in k^*$ such that $v_{\mathcal{F}}(a) > v_{\mathcal{F}}(e_1) > 0$. As $a \notin \mathfrak{p}_{\mathcal{F}}$, there exists $e_2 \in k^*$ such that for all $F \in \mathcal{F}$, there exists $x \in F$ such that $|a(x)| > |e_2|$; call this condition $(*)$. Either $e_2|_{\mathcal{F}}a$ or $a|_{\mathcal{F}}e_2$. Suppose $a|_{\mathcal{F}}e_2$. As $e_2$ is a unit in $k$, $e_2 \notin \mathfrak{p}_{\mathcal{F}}$, so there must exist $F \in \mathcal{F}$ such that for all $x \in F$, $|e_2| \leq |a(x)|$. This contradicts $(*)$. Thus, $e_2|_{\mathcal{F}}a$, or equivalently $v_{\mathcal{F}}(a) \leq v_{\mathcal{F}}(e_2)$. As $v_{\mathcal{F}}(a) > 0$, then there exists $e_1 \in k^*$ such that $v_{\mathcal{F}}(a) \geq v_{\mathcal{F}}(e_1) > 0$. Moreover, both $v_{\mathcal{F}}(e_1)$ and $v_{\mathcal{F}}(e_2)$ are in the convex subgroup $c\Gamma_{v_{\mathcal{F}}}$ of $\Gamma_{v_{\mathcal{F}}}$ generated by $\{v_{\mathcal{F}} \geq 1\}$ (because either $v_{\mathcal{F}}(e_i) \geq 1$ or $v_{\mathcal{F}}(e_i^{-1}) \geq 1$), so $v_{\mathcal{F}}(a) \in c\Gamma_{v_{\mathcal{F}}}$. Thus, $\Gamma_{v_{\mathcal{F}}} = c\Gamma_{v_{\mathcal{F}}}$, so $v_{\mathcal{F}}$ is continuous. Combining this with the fact that $v_{\mathcal{F}}(A^\circ) \leq 1$, it follows that $v_{\mathcal{F}} \in \mathrm{Spa}(A, A^\circ)$. □

**Lemma 7.26.** *For any $v \in \mathrm{Spa}(A, A^\circ)$, set*

$$d(v) := \left\{ \text{rational subsets } U \subseteq X : \begin{array}{c} \text{there is a continuous valuation } v_U \text{ on } \mathcal{O}_X(U) \\ \text{extending } v \text{ such that } v_U\left(\mathcal{O}_X(U)^\circ\right) \leq 1 \end{array} \right\}.$$

*Then, $d(v) \in \mathcal{P}(X)$.*

*Proof.* There are four conditions to check to ensure that $d(v)$ is a prime filter:

(1) If $U = \mathrm{Max}(A)$, the valuation $v$ extends to itself on $\mathcal{O}(U) = A$, so $\mathrm{Max}\, A \in d(v)$. If $U = \emptyset$, then $\mathcal{O}(U) = \frac{A\langle T\rangle}{(0 \cdot T - 1)} = 0$ and the zero ring has no valuations; in particular, $\emptyset \notin d(v)$.
(2) If $U_1, U_2 \in d(v)$, then by Proposition 7.17 we can write $U_i = R\left(\frac{f_{i,1},\ldots,f_{i,n(i)}}{g_i}\right)$ for $f_{i,1}, \ldots, f_{i,n(i)}, g_i \in A$ such that $v(f_{i,j}) \leq v(g_i)$ for all $j = 1, \ldots, n(i)$. Again using Proposition 7.17, it follows that $U_1 \cap U_2 \in \mathcal{F}$.
(3) If $U_1 \in d(v)$ and $U_2$ is a rational subset of $\mathrm{Max}(A)$ such that $U_1 \subseteq U_2$, then we must show that $U_2 \in d(v)$. As $U_1 \in d(v)$, there is a continuous valuation $v_{U_1}$ on $\mathcal{O}_X(U_1)$ extending the valuation on $A$ and such that $v_{U_1}(\mathcal{O}_X(U_1)^\circ) \leq 1$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_X(U_2) \arrow[r, "\eta"] & \mathcal{O}_X(U_1)_{v_{U_1}} \arrow[r] & \Gamma \cup \{0\} \\ & A \arrow[ul, "\pi_{U_2}"] \arrow[u, "\pi_{U_1}"] \arrow[ur, "v"'] & \end{tikzcd}$$

Define $v_{U_2} := v_{U_1} \circ \eta$, then it is an extension of $v$ by the commutativity of the diagram, and $v_{U_2}(\mathcal{O}_X(U_2)^\circ) \subseteq v_{U_1}(\mathcal{O}_X(U_1)^\circ) \leq 1$, because $\eta(\mathcal{O}_X(U_2)^\circ) \subseteq \mathcal{O}_X(U_1)^\circ$.
(4) If $U \in d(v)$ and $U_1, \ldots, U_n$ are rational subsets of $\mathrm{Max}\, A$ such that $U = U_1 \cup \ldots \cup U_n$, then we want to show that $U_i \in d(v)$ for some $i \in \{1, \ldots, n\}$. Let's replace the cover $(U_i)$ by a "better" cover of $U$ by appealing to the following result:

**Fact 7.27.** [FvdP81, Lemma III.2.5] Given an admissible cover $(U_i)_{i=1}^n$ of $X = \mathrm{Sp}(A)$, there is a *standard* admissible cover $(V_j)_{j=1}^m$ refining $(U_i)_{i=1}^n$; that is, there are $f_1, \ldots, f_m \in A$ such that $(f_1, \ldots, f_m) = A$ and if

$$V_j := \{x \in X : |f_i(x)| \leq |f_j(x)| \text{ for } i = 1, \ldots, m\}$$

then $(V_j)_{j=1}^m$ is an admissible cover of $X$ such that $V_j \subseteq U_\ell$ for some $\ell$.

By the above fact, there are $f_1, \dots, f_m \in \mathcal{O}_X(U)$ such that $(f_1, \dots, f_m) = \mathcal{O}_X(U)$ and for each $i = 1, \dots, m$, the rational subset

$$V_i := \bigcap_{j=1}^{m} \{x \in U \colon |f_j(x)| \leq |f_i(x)|\}$$

is contained in some $U_\ell$. Let $v_U$ be the extension of $v$ to $\mathcal{O}(U)$, and let $r \in \{1, \dots, m\}$ be an index such that $v_U(f_r)$ is maximal, i.e. $v_U(f_r) \geq v_U(f_i)$ for $i = 1, \dots, m$. Then, $V_r$ is a rational subset of $U$ and by Proposition 7.17 $v_U$ extends to a continuous valuation $v_{V_r}$ on $V_r$ with the usual properties, so $V_r \in d(v)$. As $V_r \subseteq U_\ell$ for some $\ell \in \{1, \dots, n\}$, it follows that $U_\ell \in d(v)$, as required.

