ABHYANKAR AND QUASI-MONOMIAL VALUATIONS

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ABSTRACT. These are the notes for a sequence of two talks given in the Student Commutative Algebra Seminar at the University of Michigan in October 2017. The goal of these notes is to introduce the classes of divisorial, quasi-monomial, and Abhyankar valuations and discuss the proof of a theorem of Ein–Lazarsfeld–Smith [ELS03, Proposition 2.8], which states that Abhyankar and quasi-monomial valuations coincide in characteristic zero. The presentation is also heavily inspired by [JM12, Dat17].

1. Basics on Valuations

Let k be a field and let K/k be a finitely-generated field extension. The setting of interest is when K/k is not finite, in which K is the function field of a quasi-projective variety[1] over k.

Definition 1.1. A *model* of K/k is a variety X over k such that $K \simeq k(X)$ as fields over k.

It is easy to see that models exist: if $y_1, \ldots, y_n \in K$ generate K as a field over k and if A denotes the k-subalgebra of K generated by $y_1, \ldots, y_n$, then $\mathrm{Spec}(A)$ is a model for K/k. In general, there is not a unique choice of a model (indeed, this is the definition of birational equivalence), but it is often psychologically helpful to make a choice.

Definition 1.2. A *valuation* on K/k is a function $v\colon K \to \mathbf{R} \cup \{+\infty\}$ such that

(1) $v(x) = +\infty$ if and only if $x = 0$;
(2) $v(xy) = v(x) + v(y)$;
(3) $v(x + y) \geq \min\{v(x), v(y)\}$;
(4) $v(a) = 0$ for all $a \in k^*$.

In the literature, a valuation as in Definition 1.2 is often called a real valuation on K over k.

Definition/Exercise 1.3. We record some basic properties on valuations, whose proof can be found e.g. in [AM69, §5]. Let v be a valuation on K/k.

(1) If $v(x) \neq v(y)$, then $v(x + y) = \min\{v(x), v(y)\}$.
(2) The *value group* of v is the (abelian) subgroup $\Gamma_v \coloneqq v(K^*)$ of $\mathbf{R}$.
(3) The *valuation ring* of v is the subring $R_v \coloneqq \{x \in K : v(x) \geq 0\}$ of K; R_v is a local ring with maximal ideal $\mathfrak{m}_v \coloneqq \{x \in K : v(x) > 0\}$.
(4) The *residue field* of v is the residue field κ_v of the local ring R_v, i.e. $\kappa_v = R_v/\mathfrak{m}_v$; the residue field κ_v is also a field extension of k.
(5) The valuation v is *discrete* if $v(K^*)$ is a discrete subgroup of $\mathbf{R}$ (and hence, isomorphic to $\mathbf{Z}$ as an abelian group). In fact, the condition that v be discrete is equivalent to the valuation ring R_v being noetherian. The valuation ring R_v of a discrete valuation v is called a *discrete valuation ring* (dvr); this class of rings admits many nice descriptions as in e.g. [AM69, Proposition 9.2].

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[1] In this note, a *variety* is an integral separated scheme of finite type over a field; in particular, a variety is irreducible.

There is a notion of equivalence of valuations: two valuations v_1 and v_2 on K/k are declared to be *equivalent* if their valuation rings R_{v_1} and R_{v_2} coincide as subrings of K. We will often implicitly only consider valuations up to equivalence. Note that specifying a valuation ring in K containing k (in the usual commutative algebraic sense) is equivalent to specifying a valuation, up to equivalence.

Example 1.4. Consider the following examples of valuations.

(1) For any extension K/k, the *trivial valuation* v_{triv} is given by
$$v_{\mathrm{triv}}(f) \coloneqq \begin{cases} 0 & f \neq 0 \\ +\infty & f = 0 \end{cases}$$
for $x \in K$. In particular, valuations always exist.

(2) If $K = k(T)/k$, consider the valuation v_0 given by
$$v_0(f) \coloneqq \max\left\{ j \in \mathbf{Z}_{\geq 0} \colon f \in (T^j) \right\}$$
for $f \in k[T]$. This formula completely determines v_0: given $f \in k(T)$, write $f = g/h$ for some $g, h \in k[T]$, and set $v_0(f) = v_0(g) - v_0(h)$.

The valuation v_0 has a nice geometric interpretation: thinking of K as the function field of the affine line $\mathbf{A}^1_k = \operatorname{Spec}(k[T])$, v_0 is the order of vanishing at the origin $0 \in \mathbf{A}^1_k$, i.e. the valuation corresponding to the discrete valuation ring $\mathcal{O}_{\mathbf{A}^1_k, 0} = k[T]_{(T)}$.

