MATH 731: TOPICS IN ALGEBRAIC GEOMETRY I – ABELIAN VARIETIES

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Course Description. The goal of the first half of this class is to introduce and study the basic structure theory of abelian varieties, as covered in (say) Mumford's book. In the second half of the course, we shall discuss derived categories and the Fourier–Mukai transform, and give some geometric applications.

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*Notes were taken by Matt Stevenson, who is responsible for any and all errors. Special thanks to Devlin Mallory for taking notes on October 12th and November 7-16. Thanks also to Attilio Castano, Raymond Cheng, Carl Lian, Devlin Mallory, Shubhodip Mondal, & Emanuel Reinecke for sending corrections. Please email stevmatt@umich.edu with any corrections.

1. September 5th

The main reference for the first part of the course is Mumford's *Abelian varieties* [Mum08]. For more modern treatments, see also [Con15, Mil08, EVdGM]. For some background material, see [BLR90, FGI$^+$05, Har77].

1.1. Group Schemes.

In this class, all schemes are noetherian. Fix a base scheme S (for all practical purposes, one can take S to be the spectrum of a field, but it is convenient to not make this restriction unless it is necessary). We work in the category (Sch/S) of schemes over S.

Definition 1.1. An *S-group scheme* (or *S-group*) is an S-scheme G along with maps of S-schemes $m\colon G\times G\to G$, $i\colon G\to G$, and $e\colon S\to G$ that encode the group law, i.e. the maps satisfy the following:

(a) m is associative, i.e. the following diagram commutes:

$$\begin{tikzcd} G\times G\times G \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd}$$

(b) e provides left and right identity elements, i.e. the following two diagrams commute:

$$\begin{tikzcd} G \arrow[r, "(e,\mathrm{id})"] \arrow[dr, "="'] & G\times G \arrow[d, "m"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id},e)"] \arrow[dr, "="'] & G\times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

Here, $e\colon G\to G$ denotes the composition of the structure map $G\to S$ with $e\colon S\to G$.

(c) i provides left and right inverses, i.e. the following two diagrams commute:

$$\begin{tikzcd} G \arrow[r, "(i,\mathrm{id})"] \arrow[d] & G\times G \arrow[d, "m"] \\ S \arrow[r, "e"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id},i)"] \arrow[d] & G\times G \arrow[d, "m"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

If m is symmetric, then G is a *commutative* S-group scheme.

Note that we are not calling a commutative group scheme an abelian group scheme! Abelian group schemes have more structure than commutative group schemes, as we will be defined later.

Definition 1.2. A map $f\colon G\to H$ of S-group schemes is a map of S-schemes that commutes with all the other structures in Definition 1.1.

Remark 1.3. [Yoneda Embedding] There is a fully faithful embedding $(\mathrm{Sch}/S)\to \mathrm{PShv}(\mathrm{Sch}/S)$ from the category of S-schemes to the category of presheaves (of sets) on (Sch/S), given by

$$X\mapsto h_X := \mathrm{Hom}_{\mathrm{Sch}/S}(-,X).$$

The functor h_X is called the *functor of points* of X, and this association $X\mapsto h_X$ commutes with products. Thus, if G is an S-group scheme, then h_G is naturally a presheaf of groups. In fact, this is reversible (because of fully faithfulness!): if G is an S-scheme, then there is a bijection

$$\{S\text{-group scheme structures on } G\}\simeq\{\text{lifts of } h_G \text{ to } (\mathrm{Sch}/S)^{\mathrm{op}}\to\mathrm{Groups}\}.$$

Thus, any abstract statement that is true for all groups will also be true for all group schemes (e.g. it suffices to check compatibility with multiplication when showing that a morphism of schemes is a homomorphism of group schemes).

The perspective of Remark 1.3 is a useful one, e.g. it will often be much easier to write down a group structure on the functor of points of a scheme rather than on a scheme itself, as demonstrated by the examples below.

Example 1.4. In the examples below, assume for simplicity that $S = \mathrm{Spec}(R)$.

(1) The *additive group* is[1] $\mathbf{G}_a = \mathbf{G}_{a,S} = \mathrm{Spec}\,(R[t])$. The functor of points is described as follows: for any S-scheme T, set

$$\mathbf{G}_a(T) = \mathrm{Hom}_S(T, \mathbf{G}_a) = \mathcal{O}(T),$$

and this is clearly a group under addition (it is even a ring!). Therefore, the obvious group structure on $\mathcal{O}(T)$, which is functorial in T, defines the structure of an S-group scheme on $\mathbf{G}_a$. Explicitly, the group operations are given by $m^*(t) = t \otimes 1 + 1 \otimes t$, $i^*(t) = -t$, and $e^*(t) = 0$.

(2) The *multiplicative group* is $\mathbf{G}_m = \mathbf{G}_{m,S} = \mathrm{Spec}\left(R[t, t^{-1}]\right)$. The functor of points is described as follows: for any S-scheme T,

$$\mathbf{G}_m(T) = \mathcal{O}(T)^\times.$$

The obvious group structure on $\mathcal{O}(T)^\times$ is functorial in T (because ring homomorphisms sends units to units), hence we obtian an S-group scheme structure on $\mathbf{G}_m$. Explicitly, the group multiplication is given by $m^*(t) = t \otimes t$.

(3) The *general linear group* is

$$\mathrm{GL}_n = \mathrm{Spec}\left(R\left[x_{ij} \colon 1 \le i, j \le n\right]\left[\frac{1}{\det}\right]\right).$$

By construction, for any S-scheme T, $\mathrm{GL}_n(T) = \mathrm{GL}_n\left(\mathcal{O}(T)\right)$, which is clearly a group, hence we get an S-group structure on GL_n. Moreover, there is a homomorphism $\det\colon \mathrm{GL}_n \to \mathbf{G}_m$ of S-group schemes. Furthermore, the group scheme GL_n is not commutative.

(4) The group of *n-th roots of unity* is $\mu_n = \mathrm{Spec}\left(R[t]/(t^n - 1)\right)$, and its functor of points is described as follows: if T is any S-scheme, then

$$\mu_n(T) = \{f \in \mathcal{O}(T)^\times \colon f^n = 1\}.$$

Multiplication of invertible global sections makes μ_n into an S-group scheme, but it is an interesting group scheme in that its behaviour depends on the characteristic of R:

(a) If $S = \mathrm{Spec}(\mathbf{C})$, then $\mu_n = \bigsqcup_{i \in \mu_n(\mathbf{C})} \mathrm{Spec}(\mathbf{C})$ and, by fixing a primitive n-th root of unity in $\mathbf{C}$, there is a non-canonical isomorphism $\bigsqcup_{i \in \mathbf{Z}/n} \mathrm{Spec}(\mathbf{C}) \simeq \underline{\mathbf{Z}/n}$ with the constant group scheme (the non-canonicity is because there is no distinguished n-root of unity in $\mathbf{C}$).

(b) If $S = \mathrm{Spec}(\mathbf{F}_p)$ and $n = p$, then

$$\mu_p = \mathrm{Spec}\left(\frac{\mathbf{F}_p[t]}{(t^p - 1)}\right) = \mathrm{Spec}\left(\frac{\mathbf{F}_p[t]}{(t-1)^p}\right),$$

which is a group scheme with nilpotents! The ring $\frac{\mathbf{F}_p[t]}{(t-1)^p}$ is non-reduced, and this is our first example of a group scheme that is not smooth (in fact, we will later see that all group schemes in characteristic zero are smooth – this is a theorem due to Oort).

In characteristic p, the group scheme μ_p has the funny feature that $\mu_p(\text{domain})$ is a point, but μ_p is not a point in general, e.g.

$$\mu_p\left(\frac{\mathbf{F}_p[\epsilon]}{(\epsilon^2)}\right) = \left\{1 + a\epsilon \in \frac{\mathbf{F}_p[\epsilon]}{(\epsilon^2)} \colon a \in \mathbf{F}_p\right\}.$$

[1] When writing the additive or multiplicative groups $\mathbf{G}_a$ or $\mathbf{G}_m$, we often suppress the subscript S, since it is really an object defined over $\mathbf{Z}$, and the additive or multiplicative group over S is obtained via base change.

This shows why it is important to work with the functor of points, as opposed to just the field-valued points of the scheme.

(5) If R has characteristic p, then consider the group scheme $\alpha_p = \operatorname{Spec}(R[t]/t^p)$. The functor of points of α_p is described as follows: for any S-scheme T, set

$$\alpha_p(T) = \{f \in \mathcal{O}(T) \colon f^p = 0\}.$$

This is obviously a group by the binomial theorem, as R has characteristic p, and hence we get a group scheme structure on α_p. Abstractly (as a scheme), α_p looks the same as μ_p (relabel t to $t-1$), but it has a different group structure. It shares with μ_p the feature that $\alpha_p(\text{domain})$ is a point, but α_p applied to a non-reduced ring will pick out the nilpotents.

Exercise 1.5.

(1) If G is an S-group scheme, then $G \to S$ is separated iff $e \colon S \to G$ is a closed immersion. Using this criterion, give an example of a non-separated group scheme (note that everything in Example 1.4 is separated).

(2) For a homomorphism $f \colon G \to H$ of S-groups with H separated, define $\ker(f)$ (as a presheaf) and show that it is representable. Then, compute the kernels of the following homomorphisms:
   (a) the homomorphism $\mathbf{G}_m \to \mathbf{G}_m$ given by $z \mapsto z^n$;
   (b) if S has characteristic p, the homomorphism $\mathbf{G}_a \to \mathbf{G}_a$ given by $x \mapsto x^p$;
   (c) if S has characteristic p, the homomorphism $\mathbf{G}_a \to \mathbf{G}_a$ given by $x \mapsto x^p - x$.

(3) Let G be a finite type group scheme over a field k. If G is geometrically reduced, then G is smooth.

Exercise 1.5(3) would, of course, be completely false if one worked instead with varieties; however, the group structure allows one to conclude that if "something funny occurs at one point", then it "happens at all points".

1.2. **Abelian Schemes.**

Definition 1.6. An *abelian scheme* over S is a group scheme A/S such that the structure map $A \to S$ is proper, smooth, and has geometrically connected fibres. If k is a field and $S = \operatorname{Spec}(k)$, an abelian scheme over S is called an *abelian variety* over k.

Notice that we never said the group law on an abelian S-scheme A is commutative! This will be a nontrivial theorem that we will prove.

Example 1.7. Abelian varieties of dimension 1 over a field k are precisely the elliptic curves over k.

Remark 1.8.

(1) Definition 1.6 is compatible with base change, i.e. if A/S is an abelian scheme and $T \to S$ is any map, then $A_T \coloneqq A \times_S T \to T$ is naturally an abelian scheme over T (this follows because properness, smoothness, and having geometrically connected fibres are all compatible with base change).

(2) If A/S is an abelian scheme, we can view $A \to S$ as a "family of abelian varieties" $A_s/\kappa(s)$, parametrized by $s \in S$; here, $\kappa(s)$ denotes the residue field of S at s.

(3) If a scheme is smooth and geometrically connected over a field, then it is geometrically integral (because a normal ring whose spectrum is connected is a domain). In particular, abelian varieties are geometrically integral. Conversely, we may weaken the smoothness assumption on $A \to S$ to flatness, provided the fibres are geometrically reduced (because for flat maps, smoothness can be detected fibrewise; see [BLR90, Proposition 2.4/8]).

We still have to address why the group law on an abelian scheme is commutative (in order for it to deserve to be called an abelian scheme!). To motivate the proof of this fact, let us have a brief discussion of the (analytic) proof of this fact for abelian varieties over $\mathbf{C}$.

Remark 1.9. If $A/\mathbf{C}$ is an abelian variety, let A^{an} be the associated complex manifold. By assumption, A^{an} is a compact, connected, complex Lie group. We claim that any such Lie group is commutative[2] If $t \in A^{\mathrm{an}}$, one gets a biholomorphic conjugation map $c_t \colon A^{\mathrm{an}} \to A^{\mathrm{an}}$, given by $g \mapsto t \cdot g \cdot t^{-1}$. This gives rise to a homomorphism

$$A^{\mathrm{an}} \to \mathrm{Aut}_{\mathbf{C}\text{-Lie groups}}(A^{\mathrm{an}}) \to \mathrm{Aut}_{\mathbf{C}\text{-linear}}\left(T_e(A^{an})\right) \simeq \mathrm{GL}_n(\mathbf{C}),$$

where $T_e(A^{\mathrm{an}})$ denotes the (holomorphic) tangent space to A^{an} at the identity e. As all maps were defined canonically, the composition $A^{\mathrm{an}} \to \mathrm{GL}_n(\mathbf{C})$ is holomorphic and any such map is constant, because A^{an} is proper and $\mathrm{GL}_n(\mathbf{C})$ is Stein (this is the analytic avatar of the fact that there are no non-constant maps from a proper variety to an affine variety). Using the exponential map $\exp \colon T_e(\mathbf{A}^{\mathrm{an}}) \to A^{\mathrm{an}}$, one sees that c_t acts trivially on an open neighbourhood of the origin $e \in A^{\mathrm{an}}$. As A^{an} is connected, any non-empty open neighbourhood in A^{an} generates A^{an} as a Lie group, hence c_t acts trivially on A^{an}. Thus, A^{an} is commutative. As $A/\mathbf{C}$ is proper, (a weak form of) GAGA implies that A is commutative.

Remark 1.9, along with classification of compact, connected, complex Lie groups, leads to the proof of the following fact.

Fact 1.10. If $A/\mathbf{C}$ is an abelian variety, then A^{an} is of the form $\mathbf{C}^g/\Lambda$ for some full-rank lattice $\Lambda \subseteq \mathbf{C}^g$.

While not all complex tori are algebraic (i.e. there exist full-rank lattices $\Lambda \subseteq \mathbf{C}^g$ such that $\mathbf{C}^g/\Lambda$ is not the analytification of an abelian variety), Riemann classified those which are algebraic (in terms of certain positivity conditions on the lattice Λ). In particular, the n-torsion is

$$A^{\mathrm{an}}[n] = \left(\frac{1}{n}\Lambda\right)/\Lambda \simeq (\mathbf{Z}/n)^{2g}.$$

If a few lectures, we will see that the n-torsion of an abelian variety admits a similar description in the algebraic world.

2. September 7th

Last time, we defined abelian schemes and we began to explain why they are commutative. We sketched a proof of this fact for abelian varieties over $\mathbf{C}$ using analytic methods. There were two key points to the proof:

(1) study the conjugation action on the tangent space $T_e(A)$ at the identity e;
(2) use the exponential map.

While (1) is mostly algebraic, (2) is certainly does not work in algebraic geometry (especially in characteristic p: there are many of examples of automorphisms that act trivially on the tangent space but not on the variety itself).

To repair this issue, we will view conjugations as a *flat family* of maps $c_t \colon A \to A$ parametrized by $t \in A$. This family has the property that, at the identity $e \in A$, the map c_e is constant. We will then prove a rigidity result, originally due to Weil, that implies that all c_t's are constant.

2.1. Rigidity Results.

The basic case of the desired rigidity result if the following:

Proposition 2.1. [Rigidity I] *If $f \colon X \to S$ is a proper flat morphism such that $\kappa(s) \overset{\simeq}{\to} H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$,*[3] *then:*

(1) the canonical map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism;
(2) if $g \colon T \to S$ is an affine map and $\pi \colon X \to T$ is an S-map, then π is constant (i.e. it factors through f).

In the special case when S is the spectrum of a field k, then the natural map $k \to H^0(X, \mathcal{O}_X)$ being an isomorphism is equivalent to the natural map $\mathcal{O}_{\mathrm{Spec}(k)} \to f_*\mathcal{O}_X$ being an isomorphism, and (2) asserts that there are no nonconstant maps from a proper k-scheme to an affine k-scheme.

[2]Without the connectedness assumption, this is not true: take a finite non-commutative group and consider the associated constant Lie group.

[3]This assumption is satisfied e.g. if all fibres of $f \colon X \to S$ are geometrically integral.

Proof. First, note that (1) implies (2). Indeed, since g is affine, we have

$$\mathrm{Hom}_S(X,T) = \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}\left(g_*\mathcal{O}_T, f_*\mathcal{O}_X\right). \tag{2.1}$$

As $\mathcal{O}_S \xrightarrow{\simeq} f_*\mathcal{O}_X$ by (1), any map $g_*\mathcal{O}_T \to f_*\mathcal{O}_X$ factors as $g_*\mathcal{O}_S \to \mathcal{O}_S \xrightarrow{\simeq} f_*\mathcal{O}_X$. Taking the inverse of (2.1) (i.e. applying $\underline{\mathrm{Spec}}$) gives (2).

It remains to prove (1) (in fact, this is a special case of cohomology and base change, but we will prove it by hand here). We may assume that $S = \mathrm{Spec}(R)$, where $(R, \mathfrak{m})$ is a complete noetherian local ring: the map $\mathcal{O}_S \to f_*\mathcal{O}_X$ of sheaves on S is an isomorphism if this is the case at every point $s \in S$, so we may assume S is the spectrum of the noetherian local ring $\mathcal{O}_{S,s}$. Moreover, the stalks of $\mathcal{O}_S$ and $f_*\mathcal{O}_X$ at s are finitely-generated $\mathcal{O}_{S,s}$-modules, so it enough to check that that the map is an isomorphism after passing to the completion $\widehat{\mathcal{O}}_{S,s}$, because the completion map $\mathcal{O}_{S,s} \hookrightarrow \widehat{\mathcal{O}}_{S,s}$ is flat.

We must show that the canonical map $R \to H^0(X, \mathcal{O}_X)$ is an isomorphism. There is a commutative diagram

$$\begin{tikzcd}
H^0(X,\mathcal{O}_X) \arrow[r, "\simeq"] & \varprojlim_n H^0(X, \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X) \\
R \arrow[u] \arrow[r, "\simeq"] & \varprojlim_n R/\mathfrak{m}^n \arrow[u]
\end{tikzcd}$$

where the map $H^0(X,\mathcal{O}_X) \xrightarrow{\simeq} \varprojlim_n H^0(X, \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X)$ is an isomorphism by the formal functions theorem, and $R \xrightarrow{\simeq} \varprojlim_n R/\mathfrak{m}^n$ is an isomorphism because R is $\mathfrak{m}$-adically complete. Note that, if $X_n \coloneqq X \times_R R/\mathfrak{m}^n$, then $R/\mathfrak{m}^n \to H^0(X, \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X) = H^0(X_n, \mathcal{O}_{X_n})$ is the natural pullback map of global sections along $X_n \to \mathrm{Spec}(R/\mathfrak{m}^n)$. In order to show that $R \to H^0(X,\mathcal{O}_X)$ is an isomorphism, the above diagram implies that it suffices to show $R/\mathfrak{m}^n \to H^0(X, \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X)$ is an isomorphism for all $n \geq 1$.

We prove the following more general statement: for any finite length R-module M, the pullback map

$$M \to H^0(X, f^*M)$$

is an isomorphism. Taking $M = R/\mathfrak{m}^n$ and using the fact that $f^*(R/\mathfrak{m}^n) = \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X$ (because the pullback functor f^* is right exact), we get that $R/\mathfrak{m}^n \to H^0(X, \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X)$ is an isomorphism.

We proceed by induction on the length $\ell(M)$ of M. If $\ell(M) = 1$, then $M \simeq \kappa(s)$, where $s \in \mathrm{Spec}(R)$ is the closed point (indeed, all length-1 modules over a noetherian local ring are isomorphic to the residue field). Thus, the claim holds by hypothesis. If $\ell(M) > 1$, there is a filtration of M

$$0 \to K \to M \to Q \to 0$$

where $\ell(K) < \ell(M)$ and $\ell(Q) < \ell(M)$ (e.g. the $\mathfrak{m}$-adic filtration has this property). As f is flat, f^* preserves exactness, so there is a short exact sequence

$$0 \to f^*K \to f^*M \to f^*Q \to 0$$

of sheaves on X. Taking global sections, we see that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\simeq"] & M \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H^0(X, f^*K) \arrow[r] & H^0(X, f^*M) \arrow[r] & H^0(X, f^*Q) &
\end{tikzcd}$$

An easy diagram chase reveals that the middle map $M \to H^0(X, f^*M)$ is also an isomorphism, which completes the proof. $\square$

Proposition 2.1 can be promoted to a stronger rigidity result, originally proved (in a different form) by Weil.

Corollary 2.2. [Rigidity II] *Let $f\colon X \to S$ be a proper flat map such that $\kappa(s) \xrightarrow{\simeq} H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$, and let $g\colon Y \to S$ be separated. If $\pi\colon X \to Y$ is an S-map and the restriction $\pi_s\colon X_s \to Y_s$ is constant for some fixed $s \in S$, then π is constant over the connected component of $s \in S$.*

As in Proposition 2.1, saying that the S-map $\pi\colon X \to Y$ is constant means that it factors through $f\colon X \to S$.

Proof. We may assume that S is affine and connected.

Step 1. There exists an open neighbourhood $V \subseteq S$ of $s \in S$ such that $\pi|_{X_V}\colon X_V \to Y_V$ is constant, where $X_V = X \times_S V$ and $Y_V = Y \times_S V$.

Indeed, choose an affine neighbourhood $U \subseteq Y$ of the point $\pi_s(X_s)$, then $\pi^{-1}(U) \supseteq X_s$. As f is proper (hence closed), there exists an open neighbourhood $V \subseteq S$ of s such that $f^{-1}(V) = X_V \subseteq \pi^{-1}(U)$. To summarize, we have

$$\begin{tikzcd} X_V \arrow[r, "\pi_V"] \arrow[dr] & U_V \arrow[r, hook] \arrow[d] & U \arrow[d] \\ & V \arrow[r, hook] & S \end{tikzcd}$$

Note that U, S, V are affine, hence U_V is affine. Thus, we can apply Proposition 2.1 to the V-map $\pi_V\colon X_V \to U_V$ to see that $\pi_V\colon X_V \to U_V$ is constant, and hence $\pi|_{X_V}\colon X_V \to Y$ is constant.

Step 2. Set $W = \{t \in S\colon X_t \xrightarrow{\pi_t} Y_t \text{ is constant}\}$. It suffices to show that $W = S$.

By Step 1, we know that W is open (because Step 1 shows that if π is constant over one point $t \in S$, then it is constant over an open neighbourhood of t). Recall that if an open subset of a noetherian topological space is closed under specialization, then it is the whole space. In particular, it suffices to show that W is closed under specialization.

Detecting specializations in S via maps from the spectra of dvr's[4], we can reduce to the case $S = \mathrm{Spec}(R)$ for some dvr R. Choose a uniformizer $t \in R$, and set $K = R[1/t]$ and $k = R/(t)$. We have an R-map $\pi\colon X \to Y$ such that the restriction $\pi|_{X_K}$ of π to the generic fibre X_K is constant, and we want to show that the restriction $\pi|_{X_k}$ of π to the special fibre X_k is constant.

As X is proper over R and Y is separated over R, $\pi(X) \subseteq Y$ is proper over R (where $\pi(X)$ is a closed subset of Y, and it is equipped with the reduced scheme structure). Replace Y with $\pi(X)$ in order to assume that Y is also proper over R and π is surjective. Moreover, as X is flat over R and $\pi\colon X \to Y$ is dominant, we also have that Y is flat over R.[5]

Now, since π_K is constant, there is a section η of $Y_K \to \mathrm{Spec}(K)$ such that π_K is the composition $X_K \to \mathrm{Spec}(K) \xrightarrow{\eta} Y_K$. By the valuative criterion of properness, there exists a unique map $\xi\colon \mathrm{Spec}(R) \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\eta"] \arrow[d] & Y_K \arrow[r, hook] & Y \arrow[d] \\ \mathrm{Spec}(R) \arrow[rru, dashed, "\exists ! \xi"'] \arrow[rr, equal] & & \mathrm{Spec}(R) \end{tikzcd}$$

[4] If X is a topological space and $x, y \in X$, recall that x *specializes* to y, written $x \rightsquigarrow y$, if y lies in the closure of $\{x\}$. If X is a (noetherian) scheme and $x \rightsquigarrow y$, then there exists a dvr R and a morphism $\mathrm{Spec}(R) \to X$ where the generic point of $\mathrm{Spec}(R)$ maps to x and the closed point of $\mathrm{Spec}(R)$ maps to y. See [Sta17, Tag 054F].