$\square$

**Proposition 7.28.** [Hub93, 7.4.3] *The map $d \colon \mathrm{Spa}(A, A^\circ) \to \mathcal{P}(X)$, given by $v \mapsto d(v)$, is a homemorphism. Moreover,*

*(1) if $V \subseteq \mathcal{P}(X)$ is a rational subset of $\mathcal{P}(X)$, then $d^{-1}(V)$ is a rational subset of* $\mathrm{Spa}(A, A^\circ)$*;*
*(2) the assignment $V \mapsto d^{-1}(V)$ gives a bijection*

$$\{\textit{rational subsets of } \mathcal{P}(X)\} \xrightarrow{\simeq} \{\textit{rational subsets of } \mathrm{Spa}(A, A^\circ)\}.$$

*Proof.* For (1), a rational subset of $\mathcal{P}(X)$ is of the form $U^\sim$, where $U = R(\frac{f_1, \dots, f_n}{g})$ is a rational subset of $X$. Then, $v \in \mathrm{Spa}(A, A^\circ)$ belongs to $d^{-1}(U^\sim)$ iff $d(v) \in U^\sim$ iff $U \in d(v)$. By Proposition 7.17, this occurs iff $v(f_i) \leq v(g)$ for all $i = 1, \dots, n$ and $v(g) \neq 0$. Thus,

$$d^{-1}(U^\sim) = \{v \in \mathrm{Spa}(A, A^\circ) \colon v(f_i) \leq v(g) \neq 0 \text{ for } i = 1, \dots, n\},$$

so $d^{-1}(U^\sim)$ is a rational subset of $\mathrm{Spa}(A, A^\circ)$. For (2), it is clear from the above that this is a bijection.

It remains to show that $d$ is a homeomorphism. For surjectivity, take $\mathcal{F} \in \mathcal{P}(X)$, then we claim that $\mathcal{F} = d(v_\mathcal{F})$. Indeed, by Proposition 7.17, $R(\frac{f_1, \dots, f_n}{g}) \in d(v_\mathcal{F})$ iff $v_\mathcal{F}(f_i) \leq v_\mathcal{F}(g)$ for all $i = 1, \dots, n$. Observe that $g \notin \mathfrak{p}_\mathcal{F}$: otherwise, $v_\mathcal{F}(g) = 0$, so $v_\mathcal{F}(f_i) = 0$ for all $i = 1, \dots, n$ and hence $A = (f_1, \dots, f_n) \subseteq \mathrm{supp}(v_\mathcal{F}) = \mathfrak{p}_\mathcal{F}$, a contradiction. Thus, $v_\mathcal{F}(f_i) \leq v_\mathcal{F}(g)$ for all $i$ is equivalent to the assertion that for all $i = 1, \dots, n$, there exists $F_i \in \mathcal{F}$ such that for all $x \in F_i$, $|f_i(x)| \leq |g(x)|$. This is equivalent to the assertion that $R(\frac{f_1, \dots, f_n}{g}) \in \mathcal{F}$. Indeed, if $R(\frac{f_1, \dots, f_n}{g}) \in \mathcal{F}$, then take $\mathcal{F}_i = R(\frac{f_1, \dots, f_n}{g})$ for all $i$. Conversely, $F_i \subseteq R(\frac{f_1, \dots, f_n}{g})$, hence belongs to $\mathcal{F}$ by property (3) of being a prime filter.

To see that $d$ is injective, take $v_1, v_2 \in \mathrm{Spa}(A, A^\circ)$ such that $d(v_1) = d(v_2)$, then we will show that $v_1$ and $v_2$ have the same order relations. Given $f, g \in A$, exactly one of the following occurs:

- if $R(\frac{f,1}{g}) \in d(v_1) = d(v_2)$, then by Proposition 7.17 $v_i(f) \leq v_i(g)$ for $i = 1, 2$;
- if $R(\frac{f,1}{g}) \notin d(v_2) = d(v_2)$, then by Proposition 7.17 $v_i(f) > v_i(g)$ for $i = 1, 2$.

That is, $v_1(f) \leq v_1(g)$ iff $v_2(f) \leq v_2(g)$. Therefore, $d$ is injective, hence bijective. As rational subsets of the domain and codomain are bases and they pull back to one another under $d$ (this is an easy check, left as an exercise), both $d$ are its inverse are continuous. Therefore, $d$ is a homeomorphism. $\square$

**Lemma 7.29.** *For any $x \in X$, set*

$$j(x) := \{\textit{rational subsets } U \subseteq X \colon x \in U\}.$$

*Then, $j(x) \in \mathcal{P}(X)$ and $j \colon X \to \mathcal{P}(X)$ is injective. Furthermore, $d^{-1}(j(X))$ is identified with the image of* $\mathrm{Max}(A)$ *in* $\mathrm{Spa}(A, A^\circ)$.

*Proof.* It is clear that $j(x)$ is a prime filter and that the map $j$ is injective. $\square$

## 7.5. Proof of the Main Theorem.

**Proposition 7.30.** [Hub93, 4.7.4] *Let $\mathcal{B}$ denote the Boolean algebra of $\mathcal{P}(X)$ generated by the rational subsets (that is, the smallest set of subsets of $\mathcal{P}(X)$ containing the rational subsets that is also closed under finite intersection and taking complements). If $\mathcal{T}$ be the topology on $\mathcal{P}(X)$ generated by $\mathcal{B}$, then*

*(1) $(\mathcal{P}(X), \mathcal{T})$ is compact (i.e. quasi-compact and Hausdorff);*
*(2) $\mathcal{B}$ consists of the sets of $\mathcal{P}(X)$ that are both open and closed in $\mathcal{T}$;*
*(3) $j(X)$ is dense in $(\mathcal{P}(X), \mathcal{T})$.*

*Proof.* For (1), let $\mathcal{R}$ denote the collection of rational subsets of $X$ and let $\mathscr{P}(\mathcal{R})$ denote the power set of $\mathcal{R}$. If $\{0,1\}$ is equipped with the discrete topology, $\mathscr{P}(\mathcal{R}) = \prod_{\mathcal{R}}\{0,1\}$ is equipped with the product topology, hence it is Hausdorff and quasi-compact (by Tychonoff's theorem). Our convention is that, in the factor of $\mathscr{P}(\mathcal{R})$ corresponding to $U \in \mathcal{R}$, 0 means that $U$ belongs to the set, and 1 means that $U$ does not belong to the set.