(3) If $K = k(T_1, T_2)/k$, one can think of K as the function field of the affine plane $\mathbf{A}^2_k = \operatorname{Spec}(k[T_1, T_2])$ and ask for a valuation that acts as "order of vanishing at the origin" $0 \in \mathbf{A}^2_k$; however, the local ring $\mathcal{O}_{\mathbf{A}^2_k, 0}$ is unfortunately not a dvr. To resolve this issue, consider the blowup $\mathrm{Bl}_0\left(\mathbf{A}^2_k\right)$ of $\mathbf{A}^2_k$ at the origin, which is given by
$$\mathrm{Bl}_0\left(\mathbf{A}^2_k\right) \coloneqq \operatorname{Proj}\left(\frac{k[T_1, T_2][S_1, S_2]}{(S_1 T_2 - T_1 S_2)}\right) \to \mathbf{A}^2_k.$$
By passing to the affine chart where $S_2 \neq 0$, we get the more concrete description as the morphism of affine varieties
$$\operatorname{Spec}\left(\frac{k[T_1, T_2][S_1]}{(S_1 T_2 - T_1)}\right) \to \operatorname{Spec}(k[T_1, T_2]).$$
In the affine chart above, the exceptional divisor $E \subseteq \mathrm{Bl}_0\left(\mathbf{A}^2_k\right)$ is cut out by the equation $T_2 = 0$.

The exceptional divisor E is irreducible, so the local ring $\mathcal{O}_{\mathrm{Bl}_0(\mathbf{A}^2_k), E}$ is a dvr and hence it comes equipped with a valuation ord_E. Moreover, the valuation ord_E on $k(T_1, T_2)$ coincides with the order of vanishing at origin as it is usually defined[2], i.e. if $f \in k(T_1, T_2)$, then $\mathrm{ord}_0(f) = \mathrm{length}(\mathcal{O}_{\mathbf{A}^2_k, 0}/(f))$.

Example 1.4(2,3) are two instances of a broader class of examples of valuations, defined below.

Definition 1.5. A valuation v on K/k is *divisorial* if there is a normal model X of K/k and a prime divisor $E \subseteq X$ such that v is equivalent to ord_E.

The aim of §3 is to characterize the class of divisorial valuations in a purely algebraic manner, which avoids having to construct a specific model in order to verify whether or not a valuation is divisorial.

From divisorial valuations, we can construct more complicated examples of valuations, as demonstrated below.

Example 1.6. Consider $K = k(T_1, T_2)/k$, thought of as the function field of the affine plane $\mathbf{A}^2_k = \operatorname{Spec}(k[T_1, T_2])$. For any $\alpha = (\alpha_1, \alpha_2) \in (\mathbf{R}_{\geq 0})^2$, consider the *monomial* valuation v_α on K/k given by
$$v_\alpha\left(\sum_{i,j \geq 0} a_{ij} T_1^i T_2^j\right) \coloneqq \min\left\{ i \cdot \alpha_1 + j \cdot \alpha_2 \colon a_{ij} \neq 0 \right\}.$$

[2] If X is an integral scheme and $E \subseteq X$ is a prime (Weil) divisor on X with generic point η, the *order of vanishing* along E is the multiplicative function $\mathrm{ord}_E \colon \mathrm{Frac}(\mathcal{O}_{X,\eta})^* \to \mathbf{Z}$ given by $f \mapsto \mathrm{length}(\mathcal{O}_{X,\eta}/(f))$ for nonzero elements $f \in \mathcal{O}_{X,\eta}$. For more details, see [Sta17, Tag 02RJ].

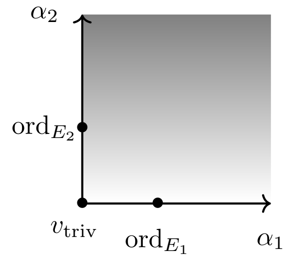

FIGURE 1. The cone of valuations v_α, as in Example 1.6.