[5] To see this, the key point is to use that flatness over a dvr (or any PID) is equivalent to torsion-freeness. More precisely, as π is dominant and Y is reduced, $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is injective, hence $\mathcal{O}_{Y,\pi(x)}$ is a sub-$R_{p(x)}$-module of $\mathcal{O}_{X,x}$ for all $x \in X$, where $p(x) \in \mathrm{Spec}(R)$ is the prime over which x lies. In particular, if $\mathcal{O}_{X,x}$ is torsion-free, then $\mathcal{O}_{Y,\pi(x)}$ is torsion-free, hence flat over $R_{p(x)}$. Thus, Y is flat over R.

In fact, one can show that ξ is an isomorphism (though this is needed to show that π is constant).[6] Using that Y is flat over R, one can show that π is the composition $X \to \mathrm{Spec}(R) \xrightarrow{\xi} Y$; thus, π is constant. $\square$

We can now start applying these rigidity results to the setting of abelian schemes, with the goal of showing that abelian schemes are commutative. First, let us introduce a piece of notation: if $b \in B(S)$, the *left translation* $t_b \colon B \to B$ by the S-point b is the composition

$$B \xrightarrow{(b,\mathrm{id})} B \times B \xrightarrow{m} B,$$

where b is thought of here as the composition $B \to S \xrightarrow{b} B$.

Corollary 2.3. *Let A and B be abelian schemes over S with identities e_A and e_B, and let $f \colon A \to B$ be an S-map. Then, there is a factorization*

$$f = t_b \circ h,$$

where $h \colon A \to B$ is a homomorphism, $b = f(e_A) \in B(S)$, and t_b is the left translation by b.

Proof. We can replace f with $t_{i(b)} \circ f$ to assume that $f(e_A) = e_B$. where $i(b)$ is the inverse of the S-point b. The goal now is to prove that f is a homomorphism, i.e. we must show that $f \circ m_A$ and $m_B \circ (f, f)$ give the same map $A \times A \to B$. Consider the map $g \colon A \times A \to B$ given by

$$g := m_B \left(f \circ m_A, i \left(m_B \circ (f, f) \right) \right).$$

The goal is to show that g is constant with image e_B. Observe that we have a diagram

$$\begin{tikzcd}
\mathrm{pr}_1^{-1}(e_A) \arrow[r] \arrow[d] & A \times_S A \arrow[r, dashed, "\exists! g'"] \arrow[d, "\mathrm{pr}_1"] \arrow[rr, bend left, "g"] & A \times_S B \arrow[r] \arrow[dl] & B \arrow[d] \\
S \arrow[r, "e_A"] & A \arrow[rr] & & S
\end{tikzcd}$$

where $\mathrm{pr}^{-1}(e_A)$ is the fibre product of e_A and pr_1, and g' is the map obtained from the universal property of the fibre product $A \times_S B$. One can check that g is constant on $\mathrm{pr}_1^{-1}(e_A)$, and hence g' is constant on $\mathrm{pr}_1^{-1}(e_A)$. Applying Corollary 2.2 to g', we see that g' is constant, and hence g is constant. It remains to show that $f \colon A \to B$ is constant. To do so, it suffices to check that there is a section σ of pr_1 such that the composition $g \circ \sigma$ is constant. Taking $\sigma = (\mathrm{id}, e_A)$, it is easy to check that $g \circ \sigma$ is constant (equal to the identity e_B) precisely because $f(e_A) = e_B$. $\square$

Corollary 2.3 is precisely the tool needed to show that abelian schemes are commutative group schemes.

Corollary 2.4. *If A/S is an abelian scheme, then A is a commutative as an S-group.*

Proof. Apply Corollary 2.3 to $i \colon A \to A$, and use the fact that groups are commutative iff the inversion map i is a homomorphism. $\square$

While the mechanics of the proof of commutativity of abelian schemes are certainly different than those present in Remark 1.9, the key point (namely, Step 1 of Proposition 2.1) is very similar to the key point in the complex-analytic proof.

As another easy application of Corollary 2.3, we see that, once an identity is fixed, there is a unique abelian scheme structure on a scheme (contrast this to the case of groups, where there can be many different groups structures on the same set).

---

[6] In order to see that $\xi \colon \mathrm{Spec}(R) \to Y$ is an isomorphism, note that it is surjective (because the restriction to the generic fibre is surjective, and hence ξ is surjective because Y is proper and flat over R). As Y is proper and flat over R, the structure map $Y \to \mathrm{Spec}(R)$ is also surjective; since ξ is a section of the structure map $Y \to \mathrm{Spec}(R)$, it follows that ξ must be a homeomorphism. Thus, $Y \simeq \mathrm{Spec}(S)$ for some finite flat R-algebra. Finally, we must have $S \simeq R$ because S has rank-1 over the generic point of R.

Corollary 2.5. *If A/S is an S-scheme and $e_A \in A(S)$, then there is at most one abelian scheme structure on A with e_A as identity.*

Proof. If m_1 and m_2 are two groups laws on A/S with identity e_A, then $\mathrm{id}\colon A \to A$ is a homomorphism $(A, m_1) \to (A, m_2)$ by Corollary 2.3, so $m_1 = m_2$. □

2.2. **Differential Properties.** We will now begin to discuss properties of abelian schemes that are more differential-geometric in nature, e.g. the behaviour of differential forms. The first such result describes how the cotangent sheaf of a group scheme is "trivial in a natural way".

Proposition 2.6. *Let $f\colon G \to S$ be a group scheme with identity $e \in G(S)$. Then, there is a natural isomorphism*

$$\Omega^1_{G/S} \simeq f^* e^* \Omega^1_{G/S}.$$

In particular, if S is the spectrum of a field, then $\Omega^1_{G/S} \simeq \mathcal{O}_G^{\oplus r}$ for some $r \geq 1$.

The sheaf $e^*\Omega^1_{G/S}$, appearing in Proposition 2.6, should be thought of as the cotangent space of G at the origin e.

Remark 2.7. Proposition 2.6 does *not* imply that group schemes over a field are smooth! In order for this to be true, one needs $\Omega^1_{G/S}$ to be free *of the correct rank*, which is not always the case.

For example, if $R = \mathbf{F}_p[x]/(x^p - 1)$ and $G = \mu_p = \mathrm{Spec}(R)$, then $\Omega^1_{G/\mathbf{F}_p} = R \cdot dx$ is free of rank 1, but G has dimension zero. In particular, G is not smooth over $\mathbf{F}_p$.

From Proposition 2.6, we will deduce many nice consequences, e.g. there is no nonconstant map from $\mathbf{P}^1_k$ to an abelian variety over k. Next class, we will discuss its proof and some consequences.

3. September 12th

Last time, we explained why abelian schemes are commutative using the rigidity lemma. For an alternative proof, one can show that conjugation acts trivially on all infinitesimal neighbourhoods of the identity, and hence on an open neighbourhood of the identity. More precisely:

Exercise 3.1. If A/k is an abelian variety, mimic the complex-analytic proof of commutativity (as in Remark 1.9) to show that conjugation acts trivially on $\widehat{\mathcal{O}}_{A,e}$. Deduce that A is commutative.

Today, we continue to discuss the differential properties of abelian schemes, with the goal of understanding the multiplication-by-n map.

3.1. Differential Properties (Continued).

Remark 3.2. Even if one is only interested in the case of abelian varieties, it will be very useful to develop the theory of abelian schemes over an arbitrary base. For example, if A is an abelian variety over k, we will later need to work with objects such as $A \times \underline{\mathrm{Pic}}^0(A) \to \underline{\mathrm{Pic}}^0(A)$, where $\underline{\mathrm{Pic}}^0(A)$ is the dual abelian variety of A. This is done by viewing $A \times \underline{\mathrm{Pic}}^0(A)$ as an abelian scheme over $\underline{\mathrm{Pic}}^0(A)$. More generally, if one needs to work with moduli spaces of abelian varieties, it is essential to consider abelian schemes over more complicated bases.

Last time, we stated, but did not prove, the following proposition.

Proposition 3.3. *If $f\colon G \to S$ is a group scheme, there exists a canonical isomorphism*

$$\Omega^1_{G/S} \simeq f^* e^* \Omega^1_{G/S}.$$

Proposition 3.3 says that there is a canonical way to trivialize the cotangent sheaf of a group scheme. For example, in the special case when $S = \mathrm{Spec}(k)$, the cotangent space $e^*\Omega^1_{G/S}$ at the origin e is simply a k-vector space, and $f^*e^*\Omega^1_{G/S}$ is the trivial vector bundle on that k-vector space.

Example 3.4. If X/k is an elliptic curve, then Proposition 3.3 implies that the canonical bundle $K_X = \Omega^1_{X/k}$ is trivial.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
G \times G \arrow[dr, "(\mathrm{pr}_1, m)"] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & & \\
& G \times G \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & G \arrow[d, "f"] \\
& G \arrow[r, "f"] & S
\end{tikzcd}$$

The "inside" square $f \circ \mathrm{pr}_2 = f \circ \mathrm{pr}_1$ is Cartesian. Moreover, it is easy to see that $G \times G \xrightarrow{(\mathrm{pr}_1, m)} G \times G$ is an isomorphism (indeed, one can check that $(\mathrm{pr}_1, m \circ (i \circ \mathrm{pr}_1, \mathrm{pr}_2))$ is an inverse[7]), and hence the "outside" square $f \circ m = f \circ \mathrm{pr}_1$ is also Cartesian.

Now, we can compute $\Omega^1_{\mathrm{pr}_1}$ in 2 different ways: since the "inside" square is Cartesian, we have $\Omega^1_{\mathrm{pr}_1} \simeq \mathrm{pr}_2^* \Omega^1_f$; similarly, the "outside" square is Cartesian, hence $\Omega^1_{\mathrm{pr}_1} \simeq m^* \Omega^1_f$. Thus, $m^*\Omega^1_{G/S} \simeq \mathrm{pr}_2^* \Omega^1_{G/S}$ as sheaves on the product $G \times G$. Pullback along $\mu \coloneqq (\mathrm{id}, e) \colon G \to G \times G$ gives isomorphisms

$$\mu^* m^* \Omega^1_{G/S} \simeq (m \circ \mu)^* \Omega^1_{G/S} \simeq \mathrm{id}^* \Omega^1_{G/S} \simeq \Omega^1_{G/S}$$

and

$$\mu^* \mathrm{pr}_2^* \Omega^1_{G/S} \simeq (\mathrm{pr}_2 \circ \mu)^* \Omega^1_{G/S} \simeq (e \circ f)^* \Omega^1_{G/S} \simeq f^* e^* \Omega^1_{G/S}.$$

Therefore, it follows that $\Omega^1_{G/S} \simeq f^* e^* \Omega^1_{G/S}$. □

Remark 3.5.

(1) As discussed in Remark 2.7, Proposition 3.3 does not imply that group schemes over a field are smooth.

(2) If S is affine, then Proposition 3.3 (along with the (f^*, f_*)-adjunction) gives a map

$$e^* \Omega^1_{G/S} \to f_* \Omega^1_{G/S} = H^0(G, \Omega^1_{G/S}),$$

whose image is precisely the *translation-invariant* 1-forms, i.e. those 1-forms $\omega \in H^0(G, \Omega^1_{G/S})$ such that for all S-schemes T and all points $g \in G(T)$, we have $t_g^* \omega = \omega$ as elements of $H^0(G_T, \Omega^1_{G_T/T})$, where $G_T \coloneqq G \times_S T \to T$ is the base change, and $t_g \colon G_T \to G_T$ is the translation by g.

(3) If $S = \mathrm{Spec}(R)$ and $G = \mathbf{G}_m = \mathrm{Spec}(R[t^{\pm 1}])$, then the map in (2) is of the form

$$I_e/I_e^2 \simeq e^* \Omega^1_{G/S} \to H^0(G, \Omega^1_{G/R}) \simeq R[t^{\pm 1}]dt$$

where $I_e = (t-1) \subseteq R[t^{\pm 1}]$ is the ideal of the origin e, and it is given explicitly by $t - 1 \mapsto \frac{dt}{t}$. The translation-invariant 1-forms in $R[t^{\pm 1}]dt$ are those that are invariant under R-scaling, which is clearly the case for $\frac{dt}{t}$.

Proposition 3.3 admits a more geometric corollary.

Corollary 3.6. *If A/k is an abelian variety, then any map $f \colon \mathbf{P}^1_k \to A$ is constant. In particular, A is not rational, unirational, rationally connected, uniruled,...*

[7] The construction of this inverse to (pr_1, m) is the only place in the proof of Proposition 3.3 where we use that G is a group as opposed to just a monoid.

Proof. We may assume $k = \overline{k}$, because a map is constant iff the base change to $\overline{k}$ is constant. Assume f is non-constant. If $C = f(\mathbf{P}^1) \subseteq A$, then C is a proper, irreducible, reduced, and possibly singular curve inside A. If $\nu\colon \widetilde{C} \to C$ is the normalization, then $\widetilde{C}$ is isomorphic to $\mathbf{P}^1_k$, since C is a rational curve. By replacing $\mathbf{P}^1 \to A$ by $\widetilde{C} \xrightarrow{\nu} C \hookrightarrow A$, we may assume that f is birational onto its image and an isomorphism over the smooth locus of C.

As k is algebraically closed, there exists a k-rational point $c \in C(k)$ in the smooth locus and it lifts uniquely to a k-rational point $\tilde{c} \in \mathbf{P}^1_k(k)$, since f is an isomorphism over the smooth locus of C. It follows that there is an injective map

$$T_{\tilde{c}}(\mathbf{P}^1_k) \xrightarrow{\simeq} T_c(C) \hookrightarrow T_c(A) \tag{$*$}$$

In particular, $(*)$ is a nonzero map. On the other hand, the pullback f^* on 1-forms is the composition

$$H^0(A, \Omega^1_{A/k}) \longrightarrow H^0(C, \Omega^1_{C/k}) \longrightarrow H^0(\mathbf{P}^1_k, \Omega^1_{\mathbf{P}^1_k/k}),$$

and this is the zero map because $H^0(\mathbf{P}^1_k, \Omega^1_{\mathbf{P}^1_k/k}) = 0$. By Proposition 3.3, $H^0(A, \Omega^1_A)$ generates Ω^1_A, i.e. the natural sheaf map

$$H^0(A, \Omega^1_{A/k}) \otimes \mathcal{O}_A \to \Omega^1_{A/k}$$

is surjective. It follows that $f^*\Omega^1_A \to \Omega^1_{\mathbf{P}^1_k/k}$ is the zero map (since it is zero on a generating set). However, taking the fibre of $f^*\Omega^1_{A/k} \to \Omega^1_{\mathbf{P}^1_k/k}$ at $\tilde{c}$ is precisely the pullback map from the cotangent space of c to the cotangent space of $\tilde{c}$; in particular, the dual gives the map $(*)$. We have shown that $(*)$ is nonzero, which contradicts the assumption that f is nonconstant. $\square$

Exercise 3.7. If X is a smooth projective surface over k and A/k is an abelian variety, then any rational map $X \dashrightarrow A$ extends to a morphism. (In fact, the same is true for smooth projective varieties of any dimension, though the proof is harder outside of the surface case; for a more general discussion, see [Mil08, §3].)

The reason for our discussion of the differential properties of abelian schemes is to understand the action of a particular class of endomorphisms, namely the multiplication-by-n maps (with the ultimate goal of understanding the n-torsion). We continue by analyzing the action of the multiplication-by-n maps on tangent spaces.

Lemma 3.8. *Let G/k be a group scheme.*

(1) There is a canonical isomorphism $T_{(e,e)}(G \times G) \simeq T_e(G) \times T_e(G)$ induced by the 2 projections.
(2) The multiplication map $m\colon G \times G \to G$ induces $(a, b) \mapsto a + b$ as a map

$$T_e(G) \times T_e(G) \simeq T_{(e,e)}(G \times G) \xrightarrow{m_*} T_e(G).$$

The proof of (2) is a "differential version" of the Eckmann–Hilton argument that one encounters in algebraic topology.

Remark 3.9. For any k-scheme X and any k-rational point $x \in X(k)$, there is a natural bijection

$$T_x(X) \simeq \left\{ f \in \mathrm{Hom}_{\mathrm{Spec}(k)}\left(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)), X\right) : \begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[dr, "x"'] & \mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \arrow[d, "f"] \\ & X \end{tikzcd} \right\},$$

where $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ is dual to the k-algebra map $k[\epsilon]/(\epsilon^2) \to k$ obtained by setting $\epsilon = 0$; see [Har77, Exercise II.2.8]. From this description, it follows that for any k-schemes X and Y and any k-rational points $x \in X(k)$ and $y \in Y(k)$, there is a natural bijection

$$T_{(x,y)}(X \times Y) \simeq T_x(X) \times T_y(Y).$$

Proof. The description in (1) follows immediately from Remark 3.9. For (2), consider the maps

$$i_1 = (\mathrm{id}, e)\colon G \to G \times G \qquad \text{and} \qquad i_2 = (e, \mathrm{id})\colon G \to G \times G,$$

which induce maps

$$(i_1)_*\colon T_e(G) \to T_{(e,e)}(G \times G) \simeq T_e(G) \times T_e(G) \qquad \text{and} \qquad (i_2)_*\colon T_e(G) \to T_{(e,e)}(G \times G) \simeq T_e(G) \times T_e(G)$$

on tangent spaces. We claim that $(i_1)_*(a) = (a, 0)$ and $(i_2)_*(a) = (0, a)$ (one can check this after composing with $(\mathrm{pr}_1)_*$ and $(\mathrm{pr}_2)_*$). Therefore, $m\colon G \times G \to G$ induces a map

$$T_e(G) \times T_e(G) \simeq T_{(e,e)}(G \times G) \xrightarrow{m_*} T_e(G),$$

which, for $(a, b) \in T_e(G) \times T_e(G)$, is given by

$$m_*(a, b) = m_*((a, 0) + (0, b)) = m_*((i_1)_*(a) + (i_2)_*(b)) = m_*(i_1)_*(a) + m_*(i_2)_*(b) = a + b.$$

□

We adopt the following notation: for any $n \in \mathbf{Z}$ and abelian scheme A/S, write $[n]\colon A \to A$ for the multiplication-by-n map on points (i.e. for any S-scheme T, $\mathrm{Hom}_S(T, A)$ is an abelian group and $[n]$ is the multiplication-by-n endomorphism on this abelian group). For example, $[0]$ is the constant map with value e, $[1]$ is the identity map id, and $[-1]$ is the inversion map i.

Corollary 3.10. *Let A/k be an abelian variety and let $[n]_*\colon T_e(A) \to T_e(A)$ be the map induced by $[n]\colon A \to A$ on the tangent space at the identity. Then, $[n]_*$ is the multiplication-by-n map on the k-vector space $T_e(A)$.*

Proof. If $\Delta_n\colon A \to A \times \ldots \times A$ is the diagonal map to n copies of A then $[n] = m \circ \Delta_n$. By Lemma 3.8(2), m_* adds the entries of a tuple in $T_e(A) \times \ldots \times T_e(A)$, and $(\Delta_n)_*$ is the diagonal map (by functoriality). Thus, $[n]_*$ is precisely multiplication-by-n. □

Both Lemma 3.8 and Corollary 3.10 hold for abelian schemes over a general base, provided the tangent space is replaced with a suitable "tangent module".

Corollary 3.11. *If A/S is an abelian scheme and $n \in \mathbf{Z}$ is such that n is invertible on S, then $[n]\colon A \to A$ is finite, étale, and surjective.*

Proof. It suffices to show that $[n]$ is étale. Indeed, since A is proper over S, $[n]$ is proper by [Sta17, Tag 01W6], hence $[n]$ has closed image. As étale morphisms are open, it follows that $[n]$ is surjective. Moreover, étale morphisms are locally quasi-finite, and proper quasi-finite morphisms are finite by Zariski's Main Theorem (see [Sta17, Tag 02LS]).

Since A is smooth over S, the fibral criterion for étaleness [BLR90, Prop 2.4/8] allows us to assume that $S = \mathrm{Spec}(k)$. By Corollary 3.10, $[n]_*\colon T_e(A) \to T_e(A)$ is the multiplication-by-n endomorphism, hence it is an isomorphism, since n is invertible in k by assumption. By translation, $[n]$ induces an isomorphism on the tangent space $T_x(A)$ for any $x \in A(\overline{k})$. As A is smooth over S, it follows that $[n]$ is étale by [BLR90, Cor 2.2/10]. □

Corollary 3.12. *If A/k is an abelian variety, $k = \overline{k}$, and n is invertible on k, then the group $A(k)$ of k-rational points is n-divisible.*

Remark 3.13. In characteristic p, $[n]$ is *not* étale if $p|n$ because $[n]$ induces the zero map on the tangent space at the origin by Corollary 3.10. However, $[n]$ will nonetheless be surjective.

4. September 14th

Last time, we saw that the multiplication-by-n map $[n]$ is finite étale surjective on abelian varieties over a field k, provided $n \in k^*$; said differently, $[n]$ behaves like a covering space map. This begs several obvious follow-up questions: what is the degree of $[n]$? Is it Galois? If so, what is the Galois group? What is the structure of the kernel (as a group scheme)?

To answer these questions, we must first understand the behaviour of line bundles on abelian varieties. In preparation, today's lecture will be a review of cohomology and base change.

4.1. **Cohomology and Base Change.** Given a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes and a quasi-coherent sheaf $\mathcal{F}$ on X, there is a natural map

$$g^* R^i f_* \mathcal{F} \to R^i f'_*(g'^* \mathcal{F}). \tag{4.1}$$

When $i = 0$, this map is constructed[8] using the pullback-pushforward adjunction ; for $i > 0$, the maps between higher pushforwards are constructed via general δ-functor nonsense. If $S' = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ then (4.1) is the map

$$H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B), \tag{4.2}$$

where $X_B = X' = X \times_S S'$ is the base change, and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B along g'. In fact, the most important special case of (4.2) is when $B = \kappa(\mathfrak{p})$ is the residue field of A at a prime $\mathfrak{p} \in \mathrm{Spec}(A)$, where it relates the cohomology of the total space to the cohomology of the fibre at the point.

The map (4.1) is not an isomorphism in general, as is illustrated by the example below (in fact, the same example will later reappear as the Poincaré bundle of an elliptic curve).

Example 4.1. Let (E, e) be an elliptic curve over a field k. The idea of the counterexample is to construct a 1-parameter family of line bundles on E, indexed by $s \in S$, consisting of non-trivial line bundles for $s \neq 0$, and specializing to the trivial line bundle $\mathcal{O}_E$ at $s = 0$.

More precisely, consider the line bundle $\widetilde{L} = \mathcal{O}_{E\times E}(\Delta - \mathrm{pr}_1^*(e))$ on the product $E \times E$, where $\Delta \subseteq E \times E$ is the diagonal divisor. For $x \in E$, we have an isomorphism $\widetilde{L}|_{E\times\{x\}} \simeq \mathcal{O}_E(x - e)$; in particular, $\widetilde{L}|_{E\times\{x\}} \simeq \mathcal{O}_E$ iff $x = e \in E(k)$, because no two k-points on an elliptic curve are linearly equivalent (one can check this using Riemann–Roch). Now, consider the Cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & E \times E \arrow[d, "\mathrm{pr}_2"] \\ S = \mathrm{Spec}(\mathcal{O}_{E,e}) \arrow[r, hook] & E \end{tikzcd}$$

and let $L := \widetilde{L}|_X$. We claim that the natural map $H^0(X, L) \otimes_{\mathcal{O}_{E,e}} k \to H^0(X_e, L_e)$ is not an isomorphism, where we write $e \in S$ for the closed point of S and we identify $k \simeq \kappa(e)$. In fact, we will show that $H^0(X, L) = 0$, but $H^0(X_e, L_e) \neq 0$. Indeed, $L_e \simeq \mathcal{O}_E$, so $H^0(X_e, L_e) \simeq k$. On the other hand, if $\eta \in S$ is the generic point, then $H^0(X, L) \to H^0(X_\eta, L_\eta)$ is injective, and $H^0(X_\eta, L_\eta) = 0$, because L_η is a non-trivial degree-0 line bundle on the elliptic curve $X_\eta = E \times_k \kappa(\eta)$ (indeed, $L_\eta \simeq \mathcal{O}_{E\times\{\eta\}}(\eta - e)$, which is not trivial since $\eta \neq e$).

Before delving further into the yoga of cohomology and base change, there are two fundamental theorems on coherent cohomology that we must recall.

[8] The construction of (4.1) when $i = 0$ is as follows: postcomposing the unit map $\mathrm{id} \to g'_* g'^*$ with f_* yields a map

$$f_* \longrightarrow f_* g'_* g'^* = (f \circ g')_* \circ g'^* = (g \circ f')_* g'^* = g_* f'_* g'^*.$$

Using the (g^*, g_*)-adjunction, one gets the desired map $g^* f_* \longrightarrow f'_* g'^*$.