The natural inclusion $\mathcal{P}(X) \hookrightarrow \mathscr{P}(\mathcal{R})$ is continuous: a base element for the topology on $\mathscr{P}(\mathcal{R})$ is of the form

$$S = \prod_{U_1,\ldots,U_n}\{0\} \times \prod_{V_1,\ldots,V_m}\{1\} \times \prod_{\mathcal{R}\backslash\{U_1,\ldots,U_n\}}\{0,1\}$$

for some $U_1, \ldots, U_n, V_1, \ldots, V_m \in \mathcal{R}$, so $S \cap \mathcal{P}(X) = U_1^{\sim} \cap \ldots \cap U_n^{\sim} \cap (\mathcal{P}(X)\backslash V_1^{\sim}) \cap \ldots \cap (\mathcal{P}(X)\backslash V_m^{\sim})$, which is an element of the base $\mathcal{B}$. As all members of $\mathcal{B}$ arise in this way, $(\mathcal{P}(X), \mathcal{T})$ agrees with the subspace topology inherited from $\mathscr{P}(\mathcal{R})$. It is clear that all members of $\mathcal{B}$ are both open and closed, so $(\mathcal{P}(X), \mathcal{T})$ is a closed subspace of $\mathscr{P}(\mathcal{R})$, hence it is compact.

For (2), this is an exercise in unraveling the definitions.

For (3), it suffices to show that $j(X)$ intersects any non-empty member of $\mathcal{B}$, a general element of which is of the form $B = U^{\sim} \cap (\mathscr{P}(X)\backslash V^{\sim})$ for rational subsets $U, V \subseteq X$. It is clear that $U^{\sim} \cap (\mathscr{P}(X)\backslash V^{\sim}) = \emptyset$ iff $U^{\sim} \subseteq V^{\sim}$, and in fact this is equivalent to $U \subseteq V$. Indeed, if $U \subseteq V$, then for any $\mathcal{F} \in \mathcal{P}(X)$ such that $U \in \mathcal{F}$, $V \in \mathcal{F}$ by property (3) of Definition 7.21. Conversely, if $U^{\sim} \subseteq V^{\sim}$, suppose there exists $x \in U\backslash V$, then $U \in j(x)$ but $V \notin j(x)$, i.e. $j(x) \in U^{\sim}\backslash V^{\sim}$, a contradiction. Thus, if $B \neq \emptyset$, there exists $x \in U\backslash V$, so $j(x) \in U^{\sim}$ and $j(x) \notin V^{\sim}$; in particular, $j(x) \in B$. Therefore, $j(X)$ is dense in $(\mathcal{P}(X), \mathcal{T})$. □

**Remark 7.31.** We will see later that $\mathcal{T}$ is the constructible topology on $\mathcal{P}(X)$.

**Proposition 7.32.** *There is a bijection*

$$\{\textit{rational subsets of } \mathcal{P}(X)\} \xrightarrow{\simeq} \{\textit{rational subsets of } X\},$$

*given by $V^{\sim} \mapsto j^{-1}(V^{\sim})$ and with inverse functor $U \mapsto U^{\sim}$.*

*Proof.* It suffices to show: if $V \subseteq X$ is a rational subset, then $j^{-1}(V^{\sim}) = V$. Indeed, $x \in j^{-1}(V^{\sim})$ iff $j(x) \in V^{\sim}$ iff $V \in j(x)$ iff $x \in V$. □

*Proof of Theorem 7.40.* Given $\mathcal{F} \in \mathrm{Shv}(X)$ and a rational subset $U^{\sim} \subseteq \mathcal{P}(X)$, set

$$\widetilde{\mathcal{F}}(U^{\sim}) := \varprojlim_{V \subseteq U} \mathcal{F}(V),$$

where the limit ranges over rational subsets $V \subseteq U \subseteq X$. This is a sheaf, so we get a functor $\mathrm{Shv}(X) \to \mathrm{Shv}(\mathcal{P}(X))$. If $\widetilde{\mathcal{G}} \in \mathrm{Shv}(\mathcal{P}(X))$ and $U \subseteq X$ is a rational subset, then set

$$\mathcal{G}(U) := \widetilde{\mathcal{G}}(U^{\sim}).$$

This is a sheaf on $\mathcal{P}(X)$, so we get an inverse functor, hence an equivalence $\mathrm{Shv}(X) \simeq \mathrm{Shv}(\mathcal{P}(X))$. Similarly, one gets an equivalence $\mathrm{Shv}(\mathcal{P}(X)) \simeq \mathrm{Shv}(\mathrm{Spa}(A, A^{\circ}))$ using Proposition 7.28. Thus, the composition

$$\mathrm{Shv}(X) \simeq \mathrm{Shv}(\mathcal{P}(X)) \simeq \mathrm{Shv}(\mathrm{Spa}(A, A^{\circ}))$$

is a canonical equivalence of topoi.

Recall Hochster's theorem: if $Z$ is a set, $\mathcal{J}$ is a quasi-compact topology on $Z$, $\mathcal{L}$ is the set of clopen subsets in $(Z,\mathcal{J})$, and $\mathcal{I}$ is a $T_0$-topology on $Z$ generated by a subset of $\mathcal{L}$, then $(Z,\mathcal{J})$ is a spectral space and $\mathcal{L}$ is the collection of constructible subsets of $(Z,\mathcal{I})$.

Take $Z=\mathcal{P}(X)$, $\mathcal{L}=\mathcal{B}$, and $\mathcal{I}$ to the usual topology on $Z$ (i.e. the topology generated by rational subsets), then Hochster's theorem asserts that $\mathcal{P}(X)$ is spectral and $\mathcal{B}$ consists of all constructible subsets of $\mathcal{P}(X)$ (so $\mathcal{T}$ is the constructible topology on $\mathcal{P}(X)$). As $\operatorname{Spa}(A,A^\circ)$ is homeomorphic to $\mathcal{P}(X)$ and the set of rational subsets are in bijection by Proposition 7.28, it follows that $\operatorname{Spa}(A,A^\circ)$ is spectral. Moreover, $j(X)$ is dense in the constructible topology, and hence dense in the constructible topology on $\operatorname{Spa}(A,A^\circ)$. ☐

**Remark 7.33.** The space $\mathcal{P}(X)$ is partially ordered by inclusion, and one can show that $\mathcal{F}_1\subseteq\mathcal{F}_2$ iff $d^{-1}(\mathcal{F}_1)$ is a specialization of $d^{-1}(\mathcal{F}_2)$ in $\operatorname{Spa}(A,A^\circ)$. Thus, if $\operatorname{Spa}(A,A^\circ)_{\max}$ denotes the set of valuation without proper horizontal specializations, then there is a bijection

$$\operatorname{Spa}(A,A^\circ)_{\max}\simeq\{\text{maximal filters in }\mathcal{P}(X)\}.$$