Consider the prime divisors $E_1 = \{T_1 = 0\}$ and $E_2 = \{T_2 = 0\}$ on $\mathbf{A}^2_k$. Notice that if $\alpha = (1, 0)$, then $v_\alpha = \mathrm{ord}_{E_1}$, and if $\alpha = (0, 1)$, then $v_\alpha = \mathrm{ord}_{E_2}$ Thus, the valuations v_α "interpolate" between the divisorial valuations ord_{E_1} and ord_{E_2}. Even weirder, if $\alpha = (0, 0)$, then $v_\alpha = v_{\mathrm{triv}}$! In the space of all valuations on K/k, the v_α's form a 2-dimensional cone, pictured in Fig. 1. For a general $\alpha \in (\mathbf{R}_{\geq 0})^2$, the valuation v_α can have odd behaviour: for example, we will see later that if α_1 and α_2 are $\mathbf{Q}$-linearly independent, then v_α is not divisorial.

We will later define a broader class of quasi-monomial valuations, which encompass both divisorial valuations and the monomial valuation of Example 1.6. There are, however, valuations that will not fit into this class. One such example is given below.

Example 1.7. Consider $K = k(T_1, T_2)/k$, thought of as the function field of $\mathbf{A}^2_k = \mathrm{Spec}(k[T_1, T_2])$. Let $q(t) \in k[[t]]$ be a power series that is not algebraic over $k[t]$ (e.g. if $k = \mathbf{C}$, then the exponential power series $q(t) = \sum_{n \geq 0} t^n/n!$ is an example). One can define an embedding

$$k(T_1, T_2) \hookrightarrow k((t))$$

given by $T_1 \mapsto t$ and $T_2 \mapsto q(t)$. The pullback v_q of the t-adic valuation on $k[[t]]$ to K gives a discrete valuation on K with residue field[3] $\kappa_{v_q} = k$. Intuitively, v_q corresponds to the "order of vanishing along the germ of the transcendental curve $T_2 = q(T_1)$ at the origin $(T_1, T_2) = (0, 0)$".

Remark 1.8. If the base field k carries a natural valuation and K/k is a finitely-generated field extension, then one ought to consider valuations on K that extend the given valuation on k (in our setting, k is equipped with the trivial valuation). In this more complicated setup, much of the story presented in this note carries over with additional work. See [Jon16, §22-25].

2. INVARIANTS OF VALUATIONS

Let K/k be a finitely-generated field extension. There are many ways to measure the complexity of a valuation on K/k, some more geometric in nature and others more algebraic. Two such invariants are introduced below.

Definition 2.1. Let v be a valuation on K/k.

(1) The *transcendence degree* of v is $\mathrm{tr.deg}(v) \coloneqq \mathrm{tr.deg}\,(\kappa_v/k)$.
(2) The *rational rank* of v is $\mathrm{rat.rk}(v) \coloneqq \dim_{\mathbf{Q}} (\Gamma_v \otimes_{\mathbf{Z}} \mathbf{Q})$.

Note that the definitions of both the transcendence degree and the rational rank of a valuation are insensitive to replacing a valuation with an equivalent one. In addition, both are invariant under replacing (K, v) with an immediate extension[4], e.g. its completion.

The rational rank of a valuation v is often defined in terms of the *divisible value group* of v, i.e. the subgroup

$$\sqrt{\Gamma_v} = \{\gamma \in \mathbf{R} \colon \exists\ n \in \mathbf{Z} \text{ such that } n\gamma \in \Gamma_v\}\,.$$

[3] The residue field κ_{v_q} contains k and is a subfield of the residue field of the t-adic valuation on $k[[t]]$, which is k. Thus, $\kappa_{v_q} = k$.
[4] An extension of valued fields is *immediate* if the value groups and residue fields are isomorphic. See [Bou89, VI, §10.2].

of $\mathbf{R}$. This does not affect the definition of $\operatorname{rat.rk}(v)$, since the natural injective map $\Gamma_v \otimes_{\mathbf{Z}} \mathbf{Q} \hookrightarrow \sqrt{\Gamma_v} \otimes_{\mathbf{Z}} \mathbf{Q}$ is, in fact, an isomorphism of $\mathbf{Q}$-vector spaces; indeed, given $x \otimes b \in \sqrt{\Gamma_v} \otimes_{\mathbf{Z}} \mathbf{Q}$, take $n \in \mathbf{Z}$ such that $nx \in \Gamma_v$, then $nx \otimes \frac{b}{n}$ maps to $x \otimes b$. This is important since, when working with valued field that are not algebraically closed, one generally works with the divisible value group as opposed to the value group.

The fundamental result concerning these invariants states that the rational rank and the transcendence degree are both bounded, and in a precise fashion.

Theorem 2.2. [The Zariski–Abhyankar Inequality] *If v is a valuation on K/k, then*

$$\operatorname{rat.rk}(v) + \operatorname{tr.deg}(v) \leq \operatorname{tr.deg}(K/k).$$

Moreover, if equality is achieved, then Γ_v is a finitely-generated abelian group and κ_v is a finitely-generated field extension of k.