Theorem 4.2. [Finiteness Theorem] *If $f\colon X \to S$ is a proper map of noetherian schemes and $\mathcal{F}$ is a coherent sheaf on X, then $R^i f_* \mathcal{F}$ is a coherent sheaf on S, for any $i \geq 0$.*

Theorem 4.2 is a global/relative version of the classical fact that the cohomology groups of a coherent sheaf on a proper variety are finite-dimension vector spaces over the base field. For a proof, consult [DG67, III_1, 3.2.1] (or, if one assumes that $f\colon X \to S$ is projective, see [Har77, III,Theorem 8.8]).

Theorem 4.3. [Formal Functions Theorem (FFT)] *Let $f\colon X \to \operatorname{Spec}(A)$ be a proper map of noetherian schemes. If $I \subseteq A$ is an ideal and $\mathcal{F}$ is a coherent sheaf on X, then there is a natural isomorphism*

$$H^i(X, \mathcal{F})^\wedge \xrightarrow{\simeq} \varprojlim_n H^i(X_n, \mathcal{F}_n),$$

where $X_n = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A/I^n)$, $\mathcal{F}_n$ is the pullback of $\mathcal{F}$ along the projection $X_n \to X$, and $H^i(X, \mathcal{F})^\wedge$ is the I-adic completion of $H^i(X, \mathcal{F})$.

For a proof of Theorem 4.3, see [Sta17, Tag 02OC]. The cohomology appearing in Theorem 4.3 will often be written differently: indeed, there are natural isomorphisms $H^i(X_n, \mathcal{F}_n) \simeq H^i(X, \mathcal{F}/I^n\mathcal{F})$ and $H^i(X, \mathcal{F})^\wedge \simeq H^i(X, \mathcal{F}) \otimes_A \widehat{A}$, where $\widehat{A}$ is the I-adic completion of A.

While we omit the proofs of Theorem 4.2 and Theorem 4.3, we will now prove some of their consequences.

Proposition 4.4. [Flat base change] *Consider the Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

If f is qcqs[9]*, g is flat, and $\mathcal{F}$ is a quasi-coherent sheaf on X, then the natural map*

$$g^* R^i f_* \mathcal{F} \xrightarrow{\simeq} R^i f'_*(g'^* \mathcal{F}).$$

is an isomorphism, for all $i \geq 0$.

Proof. The question is local on S', so we can assume $S' = \operatorname{Spec}(B')$ and $S = \operatorname{Spec}(A)$. Since f is quasi-compact and S is affine, it follows that X is quasi-compact. Assume, for simplicity, that X is separated (if $X \to S$ is only quasi-separated, there is a standard spectral sequence argument to reduce to the separated case; see [Sta17, Tag 02KH]). Thus, we need to show that the natural map $H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B)$ is an isomorphism.

Let $\mathcal{U}$ be an open affine cover of X. By passing to a subcover, we can assume $\mathcal{U}$ is finite, since X is quasi-compact. As X is separated, the natural map

$$H^i(C^\bullet(\mathcal{U}, \mathcal{F})) \to H^i(X, \mathcal{F})$$

from the Čech cohomology to the derived functor cohomology is an isomorphism. Let $\mathcal{U}_B$ denote the open affine cover of X_B obtained from $\mathcal{U}$ by base changing each member of the cover. The Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ is bounded since $\mathcal{U}$ is finite, so we have an isomorphism

$$C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B \xrightarrow{\simeq} C^\bullet(\mathcal{U}_B, \mathcal{F}_B)$$

of complexes. The functor $- \otimes_A B$ commutes with taking cohomology since $A \to B$ is flat, so there is a sequence of isomorphisms

$$H^i(X, \mathcal{F}) \otimes_A B \xleftarrow{\simeq} H^i(C^\bullet(\mathcal{U}, \mathcal{F})) \otimes_A B \xrightarrow{\simeq} H^i(C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B) \xrightarrow{\simeq} H^i(C^\bullet(\mathcal{U}_B, \mathcal{F}_B)) \xrightarrow{\simeq} H^i(X_B, \mathcal{F}_B).$$

One can check that the sequence of morphisms above gives the natural map $H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B)$, hence it is an isomorphism. □

[9] The acronym qcqs is short for "quasi-compact and quasi-separated". If one is working with noetherian schemes, then separated morphisms are qcqs, and they are the main source of interest.

Remark 4.5. Together, Proposition 4.4 and Theorem 4.3 say that we can relate $H^i(X, \mathcal{F})$ to $\{H^i(X, \mathcal{F}/I^n\mathcal{F})\}_{n\geq 1}$ and that they carry roughly the same information. While this is not a precise statement, this is often how these results are used in practice.

Proposition 4.6. *If $f\colon X \to S$ is a proper map of noetherian schemes and $\mathcal{F}$ is a coherent sheaf on X that is flat*[10] *over S, then $R^i f_* \mathcal{F} = 0$ for all $i \geq i_0$ iff $H^i(X_s, \mathcal{F}_s) = 0$ for all $i \geq i_0$ and all $s \in S$.*

Proof. Both questions are local on S, so we can assume $S = \operatorname{Spec}(A)$; thus, $R^i f_* \mathcal{F} = H^i(X, \mathcal{F})$ for all $i \geq 0$. Assume that the cohomology of $\mathcal{F}$ restricted to any fibre vanishes above degree i_0. We can use Proposition 4.4 to assume[11] that A is a complete noetherian local ring, with maximal ideal $\mathfrak{m} \subseteq A$. Write $s \in \operatorname{Spec}(A)$ for the closed point. By Theorem 4.3,

$$H^i(X, \mathcal{F}) \simeq H^i(X, \mathcal{F})^\wedge \simeq \varprojlim_n H^i(X, \mathcal{F}/\mathfrak{m}^n),$$

where the isomorphism $H^i(X, \mathcal{F}) \simeq H^i(X, \mathcal{F})^\wedge$ follows from the fact that a finite module over A is $\mathfrak{m}$-adically complete. Thus, it suffices to show that $H^i(X, \mathcal{F}/\mathfrak{m}^n) = 0$ for all $n \geq 1$ and $i \geq i_0$.

We proceed by induction on n. By assumption, $H^i(X, \mathcal{F}/\mathfrak{m}) = H^i(X_s, \mathcal{F}_s) = 0$ for all $i \geq i_0$. Now, filter $A/\mathfrak{m}^n$ by the $\mathfrak{m}$-adic filtration, so that each graded piece is a copy of the residue field $k = A/\mathfrak{m}$. As $\mathcal{F}$ is flat over A, this induces a filtration of $\mathcal{F}/\mathfrak{m}^n$ by copies of $\mathcal{F}/\mathfrak{m}$; that is, tensor the short exact sequence $0 \to A/\mathfrak{m}^{n-1} \to A/\mathfrak{m}^n \to \mathfrak{m}^{n-1}/\mathfrak{m}^n \to 0$ by $\mathcal{F}$ to obtain a short exact sequence

$$0 \to \mathcal{F}/\mathfrak{m}^{n-1} \to \mathcal{F}/\mathfrak{m}^n \to \mathcal{F} \otimes_A \mathfrak{m}^{n-1}/\mathfrak{m}^n \to 0.$$

From the long exact sequence in cohomology and the induction hypothesis, we get that $H^i(X, \mathcal{F}/\mathfrak{m}^n) = 0$.

Conversely, assume that $H^i(X, \mathcal{F}) = 0$ for all $i \geq i_0$ and $S = \operatorname{Spec}(A)$. As before, we may assume A is a complete noetherian local ring, with maximal ideal $\mathfrak{m} \subseteq A$. Choose a finite affine open cover $\mathcal{U}$ of X, then the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ is a bounded complex of flat A-modules and $H^i(X, \mathcal{F}) \simeq H^i(C^\bullet(\mathcal{U}, \mathcal{F}))$. Moreover, there is an isomorphism of complexes

$$C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A/\mathfrak{m} \simeq C^\bullet(\mathcal{U}_{A/\mathfrak{m}}, \mathcal{F}/\mathfrak{m})$$

because formation of the Čech complex commutes with base change. There is an E_2-spectral sequence of the form

$$E_2^{p,q}\colon \operatorname{Tor}^A_{-p}\left(H^q(C^\bullet(\mathcal{U}, \mathcal{F})), A/\mathfrak{m}\right) \Longrightarrow H^{p+q}\left(C^\bullet(\mathcal{U}_{A/\mathfrak{m}}, \mathcal{F}/\mathfrak{m})\right).$$

The E_2-terms that can contribute are $\operatorname{Tor}^A_{-p}(H^q(C^\bullet(\mathcal{U}, \mathcal{F})), A/\mathfrak{m})$, and these all vanish for $p + q \geq i_0$, hence $H^{p+q}(C^\bullet(\mathcal{U}_{A/\mathfrak{m}}, \mathcal{F}_{A/\mathfrak{m}})) = 0$ for all $p + q \geq i_0$. Finally, the result follows from the isomorphism

$$H^i(X_s, \mathcal{F}_s) \simeq H^i(C^\bullet(\mathcal{U}_{A/\mathfrak{m}}, \mathcal{F}_{A/\mathfrak{m}})),$$

where $s \in S$ is the closed point. $\square$

There are two more useful tools that we will introduce below.

Proposition 4.7. [Projection formula] *If $f\colon X \to S$ is a qcqs morphism, $\mathcal{E}$ is a vector bundle*[12] *on S, and $\mathcal{F}$ is a quasi-coherent sheaf on X, then for any $i \geq 0$, there is a natural isomorphism*

$$\mathcal{E} \otimes R^i f_* \mathcal{F} \xrightarrow{\simeq} R^i f_*(f^*\mathcal{E} \otimes \mathcal{F}).$$

[10]If $f\colon X \to S$ is morphism of schemes and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ is *flat over* S if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module, where $\mathcal{F}_x$ is viewed as an $\mathcal{O}_{S,f(x)}$-module via the composition $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x} \to \mathcal{F}_x$.

[11]Indeed, $R^i f_* \mathcal{F} = 0$ iff the stalk $(R^i f_* \mathcal{F})_s = H^i(X, \mathcal{F}) \otimes_A \mathcal{O}_{S,s}$ is zero for all $s \in S$. The completion $\mathcal{O}_{S,s} \hookrightarrow \widehat{\mathcal{O}}_{S,s}$ is faithfully flat, so it suffices to show that $\left(H^i(X, \mathcal{F}) \otimes_A \mathcal{O}_{S,s}\right) \otimes_{\mathcal{O}_{S,s}} \widehat{\mathcal{O}}_{S,s} \simeq H^i(X, \mathcal{F}) \otimes_A \widehat{\mathcal{O}}_{S,s}$ is zero. The map $A \to \widehat{\mathcal{O}}_{S,s}$ is flat, because it is the composition of the localization $A \to \mathcal{O}_{S,s}$ (which is flat) and the faithfully flat map $\mathcal{O}_{S,s} \hookrightarrow \widehat{\mathcal{O}}_{s,s}$. By Proposition 4.4, we have $H^i(X, \mathcal{F}) \otimes_A \widehat{\mathcal{O}}_{S,s} \simeq H^i(X_{\widehat{\mathcal{O}}_{S,s}}, \mathcal{F}_{\widehat{\mathcal{O}}_{S,s}})$, hence we are able to assume that A is a complete noetherian local ring by replacing it with $\widehat{\mathcal{O}}_{S,s}$.

[12]In this class, we use the terms "vector bundle" and "locally free sheaf" interchangeably.

Proof. To construct the map, either use Čech cohomology or a δ-functor argument. To prove that the map is an isomorphism, one can work locally on S to assume that $\mathcal{E} = \mathcal{O}_S^{\oplus n}$, in which case the claim is clear. □

Proposition 4.8. [Künneth formula] *If k is a field, X and Y are qcqs k-schemes, $\mathcal{F}$ is quasi-coherent sheaf on X, and $\mathcal{G}$ is quasi-coherent sheaf on Y, then for any $n \geq 0$, there is a natural isomorphism*

$$\bigoplus_{i+j=n} H^i(X,\mathcal{F}) \otimes H^j(Y,\mathcal{G}) \xrightarrow{\simeq} H^n(X \otimes Y, \mathcal{F} \boxtimes \mathcal{G}),$$

where $\mathcal{F} \boxtimes \mathcal{G} := \mathrm{pr}_1^* \mathcal{F} \otimes \mathrm{pr}_2^* \mathcal{G}$.

Over a general base, the Künneth formula takes the form of an E^2-spectral sequence with terms involving Tor groups of the $H^i(X,\mathcal{F})$'s and $H^j(Y,\mathcal{G})$'s, converging to $H^{i+j}(X \otimes Y, \mathcal{F} \boxtimes \mathcal{G})$.

Proof. Assume for simplicity that X and Y are separated. Choose open affine covers $\mathcal{U}$ of X and $\mathcal{V}$ of Y, then there is an isomorphism of Čech complexes

$$C^\bullet(\mathcal{U},\mathcal{F}) \otimes C^\bullet(\mathcal{V},\mathcal{G}) \xrightarrow{\simeq} C^\bullet(\mathcal{U} \times \mathcal{V}, \mathcal{F} \boxtimes \mathcal{G}),$$

where $\mathcal{U} \times \mathcal{V}$ is the open affine cover of $X \times Y$ consisting of sets of the form $U \times V$, for $U \in \mathcal{U}$ and $V \in \mathcal{V}$. To see that this map is an isomorphism, one just needs to unwind the definition of a tensor product of complexes. The isomorphism now follows from the Künneth spectral sequence, and the fact that all higher Tor's vanish since we are working over a field. See [Sta17, Tag 0BED] for the details. □

Given a morphism $f\colon X \to S$, we will begin to relate the cohomology of a coherent sheaf on X with the cohomology of the restriction of the fibres X_s for $s \in S$. To do so, we will show that one can compute cohomology using a particularly nice complex, which has nice finiteness properties (though it is not functorial). We follow the treatment of [Mum08, II, §5]. We make the following provisional definition of a perfect complex.

Definition 4.9. If A is a commutative ring, a *perfect complex* over A is a bounded complex of finite projective A-modules.

The class of perfect complexes are the "nicest" kind of complexes that one could hope to work with; one could imagine replacing "projective" with "free", but it will be important to work with projective modules (though the difference is not so big, since every projective module is a direct summand of a free module).

Example 4.10.

(1) The 2-term complex $\mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z}$ is perfect over $\mathbf{Z}$ (and it is also a resolution of $\mathbf{Z}/p$!).
(2) More generally, any finite free resolution is a perfect complex.

The goal is to prove that any cohomology group can be computed as the cohomology of a perfect complex. To accomplish this, we require the following homological algebra lemma, due to Mumford.

Lemma 4.11. [Mum08, II, §5, Lemma 1] *Let A be a noetherian ring, and let $K^\bullet$ be a complex of A-modules such that*

(1) each term K^i is A-flat, and $K^i = 0$ for all $|i| \gg 0$;
(2) each cohomology group $H^i(K^\bullet)$ is finitely-generated as an A-module.

Then, there exists a perfect complex $M^\bullet$ and a quasi-isomorphism[13] *$M^\bullet \xrightarrow{\sim} K^\bullet$.*

In fact, Mumford proves a more precise statement, namely that we only require the cohomology $H^i(K^\bullet)$ to be finitely-generated in a certain range of degrees, and the perfect complex $M^\bullet$ will be nonzero only in that specified range of degrees. However, be warned that the quasi-isomorphism $M^\bullet \xrightarrow{\sim} K^\bullet$ need not be a monomorphism of complexes, i.e. $M^\bullet$ will not necessarily be a sub-complex of $K^\bullet$.

Finally, we will record a last lemma from homological algebra that will be used next class to prove the semicontinuity theorem.

[13] A *quasi-isomorphism* is a morphism of complexes that induces isomorphisms on all cohomology groups.

Lemma 4.12. [Mum08, II, §5, Lemma 2] *If A is a commutative ring and $M^\bullet \xrightarrow{\simeq} K^\bullet$ is a quasi-isomorphism of bounded complexes of flat A-modules, then for any A-module N, $M^\bullet \otimes_A N \longrightarrow K^\bullet \otimes_A N$ is a quasi-isomorphism.*

As a good exercise, convince oneself that Lemma 4.12 is not always true without the assumptions of boundedness and flatness.

5. September 19th

5.1. **Cohomology and Base Change (Continued).** Last time, various separation hypotheses were used in different theorems on coherent cohomology. Let us record the implications between these hypotheses that one should keep in mind: if $f\colon X \to Y$ is a map of schemes and $\mathcal{F}$ is a coherent sheaf on X, then

$$\begin{array}{ccccc} & & (f \text{ is quasi-compact and separated}) & \longrightarrow & \begin{pmatrix}\text{locally on } Y, \text{ the Čech complex}\\ \text{computes } R^i f_* \mathcal{F}\end{pmatrix} \\ & & \downarrow & & \\ (X \text{ and } Y \text{ are noetherian}) & \longrightarrow & (X, Y, \text{ and } f \text{ are qcqs}) & \longrightarrow & \left(R^i f_* \text{ preserves quasi-coherence}\right) \end{array}$$

Moreover, we mentioned two commutative algebra lemmas of Mumford, which we summarize below:

Lemma 5.1. *If A is a noetherian ring and $K^\bullet$ is a complex of A-modules such that*
1. *$K^\bullet$ is bounded;*
2. *K^i is flat for all $i \in \mathbf{Z}$;*
3. *$H^i(K^\bullet)$ is finitely-generated for all $i \in \mathbf{Z}$.*

Then, there exists a perfect complex $M^\bullet$ and a quasi-isomorphism $M^\bullet \xrightarrow{\sim} K^\bullet$; furthermore, for any A-module N, the map $M^\bullet \otimes_A N \longrightarrow K^\bullet \otimes_A N$ is a quasi-isomorphism.

The statement of Lemma 5.1 produces a perfect complex $M^\bullet$ that is "universally quasi-isomorphic" to the given complex $K^\bullet$.

Example 5.2. Consider the complex $K^\bullet = \left(\mathbf{Z}[1/2] \xrightarrow{3} \mathbf{Z}[1/2]\right)$ of $\mathbf{Z}$-modules, where the far-left term is placed in degree-0. By Lemma 5.1, there is a complex $M^\bullet$ that is quasi-isomorphic to $K^\bullet$ whose terms are finite free $\mathbf{Z}$-modules: indeed, take $M^\bullet = \left(\mathbf{Z} \xrightarrow{3} \mathbf{Z}\right)$. In this particular example, $M^\bullet$ is a sub-complex of $K^\bullet$, but this is not the case in general.

The following theorem states that one can compute coherent cohomology using a perfect complex.

Proposition 5.3. [Mum08, p. 44] *If $f\colon X \to \operatorname{Spec}(A)$ is a proper map of noetherian schemes and $\mathcal{F}$ is a coherent sheaf on X that is flat over A, then there is a perfect A-complex $M^\bullet$ such that for all A-algebras B, there are canonical isomorphisms*

$$H^i(M^\bullet \otimes_A B) \simeq H^i(X_B, \mathcal{F}_B)$$

for all $i \geq 0$.

We will see that, granted Proposition 5.3, the semicontinuity theorem reduces to a statement about perfect complexes. When proving basic theorems on coherent cohomology, we had previously used the Čech complex (which involves a choice of an open affine cover, but once that choice is made, the rest of the proof would be fairly canonical). However, the perfect complex $M^\bullet$ involved in Proposition 5.3 cannot be constructed functorially (though it does have good finiteness properties).

Proof. As X is quasi-compact, we can choose a finite open affine cover $\mathcal{U}$ of X. Consider the Čech complex $K^\bullet = C^\bullet(\mathcal{U}, \mathcal{F})$ associated to $\mathcal{U}$ and $\mathcal{F}$; $K^\bullet$ is bounded because $\mathcal{U}$ is a finite cover. Furthermore, the cohomology groups of $K^\bullet$ are finitely-generated A-modules, by Theorem 4.2. Thus, $K^\bullet$ satisfies the hypotheses of Proposition 5.3, hence there exists a perfect complex $M^\bullet$ that is "universally quasi-isomorphic" to $K^\bullet$ (in the sense of Proposition 5.3). Now, the Čech complex commutes with base change, so $C^\bullet(\mathcal{U},\mathcal{F})\otimes_A B \simeq C^\bullet(\mathcal{U}_B,\mathcal{F}_B)$, where $\mathcal{U}_B$ is the cover of X_B obtained from $\mathcal{U}$ by base changing each member of the cover. Thus, we have

$$H^i(M^\bullet\otimes_A B)\simeq H^i(C^\bullet(\mathcal{U},\mathcal{F})\otimes_A B)\simeq H^i(C^\bullet(\mathcal{U}_B,\mathcal{F}_B)) = H^i(X_B,\mathcal{F}_B).$$

□

There is some flexibility in the choice of the perfect complex $M^\bullet$ involved in Proposition 5.3: if one replaces $M^\bullet$ by a quasi-isomorphic complex, the statement still holds. This is very useful when applying this result in practice.

Corollary 5.4. [Semicontinuity Theorem] *Let $f\colon X\to S=\operatorname{Spec}(A)$ be a proper map of noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X that is flat over A.*

(1) For each $i\geq 0$, the function

$$s\mapsto \dim H^i(X_s,\mathcal{F}_s)$$

is upper semicontinuous on S; that is, the set $\{s\in S\colon \dim H^i(X_s,\mathcal{F}_s)\geq k\}$ is closed for all $k\in\mathbf{Z}$.

(2) The function

$$s\mapsto \chi(X_s,\mathcal{F}_s)$$

is locally constant on S, where χ denotes the Euler characteristic.

(3) Fix $i\geq 0$. If $H^i(X_s,\mathcal{F}_s)=0$ for all $s\in S$, then $H^i(X,\mathcal{F})=0$ and $H^{i-1}(X,\mathcal{F})\otimes_A\kappa(s)\simeq H^{i-1}(X_s,\mathcal{F}_s)$ for all $s\in S$.

(4) Assume that A is reduced. Fix $i\geq 0$. The function

$$s\mapsto \dim H^i(X_s,\mathcal{F}_s)$$

is constant on S iff $H^i(X,\mathcal{F})$ is a finite projective A-module and $H^i(X,\mathcal{F})\otimes_A\kappa(s)\simeq H^i(X_s,\mathcal{F}_s)$ for all $s\in S$. Moreover, if either of the equivalent conditions are satisfied, then $H^{i-1}(X,\mathcal{F})\otimes_A\kappa(s)\simeq H^{i-1}(X_s,\mathcal{F}_s)$ for all $s\in S$.

Corollary 5.4(1) is often thought of as saying that "the dimension of cohomology jumps up when one specializes" (e.g. as was the case in Example 4.1), while (3) asserts that if the cohomology in a fixed degree vanishes for all fibres, then the global cohomology must vanish, as well. Finally, in (4), the assertion that $H^i(X,\mathcal{F})$ be a finite projective A-module means that it determines a vector bundle on S, and demanding that $H^i(X,\mathcal{F})\otimes_A\kappa(s)\simeq H^i(X_s,\mathcal{F}_s)$ for all $s\in S$ says that it is a vector bundle "of the correct rank".

In the statement of Corollary 5.4, one can replace the affine scheme $S=\operatorname{Spec}(A)$ with any noetherian scheme S (because all questions involved are local on the base), at the expense of replacing the cohomology groups H^i with the higher direct images R^if_*.

Remark 5.5. The proof of Corollary 5.4 has little to do with algebraic geometry - it is just a proof about perfect complexes, while all the serious algebro-geometric input is hidden in the finiteness theorem Theorem 4.2. See [MP] for a version of the proof phrased entirely in the framework of perfect complexes.