In [vdP82, Corollary 1.3.3], van der Put shows that such maximal filters are in bijection with the set of continuous rank-1 valuations on $A$, i.e. the Berkovich spectrum of $A$. The maps are as follows: given a maximal filter $F\in\mathcal{P}(X)$ and $f\in A$, set

$$|f|_F\coloneqq\inf_{V\in F}\sup_{x\in V}|f(x)|,$$

where one should interpret $\sup_{x\in V}|f(x)|$ as $\|\pi_V(f)\|_V$, where $\pi_V\colon A\to A_V$ is the canonical map and $\|\cdot\|_V$ is the norm on $V$. Conversely, given $x\in\mathcal{M}(A)$, set

$$F_x\coloneqq\left\{\text{rational subsets }V\subseteq X\colon \begin{array}{c}\text{there exists a rational subset }R\left(\frac{f_1,\dots,f_n}{g}\right)\subseteq V\\ \text{such that }|f_i(x)|\le|g(x)|\text{ for }i=1,\dots,n\end{array}\right\}.$$

This interpretation in terms of filters allows us to "see" the difference between the adic and Berkovich spectra. For example, if $k=k^a$, $a\in k^{\circ\circ}$, and $X=\operatorname{Sp}(k\langle T\rangle)$ is the rigid unit disc over $k$, then consider the prime filters

$$\mathcal{F}\coloneqq\{\text{rational subsets }V\subseteq X\colon\text{there exists }r\in(0,1)\text{ such that }r<|x-a|<1\}$$

and

$$\mathcal{F}_{\mathrm{Gauss}}\coloneqq\left\{\text{rational subsets }V\subseteq X\colon \begin{array}{c}\text{there exists }a_1,\dots,a_n\in k^\circ\text{ and }r_1,\dots,r_n\in[0,1]\\ \text{such that }V\supseteq D(0,1)\setminus\bigcup_{i=1}^n D(a_i,r_i)^-\end{array}\right\}.$$

Then, $\mathcal{F}\subseteq\mathcal{F}_{\mathrm{Gauss}}$ and $\mathcal{F}_{\mathrm{Gauss}}$ is a maximal filter (and corresponds to the Gauss point of $\mathcal{M}(k\langle T\rangle)$), but they are distinct! Indeed, the rational subset $V=D(0,1)\setminus D(0,1)^-=R(1/T)$ lies in $\mathcal{F}_{\mathrm{Gauss}}$ but not in $\mathcal{F}$. The point of $\operatorname{Spa}(A,A^\circ)$ corresponding to the prime filter $\mathcal{F}$ is often called a type 5 point.

## 7.6. **Another Proof of the Main Theorem.**

**Lemma 7.34.** *If $A$ is a Tate algebra over $k$, consider the space*

$$L_A\coloneqq\left\{v\in\operatorname{Spv}(A)\colon \begin{array}{c}v(a)\le 1\textit{ for all }a\in A^\circ\\ v(a)<1\textit{ for all }a\in A^{\circ\circ}\end{array}\right\}.$$

*Then, there are inclusions*

$$\operatorname{Max}A\subseteq\operatorname{Spa}(A,A^\circ)\subseteq L_A\subseteq\operatorname{Spv}(A).$$

*Moreover, if $r\colon\operatorname{Spv}(A)\to\operatorname{Spv}(A,A)$ denotes the spectral retraction, then $r^{-1}(\operatorname{Spa}(A,A^\circ))=L_A$.*

*Proof.* Everything is clear, except for the statement about the retraction, which is not needed in the proof of the main theorem. ☐

**Lemma 7.35.** *If $A\to B$ is a $k$-algebra homomorphism between Tate algebras $A,B$ over $k$, then there is an induced map $L_B\to L_A$.*

*Proof.* This follows immediately from the fact that taking power-bounded elements or topologically nilpotent elements is functorial for Banach algebras over a non-Archimedean field. ☐

The technical heart of the proof is the following result.

**Theorem 7.36.** [Hub93, Theorem 4.1] *The closure of* $\operatorname{Max} A$ *in* $\operatorname{Spv}(A)_{\text{cons}}$ *is* $L_A$*.*

It has 3 key inputs: the spectrality of $\operatorname{Spa}(A, A^\circ)$, the Weierstrass preparation theorem, and a model-theoretic fact. The proof will be added to these notes later. An easy, but important, consequence is the following.

**Theorem 7.37.** [Hub93, Corollary 4.2] *The subset* $\operatorname{Max} A$ *is dense in* $\operatorname{Spa}(A, A^\circ)_{\text{cons}}$*.*

*Proof of Theorem 7.37.* If $T$ is a nonempty constructible subset of $\operatorname{Spa}(A, A^\circ)$, we must show that $T \cap \operatorname{Max}(A) \neq \emptyset$. As $T$ is constructible, it must be a finite boolean combination of rational subsets of $\operatorname{Spa}(A, A^\circ)$. There exists a constructible subset $T' \subseteq \operatorname{Spv}(A)$ such that $T = T' \cap \operatorname{Spa}(A, A^\circ)$, and $T' \cap L_A \neq \emptyset$ because

$$T' \cap L_A \supseteq T' \cap \operatorname{Spa}(A, A^\circ) = T = \emptyset.$$

By Theorem 7.36, any constructible subset of $\operatorname{Spv}(A)$ that intersects $L_A$ must also intersect $\operatorname{Max}(A)$, hence $T' \cap \operatorname{Max}(A) \neq \emptyset$; thus,

$$T \cap \operatorname{Max}(A) = T' \cap \operatorname{Spa}(A, A^\circ) \cap \operatorname{Max}(A) = T' \cap \operatorname{Max}(A) \neq \emptyset,$$

where the second equality follows since $\operatorname{Max}(A) \subseteq \operatorname{Spa}(A, A^\circ)$. This concludes the proof. ☐

**Corollary 7.38.** [Hub93, Corollary 4.3] *If* $X_1, X_2$ *are constructible subsets of* $\operatorname{Spa}(A, A^\circ)$ *with* $X_1 \cap \operatorname{Max} A = X_2 \cap \operatorname{Max} A$*, then* $X_1 = X_2$*.*

*Proof.* The subset $X \coloneqq (X_1 \backslash X_2) \cup (X_2 \backslash X_1)$ is constructible and

$$\begin{aligned} X \cap \operatorname{Max} A &= ((X_1 \cap \operatorname{Max} A) \cap X_2^c) \cup ((X_2 \cap \operatorname{Max} A) \cap X_1^c) \\ &= \Bigg(\underbrace{(X_2 \cap \operatorname{Max} A) \cap X_2^c}_{=\emptyset}\Bigg) \cup \Bigg(\underbrace{(X_1 \cap \operatorname{Max} A) \cap X_1^c}_{=\emptyset}\Bigg) = \emptyset. \end{aligned}$$