For a proof of Theorem 2.2, see [Bou89, VI, §10.3, Cor 1] (the idea is simple: lift generators of $v(K^*)$ and of κ_v and argue that they form, possibly after making slight modifications, an algebraically independent set over k). In the above form, Theorem 2.2 is due to Zariski, and it was generalized by Abhyankar [Abh56].

Example 2.3. Let K/k be a finitely-generated field extension of transcendence degree n.

(1) If v_{triv} is the trivial valuation on K/k, then $\operatorname{rat.rk}(v_{\text{triv}}) = 0$ and $\operatorname{tr.deg}(v_{\text{triv}}) = n$. In fact, v_{triv} is the unique valuation of rational rank zero.

(2) If v is a divisorial valuation on K/k, then we claim that $\operatorname{rat.rk}(v) = 1$ and $\operatorname{tr.deg}(v) = n-1$. By assumption, there exists a normal model X of K/k and a prime divisor $E \subseteq X$ such that v is equivalent to ord_E. The valuation ring of ord_E is the dvr $\mathcal{O}_{X,E}$, so $v(K^*) \simeq \mathbf{Z}$; in particular, $\operatorname{rat.rk}(v) = 1$. Moreover, as κ_v is the function field[5] of E, the transcendence degree of κ_v/k coincides with the dimension of E, which is $n-1$.

The valuations that achieve the upper bound in Theorem 2.2 will be of particular interest, and so we give them a name.

Definition 2.4. A valuation v on K/k is *Abhyankar* if $\operatorname{rat.rk}(v) + \operatorname{tr.deg}(v) = \operatorname{tr.deg}(K/k)$.

Example 2.5. By the calculation in Example 2.3(2), any divisorial valuation is Abhyankar.

Example 2.6. Consider $K = k(T_1, T_2)/k$, thought of as the function field of $\mathbf{A}^2_k = \operatorname{Spec}(k[T_1, T_2])$.

(1) If v_α is the monomial valuation of Example 1.6 where α_1 and α_2 are $\mathbf{Q}$-linearly independent, then $\Gamma_v = \mathbf{Z} \cdot \alpha_1 \oplus \mathbf{Z} \cdot \alpha_2$, and so $\operatorname{rat.rk}(v) = 2$. By Abhyankar's inequality, it follows that $\operatorname{tr.deg}(v) = 0$ and v is Abhyankar.

(2) If v_q is as in Example 1.7, then $\Gamma_v = \mathbf{Z}$ and $\kappa_v = k$, and hence $\operatorname{rat.rk}(v_q) + \operatorname{tr.deg}(v_q) < 2$; in particular, v_q is not Abhyankar.

3. Zariski's Theorem on Divisorial Valuations

The goal of this section is to prove the following algebraic characterization of divisorial valuations, originally due to Zariski. It will serve as a warm-up to the theorem of Ein–Lazarsfeld–Smith, which provides an analogous characterization for the broader class of quasi-monomial valuations.

Theorem 3.1. [ZS60, VI, §14, Theorem 31] *If K/k is a finitely-generated field extension of transcendence degree n and v is a valuation on K/k, then the following are equivalent:*

(1) $\operatorname{rat.rk}(v) = 1$ *and* $\operatorname{tr.deg}(v) = n-1$*;*

(2) v is divisorial.

[5]By working Zariski-locally near the generic point of E, we may assume $X = \operatorname{Spec}(A)$ is affine, where the generic point of E corresponds to a height-1 prime ideal $\mathfrak{p}$ of A. Then, $\kappa_v = \mathcal{O}_{X,E}/\mathfrak{m}_E = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = (A/\mathfrak{p})_\mathfrak{p} = \operatorname{Frac}(A/\mathfrak{p})$, which is the function field of E, since E is the closed subvariety $V(\mathfrak{p})$ of X.

Using Theorem 3.1, we can determine which of the monomial valuations arising in Example 1.6 are divisorial.

Example 3.2. If v_α is the monomial valuation of Example 1.6, then $\operatorname{rat.rk}(v) = 1$ if and only if α_1 and α_2 are $\mathbf{Q}$-linearly dependent. In particular, this occurs if and only if v_α is divisorial. Said differently, the valuation v_α is divisorial, but one does not necessarily see this by thinking of it as a valuation on $\mathbf{A}^2_k$! Rather, one must perform some surgery operation (i.e. a blowup) to construct a new model on which the valuation v_α "lives".