Proof. For (1), choose a perfect complex $M^\bullet$ as in Proposition 5.3, i.e. one that "universally computes" the cohomology of $\mathcal{F}$. By shrinking $\operatorname{Spec}(A)$, we can assume that each term M^i is finite free (since projective modules are locally free). Now, the differential $d^i\colon M^i\to M^{i+1}$ is a matrix over A. The goal is to show that the function

$$s\mapsto \dim\left(\frac{\ker(d^i\otimes\kappa(s))}{\operatorname{im}(d^{i-1}\otimes\kappa(s))}\right)$$

is upper semicontinuous on S. The functions $s\mapsto k_i(s)\coloneqq \dim\left(\ker(d^i\otimes\kappa(s))\right)$ and $s\mapsto \dim\left(\operatorname{im}(d^{i-1}\otimes\kappa(s))\right)$ can take only finitely-many values, since both are integers bounded below by zero and bounded above by $\operatorname{rank}(M^i)$. Thus, it suffices to show that the functions

$$s\mapsto k_i(s)\coloneqq \dim\left(\ker(d^i\otimes\kappa(s))\right)\ \text{and}\ s\mapsto r_{i-1}(s)\coloneqq -\dim\left(\operatorname{im}(d^{i-1}\otimes\kappa(s))\right)$$

are upper semicontinuous on S. Moreover, the rank-nullity theorem says that $r_{i-1}(s) = \mathrm{rank}(M^{i-1}) - k_{i-1}(s)$, hence it suffices to show that $s \mapsto k_i(s)$ is upper semicontinuous on S. For any $c \in \mathbf{Z}_{\geq 0}$, we have

$$\begin{aligned}\{s \in \mathrm{Spec}(A) \colon k_i(s) \geq c\} &= \{s \in \mathrm{Spec}(A) \colon \mathrm{rank}(d^i \otimes \kappa(s)) < \mathrm{rank}(M^i) - c\} \\ &= \left\{\begin{matrix} \text{the vanishing locus of the} \\ (\mathrm{rank}(M^i) - c)\text{-minors of the matrix } d^i \end{matrix}\right\},\end{aligned}$$

which is closed in $\mathrm{Spec}(A)$.

For (2), we may again shrink $\mathrm{Spec}(A)$ to assume that M^i is finite free (this prevents e.g. that M^i is free of different ranks on two different connected components of $\mathrm{Spec}(A)$). Then, for any $s \in S$,

$$\chi(X_s, \mathcal{F}_s) = \chi(M \otimes_A \kappa(s)) = \sum_{i \geq 0} (-1)^i \mathrm{rank}_{\kappa(s)}(M^i \otimes \kappa(s)) = \sum_{i \geq 0} (-1)^i \mathrm{rank}_A(M^i),$$

where the last equality follows precisely because M^i is a finite free A-module. In particular, $\chi(X_s, \mathcal{F}_s)$ is independent of $s \in S$.

For (3), the argument is identical to the reverse direction of Proposition 4.6 (alternatively, see [Mum08, §5]).

For (4), we prove only the special (but illustrative) case where A is a dvr (for the general case, see [Mum08, §5, Corollary 2]; note that their proof is not via reduction to the case of a dvr). Let $K = \mathrm{Frac}(A)$ denote the fraction field, let $k = A/\mathfrak{m}$ be the residue field, and let $t \in A$ be a uniformizer. Choose a perfect complex $M^\bullet$ as in Proposition 5.3, then $M^\bullet$ is a finite free A-module, since A is a dvr. Then, the goal is to show that the equality $\dim_K H^i(M^\bullet)[1/t] = \dim_k H^i(M^\bullet/tM^\bullet)$ is equivalent to $H^i(M^\bullet)$ being a torsion-free A-module and the natural map $H^i(M^\bullet)/t \xrightarrow{\cong} H^i(M^\bullet/tM^\bullet)$ being an isomorphism. Take the standard exact sequence (i.e. the usual free resolution of the residue field k)

$$0 \to \xrightarrow{t} A \to A/t = k \to 0$$

and tensor with $M^\bullet$ in order to get the exact sequence

$$0 \to M^\bullet \xrightarrow{t} M^\bullet \to M^\bullet/tM^\bullet \to 0$$

of complexes (this is indeed exact, since the terms of $M^\bullet$ are free, hence tensoring by $M^\bullet$ is exact). Taking the long exact sequence in cohomology gives a short exact sequence

$$0 \to H^i(M^\bullet)/t \to H^i(M^\bullet/tM^\bullet) \twoheadrightarrow H^{i+1}(M^\bullet)[t] \to 0, \tag{5.1}$$

from which it follows that $\dim_K H^i(M^\bullet)[1/t] \leq \dim_k(H^i(M^\bullet)/t) \leq \dim H^i(M^\bullet/tM^\bullet)$. Thus, if we assume

$$\dim_K H^i(M^\bullet)[1/t] = \dim_k H^i(M^\bullet/tM^\bullet),$$

then $H^i(M^\bullet)$ has no t-torsion, and hence (5.1) implies that the natural map $H^i(M^\bullet)/t \to H^i(M^\bullet/tM^\bullet)$ is an isomorphism. This argument can be easily turned around to prove the opposite implication, which is left as an exercise. □

Remark 5.6. The argument given in the proof of Corollary 5.4(1) shows that, for any $k \in \mathbf{Z}$, the locus

$$\{s \in \mathrm{Spec}(A) \colon \dim H^i(X_s, \mathcal{F}_s) = k\}$$

is a constructible (in fact, locally closed) subset of $\mathrm{Spec}(A)$.

We will begin to start using the theory of cohomology and base change to understand the behaviour of line bundles on abelian varieties (and, in particular, their behaviour under $[n]$).

Lemma 5.7. *A line bundle $\mathcal{M}$ on a proper geometrically integral variety X over a field k is trivial iff we have $H^0(X, \mathcal{M}) \neq 0$ and $H^0(X, \mathcal{M}^{-1}) \neq 0$.*

Proof. Choose nonzero sections $s \in H^0(X, \mathcal{M})$ and $t \in H^0(X, \mathcal{M}^{-1})$; these can be thought of as nonzero maps $\mathcal{O}_X \xrightarrow{s} \mathcal{M}$ and $\mathcal{M} \xrightarrow{t} \mathcal{O}_X$. Multiplying, we get a map $\mathcal{O}_X \xrightarrow{t \circ s} \mathcal{O}_X$, and by integrality, the product of two nonzero global sections is again nonzero. However, $H^0(X, \mathcal{O}_X) = k$, so $t \circ s$ must be an isomorphism. Similarly, $s \circ t$ is an isomorphism. Hence, s and t must both be isomorphisms. □

In fact, Lemma 5.7 holds under the assumption that X is integral, as opposed to geometrically integral. In this case, $H^0(X, \mathcal{O}_X)$ may not be isomorphic the base field k, but it is an integral domain that is finite over k, hence it is a finite field extension of k. In particular, the nonzero element $t \circ s$ of $H^0(X, \mathcal{O}_X)$ is an isomorphism.

6. September 21st

Last time, we finished our discussion of cohomology and base change with a proof of the semicontinuity theorem. Today, we will discuss the seesaw theorem and the theorem of the cube.

6.1. **The Seesaw Theorem.**

In the following proposition, we attempt to understand the locus where the pushforward of a line bundle is a bundle (by analyzing the locus where the cohomology of the fibres is 1-dimensional).

Proposition 6.1. *If $f\colon X \to S$ is a proper flat map and $L \in \operatorname{Pic}(X)$ is a line bundle on X, then there exists a unique locally closed subscheme $Z \subseteq S$ such that*

(a) $(f_Z)_*(L_Z)$ is invertible;

(b) Z is universal with this property, i.e. for any $T \to S$ such that $(f_T)_*(L_T)$ is invertible, there exists a factorization $T \xrightarrow{g} Z \hookrightarrow S$ such that the natural map $(f_T)_*(L_T) \to g^*\left((f_Z)_*(L_Z)\right)$ is an isomorphism.

In the statement of Proposition 6.1, the morphism f_T is the base change of f along $T \to S$, and L_T is the pullback of L along f_T. Notice that the factorization $T \xrightarrow{g} Z \hookrightarrow S$ is unique if it exists, because the closed immersion $Z \hookrightarrow S$ is a monomorphism.

Remark 6.2. The locally closed subscheme $Z \subseteq S$ in Proposition 6.1 may be empty, e.g. if $S = \operatorname{Spec}(k)$ and L is a line bundle on the proper k-variety X whose space of global sections is not 1-dimensional. Moreover, the subscheme $Z \subseteq S$ need not be closed in general.

Proof. If such a Z exists, the universal property (b) tells one what the underlying point set $|Z|$ of Z must be:

$$|Z| = \left\{ s \in S \colon H^0(X_s, L_s) \text{ is 1-dimensional} \right\}.$$

The semicontinuity theorem asserts that $|Z|$ is locally closed (indeed, it is the intersection of the closed set where $\dim_{\kappa(s)} H^0(X_s, L_s) \geq 1$ and the open set where $\dim_{\kappa(s)} H^0(X_s, L_s) < 2$).

In order to specify the desired closed subscheme structure on $|Z|$, it suffices to work locally on S. Fix $s \in |Z|$ and, by passing to an affine open neighbourhood of s, assume that $S = \operatorname{Spec}(A)$.

Choose a perfect complex $M^\bullet = \left(M^0 \to M^1 \to M^2 \to \dots\right)$ as in Proposition 5.3 that "universally computes" the cohomology of L (in fact, we use Mumford's more precise statement [Mum08, II, §5, Lemma 1] to guarantee that the terms of $M^\bullet$ are nonzero only in the range where the cohomology of L is nonzero). Here, we place M^0 is degree-0. Now, we employ "Mumford's trick": consider the finitely-generated A-module

$$Q \coloneqq \operatorname{cokernel}\left((M^1)^\vee \to (M^0)^\vee\right).$$

The fact that $M^\bullet$ "universally computes" the cohomology of L translates to the following property of Q: for any finitely-generated A-module B, there are isomorphisms

$$\operatorname{Hom}_A(Q, B) \simeq H^0(M^\bullet \otimes_A B) \simeq H^0(X_B, L_B)$$

of B-modules. In particular, if $B = \kappa(s)$, then $\operatorname{Hom}_A(Q, \kappa(s)) \simeq H^0(X_s, L_s)$ is a 1-dimensional $\kappa(s)$-vector space, since $s \in |Z|$. By tensor-Hom adjunction, $\operatorname{Hom}_A(Q, \kappa(s)) \simeq \operatorname{Hom}_{\kappa(s)}(Q \otimes_A \kappa(s), \kappa(s))$, hence $Q \otimes_A \kappa(s)$ is 1-dimensional. By Nakayama's lemma, after possibly shrinking $S = \operatorname{Spec}(A)$, there is a non-canonical A-module isomorphism $Q \simeq A/I$ from Q to a cyclic A-module, for some ideal $I \subseteq A$. We claim that $Z = V(I)$ gives the correct scheme structure on $|Z|$, i.e. we have:

(a') $\mathrm{Hom}_A(Q, A/I)$ is an invertible A/I-module;
(b') for any A-algebra B, $\mathrm{Hom}_A(Q, B)$ is an invertible B-module iff $IB = 0$.

The statement (a') is clear, since $Q \simeq A/I$. To see (b'), observe that

$$\mathrm{Hom}_A(Q, B) = \mathrm{Hom}_A(A/I, B) = B[I] = \{b \in B\colon I \cdot b = 0\}$$

is the I-torsion of B. Note that the localization of an invertible module at any prime is free, and the only ideal that annihilates a free module is the zero ideal; thus, every localization of IB is zero, hence $IB = 0$ when $\mathrm{Hom}_A(Q, B) = B[I]$ is an invertible B-module. □

Given a morphism $f\colon X \to S$ and a line bundle $L \in \mathrm{Pic}(X)$ on X such that L is trivial on each fibre of f, it is too much to ask for L to be globally trivial; for example, if L is pulled back from S (i.e. there exists a line bundle $M \in \mathrm{Pic}(S)$ on S such that $L \simeq f^*(M)$, then L has this property. In the Seesaw Theorem (below), we leverage Proposition 6.1 to prove that this is all that can occur (up to reducedness concerns).

Theorem 6.3. [The Seesaw Theorem] *Let $f\colon X \to S$ be a proper flat morphism with geometrically integral fibres. If $L \in \mathrm{Pic}(X)$ is a line bundle on X, then*

(1) the set $Z \coloneqq \{s \in S\colon L_s \text{ is trivial}\}$ is closed in S;
(2) $L_{Z_{\mathrm{red}}}$ is pulled back from Z_{red}, where Z_{red} denotes the unique reduced subscheme structure on Z;
(3) there exists a unique closed subscheme structure on Z such that L_Z is pulled back from Z, and Z is universal with this property amongst all S-schemes.

Remark 6.4. What does the Seesaw Theorem have to do with seesaws? Consider the main case of interest: when $X = S \times T$ for two proper geometrically integral varieties S and T over a field k, and $f\colon X \to S$ is the first projection. Theorem 6.3 then implies that if $L_{\{s\}\times T}$ is trivial for all $s \in S$, then L is pulled back from S.

Proof. For (1), as the fibres of f are geometrically integral, Lemma 5.7 implies that Z can be rewritten as

$$Z = \{s \in S\colon H^0(X_s, L_s) \neq 0\} \cap \{s \in S\colon H^0(X_s, L_s^{-1}) \neq 0\}.$$

Both sets above are closed by the semicontinuity theorem (indeed, they are both the locus where the global sections of a certain line bundle has dimension ≥ 1), hence Z is closed.

For (2), replace S with Z_{red} in order to assume that L_s is trivial for all $s \in S$ and that S is reduced. If $M = f_*L$, then we claim that M is a line bundle on S. This problem is local on S, so we may assume that $S = \mathrm{Spec}(A)$. By Corollary 5.4(4), it follows that M is a vector bundle and $M \otimes_A \kappa(s) \simeq H^0(X_s, L_s) = H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$, where the final equality follows from the hypothesis that the map f is proper with geometrically integral fibres. Therefore, M is of rank 1, as required.

Moreover, we claim that the counit map $f^*(M) = f^*f_*L \to L$ is an isomorphism. The problem is local on S, so assume $S = \mathrm{Spec}(A)$. If one restricts the counit map to the fibre above $s \in S$, then one gets an isomorphism because L_s is trivial and $M \otimes_A \kappa(s) \xrightarrow{\simeq} H^0(X_s, L_s)$ is an isomorphism. Thus, $f^*(M) \to L$ is an isomorphism on each fibre X_s, and it is therefore an isomorphism by Nakayama's lemma.

For (3), we claim that for any S-scheme T, L_T is pulled back from T iff both $(f_T)_*L_T$ and $(f_T)_*L_T^\vee$ are invertible, and of formation compatible with base change. To prove the claim, repeat the argument given in (2).

By Proposition 6.1, there exists a maximal locally closed subscheme $W \subseteq S$ such that $(f_W)_*L_W$ and $(f_W)_*L_W^\vee$ are invertible, and of formation compatible with base change. Moreover, W is universal with this property. By the claim, W is the maximal locally closed subscheme such that L_W is pulled back from W. Therefore, it suffices to show that $W = Z$, but this immediately follows from the universal property of W. □

In summary, the idea of the proof of Theorem 6.3 is to reformulate what it means for a line bundle to be pulled back from base in terms of properties of the pushforwards, which can then be analyzed using cohomology and base change.

Remark 6.5. In [Con15, Theorem 3.1.1], Conrad gives a proof of Theorem 6.3 using the existence of Picard schemes as a black box. We summarize the idea of the argument below.

If $f\colon X \to S$ is as in Theorem 6.3, the presheaf $\underline{\mathrm{Pic}}_{X/S}\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Groups})$ given by

$$T \mapsto \mathrm{Pic}(X_T)/(f_T)^* \mathrm{Pic}(T)$$

is (often) representable by a separated S-group scheme (this is a very nontrivial theorem!). Granted this fact, Theorem 6.3 says something very natural: if $L \in \mathrm{Pic}(X)$ is a line bundle on X, it determines[14] a section $[L]\colon S \to \underline{\mathrm{Pic}}_{X/S}$ of the structure map $\underline{\mathrm{Pic}}_{X/S} \to S$. Similarly, the structure sheaf $\mathcal{O}_X$ determines the "zero section" $[0]\colon S \to \underline{\mathrm{Pic}}_{X/S}$, whose image $[0](S)$ we identify with S. The closed subscheme $Z \subseteq S$ is precisely the scheme-theoretic intersection $[L](S) \cap [0](S) \subseteq \underline{\mathrm{Pic}}_{X/S}$, viewed as living in S (note that the intersection is closed in $[0](S)$ since $\underline{\mathrm{Pic}}_{X/S} \to S$ is separated).

Remark 6.6. Let X and Y be proper, geometrically integral varieties over an algebraically closed field k, and let $L \in \mathrm{Pic}(X \times Y)$ be a line bundle on the product $X \times Y$. Assume the restrictions $L|_{X\times\{y\}}$ and $L|_{\{x\}\times Y}$ are trivial for some closed points $x \in X$ and $y \in Y$. One can ask: is L trivial?

The answer is no, L need not be trivial. Consider the setup of Example 4.1: let X and Y be an elliptic curve (E, e), and let

$$L = \mathcal{O}_{X\times Y}(\Delta - \mathrm{pr}_1^*(e) - \mathrm{pr}_2^*(e))$$

be the Poincaré bundle on $X \times Y$, where $\Delta \subseteq X \times Y$ is the diagonal. In this case, L is trivial on $X \times \{e\}$ and $\{e\} \times Y$, but it is not the trivial line bundle on $X \times Y$: indeed, if $p \in E$ is not the identity e, then $L|_{\{p\}\times Y} \simeq \mathcal{O}_E(p - e)$, which is a nontrivial degree-0 zero line bundle.

6.2. **The Theorem of the Cube.** In light of Remark 6.6, one might expect that asking the same question with more than 2 factors would also yield a negative answer, but this is not the case! This result will be called the theorem of the cube. In the language of [Mum08, II, §6, Remark], this occurs because the Picard functor is "quadratic".

In order to prove the theorem of the cube, we require the following preparatory result.

Theorem 6.7. *Let S be a connected scheme, and let $X \to S$ and $Y \to S$ be proper flat morphisms with geometrically integral fibres, Let $L \in \mathrm{Pic}(X \times_S Y)$ be a line bundle on $X \times_S Y$ and assume that:*

(1) there exist sections $e_X\colon S \to X$ and $e_Y\colon S \to Y$ such that $L|_{X\times\{e_Y\}}$ and $L|_{\{e_X\}\times Y}$ are trivial;
(2) there exists a point $s \in S$ such that L_s is trivial on $X_s \times Y_s$.

Then, L is pulled back from S.

In Remark 6.6, the second condition of Theorem 6.7 was not satisfied: indeed, the base scheme S is just the point $\mathrm{Spec}(k)$, so if assumes that the fibre above $\mathrm{Spec}(k)$ is trivial, then there is nothing to prove.

Corollary 6.8. [Theorem of the Cube] *Fix a base scheme S. Let $X \to S$ and $Y \to S$ be proper flat morphisms with geometrically integral fibres, and let Z be a connected S-scheme of finite type. If $L \in \mathrm{Pic}(X \times_S Y \times_S Z)$ is a line bundle such that there exists $x \in X(S)$, $y \in Y(S)$, and $z \in Z(S)$ with the restrictions $L|_{X\times Y\times\{z\}}$, $L|_{X\times\{y\}\times Z}$, and $L|_{\{x\}\times Y\times Z}$ all trivial, then L is trivial.*

The main case of interest in Corollary 6.8 is when $S = \mathrm{Spec}(k)$ and X, Y, Z are varieties over k. Below we give the proof of Corollary 6.8, granted Theorem 6.7.

Proof. We will apply Theorem 6.7 to the morphism

$$X \times_S Y \times_S Z = (X \times_S Z) \times_Z (Y \times_S Z) \longrightarrow Z.$$

[14] The choice of a section $S \to \underline{\mathrm{Pic}}_{X/S}$ is equivalent to the choice of a line bundle in $\mathrm{Pic}(X)$ modulo the subgroup $f^* \mathrm{Pic}(S)$, so the map $[L]\colon S \to \underline{\mathrm{Pic}}_{X/S}$ corresponds to L, and $[0]$ corresponds to $\mathcal{O}_X$.

By hypothesis, $L \in \mathrm{Pic}(X \times Y \times Z)$ is trivial when restricted to $(\{x\} \times Z) \times_Z (Y \times Z)$ and $(X \times Z) \times_Z (\{y\} \times Z)$, and it is trivial on the fibre $X \times Y \times \{z\}$ of $X \times Y \times Z$ over $\{z\}$. Thus, L is pulled back from some line bundle M on Z, by Theorem 6.7. However, $L|_{\{x\}\times\{y\}\times Z}$ is trivial, so M (and hence L) must be trivial. □

We will come back and prove Theorem 6.7 another time.

6.3. **Applications to Abelian Varieties.** Let us now apply the results of the previous sections to describe the multiplication-by-n map $[n]$ on abelian varieties. This can be formulated in the generality of abelian schemes, but here we give the simpler version.

Lemma 6.9. *Let A be an abelian variety over a field k, let Z be a k-scheme with maps $f, g, h\colon Z \to A$. For any line bundle $L \in \mathrm{Pic}(A)$, there is an isomorphism*[15]

$$(f+g+h)^*L \simeq (f+g)^*L \otimes (g+h)^*L \otimes (f+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

The formula appearing in Lemma 6.9 can be thought of as a version of the "exclusion-inclusion principle". The above lemma can be formulated for abelian schemes as well, but we only concern ourselves with the simpler setting of abelian varieties.

Proof. It suffices to consider the special case when $Z = A \times A \times A$, as this is the universal k-variety equipped with 3 maps to A. Write $m_{fgh} = f+g+h$, $m_{fg} = f+g$, and so on (because these are now just the multiplication maps on A). The goal is to show that the line bundle

$$m_{fgh}^*L^{-1} \otimes m_{fg}^*L \otimes m_{gh}^*L \otimes m_{fh}^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}$$

is trivial. Apply Corollary 6.8 to the identity sections $e \in A(k)$ to get that this is trivial on $\{e\} \times A \times A$, $A \times \{e\} \times A$, and $A \times A \times \{e\}$, and hence is trivial. (For example, if we restrict to $A \times A \times \{e\}$, then we get the trivial line bundle

$$m_{fg}^*L^{-1} \otimes m_{fg}^*L \otimes m_g^*L \otimes m_f^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \simeq \mathcal{O}_{A\times A\times\{e\}}$$

on $A \times A \times \{e\}$, as required.) □

The reason we proved this lemma is the following key result.

Corollary 6.10. *If A/k is an abelian variety, $n \in \mathbf{Z}$, $L \in \mathrm{Pic}(A)$, then $[n]^*L = L^{\frac{n^2+n}{2}} \otimes (-1)^*L^{\frac{n^2-n}{2}}$.*

Proof. For simplicity, we explain the proof only when $n = 2$: apply Lemma 6.9 to the maps $f = [1]$, $g = [1]$, and $h = [-1]$ to get an isomorphism

$$L \simeq [2]^*L \otimes [0]^*L \otimes [0]^*L \otimes L^{-1} \otimes L^{-1} \otimes L^{-1} \otimes [-1]^*L^{-1}.$$

As $[0]^*L \simeq \mathcal{O}_A$, we can rearrange the above to get that $[2]^*L \simeq L^3 \otimes [-1]^*L^{-1}$, as required. □

7. September 26th (Lecture by Zili Zhang)

The goal of this lecture and the next is to introduce the framework of derived categories that will be used later in the course.

[15] If $f, g\colon Z \to A$ are two maps from a k-scheme Z to an abelian variety A over k, then $f+g$ is the composition

$$Z \xrightarrow{\Delta_Z} Z \times Z \xrightarrow{(f,g)} A \times A \xrightarrow{m} A.$$

7.1. **Introduction to Derived Categories.** One motivation for introducing the derived category is a need for better foundations to compute derived functors. Recall that when we compute derived functors, e.g. Tor, Ext, or $R^i f_*$, then we first pick a resolution. For example, if $f\colon X \to Y$ is a morphism of schemes and $\mathcal{F}$ is a coherent sheaf on X, then $R^i f_* \mathcal{F}$ is computed by choosing an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow ...$$

of $\mathcal{F}$, and declaring $R^i f_* \mathcal{F}$ to be the i-th cohomology of the complex $f_*(\mathcal{I}^\bullet)$. In the derived category, the same data can be viewed as a morphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \mathcal{I}^2 \arrow[r] & \dots \end{tikzcd}$$

that is a quasi-isomorphism, i.e. it induces isomorphism on cohomology. Therefore, in the derived category, we do not want to just consider coherent sheaves, but cochain complexes of coherent sheaves, where morphisms are simply morphisms of cochain complexes, up to quasi-isomorphism. We will now construct the derived category associated to any abelian category.

Let $\mathcal{A}$ be an *abelian category*, i.e. $\mathcal{A}$ is an additive category (every Hom set in $\mathcal{A}$ is equipped with the structure of an abelian group and composition is a morphism of abelian groups), every morphism in $\mathcal{A}$ has a kernel and a cokernel, every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel. Let $C^*(\mathcal{A})$ be the category of *cochain complexes*, the objects of which are cochain complexes

$$\dots \longrightarrow A^{-1} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \dots$$

with terms in $\mathcal{A}$, and the morphisms are the usual morphisms of cochain complexes. Here, $*$ is a qualifier that may be empty (i.e. no conditions are imposed); $C^+(\mathcal{A})$ denotes cochain complexes that are bounded below; $C^-(\mathcal{A})$ denotes cochain complexes that are bounded above; $C^b(\mathcal{A})$ denotes cochain complexes $A^\bullet$ that are bounded, i.e. $A^i = 0$ for all $|i| \gg 0$. For more details, see [Wei94, §1.1, §A.4].