By Theorem 7.37, $\operatorname{Max}(A)$ must intersect any nonempty constructible subset, so $X = \emptyset$; that is, $X_1 = X_2$. ☐

**Corollary 7.39.** *There is a bijection*

$$\{\textit{rational subsets of } \operatorname{Spa}(A, A^\circ)\} \xrightarrow{\simeq} \{\textit{rational subsets of } \operatorname{Max} A\},$$

*given by* $U \mapsto U \cap \operatorname{Max} A$*.*

*Proof.* As rational subsets of $\operatorname{Spa}(A, A^\circ)$ are constructible, the injectivity of the above map is immediate from Corollary 7.38. For surjectivity, take a rational subset $R\left(\frac{f_1, \ldots, f_n}{g}\right)$ of $\operatorname{Max}(A)$, for some $f_1, \ldots, f_n, g \in A$. Set $T = \{f_1, \ldots, f_n\}$ and $s = g$. ☐

Arguing as in the first proof, we recover:

**Theorem 7.40.** [Hub93, Corollary 4.4] *There is an equivalence of topoi* $\operatorname{Shv}(\operatorname{Sp}(A)) \simeq \operatorname{Shv}(\operatorname{Spa}(A, A^\circ))$*.*

## 8. April 14th – Stably uniform affinoids are sheafy, after Buzzard and Verberkmoes (Axel Stäbler)

Today's[6] goal is to prove the statement in the title, following [BV14, §§2–3]. Next time, we will present some counterexamples for sheafiness from [BV14, §4].

Let $R$ be a Tate ring, with couple of definition $(R_0, \varpi)$. We denote its completion by

$$\widehat{R} = \lim_n R/\varpi^n R_0,$$

[6]Today's notes were taken by Takumi Murayama.

and denote $i\colon R \to \widehat{R}$ to be the canonical map. If $U \subseteq R$ is an open abelian group, we denote $\widehat{U} = \overline{i(U)}$ to be the closure of the image of $U$ in $\widehat{R}$.

We have the following key properties of the completion, which we will not prove.

**Lemma 8.1** [BV14, Lem. 1]**.**

*(i)* $\widehat{\varpi^n R_0} = \varpi^n \widehat{R_0}$;

*(ii)* *If* $U \subseteq R$ *is an open subgroup, then* $i^{-1}(\widehat{U}) = U$;

*(iii)* $\widehat{R}$ *is a Tate ring with couple of definition* $(\widehat{R_0}, i(\varpi))$, *and* $\widehat{R^\circ} = (\widehat{R})^\circ$.

*(iii)* is especially important: it implies that $R$ is uniform if and only if $\widehat{R}$ is uniform [BV14, p. 7].

Our first goal is to prove that the sheaf property holds for a very special kind of covering. We first setup some notation. Let $X = \operatorname{Spa}(R, R^+)$, and let $\mathcal{O}_X$ be the presheaf of complete topological rings defined in [Hub94, §1]. If $t \in R$, then we consider the following special kind of covering:

$$\begin{aligned} U &:= X(\tfrac{t}{1}) = \{t \in X \mid |t(x)| \le 1\} \\ V &:= X(\tfrac{1}{t}) = \{t \in X \mid |t(x)| \ge 1\} \end{aligned} \tag{8.2}$$

Note that $U$, $V$, and $U \cap V$ are rational subsets, and that $X = U \cup V$ is an open cover. Since our eventual goal is to prove that $\mathcal{O}_X$ is a sheaf of complete topological rings, we have to describe its sections on these rational subsets, and the topology on these rings of sections.

Let

$$A = R \qquad \text{and} \qquad A_0 = R_0[t]$$

which is topologized by $(A_0, \varpi)$. Letting $\phi\colon R \to R[\frac{1}{t}]$ be the canonical localization map, we denote

$$B = R[\tfrac{1}{t}] \qquad \text{and} \qquad B_0 = \phi(R_0)[\tfrac{1}{t}],$$

which is topologized by $(B_0, \varpi)$. Finally, we set

$$C = R[t, \tfrac{1}{t}] \qquad \text{and} \qquad C_0 = \phi(R_0)[t, \tfrac{1}{t}].$$

We can then look at the corresponding sequence of abelian groups:

$$\begin{array}{ccccccccc} 0 \longrightarrow & R & \xrightarrow{\varepsilon} & A \oplus B & \xrightarrow{\delta} & C & \longrightarrow 0 \\ & r & \longmapsto & (r, \phi(r)) & & & \\ & & & (a,b) & \longmapsto & b - \phi(a) & \end{array} \tag{$**$}$$

This is split exact, and you can check that the maps are continuous. Thus, we can view the sequence $(**)$ as a sequence of topological rings. Notice that after completion, the sequence $(**)$ becomes the equalizer sequence for the cover $X = U \cup V$. Thus, we need to understand when the sequence $(**)$ stays exact after completion. To do so, we first define the notion of strictness, which relates the quotient and subspace topologies of the image of a ring homomorphism.

**Definition 8.1.** If $\psi\colon V \to W$ is a continuous map of topological spaces, then $\psi$ is called *strict* if the quotient topology on the image $\psi(V)$ coincides with the subspace topology on the image $\psi(V)$.

We verify this for the map $\delta$ in $(**)$. One can check that the map $\delta$ is continuous and surjective, hence $\delta$ is strict if and only if it is open. Since $\varpi$ is the ideal of definition, it suffices to note that

$$\delta(\varpi^n A_0 \oplus \varpi^n B_0) = \varpi^n C_0,$$

and so $\delta$ is open, hence strict.

It will turn out that $\varepsilon$ is not always strict: the proof that $\varepsilon$ is strict will use uniformity in an important way, and a general form of the Banach open mapping theorem.

**Lemma 8.3** [BV14, Lem. 2]**.** *The following are equivalent:*

(*i*) *The sequence*

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V) \tag{$*$}$$

*is exact;*

(*ii*) *The sequence* ($*$) *is exact, and the map*

$$\mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$$

*is surjective, i.e., the sequence* ($*$) *extends to a short exact sequence on the right;*

(*iii*) *There exists* $n \geq 0$ *such that*

$$\varpi^n(A_0 \cap \phi^{-1}(B_0)) \subseteq R_0,$$

*i.e.,* $A_0 \cap \phi^{-1}(B_0)$ *is bounded in* $R$*;*

(*iv*) *The map* $\varepsilon$ *in* ($**$) *is strict.*

In the course of the proof, we will use the following:

**Fact 8.4** [Bou98, Ch. III, §2, n° 12, Lem. 2]**.** *Let*

$$A \longrightarrow B \longrightarrow C$$

*be an exact sequence of strict homomorphisms of topological groups whose identity elements admit countable fundamental systems of neighborhoods. Then, the sequence*

$$\widehat{A} \longrightarrow \widehat{B} \longrightarrow \widehat{C}$$

*is an exact sequence of strict homomorphisms of topological groups.*

*Proof.* $(ii) \Rightarrow (i)$ is trivial, and $(i) \Rightarrow (ii)$ follows by Fact 8.4, since the sequence ($**$) is already exact on the right, and that $\delta$ is strict. Fact 8.4 also implies $(iv) \Rightarrow (i)$ by using that ($**$) is exact.