Remark 3.3. If one fixes a model X of K/k and v admits a centre on X (see Definition 5.1), then Theorem 3.1 can be recast as follows: a valuation v on $k(X)/k$ is divisorial if and only if there is a normal variety Y over k, a proper birational morphism $Y \to X$, and a prime divisor $E \subseteq Y$ such that v is equivalent to ord_E. This is the formulation of Theorem 3.1 that is generalized to the case of Abhyankar valuations by Ein–Lazarsfeld–Smith.

Proof of Theorem 3.1. If v is divisorial, then we have shown in Example 2.3(2) that the rational rank and the transcendence degree of v have the correct values. Conversely, let v be a valuation on K/k with $\operatorname{rat.rk}(v) = 1$ and $\operatorname{tr.deg}(v) = n-1$. The idea of the proof is to break up K/k into a purely transcendental extension followed by a finite extension, and analyze the behaviour of v separately on each piece.

Pick $x_1, \dots, x_{n-1} \in R_v$ whose images in the residue field κ_v form a transcendence basis over k. They are necessarily algebraically independent over k (indeed, any algebraic relation would descend to κ_v). Take $x_n \in R_v$ such that $x_1, \dots, x_{n-1}, x_n$ forms a transcendence basis of K/k. Consider the subfield $\ell \coloneqq k(x_1, \dots, x_{n-1}, x_n)$ of K, and let w denote the restriction of v to a valuation on ℓ/k.

If $S \coloneqq k[x_1, \dots, x_{n-1}, x_n]$, then $w(f) \geq 0$ for all $f \in S$, since it holds for each x_i. Our first goal is to construct the valuation ring of w in ℓ as a localization of S. Consider the prime ideal $\mathfrak{q}_w \coloneqq \{g \in S \colon w(g) > 0\}$ (i.e. the *centre* of w on S); one can show that $\mathfrak{q}_w$ has height 1 on S. Moreover, observe that the quotient $S/\mathfrak{q}_w$ is an integral domain such that

$$\operatorname{tr.deg}(\operatorname{Frac}(S/\mathfrak{q}_w)/k) = n-1,$$

because $x_1, \dots, x_{n-1}$ are algebraically independent over k and $x_n \in \mathfrak{q}_w$. Thus, for any prime ideal $\mathfrak{p}$ of S such that $\mathfrak{p} \subsetneq \mathfrak{q}_w$, we must have

$$n-1 = \operatorname{tr.deg}(\operatorname{Frac}(S/\mathfrak{q}_w)/k) < \operatorname{tr.deg}(\operatorname{Frac}(S/\mathfrak{p})/k) \leq \operatorname{tr.deg}(\operatorname{Frac}(S)/k) = n,$$

and hence $\mathfrak{p} = 0$. It follows that $\mathfrak{q}_w$ is a height-1 prime on the noetherian normal domain S, so the localization $S_{\mathfrak{q}_w}$ is a dvr in ℓ. By construction, $w \geq 0$ on $S_{\mathfrak{q}_w}$, so $S_{\mathfrak{q}_w}$ is contained in the valuation ring of w. As dvr's are maximal with respect to dominance, it follows that $S_{\mathfrak{q}_w}$ is the valuation ring of w.

Now, K is a finite extension of ℓ, so v is the unique extension of w from a valuation on ℓ/k to a valuation on K/k; in particular, v is discrete. If R denotes the integral closure of S in K, then $X \coloneqq \operatorname{Spec}(R)$ is a normal variety over k and $v \geq 0$ on R (indeed, the integral closure is the intersection of all valuation rings containing S; in particular, $S \subseteq R_v$, and hence $R \subseteq R_v$). If $\mathfrak{p}_v \coloneqq \{f \in R \colon v(f) > 0\}$, then repeating the same argument as above yields that $\mathfrak{p}_v$ is a height 1 prime ideal of R. Let $E \coloneqq V(\mathfrak{p}_v)$ be the corresponding prime divisor on X. As $v \geq 0$ on $R_{\mathfrak{p}_v}$, we have $R_{\mathfrak{p}_v} \subseteq R_v$; however, $R_{\mathfrak{p}_v}$ is a dvr, so we must equality. Thus, $R_v = \mathcal{O}_{X,E}$, i.e. v is equivalent to ord_E. □

Remark 3.4. To the reader unfamiliar with the book [ZS60], the terminology there differs from certain modern definitions in commutative algebra. If one is solely interested in the proof of [ZS60, VI, §14, Theorem 31], the most notable distinction is that a 'minimal prime' in the sense of [ZS60] is a 'height 1 prime' in the modern sense.