From the category of cochain complexes, one can construct the *homotopy category* $K(\mathcal{A})$ of $\mathcal{A}$: the objects of $K(\mathcal{A})$ are the cochain complexes with terms in $\mathcal{A}$ (the objects are the same as in $C(\mathcal{A})$!); if $A^\bullet$ and $B^\bullet$ are cochain complexes, then the morphisms between them in $K(\mathcal{A})$ are

$$\operatorname{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \operatorname{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet) \Big/ \operatorname{Hom}^0_{C(\mathcal{A})}(A^\bullet, B^\bullet),$$

where $\operatorname{Hom}^0_{C(\mathcal{A})}(A^\bullet, B^\bullet)$ denotes the subgroup of the morphisms in $\operatorname{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)$ that homotopic to zero. Recall that a morphism $f\colon A^\bullet \to B^\bullet$ of cochain complexes is *homotopic to zero* if there are morphisms $h\colon A^* \to B^{*-1}$ such that $f = dh + hd$, where d denotes both the differential of $A^\bullet$ and of $B^\bullet$. The setup is summarized in the diagram below:

$$\begin{tikzcd} \dots \arrow[r, "d"] & A^{-1} \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed, "\exists h"'] & A^0 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed, "\exists h"'] & A^1 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed, "\exists h"'] & \dots \arrow[dl, dashed, "\exists h"'] \\ \dots \arrow[r, "d"] & B^{-1} \arrow[r, "d"] & B^0 \arrow[r, "d"] & B^1 \arrow[r, "d"] & \dots \end{tikzcd}$$

More generally, two morphisms f and g in $C(\mathcal{A})$ are *homotopic* if $f - g$ is homotopic to zero. For more details, see [Wei94, §1.4, §10.1].

One can show that if f and g are homotopic maps in $C(\mathcal{A})$, then they induce the same map on all homology groups; see [Wei94, Lemma 1.4.5]. In particular, the notion of a quasi-isomorphism makes sense in the homotopy category. To construct the derived category of $\mathcal{A}$ from the homotopy category $K(\mathcal{A})$, we want to declare the quasi-isomorphisms in $K(\mathcal{A})$ to be isomorphisms in the derived category.

The *derived category* $\mathrm{D}(\mathcal{A})$ is the category whose objects are cochain complexes with terms in $\mathcal{A}$ (so the objects are still same as in $C(\mathcal{A})$!), but a morphism $A^\bullet \to B^\bullet$ between two cochain complexes is a diagram of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "h"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where f and h are maps in $K(\mathcal{A})$, and h is a quasi-isomorphism. Such a diagram is often called a "roof".

Note that we have inverted the quasi-isomorphisms of the homotopy category! However, there is much to check in order to show that this definition works. For example, different pairs (h, f) and (h', f') as in the diagram may give the same morphism in $\mathrm{D}(\mathcal{A})$. In addition, it is not trivial to verify that the composition of two morphisms is again a morphism, or that there is associativity.

If one attempts to construct the derived category $\mathrm{D}(\mathcal{A})$ directly from the category $C(\mathcal{A})$ of cochain complexes, it is not clear that the composition of morphisms is a morphism, as explained in [Huy06, §2.1, p.32-33]. For further details on the construction of the derived category of an abelian category, see [Huy06, §2.1] or [Wei94, §10.4].

The derived category $\mathrm{D}(\mathcal{A})$ is equipped with a *shift functor* $[n]\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ given by

$$\begin{cases} (A^\bullet[n])^i \coloneqq A^{i+n} \\ d^i_{A^\bullet[n]} \coloneqq (-1)^n d^{i+n}_{A^\bullet} \end{cases}$$

The shift functor already allows us too see some of the connections between the derived category and the derived functors that we are used to.

Example 7.1. If $A, B \in \mathcal{A}$ are viewed as complexes placed in degree 0, then $\mathrm{Ext}^n_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[n])$.

To see how one might prove this, consider the case $n = 1$: we know $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$ is parametrized by equivalence classes of extensions E of A by B, i.e. equivalence classes of short exact sequences $0 \to B \to E \to A \to 0$ (see [Wei94, §3.4] for this description of Ext). This can be rewritten as a quasi-isomorphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where E and A are placed in cohomological degree-0. Thus, we have a diagram

$$\begin{tikzcd} & (0 \to B \to E \to 0) \arrow[dl, "\text{quasi-isomorphism}"'] \arrow[dr] & \\ A = (0 \to 0 \to A \to 0) & & (0 \to B \to 0 \to 0) = B[1] \end{tikzcd}$$

i.e. a morphism $A \to B[1]$ in $\mathrm{D}(\mathcal{A})$. This process can be reversed, as well.

The derived category is always an additive category, but it is not abelian. It does have some additional structure, as we will see later.

7.2. **The Derived Category of Coherent Sheaves.** For us, the most interesting case is when $\mathcal{A}$ is the category $\mathrm{Coh}(X)$ of coherent sheaves on a noetherian scheme X (the requirement that X be noetherian is crucial here, otherwise $\mathrm{Coh}(X)$ may not be an abelian category). In this case, the unbounded derived category $\mathrm{D}(\mathrm{Coh}(X))$ is too large and complicated, so we often choose to work with the bounded derived category $\mathrm{D}^b(\mathrm{Coh}(X))$ instead. However, using $\mathrm{D}^b(\mathrm{Coh}(X))$ is problematic, because the only allowable complexes in this category have only finitely-many nonzero terms; in particular, it does not necessarily contain[16] an injective resolution of a coherent sheaf (viewed as a complex by placing it in cohomological degree-0). The solution is to use a "different model", i.e. work with a category equivalent to $\mathrm{D}^b(\mathrm{Coh}(X))$ that is more amenable to our purposes.

[16]The problem is twofold: an injective resolution may be unbounded, and also an injective sheaf is rarely coherent.

The new model will be denoted $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$. In order to define this category, first consider the category $\mathrm{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules on X, and take the associated unbounded derived category $\mathrm{D}(\mathcal{O}_X) \coloneqq \mathrm{D}(\mathrm{Mod}(\mathcal{O}_X))$. Now, we will add certain restrictions on which complexes in $\mathrm{D}(\mathcal{O}_X)$ are allowed: consider only complexes $\mathcal{F}^\bullet \in \mathrm{D}(\mathcal{O}_X)$ such that

- the cohomology sheaf[17] $H^i(\mathcal{F}^\bullet)$ is coherent for all $i \in \mathbf{Z}$;
- the cohomology sheaf $H^i(\mathcal{F}^\bullet)$ is zero for $|i| \gg 0$.

That is, we have no control on where the non-zero terms of $\mathcal{F}^\bullet$ appear, but we do control the nonzero terms of the cohomology. Let $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$ denote the full subcategory of $\mathrm{D}(\mathcal{O}_X)$ consisting of such complexes.

There is a natural inclusion $\mathrm{D}^b(\mathrm{Coh}(X)) \hookrightarrow \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$ and, since X is noetherian, this is an equivalence of categories. It is easy to see that it is fully faithful, but essential surjectivity is the difficult part: one must show that an infinite complex of $\mathcal{O}_X$-modules can be "replaced" by a complex with finitely-many coherent terms, throwing out infinitely-many redundant terms. See [Huy06, Proposition 3.5].

One can construct other models of $\mathrm{D}^b(\mathrm{Coh}(X))$ built from injective or flat resolutions. Let $\mathrm{Inj}(\mathcal{O}_X)$ denote the category of injective $\mathcal{O}_X$-modules; this is an additive category, but it is not abelian (e.g. the kernel of a morphism between injective $\mathcal{O}_X$-modules need not be injective). Nonetheless, we can form the unbounded derived category $\mathrm{D}(\mathrm{Inj}(\mathcal{O}_X))$ using the same method. Since we would like to take injective resolutions of (coherent) sheaves, we consider instead the subcategory $\mathrm{D}^+(\mathrm{Inj}(\mathcal{O}_X))$.

There is a natural embedding $\mathrm{D}^+(\mathrm{Inj}(\mathcal{O}_X)) \hookrightarrow \mathrm{D}^+(\mathcal{O}_X)$, which is again a natural equivalence under mild assumptions. It is easy to see that it is fully faithful. To check that it is essentially surjective, take a complex $\mathcal{F}^\bullet \in \mathrm{D}^+(\mathcal{O}_X)$ and construct an injective resolution of each term to get a bicomplex. Totalizing the bicomplex gives a complex in $\mathrm{D}^+(\mathrm{Inj}(\mathcal{O}_X))$ that is quasi-isomorphic to $\mathcal{F}^\bullet$.

One can restrict the equivalence $\mathrm{D}^+(\mathrm{Inj}(\mathcal{O}_X)) \xrightarrow{\simeq} \mathrm{D}^+(\mathcal{O}_X)$ onto the "coherent part" to get an equivalence

$$\mathrm{D}^b_{\mathrm{coh}}(\mathrm{Inj}(\mathcal{O}_X)) \xrightarrow{\simeq} \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X).$$

Note that, even though $\mathrm{Inj}(\mathcal{O}_X)$ is not an abelian category, we can still consider the cohomology sheaf $H^i(\mathcal{F}^\bullet)$ of a complex $\mathcal{F}^\bullet \in \mathrm{D}^b_{\mathrm{coh}}(\mathrm{Inj}(\mathcal{O}_X))$ (though it may not be an injective $\mathcal{O}_X$-module!) and ask whether or not it is coherent. The equivalence $\mathrm{D}^b_{\mathrm{coh}}(\mathrm{Inj}(\mathcal{O}_X)) \xrightarrow{\simeq} \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$ says, more or less, that any "reasonable sheaf" admits an injective resolution.

The analogous assertions hold if one replaces injective modules with flat modules. Moreover, when X is non-singular, one can also use locally free modules.

7.3. **Triangulated Categories.** So far, we only know that the derived category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is an additive category (that is not abelian!), but it still has some additional structure – it is a triangulated category. There is a long list of axioms that define a triangulated category; instead of listing them, let us highlight the structures that are present.

Given a map $f\colon A^\bullet \to B^\bullet$ of cochain complexes, the *mapping cone* $C^\bullet(f)$ of f has terms $C^n(f) \coloneqq B^n \oplus A^{n+1}$ and the differential $d\colon B^n \oplus A^{n+1} \to B^{n+1} \oplus A^{n+2}$ is given by

$$d(b,a) = (d(b) + f(a), -d(a))).$$

The minus sign present in the second factor of the differential ensures that $C^\bullet(f)$ is a complex.

Directly from the definition, one can see that there is an exact sequence of cochain complexes

$$0 \longrightarrow B^\bullet \xrightarrow{u} C^\bullet(f) \xrightarrow{v} A^\bullet[1] \longrightarrow 0,$$

where $u(b) = (b,0)$ and $v(b,a) = a$. This can be extended by appending $f\colon A^\bullet \to B^\bullet$ on the left (and this becomes a complex after passing to the derived category). In addition, it can be extended to the right by

[17] If $\mathcal{F}^\bullet \in \mathrm{D}(\mathcal{O}_X)$, the *cohomology sheaf* $H^i(\mathcal{F}^\bullet)$ denotes the sheaf of $\mathcal{O}_X$-modules given by $\ker(d^i)/\mathrm{im}(d^{i-1})$, where $d^\bullet$ are the differentials of the complex $\mathcal{F}^\bullet$. The same construction can be made for any cochain complex with terms in an abelian category.

appending $f[1]\colon A^\bullet[1] \to B^\bullet[1]$. This can be continued, and this data is called a *triangle*. This data is often summarized in the diagram

$$\begin{tikzcd} A^\bullet \arrow[rr, "f"] & & B^\bullet \arrow[dl] \\ & C^\bullet(f) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

A *distinguished triangle* is a triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ that is isomorphic in $\mathrm{D}(\mathcal{A})$ to one coming from the above construction using the mapping cone. See [Wei94, §1.5] for more details.

A *triangulated category* is an additive category with a collection of distinguished triangles that satisfy certain axioms, e.g. any morphism can be completed to a distinguished triangle, the shift of a distinguished triangle is a distinguished triangle, an isomorphism gives rise a distinguished triangle. There are also more complicated axioms, e.g. the octahedral axiom, which is the analogue of the isomorphism $(A/B)/(B/C) \simeq A/B$ from an abelian category.

One can check that, given a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in the derived category $\mathrm{D}(\mathcal{A})$, then taking cohomology gives a long exact sequence

$$\ldots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow \ldots$$

This fact suggests that distinguished triangles are the analogue/replacement for short exact sequences in the derived category. This leads to the the derived/triangulated category analogue of the left and right exact functors in abelian categories.

An additive functor $F\colon \mathcal{C} \to \mathcal{D}$ between triangulated category is *exact* if F sends distinguished triangles to distinguished triangles. From the above remark, if $\mathcal{C}$ and $\mathcal{D}$ are the derived categories of abelian categories $\mathcal{A}$ and $\mathcal{B}$, then given a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$, applying the functor F gives another distinguished triangle

$$F(A^\bullet) \longrightarrow F(B^\bullet) \longrightarrow F(C^\bullet) \longrightarrow F(A^\bullet)[1]$$

and taking cohomology yields a long exact sequence

$$\ldots \longrightarrow H^i(F(A^\bullet)) \longrightarrow H^i(F(B^\bullet)) \longrightarrow H^i(F(C^\bullet)) \longrightarrow H^{i+1}(F(A^\bullet)) \longrightarrow \ldots$$

In particular, an exact functor between derived categories sends long exact sequences to long exact sequences (which meets our expectation of an exact functor!).

7.4. **Derived Functors.** We now return to the setting of $\mathrm{D}^b_{\mathrm{coh}}(\mathscr{O}_X)$ for a scheme X, and we will discuss the "6 functor formalism", i.e. the derived tensor product $\otimes^{\mathbb{L}}$, the derived Hom $\mathscr{H}$, pushforward, pullback, proper/exceptional pushforward, and proper/exceptional pullback. One key fact that will be used repeatedly is the following:

Fact 7.2. If X is a noetherian, integral, separated, regular scheme, then every coherent sheaf on X admits a finite length resolution by locally free sheaves.[18]

We will begin by discussing the usual pushforward functor. Recall that when defining the (higher) pushforward of a coherent sheaf classically (e.g. as in [Har77, III.8]), one takes an injective resolution, applies the pushforward functor and takes the cohomology of the resulting complex. This is aritificial, in the sense that it involves a choice of an injective resolution, but in the derived category it can be defined in a universal way. This will be discussed next time.

[18] By [Har77, Exercise III.6.8], every coherent sheaf $\mathcal{F}$ on X admits a locally free resolution. If $\mathrm{hd}(\mathcal{F})$ denotes the homological dimension of $\mathcal{F}$ (i.e. the minimal length of a finite locally free resolution), then it may be computed as $\mathrm{hd}(\mathcal{F}) = \sup_{x\in X} \mathrm{pd}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$, where $\mathrm{pd}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$ denotes the projective dimension of $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module. As $\mathcal{O}_{X,x}$ is a regular local ring and $\mathcal{F}_x$ is a finite $\mathcal{O}_{X,x}$-module, we have $\mathrm{pd}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \le \dim \mathcal{O}_{X,x}$ by [Sta17, Tag 00O7]; in particular, $\mathrm{hd}(\mathcal{F}) \le \sup_{x\in X} \dim \mathcal{O}_{X,x} = \dim X < +\infty$ by [Sta17, Tag 04MU], where $\dim X < +\infty$ follows from quasi-compactness.

8. September 28th (Lecture by Zili Zhang)

8.1. **Derived Functors (Continued).** If X is a scheme, the goal of today is to discuss the "6-functor formalism" of the derived functors on the bounded derived category $\mathrm{D}^b(\mathrm{Coh}(X))$ of coherent sheaves on X.

Let $f\colon X \to Y$ be a morphism of schemes that is quasi-compact and separated (in particular, pushing forward along f preserves quasi-coherence). The first goal is to define the derived pushforward

$$Rf_*\colon \mathrm{D}^+(\mathrm{QCoh}(X)) \longrightarrow \mathrm{D}^+(\mathrm{QCoh}(Y))$$

One might first ask: why is the derived functor Rf_* better than the higher pushforwards? Consider the following: if $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms of schemes as above and $\mathcal{F}$ is a coherent sheaf on X, then it is difficult to say much about $R^{i+j}(g \circ f)_* \mathcal{F}$ granted $R^j g_* \left(R^i f_* \mathcal{F}\right)$; one simply knows that there is an E_2-spectral sequence containing the $R^j g_* \left(R^i f_* \mathcal{F}\right)$ and abutting to the $R^{i+j}(g \circ f)_* \mathcal{F}$'s. However, in the derived category setting, there is a natural isomorphism

$$Rg_* \left(Rf_* \mathcal{F}\right) \xrightarrow{\simeq} R(g \circ f)_* \mathcal{F}\,.$$

That having been said, if one wants to actually compute one of these quantities, one needs to use the spectral sequence.

An important special case is when Z is a point, in which case pushing forward along g_* is the same as taking cohomology. Thus, the E_2-spectral sequence referred to above is just the Leray spectral sequence

$$H^i(Y, R^j f_* \mathcal{F}) \Longrightarrow H^{i+j}(X, \mathcal{F}).$$

In the derived category setting, this can be written as $R\Gamma(Y, Rf_* \mathcal{F}) \simeq R\Gamma(X, \mathcal{F})$.

One way to define Rf_* is as follows: the inclusion $\mathrm{InjQCoh}(X) \hookrightarrow \mathrm{QCoh}(X)$ induces an equivalence

$$\mathrm{D}^+_{\mathrm{qcoh}}(\mathrm{InjQCoh}(X)) \overset{\simeq}{\hookrightarrow} \mathrm{D}^+(\mathrm{QCoh}(X)),$$

where $\mathrm{InjQCoh}(X)$ is the full subcategory of injective quasi-coherent $\mathcal{O}_X$-modules, and $\mathrm{D}^+_{\mathrm{qcoh}}(\mathrm{InjQCoh}(X))$ is the full subcategory of $\mathrm{D}^+(\mathrm{InjQCoh}(X))$ consisting of those complexes whose cohomology sheaves are quasi-coherent in each degree and vanish in small enough degrees. This equivalence is fundamentally due to the fact that every quasi-coherent sheaf on X admits an injective resolution.

Using the isomorphism $\mathrm{D}^+_{\mathrm{qcoh}}(\mathrm{InjQCoh}(X)) \overset{\simeq}{\hookrightarrow} \mathrm{D}^+(\mathrm{QCoh}(X))$, it suffices to specify $Rf_*\mathcal{I}^\bullet$ for injective quasi-coherent sheaves $\mathcal{I}^\bullet$. As injective are acyclic for the pushforward functor, define this to be the complex

$$f_*\mathcal{I}^\bullet = \left(\ldots \longrightarrow f_*\mathcal{I}^{-1} \longrightarrow f_*\mathcal{I}^0 \longrightarrow f^*\mathcal{I}^1 \longrightarrow \ldots\right),$$

which is a complex that only lives in $\mathrm{D}^+(\mathrm{QCoh}(Y))$.

In the above definition of Rf_*, we are still implicitly choosing an injective resolution by using the isomorphism $\mathrm{D}^+_{\mathrm{qcoh}}(\mathrm{InjQCoh}(X)) \overset{\simeq}{\hookrightarrow} \mathrm{D}^+(\mathrm{QCoh}(X))$. One can instead make a "canonical choice" of a resolution, namely the Godement resolution (this is a resolution by flasque[19] sheaves as opposed to injective ones, but flasque sheaves are also acyclic with respect to the pushforward functor).

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules (or abelian groups) on X, the *Godement resolution* $G^\bullet(\mathcal{F})$ of $\mathcal{F}$ is constructed as follows:

- Set $G^0(\mathcal{F})$ to be the sheaf of "discontinuous sections" of $\mathcal{F}$, i.e. for any open set $U \subseteq X$,

$$G^0(\mathcal{F})(U) := \prod_{x \in U} \mathcal{F}_x\,.$$

  In addition, there is an obvious map $d^0\colon \mathcal{F} \longrightarrow G^0(\mathcal{F})$.
- For $n > 0$, set $G^n(\mathcal{F}) := G^0(\mathrm{coker}(d_{n-1}))$ and let $d^n\colon G^{n-1}(\mathcal{F}) \longrightarrow G^n(\mathcal{F})$ be the obvious map.

[19] If X is a topological space and $\mathcal{F}$ is a sheaf of sets on X, then $\mathcal{F}$ is *flasque* if every restriction map is surjective, i.e. for any open sets $U \subseteq V$, the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective. The adjective *flabby* is also often used in place of flasque.

It is not hard to check that $G^0(\mathcal{F})$ is a flasque sheaf (and hence $G^n(\mathcal{F})$ is flasque for all $n \geq 0$). The resulting complex

$$G^\bullet(\mathcal{F}) = \left(0 \longrightarrow \mathcal{F} \xrightarrow{d^0} G^0(\mathcal{F}) \xrightarrow{d^1} G^1(\mathcal{F}) \xrightarrow{d^2} \dots\right)$$

is a flasque resolution of $\mathcal{F}$. Thus, we can define

$$Rf_* \mathcal{F} := f_* \left(G^\bullet(\mathcal{F})\right).$$

In the above construction, $\mathcal{F}$ was a sheaf of $\mathcal{O}_X$-modules; however, the same process can be performed for a complex of sheaves of $\mathcal{O}_X$-modules, namely construct the Godement resolution of each term of the complex to get a bicomplex, and define the derived pushforward to be the pushforward of the totalization of this bicomplex.

There is a more functorial definition of derived functors, due to Deligne, which works in a far more general context. For a full treatment, see [Sta17, Tag 05S7], or [AGV71, Exposé XVII] for Deligne's original exposé. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive left-exact functor. We will define the right-derived functor RF on the derived category $\mathrm{D}(\mathcal{A})$.

We say that RF *is defined at* $X^\bullet \in \mathrm{D}(\mathcal{A})$ if the ind-object[20]

$$\left(X^\bullet \xrightarrow{\text{quasi-isomorphism}} X'^\bullet\right) \mapsto F(X'^\bullet)$$

is essentially constant. In this case, set

$$RF(X^\bullet) := \operatorname*{colim}_{(X^\bullet \to X'^\bullet)} F(X'^\bullet),$$

where the colimit is taken over all quasi-isomorphisms $X^\bullet \to X'^\bullet$ in $\mathrm{D}(\mathcal{A})$. The left-exactness of F guarantees that the zeroth term $R^0F(X^\bullet)$ of the complex $RF(X^\bullet)$ coincides with $F(X^\bullet)$.

Assume there is a collection $\mathcal{I}$ of objects of $\mathcal{A}$ satisfying:

(1) every object in $\mathcal{A}$ admits an injection into an object of $\mathcal{I}$;
(2) if $0 \to P \to Q \to R \to 0$ is an exact sequence in $\mathcal{A}$ such that $P, Q \in \mathcal{I}$, then $R \in \mathcal{I}$ and the sequence $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.

For example, if $\mathcal{A}$ is the category of $\mathcal{O}_X$-modules on a scheme X, then the collection of injective sheaves does not satisfy (1) and (2), but the collection of flasque sheaves does satisfy these two conditions.

Proposition 8.1. [Sta17, Tag 06XN] *If there is a collection $\mathcal{I}$ of objects in $\mathcal{A}$ satisfying (1) and (2), then RF is defined on every object of* $\mathrm{D}^+(\mathcal{A})$.

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[20] If $\mathcal{C}$ is a category, the *Ind-completion* $\mathrm{Ind}(\mathcal{C})$ of $\mathcal{C}$ is the minimal subcategory of the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ that contains the image of $\mathcal{C}$ under the Yoneda embedding, and is closed under filtered colimits (said differently, $\mathrm{Ind}(\mathcal{C})$ is the universal category with all filtered colimits that receives a functor from $\mathcal{C}$).