We now show $(iii) \Leftrightarrow (iv)$ by exhibiting a sequence of equivalences. By "the subspace topology," we will mean the subspace topology induced by $A \oplus B$ on the subgroup $R \subseteq A \oplus B$. First, we have

$$\begin{aligned} \varepsilon \text{ is strict} &\iff \text{the topology on } R \text{ coincides with the subspace topology} \\ &\iff \varpi^m R_0 \text{ is open for all } m \text{ with respect to the subspace topology} \end{aligned}$$

where the first equivalence is by the definition of strictness (Definition 8.1), and the second is by the fact that $(R_0, \varpi)$ defines the topology on $R$. Fixing $m$, we have that

$$\begin{aligned} &\varpi^m R_0 \text{ is open with respect to the subspace topology} \\ &\qquad \iff \text{there exist } a, b \text{ such that } \varepsilon^{-1}(\varpi^a A_0 \oplus \varpi^b B_0) \subseteq \varpi^m R_0 \\ &\qquad \iff \text{there exists } c \text{ such that } \varepsilon^{-1}(\varpi^c A_0 \oplus \varpi^c B_0) \subseteq \varpi^m R_0 \end{aligned}$$

where the first equivalence is by definition of the topology on $A \oplus B$, and the second is by choosing $c = \max\{a, b\}$ in the forward direction, and conversely by letting $a = b = c$. But then,

$$\varepsilon^{-1}(\varpi^c A_0 \oplus \varpi^c B_0) \subseteq \varpi^m R_0 \iff \varepsilon^{-1}(A_0 \oplus B_0) \subseteq \varpi^{m-c} R_0,$$

which is exactly the condition for boundedness.

We now need to show that $(ii) \Rightarrow (iii)$. We will define a topology on $R$ so that $\varepsilon$ is automatically strict, and then compare this new topology with the old one. Let $S = R$ and $S_0 = A_0 \cap \phi^{-1}(B_0)$, which we topologize via $(S_0, \varpi)$. Then, the map

$$\widetilde{\varepsilon}\colon S \longrightarrow A \oplus B$$

is strict, and so we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{R} \arrow[r, "\varepsilon"] \arrow[d, "\varphi"'] & \widehat{A \oplus B} \arrow[r, "\delta"] & \widehat{C} \arrow[r] & 0 \\
0 \arrow[r] & \widehat{S} \arrow[ur, "\widetilde{\varepsilon}"'] & & &
\end{tikzcd}$$

Note Fact 8.4 implies the top row is strict and exact, and $\varphi\colon \widehat{R} \to \widehat{S}$ is the map induced by the continuous map $R \to S$, which is the identity map on rings. Since $\widehat{R}, \widehat{S}$ are both the kernel of the map $\delta\colon \widehat{A \oplus B} \longrightarrow \widehat{C}$ as abelian groups, we see that $\varphi$ is continuous and bijective. But a general version of the open mapping theorem[7] implies that $\varphi$ is open. Thus, there exists $n$ such that $\varpi^n \widehat{S_0} \subseteq \varphi(\widehat{R_0})$. Restricting to $R$, this inclusion implies $\varpi^n S_0 \subseteq R_0$ by Lemma 8.1($ii$). This inclusion is exactly what is needed in ($iii$). $\square$

Lemma 8.3 is crucial: to get exactness of $(*)$, we will verify that the morphism $\varepsilon$ in $(**)$ is strict.

We have to prove one more Lemma, after which we can move on to the proof of the main result.

**Recall 8.5.** We say that a Tate ring $R$ is *uniform* if $R^\circ$ is bounded, and that a Tate affinoid ring $(R, R^+)$ is *stably uniform* if for all rational subsets $U \subseteq X = \operatorname{Spa}(R, R^+)$, the ring $\mathcal{O}_X(U)$ is uniform.

**Key Lemma 8.6** [BV14, Lem. 3]**.** *Let $R$ be a Tate ring with couple of definition $(R_0, \varpi)$. Let $t_1, \ldots, t_n \in R$ be a set of generators for the unit ideal in $R$, and denote*

$$\phi_i\colon R \longrightarrow R[\tfrac{1}{t_i}]$$

*to be the natural localization maps. Then,*

$$\bigcap_{i=1}^{n} \phi_i^{-1}\big(\phi_i(R_0)[\tfrac{t_1}{t_i}, \ldots, \tfrac{t_n}{t_i}]\big) \subseteq R^\circ.$$

**Remark 8.2.** This seems to follow by the fact that $R^+$ is the set of elements in $R$ that have valuation $\leq 1$ at every point, and since $R^+ \subseteq R^\circ$, which hold by definition of $\operatorname{Spa}(R, R^+)$ in [**?**, §3].

*Proof of Key Lemma 8.6.* Fix $r$ in the left-hand side. For each $i$, there is a homogeneous polynomial $f_i \in R_0[T_1, \ldots, T_n]$ such that

$$\phi_i(r) = t_i^{-\deg f_i} \phi_i(f_i(t_1, \ldots, t_n)).$$

This implies that

$$t_i^{\deg f_i} r - f_i(t_1, \ldots, t_n) \in \ker \phi_i,$$

and so there exists $c_i \geq 0$ such that

$$t_i^{c_i}(t_i^{\deg f_i} r - f_i(t_1, \ldots, t_n)) = 0.$$

Rearranging terms, we obtain

$$r t_i^{c_i + \deg f_i} - t_i^{c_i} f_i(t_1, \ldots, t_n) = 0. \tag{8.7}$$

Now let $d_i = c_i + \deg f_i$, so that $g_i(t_1, \ldots, t_n) = t_i^{c_i} f_i(t_1, \ldots, t_n)$ is homogenous of degree $d_i$.

Let $N = d_1 + \cdots + d_n$, and take $A \geq 0$ such that $\varpi^A \cdot t_i \in R_0$ for all $i$. We then claim:

**Claim 8.3.** For all $h \in R_0[T_1, \ldots, T_n]$ homogeneous of degree $n$, and all $m \geq 0$, we have

$$\varpi^{NA} h(t_1, \ldots, t_n) r^m \in R_0.$$

[7] One may either check that the argument of [Bou03, Ch. I, §3, n° 3, Thm. 1] applies, or use [Hen14, Thm. 1.6].