4. Quasi-Monomial Valuations

Let K/k be a finitely-generated field extension. We introduce below a class of geometrically-defined valuations on K/k, known in the literature as quasi-monomial valuations, that generalizes the class of divisorial valuations.

Definition 4.1. A valuation v on K/k is *quasi-monomial* (qm) if there exists

(1) a smooth[6] model X of K/k,
(2) a (not necessarily closed) point $x \in X$,
(3) a regular system of parameters $y = (y_1, \dots, y_d)$ of the local ring $\mathcal{O}_{X,x}$ at x,

such that $v(y_1), \dots, v(y_d)$ freely generate Γ_v as an abelian group. If the model X of K/k is fixed, then v is a *monomial valuation on* X.

The prototypical example of quasi-monomial valuations are those constructed in Example 1.6. More generally, all quasi-monomial valuations arise via a similar procedure, as described below.

Remark 4.2. It is easy to check that all quasi-monomial valuations on K/k arise via the following geometric construction. Given X, x, and y as in Definition 4.1, choose $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbf{R}_{\geq 0})^d$ and define a valuation $v_{y,\alpha}$ on K/k as follows: the inclusion $k \hookrightarrow \mathcal{O}_{X,x}$ induces an isomorphism $k[[y_1, \dots, y_d]] \simeq \widehat{\mathcal{O}}_{X,x}$ by the Cohen structure theorem, so any $f \in \mathcal{O}_{X,x}$ can be written as

$$f = \sum_{\mu \in (\mathbf{Z}_{\geq 0})^d} a_\mu y^\mu$$

for some $a_\mu \in k$, and we set

$$v_{y,\alpha}(f) \coloneqq \min \{ \langle \alpha, \mu \rangle \colon a_\mu \neq 0 \},$$

where $\langle \alpha, \mu \rangle \coloneqq \alpha_1\mu_1 + \ldots + \alpha_d\mu_d$ is the usual dot product. Thinking of the regular system of parameters $y_1, \dots, y_d$ as (analytic) local coordinates at x, the valuation $v_{y,\alpha}$ is the minimal one that assigns the "weight" α_i to the coordinate y_i. See [JM12, Proposition 3.1] for further details.

Example 4.3. Let K/k be a finitely-generated field extension.

(1) For any model X of K/k, any point $x \in X$, and any regular system of parameters y at x, the valuation $v_{y,0}$ is the trivial valuation v_{triv} on K/k; in particular, the trivial valuation is quasi-monomial.
(2) For any smooth model X of K/k and any prime divisor $E \subseteq X$, take a point $x \in E$. In the local ring $\mathcal{O}_{X,x}$, the ideal cutting out E is generated by a single element $g_E \in \mathfrak{m}_x$. Extend g_E to a regular system of parameters $y = (g_E, y_2, \dots, y_d)$ of $\mathcal{O}_{X,x}$, then the divisorial valuation ord_E corresponds to the quasi-monomial valuation $v_{y,\alpha}$, where $\alpha = (1, 0, \dots, 0)$.
(3) Example 1.6 is the classic example of a quasi-monomial valuation, and Fig. 1 illustrates that the quasi-monomial valuations are "linear interpolations" of the divisorial valuations ord_{E_1} and ord_{E_2}.

Proposition 4.4. *Any quasi-monomial valuation is Abhyankar.*

Proof. Let K/k be a finitely-generated field extension of transcendence degree n, and let v be a quasi-monomial valuation on K/k. With notation as in Definition 4.1, we can assume (after reindexing) that $v(y_1), \dots, v(y_r)$ are nonzero and $v(y_{r+1}), \dots, v(y_d)$ are zero, where $r = \mathrm{rat.rk}(v)$. As $v \geq 0$ on $\mathcal{O}_{X,x}$ by definition, there is a local homomorphism $\mathcal{O}_{X,x} \hookrightarrow R_v$, which induces an inclusion $\kappa(x) \hookrightarrow \kappa_v$ of residue fields. The elements $y_{r+1}, \dots, y_d$ descend to give algebraically independent elements of κ_v by assumption, so it follows that

$$\mathrm{tr.deg}(v) = \mathrm{tr.deg}(\kappa_v/k) \geq d - r,$$

and hence $\mathrm{rat.rk}(v) + \mathrm{tr.deg}(v) \geq d$. However, since X is smooth over k (hence a regular scheme), the dimension of X is equal to both the dimension of any of its local rings and the transcendence degree of K/k. As $d = \dim \mathcal{O}_{X,x}$, the Zariski–Abhyankar inequality implies that v is Abhyankar. □

It is a very deep theorem due to Knaf–Kuhlmann [KK05, Theorem 1.1] that the converse holds over perfect fields (this is known in the literature as "local monomialization (of Abhyankar valuations)"). For a nice discussion of related results, see [Dat17, §2.3]. The goal of the next section is to prove this converse result over a field of characteristic zero, following [ELS03, Proposition 2.8].