Let $\mathcal{A}$ be an abelian category and let $K(\mathcal{A})$ be the homotopy category of $\mathcal{A}$. Any functor $F\colon K(\mathcal{A}) \to \mathcal{C}$ induces a functor $r_F\colon K(\mathcal{A}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ given by

$$K(\mathcal{A}) \ni X^\bullet \mapsto \left(\mathcal{C} \ni Y \mapsto \operatorname*{colim}_{X^\bullet \xrightarrow{\simeq} X'^\bullet} \mathrm{Hom}_{\mathcal{C}}(Y, F(X'^\bullet))\right), \tag{$**$}$$

where the colimit runs over all quasi-isomorphisms $X^\bullet \xrightarrow{\simeq} X'^\bullet$ in $K(\mathcal{A})$. By the universal property of the Ind completion, the image of r_F lands in $\mathrm{Ind}(\mathcal{C})$. In fact, if $F = \mathrm{id}_{K(\mathcal{A})}$ is the identity functor on $K(\mathcal{A})$, then the essential image of $r_{\mathrm{id}_{K(\mathcal{A})}}$ is isomorphic to the derived category $\mathrm{D}(\mathcal{A})$.

Now, when attempting to construct the derived functor $RF\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ of $F\colon \mathcal{A} \to \mathcal{B}$, consider the composition of functors

$$K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B}) \xrightarrow{r_{\mathrm{id}_{K(\mathcal{B})}}} \mathrm{Ind}(\mathrm{D}(\mathcal{B}))$$

This gives rise to a functor $\mathrm{D}(\mathcal{A}) \to \mathrm{Ind}(\mathrm{D}(\mathcal{B}))$. If $X^\bullet \in \mathrm{D}(\mathcal{A})$, the condition that "$RF$ is defined at $X^\bullet$" (as defined above) guarantees that the image of $X^\bullet$ under this functor $\mathrm{D}(\mathcal{A}) \to \mathrm{Ind}(\mathrm{D}(\mathcal{B}))$ lands in (the Yoneda embedding of) $\mathrm{D}(\mathcal{B})$; if RF is defined at $X^\bullet$, then the functor on $\mathrm{D}(\mathcal{B})$ defined in ($**$) is representable by some $Z^\bullet \in \mathrm{D}(\mathcal{B})$, and we set $RF(X^\bullet) := Z^\bullet$.

For a full and very nice treatment of this construction, see [Hai14, §4].

The above story "dualizes" so that one can define the left-derived functor LF of a right-exact additive functor F between abelian categories.

Let us now specialize to the geometric setting: if $\mathcal{A}$ and $\mathcal{B}$ are both the category $\mathrm{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules and $\mathcal{I}$ is the collection of flasque sheaves, then Proposition 8.1 implies that the functors Rf_* and $R\Gamma$ are defined on the derived category $\mathrm{D}(\mathcal{O}_X)$. Similarly, taking $\mathcal{I}$ to be collection of flat sheaves, one gets that the derived tensor product $\otimes^{\mathbb{L}}$ is defined everywhere.

Granted the existence of the derived tensor product, we can construct the derived pullback. If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, recall that the pullback $f^*\mathcal{F}$ is defined as $f^*\mathcal{F} \coloneqq f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, where f^{-1} denotes the inverse image of abelian sheaves. Recall that f^{-1} is always exact; thus, if $\mathcal{F}^\bullet \in \mathrm{D}(\mathcal{O}_Y)$ is a complex of $\mathcal{O}_Y$-modules, then one can define the derived pullback $Lf^*\mathcal{F}^\bullet$ by the formula

$$Lf^*\mathcal{F}^\bullet \coloneqq f^{-1}\mathcal{F}^\bullet \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

As in the classical case, there is an adjunction between the derived pushforward and the derived pullback: if $\mathcal{F}^\bullet \in \mathrm{D}(\mathcal{O}_Y)$ and $\mathcal{G}^\bullet \in \mathrm{D}(\mathcal{O}_X)$, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(Lf^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet).$$

To prove this, replace $\mathcal{G}^\bullet$ by an injective resolution and $\mathcal{F}^\bullet$ by a flat resolution, and use the usual pushforward-pullback adjunction.

Now that we have constructed certain derived functors, we can revisit some classical theorems and discuss them in the derived setting. In the derived category, this becomes:

Theorem 8.2. [Projection Formula–Derived Version] *Let $f\colon X \to Y$ be a qcqs morphism of schemes. If $\mathcal{E}^\bullet \in \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_Y)$ and $\mathcal{F}^\bullet \in \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$, then the natural map*

$$Rf_*(\mathcal{F}^\bullet \otimes^{\mathbb{L}} Lf^*\mathcal{E}^\bullet) \longrightarrow Rf_*\mathcal{F}^\bullet \otimes^{\mathbb{L}} \mathcal{E}^\bullet$$

is an isomorphism.

Sketch. We sketch the proof with the additional assumption that $\mathcal{E}^\bullet$ admits a finite locally free resolution $\mathcal{P}^\bullet \to \mathcal{E}^\bullet$ (this holds e.g. if Y is regular by Fact 7.2). If one replaces $\mathcal{F}^\bullet$ by an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$, then

$$Rf_*(\mathcal{F}^\bullet \otimes^{\mathbb{L}} Lf^*\mathcal{E}^\bullet) = Rf_*(\mathcal{I}^\bullet \otimes^{\mathbb{L}} f^*\mathcal{P}^\bullet) = f_*(\mathcal{I}^\bullet \otimes f^*\mathcal{P}^\bullet),$$

where $f^*\mathcal{P}^\bullet$ is not derived, since it is a complex of locally free (hence flat) objects. One can then use the classical projection formula to conclude. □

There are other version of the derived projection formula that appear in the literature, e.g. one can allow $\mathcal{E}^\bullet$ to be a perfect object of the derived category; see [Sta17, Tag 0B54].

8.2. **Duality.** We will now promote Serre duality to the derived category and discuss a special case of Grothendieck duality. Recall the statement of (a special case of) Serre duality: if $\mathcal{F}$ is a coherent sheaf on a smooth projective variety X of dimension n over an algebraically closed field k, then the natural map

$$\mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X) \longrightarrow H^i(X, \mathcal{F})^\vee.$$

is an isomorphism for all $i \geq 0$. Here, $H^i(X,\mathcal{F})^\vee = \mathrm{Hom}_k(H^i(X,\mathcal{F}), k)$. This is the statement that we will translate into the derived category language.

From Example 7.1, we know what the derived-category interpretation of the Ext group is, namely

$$\mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X) = \mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}(X))}(\mathcal{F}, \omega_X[n-i]) = \mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}(X))}(\mathcal{F}[i], \omega_X[n]).$$

In addition, there are natural isomorphisms

$$H^i(X, \mathcal{F}) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}) = \mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}(X))}(\mathcal{O}_X, \mathcal{F}[i]),$$

for all $i \geq 0$, and hence we can interpret Serre duality as an assertion about a pairing between the Hom groups $\mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}(X))}(\mathcal{F}[i], \omega_X[n])$ and $\mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}(X))}(\mathcal{O}_X, \mathcal{F}[i])$, followed by a trace map.

If we work in the derived category, we can replace $\mathcal{F}[i]$ with any complex, rather than just a coherent sheaf placed in a fixed degree; this leads to the following special case of Grothendieck duality.

Theorem 8.3. [Derived Serre Duality] *Let X be a smooth projective variety of dimension n over an algebraically closed field k. If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$, then the natural map*

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \omega_X[n]) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{G}^\bullet, \mathcal{F}^\bullet)^\vee.$$

is an isomorphism for all $i \geq 0$.

A priori, the tensor product appearing in Theorem 8.3 ought to be the derived tensor product $\mathcal{G}^\bullet \otimes^{\mathbb{L}}_{\mathcal{O}_X} \omega_X[n]$; however, as X is smooth over k, ω_X is locally free so the derived tensor product becomes the usual tensor product of complexes. Furthermore, just as for the classical Serre duality, Theorem 8.3 holds also for mildly singular schemes, provided the canonical sheaf is replaced with the dualizing sheaf.

Let us now proceed to a generalization of derived Serre duality, known as Grothendieck duality. The statement involves a functor $f^!$, associated to a proper map $f\colon X \to Y$ of schemes, that we have yet to introduce; Grothendieck duality asserts that $f^!$ is right adjoint to the derived pushforward Rf_* (previously we had defined its left adjoint Lf^*). The functor $f^!$ lives only in the derived category, that is, it is not the derived functor of any half-exact functor on the original abelian category. Nonetheless, $f^!$ is often written $Rf^!$ as an abuse of notation.

If $f\colon X \to Y$ is a proper map of schemes, the definition of $f^!\colon \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_Y) \to \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$, often called the exceptional pullback, is quite involved; see [Har66]. We concern ourselves only with a the case where X and Y are smooth projective varieties over a field k (so any morphism between X and Y is automatically proper). In this setting, $f^!$ admits a simple description, originally appearing in [BK89].

Theorem 8.4. *Let X and Y be smooth schemes over a field k and let $f\colon X \to Y$ be a proper morphism. If $\mathcal{G}^\bullet \in \mathrm{D}^b_{\mathrm{coh}}(\mathcal{O}_Y)$, then there is a natural isomorphism*

$$f^!\,\mathcal{G}^\bullet \simeq \omega_X[\dim X] \otimes Lf^*\left(\mathcal{G}^\bullet \otimes \omega_Y^{-1}[-\dim Y]\right).$$

For a full treatment of Grothendieck duality for smooth schemes over a field, see [Huy06, §3.4]; for the story in complete generality, see [Har66].

The flat base change theorem Proposition 4.4 can also be promoted to the setting of derived categories.

Theorem 8.5. [Flat Base Change–Derived Version] *Consider the diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of morphisms of schemes and assume that f is quasi-compact and separated, and g is flat. If $\mathcal{F}^\bullet \in \mathrm{D}^b(\mathrm{QCoh}(X))$, then the natural morphism

$$g^* Rf_*\,\mathcal{F}^\bullet \longrightarrow Rf'_* g'^*\,\mathcal{F}^\bullet$$

is an isomorphism.

Note that we use g^* as opposed to Lg^* in Theorem 8.5 since g is assumed to be flat. In fact, we may weaken the flatness hypothesis on g so that the statement of Theorem 8.5 still holds (at the expense of replacing g^* with Lg^*). The correct conditions that replaces flatness is that f and g are *Tor independent* over Y: for every $x \in X$ and $y' \in Y'$ such that $f(x) = y = g(y')$, the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y',y'}$ are *Tor independent* as modules over $\mathcal{O}_{Y,y}$, i.e. $\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0$ for all $i > 0$. For a proof of Theorem 8.5 under these weakened hypotheses[21], see [Sta17, Tag 08IB].

[21] In class, it was claimed that one needs the assumption that $\mathcal{F} \in \mathrm{D}^b(\mathrm{QCoh}(X))$ has *Tor amplitude* in some interval $[m,n] \subseteq \mathbf{Z}$ as a complex of $f^{-1}\mathcal{O}_Y$-modules; that is, for any $\mathcal{G}^\bullet \in \mathrm{D}^-(f^{-1}\mathcal{O}_Y)$ and any $i \in [m,n]$, $\mathrm{Tor}_i^{f^{-1}\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = 0$. Based on the statement of [Sta17, Tag 08IB], this assumption does not seem necessary. This weaker version appears as [BGI67, IV, Theorem 3.1.0].

9. October 3rd

If one is interested in the derived category framework covered in the last two lectures, one should read the (very short) paper of Beilinson [BGG78], which is one of the most influential on the subject of derived categories. Now, we return to our discussion of line bundles on abelian varieties, which we return to with the aim of moving towards proving the projectivity of abelian varieties.

9.1. **The Theorem of the Cube (Continued).**

Theorem 9.1. *Let S be a connected noetherian scheme, and let $X \to S$ and $Y \to S$ be proper flat morphisms with geometrically integral fibres. Let $L \in \operatorname{Pic}(X \times_S Y)$. Assume*

(1) there exist sections $e_X \in X(S)$ and $e_Y \in Y(S)$ such that $L|_{\{e_X\}\times_S Y}$ and $L|_{X\times_S\{e_Y\}}$ are trivial;
(2) there exists a point $s \in S$ such that $L|_{X_s\times_s Y_s}$ is trivial.

Then, L is pulled back from S, i.e. there exists a line bundle $M \in \operatorname{Pic}(S)$ such that L is isomorphic to the pullback of M along $X \times_S Y \to S$.

We saw in Remark 6.6 an example of a line bundle on a product of elliptic curves that shows the first hypothesis of Theorem 9.1 is not enough.

Proof. The proof is deformation-theoretic and follows a standard method to analyze line bundles in families: we consider a trivialization in an infinitesimal neighbourhood of s and extend this trivialization on fatter and fatter neighbourhoods.

Let $\pi\colon P \coloneqq X \times_S Y \to S$. Let $Z \subseteq S$ the universal closed subscheme such that L_{P_Z} is pulled back from Z, as constructed in the Seesaw Theorem (Theorem 6.3). Our goal is to show that $Z = S$. As Z is non-empty (because $s \in Z$), $Z \subseteq S$ is closed, and S is connected, it suffices to show that Z is closed under generalization. We may assume $S = \operatorname{Spec}(R)$, where $(R, \mathfrak{m})$ is a noetherian local ring, and $Z = \operatorname{Spec}(R/I)$, where $I \subseteq R$ is a non-unit ideal (since Z is non-empty). In fact, we could take R to be a dvr. The goal is now to show that $I = 0$, and hence $Z = S$.

If $I \neq 0$, then there exists an ideal $J \subseteq I$ such that $I/J \simeq k = R/\mathfrak{m}$ (e.g. J could be the preimage of any hyperplane in $I/\mathfrak{m}I$). If $W = \operatorname{Spec}(R/J) \subseteq S$, then $Z \subsetneq W \subseteq S$, where $Z \neq W$ follows since the difference between the ideals I and J is nonzero by construction. Now, we must leverage the fact that Z is maximal with some property, and hence W cannot also have this property.

As L_{P_Z} is pulled back from Z, it must be the pullback of the trivial line bundle (and hence, itself trivial) because any line bundle on the spectrum of a local ring is trivial. We shall check that L_{P_W} is trivial, which contradicts the maximality of Z. To do so, consider the following short exact sequence on Z:

$$0 \to k \to R/J \to R/I \to 0 \tag{9.1}$$

Pulling (9.1) back to P and tensoring with L gives a short exact sequence

$$0 \to L_{P_s} \to L_{P_W} \to L_{P_Z} \to 0 \tag{9.2}$$

on P, where exactness follows from the flatness of π, and the fact that tensoring with any line bundle is exact. Choose a trivialization s of L_{P_Z}, i.e. a section of L_{P_Z} such that $L_{P_Z} \simeq \mathcal{O}_{P_Z} \cdot s$. Observe that L_{P_W} is trivial iff $s \in H^0(P_Z, L_{P_Z})$ lifts to $H^0(P_W, L_{P_W})$. Thus, if $\delta\colon H^0(P_Z, L_{P_Z}) \to H^1(P_s, L_{P_s})$ is the boundary map, it suffices to show that $\delta(s) = 0$. The same holds in the other factor, so $\delta(s)$ maps to zero under the map

$$H^1(P_s, L_{P_s}) \longrightarrow H^1(X_s, L_{X_s}) \times H^1(Y_s, L_{Y_s}).$$

As $L|_{P_s} \simeq \mathcal{O}_{P_s}$ is the trivial line bundle and the fibres are geometrically integral (hence the global sections of the structure sheaf of each fibre are just k), this map is bijective by the Künneth formula. Thus, $\delta(s) = 0$, which completes the proof. $\square$

Theorem 9.1 is the key ingredient necessary to deduce the Theorem of the Cube, which we state below in the case where the base is a point (as this will be the only case where the theorem is applied).

Corollary 9.2. *Let k be a field. Let X and Y be proper, geometrically integral schemes of finite type over k and let Z be a connected scheme of finite type over k. Let $L \in \mathrm{Pic}(X \times Y \times Z)$. If there are points $x \in X(k)$, $y \in Y(k)$, and $z \in Z(k)$ such that L is trivial on each of $X \times Y \times \{z\}$, $X \times \{y\} \times Z$, and $\{x\} \times Y \times Z$, then L is trivial.*

The proof of Corollary 9.2, granted Theorem 9.1, was previously discussed (see Corollary 6.8). In addition, we deduced the following about the behaviour of line bundles under pullback by multiplication-by-n.

Corollary 9.3. *If A is an abelian variety over k and L is a line bundle on A, then there is an isomorphism*

$$[n]^*L \simeq L^{\frac{n^2+n}{2}} \otimes [-1]^*L^{\frac{n^2-n}{2}}.$$

These tools can now be used to prove the Theorem of the Square, which describes the behaviour of a line bundle under pullback by translations. It's unclear why it has its name.

Corollary 9.4. [Theorem of the Square] *If A is an abelian S-scheme and $L \in \mathrm{Pic}(A)$ is a line bundle, then for any two points $x, y \in A(S)$, there is an isomorphism*

$$t_x^*(L) \otimes t_y^*(L) \simeq t_{x+y}^*(L) \otimes L^{-1}$$

up to line bundles pulled back from S, where t_* is the translation map on A.

The isomorphism appearing in Corollary 9.4 only holds up to pullbacks of line bundles from S. For the precise correction term, see [Con15, Corollary 3.2.3].

Proof. Write c_x and c_y for the constant maps $A \to A$ with values x and y, i.e. the compositions $A \to S \xrightarrow{x} A$ and $A \to S \xrightarrow{y} A$. Similarly, let c_e denote the constant map at the identity point e. Applying Lemma 6.9 with $f = c_x$, $g = c_y$, and $h = c_e$, we see that

$$t_{x+y}^*(L) = c_{x+y}^*(L) \otimes t_y^*(L) \otimes t_x^*(L) \otimes c_x^*(L^{-1}) \otimes c_y^*(L^{-1}) \otimes L^{-1} \otimes c_e^*(L)$$

Ignoring the pullbacks from S, we obtain $t_{x+y}^*(L) \simeq t_y^*(L) \otimes t_x^*(L) \otimes L^{-1}$. □

The statement of Corollary 9.4 may seem strange, because it is not very symmetric. This asymmetry is explained by the observation below.

Remark 9.5. If A is an abelian S-scheme and $L \in \mathrm{Pic}(A)$ is fixed, Corollary 9.4 can be formulated as the assertion that the map $A \to \underline{\mathrm{Pic}}(A)$, given by

$$x \mapsto t_x^*(L) \otimes L^{-1},$$

is a homomorphism of presheaves. This map is generally denoted ϕ_L, and we will later see that its kernel is the dual abelian scheme $\mathrm{Pic}^0(A)$.

9.2. **The Mumford Bundle.** In this section, we attempt to characterize the ampleness of a line bundle on an abelian variety in terms of other data, with the eventual goal of proving that abelian varieties are projective. A reference for this section is [EVdGM].

Definition 9.6. If A/k is an abelian variety and $L \in \mathrm{Pic}(A)$, then the *Mumford bundle* of L is

$$\Lambda(L) \coloneqq m^*(L) \otimes \mathrm{pr}_1^*(L^{-1}) \otimes \mathrm{pr}_2^*(L^{-1}) \in \mathrm{Pic}(A \times A).$$

Here, we think of the product $A \times A$ as a family of abelian varieties, indexed by A. Applying the Seesaw Theorem to the first projection $\mathrm{pr}_1 \colon A \times A \to A$, we can construct the maximal closed subscheme $K(L) \subseteq A$ of A such that $\Lambda(L)|_{K(L) \times A}$ is pulled back from $K(L)$. If one is interested only in $K(L)$, then one could ignore the $\mathrm{pr}_1^*(L^{-1})$ factor appearing in the definition of $\Lambda(L)$, but it is better to leave it for normalization purposes, as we will see later.

Fix a morphism $x \colon T \to A$ from a k-scheme T. One can ask: when does this map factor through $K(L)$? There is, of course, the interpretation of $K(L)$ coming from Theorem 6.3, but there is also a much more concrete description. If (x, id) is defined as in the diagram

$$\begin{tikzcd} T \times A \arrow[r, "(x,\mathrm{id})"] \arrow[d, "\mathrm{pr}_1"'] & A \times A \arrow[d, "\mathrm{pr}_2"] \\ T \arrow[r, "x"] & A \end{tikzcd}$$

then the pullback of $\Lambda(L)$ along (x, id) is $t_x^*(L_T) \otimes \mathrm{pr}_1^* x^*(L^{-1}) \otimes L_T^{-1}$, where L_T denotes the pullback of L to T along x. Here, t_x is thought of as an operation on the abelian T-scheme $T \times A$.

Corollary 9.7. *A map $x\colon T \to A$ factors through $K(L)$ exactly when $t_x^*(L_T) \otimes L_T^{-1}$ is pulled back from T.*

Thus, Corollary 9.7 asserts that $K(L)$ describes the points where the line bundle L is translation invariant. In particular, the underlying point set of $K(L)$ is

$$|K(L)| = \{x \in A\colon t_x^*(L) \simeq L\},$$

and $|K(L)|$ has been equipped with a fancy scheme structure from coming from Theorem 6.3.

In addition, this construction $L \mapsto K(L)$ gives a mechanism to product abelian subvarieties of A, as described in Lemma 9.8 and Corollary 9.9 below.

Lemma 9.8. *$K(L)$ is a subgroup scheme of A.*

Proof. This is an easy application of Corollary 9.4, which we leave as an exercise. □

Corollary 9.9. *If k is an algebraically closed field, then*

(1) $B := (K(L)^\circ)_{\mathrm{red}}$, where $K(L)^\circ$ is the connected component of $K(L)$ containing the identity[22]*;*

(2) $L|_B \otimes [-1]^* L|_B$ is trivial.

In particular, if k is any field and L is ample, then $K(L)$ is finite (i.e. the structure map $K(L) \to \mathrm{Spec}(k)$ is finite).

Later, we will prove the converse to this finiteness condition.

Proof. For (1), note that $(K(L)^\circ)_{\mathrm{red}}$ is a proper, reduced, connected group scheme over the algebraically closed field k, hence it is smooth (indeed, if a scheme over an algebraically closed field is reduced, then the smooth locus is non-empty; the group structure can then be used to "move the smooth locus around" to cover the whole scheme). Thus, $(K(L)^\circ)_{\mathrm{red}}$ is an abelian variety (this gives us a cheap way of producing abelian subvarieties of an abelian variety).

For (2), set $M = L|_B \in \mathrm{Pic}(B)$ and consider the following two claims:

- $\Lambda(M) = \Lambda(L)|_{B\times B}$;
- $\Lambda(L)|_{K(L)\times A}$ is trivial.

The first claim is obvious. To see the second claim, we know that $\Lambda(L)|_{K(L)\times A}$ is pulled back from $K(L)$, so it suffices to check triviality after pullback along a section $\sigma\colon K(L) \to K(L) \times A$ of the first projection. There is an obvious section σ that we can use, namely (id, e): compute that

$$\sigma^*\Lambda(L)|_{K(L)\times A} = i^*(L) \otimes i^*(L^{-1}) \otimes c_e^*(L^{-1}),$$

where $i\colon K(L) \hookrightarrow A$ is the inclusion. As we are working over a field, the $c_e^*(L^{-1})$ term is trivial, and hence $\sigma^*\Lambda(L)|_{K(L)\times A}$ is trivial.

Now, the first claim implies that $\Lambda(L)|_{B\times B} \simeq \Lambda(M)$, and thus $\Lambda(L)|_{B\times B}$ is trivial by the second claim. Therefore, $\Lambda(M)$ is trivial. Pulling back along the section $\tau = (\mathrm{id}, -\mathrm{id})\colon B \to B \times B$ gives

$$\tau^*(M) = \tau^* m^*(M) \otimes \tau^* \mathrm{pr}_1^*(M^{-1}) \otimes \tau^* \mathrm{pr}_2^*(M^{-1}) = M \otimes [-1]^* M,$$

and this gives the claim, since $\tau^*(M)$ is trivial.

[22] What exactly is the difference between B and $K(L)$? As $K(L)$ is proper, it is quasi-compact, and so we have just extracted one of its finitely-many connected components. In particular, $K(L)$ is finite iff $B = 0$.

It remains to show that over any field k, if L is ample, then $K(L)$ is finite. By passing base changing to the algebraic closure $\overline{k}$ and using the fact that the formation of $K(L)$ commutes with base change, we may assume that k is algebraically closed. Note that showing $K(L)$ is finite is equivalent to showing that $B = 0$, which is in turn equivalent to the assertion that $\dim B = 0$, since B is an abelian variety. Now, L being ample implies that M is ample, since M is simply the restriction of L to B. (because M is just the restriction to B), then In addition, it follows that $[-1]^*M$ is ample, because $[-1]$ is an automorphism. Thus, $M \otimes [-1]^*M$ is ample, but we also shown that it is trivial, hence we must have $\dim B = 0$. $\square$

Next time, we'll come back and try to use this to prove that abelian varieties are projective.