*Proof of Claim 8.3.* The proof is by induction on $m$. By splitting $h$ up into monomials, it suffices to consider the case when $h$ is a monomial. The case $m = 0$ holds by choice of $A$. Next, assume that $m > 0$. Write $h = T_1^{e_1} \cdots T_n^{e_n}$. Since

$$e_1 + \cdots + e_n = N = d_1 + \cdots + d_n,$$

there exists some $i$ such that $e_i \geq d_i$; after permuting coordinates, we may assume that $i = 1$. Write

$$\varpi^{NA} t_1^{e_1} \cdots t_n^{d_n} \cdot r^m = \varpi^{NA} t_1^{e_1 - d_1} t_2^{e_2} \cdots t_n^{e_n} g_1(t_1, \ldots, t_n) \cdot r^{m-1} \in R_0$$

by (8.7) and the inductive hypothesis, giving Claim 8.3. □

Returning to the proof of Key Lemma 8.6, let $a_1, \ldots, a_n$ such that $\sum a_i t_i = 1$. Fix $B \geq 0$ such that $\varpi^B a_i \in R_0$. We then apply Claim 8.3 to the polynomial

$$h = (\varpi^B a_1 T_1 + \cdots + \varpi^B a_n T^n)^N$$

to obtain

$$\varpi^{BN+NA} h\Big(\frac{t_1}{\varpi^B}, \ldots, \frac{t_n}{\varpi^B}\Big) r^m = \varpi^{BN+NA} r^m \in R_0$$

for all $m \geq 0$, hence $r \in R^\circ$. □

The next step is to show the sheaf axiom for a special cover as in (8.2).

**Remark 8.4** (Comparison with rigid analytic setting)**.** In the rigid analytic setting, one of the main difficulties is to reduce to the case of rational subsets, which you do using the Gerritzen–Grauert theorem. We can skip this step since adic spaces are spectral, and rational subsets form a basis. Afterward, a formal argument allows one to reduce to a special cover as in (8.2), which can be analyzed directly. Some things are harder in the adic setting, however, since there are no noetherian hypotheses: our rings are very general.

**Remark 8.5.** Stable uniformity will be used in two places:

(1) To show $\mathcal{O}_X$ is a sheaf of abelian groups; and
(2) To check that it is moreover a sheaf of topological rings.

**Remark 8.6.** The reduction to the case of the special cover in (8.2) is something very general, and appears in [BGR84, 8.1.4]. This indicates that the argument we are presenting here likely also proves the Tate acyclicity theorem for stably uniform Tate rings.

**Corollary 8.8** [BV14, Cor. 4]**.** *Let $(R, R^+)$ be a uniform Tate ring, and denote $X = \operatorname{Spa}(R, R^+)$. Let $t \in R$, and consider $U = X(\frac{t}{1})$, $V = X(\frac{1}{t})$. Then, the sequence*

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V) \longrightarrow 0$$

*is exact.*

*Proof.* The assertion is Lemma 8.3($ii$), and so it suffices to verify Lemma 8.3($iii$). We apply Key Lemma 8.6 to the elements $t_1 = 1$ and $t_2 = t$, in which case $\phi_1 = \mathrm{id}$ and $\phi_2 = \phi$. Key Lemma 8.6 gives

$$A_0 \cap \phi^{-1}(B_0) \subseteq R^\circ.$$

Now by uniformity, $R^\circ$ is bounded, so this inclusion implies Lemma 8.3($iii$) is satisfied. □

Even though Corollary 8.8 does not require stable uniformity, it is necessary in the reduction to the case of the special cover (8.2), since we will need to pass to rational refinements of any given cover.

**Corollary 8.9** [BV14, Cor. 5]**.** *Let $\operatorname{Spa}(R, R^+) = X$, and let $f \in \mathcal{O}_X(X)$. If $(U_i)_i$ is such that $f|_{U_i} = 0$, then $f$ is topologically nilpotent.*

This is true if $\mathcal{O}_X$ is already a sheaf, but of course that is what we are trying to prove. A stronger version of this was proved in [Bha17, Cor. 17.3].

*Proof.* Key Lemma 8.6 (after refining to a special rational covering as in Lemma 8.11($i$)) implies that

$$\{(\varpi^{-1}f)^n \mid n \in \mathbf{N}\}$$

is bounded, i.e., it is contained in $\varpi^N R_0$ for some $N \leq 0$. Thus $f^m \in \varpi^{N+m}R_0$ for all $m \in \mathbf{N}$, which implies $f$ is topologically nilpotent. □

**Definition 8.7.** Let $R$ be a Tate ring. Then, $\mathrm{Spa}(R, R^+)$ is *stably uniform* if for every rational subset $U$, the ring $\mathcal{O}_X(U)$ is uniform.

We now state the main result:

**Theorem 8.10** [BV14, Thm. 7]**.** *If $(R, R^+)$ is a stably uniform Tate ring, then $\mathcal{O}_X$ is a sheaf.*

We now define a few special types of coverings; Lemma 8.11 describes how they are related.

**Definition 8.8** (Special types of rational covers)**.** Let $t_1, \dots, t_n \in R$ be generators for the unit ideal in $R$. Then, the rational covering

$$\Big\{U_i \coloneqq X\Big(\frac{t_1, \dots, t_n}{t_i}\Big) \;\Big|\; i \in \{1, \dots, n\}\Big\}$$

of $X$ is called the *rational covering associated to* $t_1, \dots, t_n$. Moreover, if all the $t_i$ are units, then we call $\{U_i\}$ as above a *rational covering associated to units*.

Given a rational covering associated to $t_1, \dots, t_n$, the rational cover

$$\Big\{U_I \coloneqq \bigcap_{i \in I} X(\tfrac{t_i}{1}) \cap \bigcap_{i \notin I} X(\tfrac{1}{t_i}) \;\Big|\; I \subseteq \{1, \dots, n\}\Big\}$$

of $X$ is called a *Laurent cover*. The $U_I$ are rational subsets since they are finite intersections of rational subsets, and satisfy

$$U_I = \left\{x \in X \;\middle|\; \begin{matrix} |t_i(x)| \leq 1\ \forall i \in I, \\ |t_i(x)| \geq 1\ \forall i \notin I \end{matrix}\right\}.$$

**Fact 8.9.** $U_I$ is the rational cover associated to

$$\Big\{\prod_{j \in J} t_j \;\Big|\; J \subseteq \{1, \dots, n\}\Big\}.$$

Now the strategy for Theorem 8.10 is to start with an arbitrary covering of $X$, and to refine it until it is of one of the forms in Definition 8.8. The following Lemma allows us to do this.