[6] Alternatively, one could allow normal models X of K/k and regular points $x \in X$, but these yield equivalent notions of quasi-monomial valuations since a regular point of a variety has a smooth open neighbourhood.

5. Abhyankar Valuations are Quasi-Monomial in Characteristic Zero

Let K/k be a finitely-generated field extension. The converse to Proposition 4.4 that we will prove is a generalization of Remark 3.3; in particular, it requires us to fix a model of K/k. This necessitates the introduction of the notion of a centre of a valuation.

Definition 5.1. Let X be a model of K/k. A valuation v on K/k is *centered* on X if there is a point $x \in X$ and a local inclusion $\mathcal{O}_{X,x} \hookrightarrow R_v$ of local rings. In this case, the point $x \in X$ is called the *centre* of v on X, and it is denoted by $c_X(v)$.

The valuative criteria of separatedness and properness guarantee that, if a valuation v admits a centre on a model X, then the centre is unique, and every valuation admits a centre on a proper model.

Example 5.2.

(1) If X is a model of K/k, then the centre $c_X(v_{\mathrm{triv}})$ of the trivial valuation v_{triv} is the generic point of X.
(2) If X is a normal model of K/k and $E \subseteq X$ is a prime divisor, then the centre $c_X(\mathrm{ord}_E)$ of the divisorial valuation ord_E on X is the generic point of E. Indeed, the local ring $\mathcal{O}_{X,E}$ is exactly the valuation ring of ord_E.
(3) In Example 1.6, the origin $0 \in \mathbf{A}^2_k$ is the centre of v_α on $\mathbf{A}^2_k$, provided $\alpha \neq (0,0)$.
(4) If $X = \mathbf{A}^1_k = \mathrm{Spec}(k[T])$ and if ord_∞ is the discrete valuation on $k(T)/k$ with valuation ring $k[T^{-1}]_{(T^{-1})}$, then $c_X(\mathrm{ord}_\infty)$ does not exist!

Remark 5.3. As demonstrated by Example 5.2(2), the centre $c_X(v)$ of a valuation v on X need not be a closed point. For this reason, we often say that the closure $\overline{\{c_X(v)\}}$ is the centre of v on X.

Remark 5.4. If X is affine, say there exists an integral k-algebra A such that $X \simeq \mathrm{Spec}(A)$, then a valuation v on $K = \mathrm{Frac}(A)$ over k admits a centre on X iff $v|_A \geq 0$. In this case, the centre corresponds to the prime ideal $\{f \in A \colon v(f) > 0\}$ of A.

In the sequel, we write $\dim(c_X(v))$ for the dimension of the local ring $\mathcal{O}_{X,c_X(v)}$, or equivalently for the dimension of the closed subscheme $\overline{\{c_X(v)\}}$ of X. The dimension $\dim(c_X(v))$ records information about the transcendence degree $\mathrm{tr.deg}(v)$ of a valuation v, as is made precise below.

Lemma 5.5. *If v is a valuation on K/k, then*

$$\mathrm{tr.deg}(v) = \max_X \dim(c_X(v)),$$

where the maximum ranges over all models X of K/k on which v admits a centre.

As demonstrated by Example 5.2(3), it is not necessarily the case the transcendence degree of a valuation v on K/k is given by the dimension of the centre on an arbitrary model. The proof of Lemma 5.5 is left as an exercise to the reader.

Remark 5.6. In fact, more can be said about the maximum occuring in Lemma 5.5: if v is a valuation on K/k, X is a model of K/k on which v admits a centre, and $Y \to X$ is a proper birational morphism, then it follows immediately from the valuative criterion of properness that v admits a centre on Y and

$$\dim(c_X(v)) \leq \dim(c_Y(v)).$$

We are now prepared to prove the converse to Proposition 4.4, the proof of which is morally similar to the proof of Theorem 3.1 but crucially also uses Hironaka's theorem on resolution of singularities.