10. October 5th

10.1. **The Mumford Bundle (Continued).** Last time, we defined the Mumford bundle $\Lambda(L)$ associated to a line bundle L on an abelian variety, as well as the group scheme $K(L)$. To contextualize these constructions, we can discuss the relations with the Picard functor, as defined below.

Definition 10.1. If A is an abelian S-scheme, the *Picard functor* of A is the functor[23] $\operatorname{Pic}_{A/S} \colon (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Ab}$ given by

$$(T \to S) \mapsto \operatorname{Pic}(A_T)/\operatorname{Pic}(T).$$

The Picard functor is intimately related to the constructions from last class: observe that if $L \in \operatorname{Pic}(A)$, there is a map $\phi_L \colon A \to \operatorname{Pic}_{A/S}$ of presheaves which, to a T-valued point $x \colon T \to A$ of A, associates the class of the line bundle $\phi_L(x) := t_x^*(L_T) \otimes L_T^{-1}$ on $A \times_S T$ (here, we denote $L_T = \mathrm{pr}_1^*(L)$ on $A \times_S T$). In this context, the Theorem of the Square Corollary 9.4 asserts that ϕ_L is a homomorphism.

Moreover, the group scheme $K(L)$, as defined via $\Lambda(L)$ last time, is canonically isomorphic to $\ker(\phi_L)$ (where the kernel is taken at the level of presheaves). This is a reformulation of the assertion that a map $x \colon T \to A$ factors through $\ker(\phi_L)$ iff $t_x^*(L_T) \otimes L_T^{-1}$ is pulled back from T, because $K(L)$ admits the same characterization, as shown in Corollary 9.7.

Remark 10.2. The upshot of the Mumford bundle construction is that it guarantees that $K(L) = \ker(\phi_L)$ is a scheme, as opposed to just a presheaf. If one knew that the Picard functor $\operatorname{Pic}_{A/S}$ was representable, then it would immediately that $\ker(\phi_L)$ is a closed subgroup scheme of A. This is the approach taken in [Con15]. We will eventually prove that $\operatorname{Pic}_{A/S}$ is representable by a separated group scheme such that the connected component of the identity is an abelian scheme.

Finally, we showed in Corollary 9.9 that if L is ample, then $K(L)$ is finite as a scheme over S. The goal of today's class is to obtain a result in the opposite direction, i.e. we would like to produce ample line bundles from some group-theoretic data.

The literal converse to Corollary 9.9 is false for trivial reasons: observe that

$$\phi_{L^{-1}}(x) = t_x^*(L_T^{-1}) \otimes L_T = -\phi_L(x),$$

where $-\phi_L$ denotes inversion in the group law of the Picard scheme. It follows that $K(L) = K(L^{-1})$. Thus, the finiteness of $K(L)$ cannot characterize the ampleness L, but the obvious extra condition to demand is effectivity.

10.2. **Projectivity of Abelian Varieties.** If k is a field, the goal of this section is to show that an abelian variety over k is projective. To simplify our arguments, we will frequently employ the following fact.

Exercise 10.3. If X is a scheme over k, then X is projective over k iff $X_{\overline{k}}$ is projective over $\overline{k}$.

For a careful proof of Exercise 10.3, see [Con15, Proposition 3.4.2]. We now assume that $k = \overline{k}$ in the sequel.

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[23]The Picard functor as defined in Definition 10.1 is only a presheaf on the category $(\mathrm{Sch}/S)^{\mathrm{op}}$, but often the term 'Picard functor' refers to the sheafification of this presheaf with respect to the fppf topology on (Sch/S), as in [Sta17, 09BL].

Proposition 10.4. *Let A be an abelian variety over k and let $f\colon A \to Y$ be a morphism of varieties over k. For each closed point $a \in A(k)$, set $F_a \coloneqq \big(f^{-1}(f(a))\big)^{\circ}_{\mathrm{red}}$, where $\big(f^{-1}(f(a))\big)^{\circ}$ denotes the connected component of $f^{-1}(f(a))$ containing a. Then, F_e is an abelian subvariety, and $F_a = a + F_e = t_a(F_e)$ for all $a \in A(k)$.*

This remarkable result seems to have first appeared as [EVdGM, Proposition 2.20].

Proof. The proof is based on the Rigidity Lemma (Corollary 2.2). Consider the map $\phi \coloneqq f \circ m\colon A \times F_a \to Y$, where we think of $A \times F_a$ as being fibered over A via the first projection $\mathrm{pr}_1\colon A \times F_a \to A$. The fibre $\mathrm{pr}_1^{-1}(e)$ is collapsed onto $f(a)$ by ϕ, so Corollary 2.2 implies that all of $A \times F_a$ collapses, i.e. ϕ factors as

$$\begin{tikzcd} A \times F_a \arrow[r, "\phi"] \arrow[d, "\mathrm{pr}_1"] & Y \\ A \arrow[ur, dashed, "\exists\overline{\phi}"'] & \end{tikzcd}$$

If $\sigma\colon A \to A \times F_a$ denotes the section of pr_1 given by $b \mapsto (b, a)$. then we have $\overline{\phi}(b) = \phi(b, a) = f(b + a)$ by the commutativity of the diagram.

Now, observe that

$$f(b - a + F_a) = \phi(b - a, F_a) = \overline{\phi}(b - a) = f(b - a + a) = f(b).$$

In particular, taking $a = e$ gives that $f(b + F_e) = f(b)$, so $b + F_e \subseteq F_b$. Similarly, taking $b = e$ implies that $f(-a + F_a) = f(e)$, so $-a + F_a \subseteq F_e$ and hence $F_a \subseteq a + F_e$. Combining these two inclusions gives $F_a = a + F_e$ for all $a \in A(k)$, as required.

It remains to verify that F_e is an abelian subvariety, i.e. $F_e \subseteq A$ is closed under addition. However, this follows from the work done above: if $b \in F_e$, then $b + F_e \subseteq F_b = F_e$ by the above calculations. □

In the proof of Proposition 10.4, we use that $k = \overline{k}$ in order to argue only at the level of closed points; however, it does not seem that the argument fundamentally needs this requirement.

Intuitively, Proposition 10.4 asserts that maps out of abelian varieties are essentially just quotient maps by abelian subvarieties. This is the 'dual' idea to the fact that a map from a projective variety to an abelian variety factors into a map between abelian varieties and an intrinsic map (namely, the map from the projective variety to its Albanese).

We now have to mechanisms to construct abelian subvarieties of a given abelian variety, which we will play off one another to produce an ample line bundle on the given abelian variety.

Proposition 10.5. *If A is an abelian variety over k and $L = \mathcal{O}_A(D)$ for an effective Cartier divisor $D \subseteq A$, then L is semi-ample*[24]*. In fact, $L^{\otimes 2}$ is globally-generated.*

The conclusion of Proposition 10.5 is totally false on a general variety: for example, blow up a closed point on a projective surface, then the exceptional divisor is effective but not semi-ample.

Proof. Fix $a \in A(k)$. We must show that there exists an effective divisor $E \in |2D|$ such that $a \notin E$. Consider the dense open subset $U \coloneqq (A \backslash D) + a$ of A. Then, $U \cap [-1]^*U$ is also a dense open subset of A, because A is irreducible. Pick $b \in U \cap [-1]^*U$ and notice that

- as $b \in U$, $b + a \in A \backslash D$;
- as $b \in [-1]^*U$, $-b \in U$ and hence $a - b \in A \backslash D$.

Thus, $a \notin b + D$ and $a \notin -b + D$; in particular, $a \notin t_{-b}(D) \cup t_b(D)$. If $E \coloneqq t_b(D) + t_{-b}(D)$, then E is an effective divisor on A and it is linearly equivalent to $2D$ by Corollary 9.4. □

[24] A line bundle L is *semi-ample* if some power of L is globally-generated.

Remark 10.6. If A is an abelian variety over k and $L = \mathcal{O}_A(D)$ is an effective line bundle on A, then Proposition 10.5 implies that that there is a morphism

$$f\colon A \to \mathbf{P} \coloneqq \mathbf{P}(H^0(A, L^{\otimes 2})^\vee)$$

such that $f^*\mathcal{O}_{\mathbf{P}}(1) = L^{\otimes 2}$. By Proposition 10.4, one can construct the abelian subvariety F_e of A as the connected component of the identity e in the fibre $f^{-1}(f(e))$, equipped with the reduced subscheme structure. On the other hand, one can construct the closed subgroup scheme $K(L) \subseteq A$ closed subgroup scheme, and from this one produces the abelian subvariety $B = (K(L)^\circ)_{\mathrm{red}}$ of A. That is, we have two possibly different abelian subvarieties of A attached to L. We claim that $F_e = B$.

To see the inclusion $F_e \subseteq B$, it suffices to show that $F_e \subseteq K(L)$. As both subschemes are reduced, it suffices to check this inclusion at the level of closed points. For fixed $x \in F_e(k)$, we must show that $t_x^*(L) \simeq L$; in fact, we will show something stronger, namely that there is an equality at the level of divisors. Since $x \in F_e$, we know that f is equivariant for the translation by x, i.e. $f \circ t_x = f$. If $s \in H^0(A, L)$ denotes the section of L corresponding to the divisor D, then $s^2 \in H^0(A, L^{\otimes 2})$ corresponds to $2D$, and so there is a hyperplane $H \subseteq \mathbf{P}$ such that $2D = f^{-1}(H)$. As $f \circ t_x = f$, we have $t_x^{-1}(2D) = 2D$ as subschemes of A, and thus $t_x^{-1}(D) = D$ as subschemes or as Cartier divisors. By passing to the corresponding line bundles, we see that $t_x^*(L) = t_x^*(\mathcal{O}_A(D)) \simeq \mathcal{O}_A(D) = L$, as required.

For the opposite inclusion $B \subseteq F_e$, set $M = L|_B$. As f is defined by the line bundle $L^{\otimes 2}$, it is enough to show that $L^{\otimes 2}|_B = M^{\otimes 2}$ is trivial. By Corollary 9.9, $M \otimes [-1]^*M$ is trivial, hence $M^{-1} \simeq [-1]^*M$. As M is globally-generated, so too is $[-1]^*M$. As B is irreducible (indeed, it is an abelian variety), and the line bundles M and M^{-1} both have sections, we have shown that $M \simeq \mathcal{O}_B$ by Lemma 5.7.

Notice that for both inclusions, we show a statement that is slightly stronger than is necessary. This has to do with the fact that we have taken the reduced structure on (the underlying sets of) B and F_e.

Remark 10.7. The proof of the claim in Remark 10.6 gives the following stronger statement:

$$F_e(k) \subseteq \mathrm{H}(D) \coloneqq \{x \in A(k) \colon t_x^{-1}(D) = D \text{ as divisors}\},$$

where the notation is as in Remark 10.6. In particular, we have a sequence of inclusions

$$B(k) = K(L)^\circ(k) \subseteq F_e(k) \subseteq \mathrm{H}(D).$$

This fact will be useful in the proof of projectivity of abelian varieties.

Proposition 10.8. *If A is an abelian variety over k and L is an effective line bundle such that $K(L)$ is finite, then L is ample.*

This is the promised "converse" to Corollary 9.9.

Proof. The line bundle $L^{\otimes 2}$ is globally-generated by Proposition 10.5, so there is a morphism $f\colon A \to \mathbf{P} \coloneqq \mathbf{P}(H^0(A, L^{\otimes 2})^\vee)$ such that $f^*\mathcal{O}_{\mathbf{P}}(1) = L^{\otimes 2}$. As $\mathcal{O}_{\mathbf{P}}(1)$ is ample, it suffices to show that f is finite, since the pullback of an ample line bundle along a finite map is again ample[25]. As A and $\mathbf{P}$ are both proper over k, it suffices to show that f has finite fibres (since a proper quasi-finite morphism is finite). By Proposition 10.4, it suffices to show that F_e is finite (as a scheme over k). The proof of the claim in Remark 10.6 asserts that $F_e \subseteq K(L)$ and $K(L)$ is finite by hypothesis, hence F_e is finite. $\square$

The tools are now in place to prove the projectivity of abelian varieties, as we only need to produce a line bundle as in the statement of Proposition 10.8.

Theorem 10.9. *Abelian varieties are projective.*

[25]Let $f\colon X \to Y$ be a finite morphism and let L be an ample line bundle on Y. By the cohomological criterion for ampleness ([Har77, Proposition III.5.3]), we must show that for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 \in \mathbf{Z}_{\geq 0}$ such that for all $i > 0$ and all $n \geq n_0$, $H^i(X, \mathcal{F} \otimes (f^*L)^{\otimes n}) = 0$. As f is finite, we have $R^i f_*(\mathcal{F} \otimes f^*L^{\otimes n}) = 0$ for all $i > 0$. Thus, combining the Leray spectral sequence $H^p(Y, R^q f_*(\mathcal{F} \otimes f^*L^{\otimes n})) \Rightarrow H^{p+q}(X, \mathcal{F} \otimes f^*L^{\otimes n})$ with the corresponding cohomological criterion for ampleness of L on Y, one can see that f^*L is ample.

There are relative versions of the projectivity of abelian varieties (see [Ray70, XI.1.4]), but they involve subtle distinctions between schemes and algebraic spaces, so we will ignore these more general results.

The proof of Theorem 10.9 uses the following general result, in a crucial way.

Lemma 10.10. *If X is a noetherian separated scheme and $U \subseteq X$ is a dense affine open, then each generic point of $X \backslash U$ has codimension-1 in X; in particular, $X \backslash U$ is a (Weil) divisor.*

Proof. For simplicity, we will prove the lemma under the added hypothesis that X is normal (for the general case, see [Sta17, Tag 0BCU]). Pick a generic point $x \in X \backslash U$ and consider the diagram

$$\begin{tikzcd} U_x \arrow[r, hook] \arrow[d] & \operatorname{Spec}(\mathcal{O}_{X,x}) = X_x \arrow[d] \\ U \arrow[r, hook] & X \end{tikzcd}$$

As x is a generic point, we have $X_x \backslash \{x\} = U_x$. If $\operatorname{codim}(x, X) \geq 2$, then $H^0(X_x, \mathcal{O}_{X,x}) \simeq H^0(U_x, \mathcal{O}_{U_x})$ by the normality of X. As X is separated, U_x is affine, and hence the isomorphism $H^0(X_x, \mathcal{O}_{X_x}) \simeq H^0(U_x, \mathcal{O}_{U_x})$ implies that $U_x = X_x$; however, this is impossible, since x lies in the complement $X_x \backslash U_x$. If $\operatorname{codim}(x, X) = 0$, then $\mathcal{O}_{X,x}$ is a field; in particular, U_x is empty, which contradicts the density of U in X. Thus, $\operatorname{codim}(x, X) = 1$. □

Proof of Theorem 10.9. Choose an affine open subset $U \subseteq A$ containing the identity e. By Lemma 10.10, the complement $D \coloneqq A \backslash U$ of U is a Cartier divisor. The line bundle $L \coloneqq \mathcal{O}_A(D)$ is thus effective by construction. Moreover, by Remark 10.7, it suffices to show that $\mathrm{H}(D)$ is contained in a non-empty affine open subset of A (indeed, if so, then $K(L)^\circ$ is a proper over k and it is contained in a non-empty affine variety over k, hence it is finite). In fact, we claim that $\mathrm{H}(D) \subseteq U$. If $x \in \mathrm{H}(D)$, then $t_x(U) = U$, which implies that $x = x + e \in U$. This completes the proof. □

11. October 10th

11.1. **Projectivity of Abelian Varieties (Continued).** Last time, we proved that abelian varieties are projective, following [EVdGM]. One of the key ingredients was Proposition 10.4, which asserted that a map out of an abelian variety is just collapsing an abelian subvariety. In fact, this tool can be used to give a simpler proof of projectivity, which does not seem to appear in the literature.

Theorem 11.1. *If A is an abelian variety over k, then A is projective over k.*

While the idea of the proof is similar to the one given last time, it avoids the jargon of Mumford bundles and the group schemes $K(L)$.

Proof. If $U \subseteq A$ is a non-empty affine open then $D \coloneqq A \backslash U$ is an effective Cartier divisor by Lemma 10.10. Set $L = \mathcal{O}_A(D)$, then we claim that L is ample. By Proposition 10.5, $L^{\otimes 2}$ is globally-generated, so there is a morphism $f \colon A \to \mathbf{P}^m$ such that $f^*\mathcal{O}_{\mathbf{P}^m}(1) = L^{\otimes 2}$. As any section of $L^{\otimes 2}$ is pulled back from a section of $\mathcal{O}_{\mathbf{P}^m}(1)$, there exists a hyperplane $H \subseteq \mathbf{P}^m$ such that $f^{-1}(H) = 2D$. For any closed point $x \in \mathbf{P}^m \backslash H$, $f^{-1}(x) \subseteq A \backslash D = U$ (moreover, there exists such an x with $f^{-1}(x)$ is non-empty, because f is defined by a complete linear system). If $f^{-1}(x)$ is non-empty, then since U is affine and $f^{-1}(x)$ is proper, it follows that $f^{-1}(x)$ is finite; Proposition 10.4 then implies that f is quasi-finite. By Zariski's main theorem, f is finite, and hence L is ample, since $L^{\otimes 2}$ is the pullback of the ample line bundle $\mathcal{O}_{\mathbf{P}^m}(1)$ by the finite map f. □

A fun application of Theorem 11.1 is the following result about lower bounds on the dimension of projective spaces in which abelian varieties can be embedded.

Corollary 11.2. *If A is an abelian variety over k of dimension g, then A cannot be embedded (as a closed subscheme) in $\mathbf{P}^{2g-1}$.*

In fact, more careful arguments in [EVdGM, Theorem 2.27] (using the Grothendieck–Riemann–Roch theorem) show that an abelian variety of dimension g cannot be embedded in $\mathbf{P}^{2g}$ when $g \geq 2$. As any projective variety of dimension d can be embedded in $\mathbf{P}^{2d+1}$ (for the case of smooth curves, see [Har77, Corollary IV.3.6]; a similar argument holds for any smooth projective variety), this stronger result shows that an abelian variety of dimension g can be embedded in $\mathbf{P}^{2g+1}$, but not in any lower-dimensional projective space.

The proof of Corollary 11.2 uses Chern classes as a key tool, whose properties we recall here: if H^* is any Weil cohomology theory[26], then to any coherent sheaf $\mathcal{E} \in \mathrm{Coh}(\mathbf{P}^m)$, we associate the i-th *Chern class* $c_i(\mathcal{E}) \in H^{2i}(\mathbf{P}^m)$; this rule satisfies the following conditions:

- the assignment $\mathcal{E} \mapsto c_i(\mathcal{E})$ is compatible with pullback;
- the *total Chern class* of $\mathcal{E}$
$$c_{\mathrm{tot}}(\mathcal{E}) \coloneqq \sum_{i=0}^{\infty} c_i(\mathcal{E}) \in H^{2*}(\mathbf{P}^m) \coloneqq \bigoplus_{i=0}^{\infty} H^{2i}(\mathbf{P}^m)$$
gives a map $K_0(\mathbf{P}^m) \to H^{2*}(\mathbf{P}^m)$ from the K-theory[27] to the (even-dimensional) cohomology ring, which carries addition to multiplication.
- if $\mathcal{E}$ is a vector bundle of rank r on $\mathbf{P}^m$, then $c_i(\mathcal{E}) = 0$ for all $i > r$;
- if $h \coloneqq c_1(\mathcal{O}(1)) \in H^2(\mathbf{P}^m)$ is the hyperplane class, then for any irreducible subvariety $X \subseteq \mathbf{P}^m$ of dimension d, the class $(h|_X)^d \in H^{2d}(X)$ is nonzero (this class is known as the *degree* of $X \subseteq \mathbf{P}^m$).

The construction of Chern classes is not exclusive to projective space; indeed, it works for any smooth projective variety. For full details, see [Ful84, §3.2].

Proof of Corollary 11.2. The idea of the proof is to leverage the vanishing of Chern classes against the non-vanishing for a specific bundle on the abelian variety. More precisely, we consider the Chern classes of the conormal bundle, and study them using adjunction.

Suppose there exists a closed immersion $i\colon A \hookrightarrow \mathbf{P}^m$, then we must show that $m \geq 2g$. Consider the conormal exact sequence
$$0 \to \mathcal{I}_A/\mathcal{I}_A^2 \to \Omega^1_{\mathbf{P}^m}|_A \to \Omega^1_A \to 0, \tag{11.1}$$
where $\mathcal{I}_A$ denotes the ideal that defines (the image of) A inside $\mathbf{P}^m$. As A is smooth over k, the conormal bundle $\mathcal{I}_A/\mathcal{I}_A^2$ is a vector bundle of rank $m-g$; in particular, $c_i\left(\mathcal{I}_A/\mathcal{I}_A^2\right) = 0$ for all $i > m-g$. Applying the total Chern class to (11.1) gives
$$c_{\mathrm{tot}}\left(\Omega^1_{\mathbf{P}^m}|_A\right) = c_{\mathrm{tot}}\left(\mathcal{I}_A/\mathcal{I}_A^2\right) \cdot c_{\mathrm{tot}}\left(\Omega^1_A\right).$$
As $\Omega^1_A \simeq \mathcal{O}_A^{\oplus g}$, $c_{\mathrm{tot}}(\Omega^1_A) = c_{\mathrm{tot}}(\mathcal{O}_A)^g = 1$, where the equality $c_{\mathrm{tot}}(\mathcal{O}_A) = 1$ follows from the fact that Chern classes are compatible with pullback and $\mathcal{O}_A$ arises via pullback from a point. Thus, $c_{\mathrm{tot}}\left(\Omega^1_{\mathbf{P}^m}|_A\right) = c_{\mathrm{tot}}\left(\mathcal{I}_A/\mathcal{I}_A^2\right)$. Both of these total Chern classes are elements of the *graded* cohomology ring, so we conclude that $c_i\left(\Omega^1_{\mathbf{P}^m}|_A\right) = 0$ for all $i > m-g$.

On the other hand, the Chern classes of $\Omega^1_{\mathbf{P}^m}|_A$ can be computed as follows: consider the Euler exact sequence
$$0 \to \Omega^1_{\mathbf{P}^m} \to \mathcal{O}_{\mathbf{P}^m}(-1)^{\oplus(m+1)} \to \mathcal{O}_{\mathbf{P}^m} \to 0$$
Taking total Chern classes gives
$$c_{\mathrm{tot}}\left(\Omega^1_{\mathbf{P}^m}\right) \cdot \underbrace{c_{\mathrm{tot}}\left(\mathcal{O}_{\mathbf{P}^m}\right)}_{=1} = c_{\mathrm{tot}}\left(\mathcal{O}_{\mathbf{P}^m}(-1)^{\oplus(m+1)}\right) = c_{\mathrm{tot}}\left(\mathcal{O}_{\mathbf{P}^m}(-1)\right)^{m+1} = (1-h)^{m+1}. \tag{11.2}$$

[26]For example, if $k = \mathbf{C}$, the singular cohomology of the analytification is a Weil cohomology theory; if one is working in positive characteristic, ℓ-adic or crystalline cohomology are Weil cohomology theories.

[27]If X is a noetherian scheme, the *K-theory* $K_0(X)$ of X is the free abelian group on the isomorphism classes of coherent sheaves on X modulo the following relations: if there is a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent sheaves, then $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ in K-theory (here, $[\cdot]$ denotes the isomorphism class of the coherent sheaf).

The equality (11.2) remains true after restricting to A, and so $h^i|_A = 0$ for $i > m-g$ by our previous calculation (since a Weil cohomology theory has characteristic zero coefficients). However, $h^g|_A \neq 0$, so we must have $m - g \geq g$, i.e. $m \geq 2g$. □

11.2. **Torsion subgroups.** Now that we have established the projectivity of abelian varieties, we would like to use these tools to understand the (group-theoretic and scheme-theoretic) structure of the n-torsion subgroup of an abelian variety.

Theorem 11.3. *If A is an abelian scheme over S and $n \in \mathbf{Z}_{\neq 0}$, then $[n]\colon A \to A$ is finite flat of degree n^{2g}. In particular, for any algebraically closed S-field k, $A(k)$ is divisible.*

With the additional hypothesis that n is invertible on S, we proved in Corollary 3.11 that $[n]$ is finite étale using local arguments. The proof of Theorem 11.3 will give a uniform approach using global methods, which will ultimately allow us to calculate the degree, as well.