**Lemma 8.11** [BV14, Lem. 8] (Huber)**.** *Let $R$ be a complete Tate ring, and let $X = \mathrm{Spa}(R, R^+)$.*

($i$) *For every cover $\mathcal{U}$ of $X$, there exists a rational cover $\mathcal{V}$ associated to $(t_1, \dots, t_n)$, which is a refinement of $\mathcal{U}$.*
($ii$) *If $\mathcal{U}$ is a rational cover of $X$, then there exists a Laurent cover $\mathcal{V}$ such that for all $V \in \mathcal{V}$, the cover $\{U \cap V \mid U \in \mathcal{U}\}$ of $V$ is a rational covering generated by units.*
($iii$) *If $\mathcal{U}$ is a rational cover of $X$ generated by units, then there exists a Laurent cover $\mathcal{V}$ of $X$, which is a refinement of $\mathcal{U}$.*

*Proof.* For ($i$), you may first assume that $\mathcal{U}$ is a rational covering, since rational subsets form a basis for $X$. Then, use [Hub94, Lem. 2.6].

($iii$) is purely combinatorial, and can be shown by the argument in the proof of [BGR84, Lem. 8.2.2/4] or [Bos14, Lem. 4.3/6]. The point is that the Laurent covering associated to all products $t_i t_j^{-1}$ for $0 \leq i < j \leq n$ works.

For ($ii$), first assume that $\mathcal{U}$ is a rational cover associated to $t_1, \dots, t_n$ (by using ($i$)). Then, there exist $a_i$ such that $\sum a_i t_i = 1$, and since $R^+$ is open, there exists some $B \geq 0$ such that $\varpi^B a_i \in R^+$ for all $i$. This implies that $|\varpi^B a_i(x)| \leq 1$, and so

$$|\varpi^{B+1} a_i(x)| < 1.$$

Now the strict triangle inequality implies

$$|\varpi^{B+1}(x)| < \max_i |t_i(x)|. \tag{8.12}$$

We can define our Laurent cover $\mathcal{V}$ as that associated to $\varpi^{-B-1}t_1, \ldots, \varpi^{-B-1}t_n$.

We have to show that $V_I \cap U_{i_0}$ is a rational subset generated by units in $V_I$, for all $U_{i_0} \in \mathcal{U}$. First,

$$\begin{aligned} V_I &= \left\{ x \in X \;\middle|\; \begin{array}{l} |t_i(x)| \le |\varpi^{B+1}(x)| \ \forall i \in I, \\ |t_i(x)| \ge |\varpi^{B+1}(x)| \ \forall i \notin I \end{array} \right\}, \\ U_{i_0} &= \{x \in X \mid |t_j(x)| \le |t_{i_0}(x)| \ \forall j = 1, \ldots, n\} \\ &= \left\{ x \in X \;\middle|\; \max_j |t_j(x)| = |t_{i_0}(x)| \ \forall j = 1, \ldots, n \right\}. \end{aligned}$$

If $i_0 \in I$, then for all $x \in V_I \cap U_{i_0}$, we must have

$$|t_{i_0}(x)| \le |\varpi^{B+1}(x)| < \max_i |t_i(x)|$$

by (8.12), which is a contradiction, hence $V_I \cap U_{i_0} = \emptyset$. On the other hand, suppose $i_0 \notin I$. Then, for all $x \in V_I \cap U_{i_0}$, we have

$$|t_{i_0}(x)| = \max_j |t_j(x)| \ge |\varpi^{B+1}(x)|,$$

and so the conditions $|t_j(x)| \le |t_{i_0}(x)|$ defining $U_{i_0}$ for $j \in I$ may be ommitted. Thus,

$$V_I \cap U_{i_0} = V_I\left( \frac{t_i}{t_{i_0}} \;\middle|\; i \notin I \right). \tag{8.13}$$

Now for all $i \notin I$, we have $|t_i(x)| \ge |\varpi^{B+1}(x)| \ne 0$ for all $x \in V_I$, and so the $t_i$ involved on the right-hand side of (8.13) are units by [Bha17, Prop. 16.1(4)]. $\square$

We are now ready to prove our main result.

*Proof of Theorem 8.10.* We will first show $\mathcal{O}_X$ is a sheaf of abelian groups, and then worry about the topological condition.

**Step 1.** If $U \subseteq X$ is rational, and $\mathcal{U}$ is a Laurent cover of $U$, then $\mathcal{O}_X$ is a $\mathcal{U}$-sheaf (i.e., it satisfies the sheaf axiom for this covering).

We proceed by induction on the number of elements in the covering. The case $n = 2$ is Corollary 8.8. For the inductive step, use [BGR84, Lem. 8.1.4/4].

**Step 2.** A cover by rational subsets $U \subseteq X$ satisfies the sheaf condition, hence $\mathcal{O}_X$ is a sheaf of abelian groups.

Use Lemma 8.11($i$)–($ii$) to reduce to the case of a Laurent cover. Now use Step 1. The last assertion follows since rational subsets form a basis for $X$.

**Step 3.** $\mathcal{O}_X$ is a sheaf of complete topological rings.

Let $U$ be a rational set and let $\bigcup_i U_i = U$ be the rational covering associated to $t_1, \ldots, t_n$. Then, we need to show

$$\mathcal{O}_X(U) \xrightarrow{\phi} \prod_i \mathcal{O}_X(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)$$

is an equalizer diagram, i.e., we need to show that if $A$ is any complete topological ring as in the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r, "\phi"] & \prod_i \mathcal{O}_X(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \\ A \arrow[u, dashed, "\exists!"] \arrow[ur] & & \end{tikzcd}$$

such that the two composed maps to $\prod_{i,j} \mathcal{O}_X(U_i \cap U_j)$ are equal, then there exists a unique factorization through $\mathcal{O}_X(U)$ as above. By [Hub94, §2, first paragraph], it suffices to show that the map $\phi$ is strict (if $\phi$ is only continuous, then $\mathcal{O}_X(U)$ could have a finer topology than the subspace topology, in which case the inclusion of $A = \mathcal{O}_X(U) \subseteq \prod_i \mathcal{O}_X(U_i)$ with the subspace topology would not factor through $\mathcal{O}_X(U)$).

To show $\phi$ is strict, Lemma 8.11($i$) implies that there exists a rational cover $U = \bigcup_j V_j$ that refines $U = \bigcup_i U_i$. Key Lemma 8.6 and the uniformity of $\mathcal{O}_X(V_j)$ imply that the map $\mathcal{O}_X(U) \to \prod_j \mathcal{O}_X(V_j)$ is strict. Since this map factors though $\phi$, we see that $\phi$ must also be strict. $\square$

This completes the proof, although there are some formal steps we did not prove. Note that Lemma 8.3, Key Lemma 8.6, and Corollary 8.8 are the key inputs, and that reducing to the special covering in (8.2) was purely formal.

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