Theorem 5.7. [ELS03, Proposition 2.8] *Let k be a field of characteristic zero. If X is a model of K/k and v is an Abhyankar valuation on K/k that admits a centre on X, then there exists*

(1) a smooth model Y of K/k,
(2) a proper birational morphism $Y \to X$,

(3) a regular system of parameters $y = (y_1, \dots, y_d)$ of $\mathcal{O}_{Y,c_Y(v)}$,

such that $v(y_1), \dots, v(y_d)$ freely generate Γ_v. In particular, v is quasi-monomial.

With the terminology introduced in Definition 4.1, if one could find a regular system of parameters of $\mathcal{O}_{X,c_X(v)}$ as above, then v would be monomial on X. This is, of course, not possible in general - one must first pass to the "higher model" $Y \to X$ of K/k.

Proof. If $r = \operatorname{rat.rk}(v)$, take $f_1, \dots, f_r \in K$ such that $v(f_1), \dots, v(f_r)$ generate Γ_v; by replacing f_i with f_i^{-1}, we may assume that $v(f_i) > 0$ for all $i = 1, \dots, r$. Assume, for simplicity, that there exists an open neighbourhood $U \subseteq X$ of $c_X(v)$ such that

- $f_1, \dots, f_r \in \mathcal{O}_X(U)$;
- $\operatorname{tr.deg}(v) = \dim(c_X(v))$.

In general, one can reduce to this case by performing suitable blowups; see (the second paragraph of the proof of) [ELS03, Proposition 2.8].

Consider the effective divisor $H_U \coloneqq \{f_1 \cdots f_r = 0\}$ on U. By Hironaka's theorem on resolution of singularities, there exists a smooth variety Y over k and a proper birational morphism $\pi\colon Y \to X$ such that $\pi^{-1}(U) \to U$ is a log resolution of H_U; that is, the effective divisor $\pi^*(H_U) = \{(\pi^* f_1) \cdots (\pi^* f_r) = 0\}$ has simple normal crossings[7] on $\pi^{-1}(U)$. Thus, there exists a regular system of parameters $y_1, \dots, y_d \in \mathcal{O}_{Y,c_Y(v)}$ such that

$$(\pi^* f_1) \cdots (\pi^* f_r) = u y_1^{a_1} \cdots y_d^{a_d}$$

for some non-negative integers $a_1, \dots, a_d \in \mathbf{Z}_{\geq 0}$ and some unit $u \in \mathcal{O}_{Y,c_Y(v)}^\times$.

As Y is smooth over k, the local ring $\mathcal{O}_{Y,c_Y(v)}$ is regular, hence it is a unique factorization domain by the Auslander-Buchsbaum theorem; thus for each $i = 1, \dots, r$, we can write

$$f_i = u_i y_1^{a_{1,i}} \cdots y_d^{a_{d,i}}$$

for some non-negative integers $a_{1,i}, \dots, a_{d,i} \in \mathbf{Z}_{\geq 0}$ and some unist $u_i \in \mathcal{O}_{Y,c_Y(v)}^\times$. By definition of $c_Y(v)$, there is a local inclusion $\mathcal{O}_{Y,c_Y(v)} \subseteq R_v$, hence $v(u_i) = 0$ for all $i = 1, \dots, r$. It follows that

$$v(f_i) = \sum_{j=1}^{d} a_{j,i} v(y_j).$$

In particular, $v(y_1), \dots, v(y_d)$ generate the value group Γ_v. In order to show that they *freely* generate Γ_v, it suffices to show that $d = r$.

As r is the rational rank of v, we must have that $d \geq r$. To see that we must have equality, Theorem 2.2 implies that

$$\operatorname{tr.deg}(K/k) - r = \operatorname{tr.deg}(v) = \dim(c_Y(v)) = \dim(Y) - \dim(\mathcal{O}_{Y,c_Y(v)}) = \operatorname{tr.deg}(K/k) - d,$$

and hence $d = r$. The second equality follows by combining Lemma 5.5 and Remark 5.6 with the hypothesis that $\operatorname{tr.deg}(v) = \dim(c_X(v))$. This completes the proof of Theorem 5.7. $\square$

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[7] If X is a regular scheme and D is an effective divisor on X, then D has *simple normal crossings* if at every point $\xi \in X$, there is a regular system of parameters $x_1, \dots, x_r$ of $\mathcal{O}_{X,\xi}$ such that D is defined at ξ by $x_1^{a_1} \dots x_r^{a_r}$ for some $a_1, \dots, a_r \in \mathbf{Z}_{\geq 0}$.

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