Proof of finite flatness. Recall that the flatness of a map between two smooth S-schemes can be checked fibrewise, so we may assume that $S = \operatorname{Spec}(k)$, where k is an algebraically closed field. Pick an ample line bundle $L \in \operatorname{Pic}(A)$ and set $M \coloneqq L \otimes [-1]^*L$. Then, M is ample and it is symmetric (i.e. $[-1]^*M \simeq M$). By Corollary 6.10, we have a decomposition

$$[n]^*M \simeq M^{\frac{n^2+n}{2}+\frac{n^2-n}{2}} \simeq M^{n^2}.$$

In particular, M^{n^2} is ample, so $[n]$ must be finite (indeed, if the pullback of an ample line bundle along a map is again ample, then the map must be finite; otherwise, some fibre is a positive-dimensional variety with an ample trivial line bundle, which does not exist). Moreover, by (what Brian Conrad refers to as) the Miracle Flatness Lemma (see [Mat89, Theorem 23.1]), it follows that $[n]$ is flat. □

Remark 11.4. If A is an abelian variety over $\mathbf{C}$, then A^{an} is homeomorphic to $(S^1)^{2g}$, so $A[n](\mathbf{C}) = (\mathbf{Z}/n)^{2g}$; in particular, $\deg([n]) = n^{2g}$. This argument, of course, does not work over an arbitrary field.

In order to compute the degree of $[n]$, we require the following general construction: if X is an irreducible projective variety of dimension g, $L \in \operatorname{Pic}(X)$ is a line bundle on X, and $\mathcal{F} \in \operatorname{Coh}(X)$ is a coherent sheaf on X, then the function

$$n \mapsto \chi(\mathcal{F} \otimes L^{\otimes n})$$

is a polynomial; denote it by $P_{\mathcal{F},L}(n)$. The *degree* of $\mathcal{F}$ with respect to L, denoted $d_L(\mathcal{F})$, is $g!$ times the coefficient of n^g in $P_{\mathcal{F},L}(n)$. The *degree* of L is $\deg(L) \coloneqq d_L(\mathcal{O}_X)$. Some basic facts about the degree are summarized in the exercise below.

Exercise 11.5. With notation as above,

(1) the assignment $\mathcal{F} \mapsto d_L(\mathcal{F})$ is additive in short exact sequences;
(2) for any $k \in \mathbf{Z}_{>0}$, $\deg(L^k) = k^g \cdot \deg(L)$;
(3) if L is ample, then $d_L(\mathcal{F}) \geq 0$ for all $\mathcal{F}$, and $d_L(\mathcal{F}) > 0$ iff $\dim(\mathcal{F}) \coloneqq \dim(\operatorname{Supp}(\mathcal{F})) = g$.

Proposition 11.6. *With notation as above,*

(1) $d_L(\mathcal{F}) = \mathit{rank}(\mathcal{F}) \cdot d_L(\mathcal{O}_X) = \mathit{rank}(\mathcal{F}) \cdot \deg(L)$;
(2) if Y is an irreducible projective variety of dimension g and $f\colon Y \to X$ is a finite morphism, then

$$\deg(f^*L) = \deg(f) \cdot \deg(L).$$

Recall that the rank of a coherent sheaf on an irreducible variety is the rank of the sheaf at the generic point, and the degree of a finite morphism of irreducible varieties is the degree of the corresponding extension of function fields.

Proof. For (1), choose a short exact sequence

$$0 \to \mathcal{I}^{\mathrm{rank}(\mathcal{F})} \to \mathcal{F} \to \mathcal{Q} \to 0 \tag{11.3}$$

where $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal sheaf and $\mathcal{Q}$ is a torsion sheaf (it is left as an exercise to show that such an exact sequence exists). Applying d_L to (11.3) yields $d_L(\mathcal{F}) = d_L(\mathcal{Q}) + d_L(\mathcal{I}^{\mathrm{rank}(\mathcal{F})}) = d_L(I)^{\mathrm{rank}(\mathcal{F})}$, where $d_L(\mathcal{Q}) = 0$ because $\mathcal{Q}$ is supported on lower-dimensional subvariety of X. Similarly, use the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$ that defines $\mathcal{I}$ to see that $d_L(\mathcal{O}_X) = d_L(\mathcal{I})$, from which the conclusion follows.

For (2), observe that

$$P_{\mathcal{O}_Y, f^*L}(n) = \chi(f^*L^n) = \chi(f_*f^*L^n) = \chi((f_*\mathcal{O}_Y) \otimes L^n) = P_{f_*\mathcal{O}_Y, L}(n),$$

where the second equality follows from the fact that cohomology may be computed after pushing forward along a finite morphism, and the second-to-last equality follows from the projection formula Thus, comparing leading coefficients, we find that

$$\deg(f^*L) = d_{f^*L}(\mathcal{O}_Y) = d_L(f_*\mathcal{O}_Y) = \mathrm{rank}(f_*\mathcal{O}_Y) \cdot d_L(\mathcal{O}_X) = \deg(f) \cdot \deg(L)$$

where the middle equality follows from (1). □

Proof of Theorem 11.3. It remains to show that $\deg([n]) = n^{2g}$. As in the proof that $[n]$ is finite flat, choose a symmetric ample line bundle $M \in \mathrm{Pic}(A)$. Applying Proposition 11.6(2) to $[n]$ gives the equality

$$\deg(M^{n^2}) = \deg([n]^*M) = \deg([n]) \cdot \deg(M).$$

By Exercise 11.5(2), we have $\deg(M^{n^2}) = (n^2)^g \deg(M)$. Finally, as M is ample, $\deg(M) > 0$, so it follows that $\deg([n]) = n^{2g}$. □

It follows from Theorem 11.3 that the cardinality of the n-torsion subgroup of an abelian variety A of dimension g is n^{2g}, but one can further analyze its structure and determine the abelian group structure on $A[n]$.

Theorem 11.7. *Let A be an abelian variety of dimension g over an algebraically closed field k, let $n \in \mathbf{Z}_{\neq 0}$, and consider the group scheme $A[n] \coloneqq \ker([n]\colon A \to A)$.*

(1) If n is invertible on k, then $A[n] \simeq \underline{(\mathbf{Z}/n)}^{2g}$ is the constant group scheme[28]; in particular,

$$A[n](k) \simeq (\mathbf{Z}/n)^{2g}.$$

(2) If char$(k) = p > 0$, then there is a unique integer $0 \leq i \leq g$, called the p-rank of A, *such that*

$$A[p^m](k) \simeq (\mathbf{Z}/p^m)^i.$$

Example 11.8. A *supersingular* elliptic curve is one with p-rank equal to zero.

Proof of Theorem 11.7(1). If $n \in k^*$, then $[n]\colon A \to A$ is finite étale by Corollary 3.11, and so the $A[n]$ is finite étale over k. However, a finite étale scheme over an algebraically closed field is just a finite set, i.e. there is an equivalence of categories

$$\{\text{finite étale } k\text{-schemes}\} \xrightarrow{\simeq} \{\text{finite sets}\}$$

given by $X \mapsto X(k)$, so it suffices to show that $G \coloneqq A[n](k) \simeq (\mathbf{Z}/n)^{2g}$ as abelian groups. We have shown that G is abelian, $\#G = n^{2g} = \deg([n])$, and $n \cdot G = 0$, but we know more: for all m dividing n, $G[m]$ has the same properties, i.e. $\#G = m^{2g}$ and $m \cdot G = 0$. By the classification of finite abelian groups, we must have $G \simeq (\mathbf{Z}/n)^{2g}$. □

In order to prove Theorem 11.7(2), we require the following lemma relating the kernel of the de Rham differential to the image of the Frobenius map.

[28] If G is a finite group, the associated *constant group scheme* $\underline{G}$ over k is the k-scheme $\coprod_{g \in G} \mathrm{Spec}(k)$, with multiplication determined by the multiplication in G.

Lemma 11.9. *If k is a perfect field of characteristic p and R is a smooth k-algebra, then* $\ker(d\colon R \to \Omega^1_{R/k})$ *is the subring $R^p \subseteq R$ of p-powers.*

The conclusion of Lemma 11.9 is very much a feature of positive characteristic, as opposed to in characteristic zero, where the kernel of the differential d is generally just the constants. The proof of Lemma 11.9 is left as an exercise.

Remark 11.10. If k and R are as in Lemma 11.9, the conclusion of the lemma can be reformulated as follows: if $f\colon S \to R$ is a map of k-algebras such that $\Omega^1_{S/k} \to \Omega^1_{R/k}$ is the zero map, then f factors uniquely as $S \to R^p \subseteq R$.

Remark 11.11. If k is a field of characteristic p and R is a k-algebra, let $R^{(1)}$ be the k-algebra obtained as the Cartesian product

$$\begin{tikzcd}
k \arrow[r, "\mathrm{Frob}_k"] \arrow[d] & k \arrow[d] \\
R \arrow[r, "\mathrm{Frob}_k \otimes 1"] & R^{(1)}
\end{tikzcd}$$

The *relative Frobenius map* $\mathrm{Frob}_{R/k}\colon R^{(1)} \to R$ is the k-algebra map defined by the diagram

$$\begin{tikzcd}
k \arrow[r, "\mathrm{Frob}_k"] \arrow[d] & k \arrow[d] \arrow[ddr, bend left, "f"] & \\
R \arrow[r, "\mathrm{Frob}_k \otimes 1"] \arrow[drr, bend right, "\mathrm{Frob}_R"'] & R^{(1)} \arrow[dr, dashed, "\exists!\, \mathrm{Frob}_{R/k}"'] & \\
& & R
\end{tikzcd}$$

Note that the usual Frobenius map $\mathrm{Frob}_R\colon R \to R$ is not a k-algebra map, whereas the relative Frobenius morphism $\mathrm{Frob}_{R/k}\colon R^{(1)} \to R$ is a k-algebra map.

In particular, if R is a smooth k-algebra, then any morphism $f\colon S \to R$ such that $\Omega^1_{S/k} \to \Omega^1_{R/k}$ is the zero map factors uniquely (as a k-algebra map) over the relative Frobenius, i.e. there is a unique factorization

$$S \to R^{(1)} \xrightarrow{\mathrm{Frob}_{R/k}} R$$

While this is another reinterpretation of Remark 11.10, it is often very useful to describe the ring R^p of p-powers in this way.

12. October 12th

12.1. **Torsion Subgroups (Continued).** Last time, we defined the relative Frobenius morphism in the case of affine schemes, in order to understand the structure of the torsion subgroups of abelian varieties. In general, the relative Frobenius is constructed as follows: if k is a perfect field of characteristic p and X is a k-scheme, then there is a diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\mathrm{Frob}_{X/k}"] \arrow[drr, bend left, "\mathrm{Frob}_X"] \arrow[ddr, bend right, "f"'] & & \\
& X^{(1)} \arrow[r] \arrow[d, "f^{(1)}"] & X \arrow[d, "f"] \\
& \mathrm{Spec}(k) \arrow[r, "\mathrm{Frob}_k"] & \mathrm{Spec}(k)
\end{tikzcd}$$

The global analogue of Remark 11.11 is then the following result.

Fact 12.1. If X is a smooth k-scheme and $g : X \to Y$ is a map of k-schemes such that $g^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ is the zero map, then there is a unique factorization of g as

$$X \xrightarrow{\mathrm{Frob}_{X/k}} X^{(1)} \longrightarrow Y$$

as morphisms of k-schemes.

The proof of Fact 12.1 is left as an exercise. From last time, it remained to understand the structure of the torsion subgroups of an abelian variety in positive characteristic. More precisely, we must prove the following:

Theorem 12.2. *If A is an abelian variety of dimension g over an algebraically closed field k of characteristic p, then there is a unique $1 \le i \le g$, called the* p-rank of A, *such that*

$$A[p^m](k) = (\mathbf{Z}/p^m\mathbf{Z})^i.$$

Proof. We proceed by induction on m. Consider first the case $m = 1$: we claim that the map $[p]\colon A \to A$ factors over the relative Frobenius $\mathrm{Frob}_{A/k}$ as

$$A \xrightarrow{\mathrm{Frob}_{A/k}} A^{(1)} \longrightarrow A.$$

Label the second map as $V\colon A^{(1)} \to A$. To see this, note that since $\Omega^1_{A/k}$ is free, Fact 12.1 implies that it suffices to check that the map induced by $[p]$ on global sections is zero. However, we already know that the induced map $T_e([p])\colon T_e(A) \to T_e(A)$ on tangent spaces is multiplication-by-p, hence zero. Dualizing gives that $[p]$ induces the zero map on the cotangent space at the identity, from which it follows that $[p]$ induces the zero map on the global sections of $\Omega^1_{A/k}$.

As $\mathrm{Frob}_{A/k}$ is a homeomorphism, there is a bijection between the p-torsion k-points $A[p](k) = [p]^{-1}(e)(k)$ and the k-points $V^{-1}(e)(k)$. Since $\deg([p]) = p^{2g}$ and $\deg(\mathrm{Frob}_{A/k}) = p^g$, we must have $\deg V = p^g$; in particular, the cardinality of $A[p](k)$ is bounded above by $\#V^{-1}(e)(k) \le p^g$ (indeed, the degree is the size of the scheme-theoretic fibre, which is at least the degree of the set-theoretic fibre). However, $A[p](k)$ is a p-torsion abelian group, and thus there exists $0 \le i \le g$ such that $A[p](k) \simeq (\mathbf{Z}/p\mathbf{Z})^i$.

If $m > 1$, we can use that $[p^m]\colon A \to A$ is surjective by Theorem 11.3 (and hence that $A(k)$ is divisible) to construct a short exact sequence

$$0 \longrightarrow A[p](k) \longrightarrow A[p^m](k) \xrightarrow{[p]} A[p^{m-1}](k) \longrightarrow 0.$$

By induction on m, we have isomorphisms $A[p](k) \simeq (\mathbf{Z}/p\mathbf{Z})^i$ and $A[p^{m-1}](k) \simeq (\mathbf{Z}/p^{m-1}\mathbf{Z})^i$, so

$$\#A[p^m](k) = \#(\mathbf{Z}/p\mathbf{Z})^i + \#(\mathbf{Z}/p^{m-1}\mathbf{Z})^i = \#(\mathbf{Z}/p^m\mathbf{Z})^i,$$

i.e. $A[p^m](k)$ has the correct cardinality. Now, observe that $A[p^m](k)/p^{m-1} \simeq A[p](k)$, so Nakayama's lemma implies that $A[p^m](k)$ is a quotient of $(\mathbf{Z}/p^m\mathbf{Z})^i$. As $A[p^m](k)$ and $(\mathbf{Z}/p^m\mathbf{Z})^i$ have the same cardinality, it follows that they must be isomorphic. $\square$

12.2. **The Dual Abelian Variety.** In order to construct the dual abelian variety, we could proceed by abstract nonsense (i.e. by verifying certain axioms about the representability of a particular functor), but we follow Mumford's more direct approach from [Mum08, §II.8], which is more enlightening. We begin by discussing line bundles of degree zero. In general, one can define the degree using intersection theory, but there is a more concrete description on an abelian variety.

Let A be an abelian variety over an algebraically closed field k (while all results hold over an arbitrary field, they can easily be reduced to the case where the base field is algebraically closed, so we make this assumption for simplicity). Recall that the *Picard scheme* of A over k has the functor of points

$$T \mapsto \mathrm{Pic}_{A/k}(T) \coloneqq \mathrm{Pic}(A \times_k T)/\mathrm{Pic}(T),$$

for all k-schemes T. Moreover, recall that there is a map

$$\mathrm{Pic}(A) \longrightarrow \mathrm{Hom}(A, \mathrm{Pic}_{A/k}),$$

given by $L \mapsto \phi_L$, where $\phi_L\left(T \xrightarrow{x} A\right) := t_x^*(L_T) \otimes L_T^{-1}$. The map $L \mapsto \phi_L$ is a homomorphism by Corollary 9.4. Note that $\phi_L = 0$ if and only if for all $x\colon T \to A$, $t_x^*(L_T) \cong L_T$.

Definition 12.3. Define $\mathrm{Pic}^0(A) := \{L \in \mathrm{Pic}(A) : \phi_L = 0\}$ as a subgroup of $\mathrm{Pic}(A)$.

The set $\mathrm{Pic}^0(A)$ will ultimately be the underlying set of the dual abelian variety of A (i.e. its k-points, since k is assumed to be algebraically closed). In order to realize $\mathrm{Pic}^0(A)$ as a geometric object, we must first consider the structure of line bundles that belong to $\mathrm{Pic}^0(A)$.

Lemma 12.4. *For any $L \in \mathrm{Pic}(A)$ and any $x \in A(k)$, we have*

$$\phi_L(x) = t_x^*(L) \otimes L^{-1} \in \mathrm{Pic}^0(A).$$

In particular, there is a well-defined map $\mathrm{Pic}(A)/\mathrm{Pic}^0(A) \to \mathrm{Hom}(A(k), \mathrm{Pic}^0(A))$.

Proof. It suffices to check that for any $y\colon T \to A$, $t_y^*(t_x^*(L) \otimes L^{-1}) \simeq t_x^*(L) \otimes L^{-1}$; for simplicity, we check only the case when $y \in A(k)$, but the same argument works in general. Observe that

$$t_y^*(t_x^*(L) \otimes L^{-1}) \simeq t_{x+y}^*(L) \otimes t_y^*(L^{-1}) \simeq t_x^*(L) \otimes t_y^*(L) \otimes L^{-1} \otimes t_y^*(L^{-1}) \simeq t_x^*(L) \otimes L^{-1},$$

as required. □

The upshot of Lemma 12.4 is that we have a method of producing points of $\mathrm{Pic}^0(A)$, given a point of A and a line bundle on A.

In fact, the membership in $\mathrm{Pic}^0(A)$ can be characterized in terms of Mumford bundles. Recall that the Mumford bundle $\Lambda(L)$ of $L \in \mathrm{Pic}(A)$ is defined to be

$$\Lambda(L) := m^*(L) \otimes \mathrm{pr}_1^*(L^{-1}) \otimes \mathrm{pr}_2^*(L^{-1}) \in \mathrm{Pic}(A \times A).$$

Lemma 12.5. *If $L \in \mathrm{Pic}\, A$, then $L \in \mathrm{Pic}^0 A$ iff $\Lambda(L)$ is trivial on $A \times A$.*

Proof. Let $K(L) \subset A$ be the maximal closed subscheme of A such that $\Lambda(L)$ arises via pullback along pr_1, as defined in § 9.2. By Corollary 9.7, we have that $L \in \mathrm{Pic}^0(A)$ iff $K(L) = A$, and hence $\Lambda(L)$ arises via pullback along pr_1. If $\sigma\colon A \to A \times A$ is the section of pr_1 given by (id, e), then pulling back $\Lambda(L)$ along σ gives the line bundle $L \otimes L^{-1} \otimes \mathcal{O}_A \simeq \mathcal{O}_A$, so $\Lambda(L)$ must be trivial. □

Lemma 12.6. *If $L \in \mathrm{Pic}^0(A)$, T is a k-scheme, and $x, y \in A(T)$, then*

$$(x+y)^*L \simeq x^*L \otimes y^*L.$$

In particular, $[n]^*L \simeq L^n$.

Lemma 12.6 asserts that "degree-zero" line bundles behave linearly with respect to $[n]$, while general line bundles behave "quadratically" by Corollary 6.10; in fact, if $L \in \mathrm{Pic}(A)$ is symmetric (i.e. $[-1]^*L \simeq L$), then we have $[n]^*L \simeq L^{n^2}$.

Proof. By Lemma 12.5, we have $m^*L \simeq \mathrm{pr}_1^*L \otimes \mathrm{pr}_2^*(L)$ on $A \times A$. Pulling back this isomorphism along the map $(x, y)\colon T \to A \times A$ gives the desired isomorphism. □

Lemma 12.7. *For any $L \in \mathrm{Pic}(A)$, there exists $M \in \mathrm{Pic}^0(A)$ such that $[n]^*L \simeq L^{n^2} \otimes M$.*

Proof. By Corollary 6.10, we have isomorphisms

$$[n]^*L \simeq L^{\frac{n^2+n}{2}} \otimes [-1]^*L^{\frac{n^2-n}{2}} \simeq L^{n^2} \otimes \left(L \otimes [-1]^*L^{-1}\right)^{\frac{n^2-n}{2}}.$$

It suffices to show that $L \otimes [-1]^*L^{-1} \in \mathrm{Pic}^0(A)$, as then $M = \left(L \otimes [-1]^*L^{-1}\right)^{\frac{n^2-n}{2}}$ lies in $\mathrm{Pic}^0(A)$, because $\mathrm{Pic}^0(A)$ is a subgroup of $\mathrm{Pic}(A)$ under the tensor product. As in the proof of Lemma 12.4, we only test membership using k-points, with the general case being similar. If $x \in A(k)$, then

$$t_x^*\left(L \otimes [-1]^*L^{-1}\right) \simeq t_x^*(L) \otimes [-1]^*t_{-x}^*(L^{-1}). \simeq t_x^*(L) \otimes [-1]^*\left(L \otimes t_{-x}^*(L^{-1})\right) \otimes [-1]^*L^{-1}. \tag{12.1}$$

By Lemma 12.4, the middle factor lies in $\mathrm{Pic}^0(A)$, and so it simplifies by Lemma 12.6 to $L^{-1} \otimes t_{-x}^*(L)$. Thus, (12.1) becomes

$$t_x^*(L) \otimes L^{-1} \otimes t_{-x}^*(L) \otimes [-1]^* L^{-1} \simeq t_e^*(L) \otimes [-1]^* L^{-1} \simeq L \otimes [-1]^* L^{-1}$$

where the second-to-last isomorphism follows from Corollary 9.4. Thus, $L \otimes [-1]^* L^{-1} \in \mathrm{Pic}^0(A)$, as required. □

Lemma 12.8. *If $L \in \mathrm{Pic}\, A$ has finite order, then $L \in \mathrm{Pic}^0(A)$.*

Proof. Consider the group homomorphism $\mathrm{Pic}(A) \to \mathrm{Hom}(A, \mathrm{Pic}_{A/k})$ given by $L \mapsto \phi_L$. If $L^n \simeq \mathcal{O}_A$, then $n\phi_L = 0$, so $0 = n\phi_L(x) = \phi_L(nx)$ for any $x \in A(k)$. As $A(k)$ is divisible by Theorem 11.3, we have $\phi_L = 0$, i.e. $L \in \mathrm{Pic}^0(A)$. (To be precise, one must check this not only on the k-points of A, but rather on all test schemes; to do so, use that $[n]$ is surjective by Theorem 11.3). □

Lemma 12.9. *If S is an connected scheme of finite type over k, $L \in \mathrm{Pic}(A \times S)$, and $s, t \in S$, then*

$$L_s \otimes L_t^{-1} \in \mathrm{Pic}^0(A),$$

where $L_s := L|_{A \times \{s\}}$ and $L_t := L|_{A \times \{t\}}$.

In particular, if there exists a single point $s \in S$ such that the restriction L_s belongs to $\mathrm{Pic}^0(A)$, then it follows that $L_t \in \mathrm{Pic}^0(A)$ for all $t \in S$.

Proof. By shrinking S, we may assume $L|_{\{e\} \times S}$ is trivial. Furthermore, upon replacing L by $L \otimes \mathrm{pr}_1^*(L_s^{-1})$, we may assume that L_s is trivial. We now claim that L_t lies in $\mathrm{Pic}^0(A)$ for all $t \in S$. By Lemma 12.5, it suffices to show that $\Lambda(L_t)$ is trivial. Consider the line bundle bundle M on $S \times A \times A$ given by the formula

$$M = \mu^* L \otimes \mathrm{pr}_{12}(L^{-1}) \otimes \mathrm{pr}_{13}(L^{-1}),$$

where $\mu : S \times A \times A \to S \times A$ is given by $(s, a, b) \mapsto (s, a + b)$. Note that $M = \Lambda(L)$, i.e. M is the Mumford bundle for L on the abelian scheme $A \times S \to S$. Moreover, for each $t \in S$, $M|_{\{t\} \times A \times A} \simeq \Lambda(L_t)$. It now suffices to show that M is trivial, since its fibre over $\{s\} \times A \times A$ is trivial by assumption. As M is trivial on the fibres $S \times \{e\} \times A$ and $S \times A \times \{e\}$ by construction, the Theorem of the Cube (Corollary 6.8) implies that M is trivial, which completes the proof. □

Lemma 12.10. *If $L \in \mathrm{Pic}^0(A)$ is non-trivial, then $H^i(A, L) = 0$ for all $i \in \mathbf{Z}_{\geq 0}$.*

Proof. Consider first when $i = 0$: if $H^0(A, L) \neq 0$, then we can write $L \simeq \mathcal{O}_A(D)$ for some *effective* Cartier divisor on A. By Lemma 12.6, $[-1]^* L \simeq L^{-1}$, so L^{-1} is also effective, and thus L is trivial, a contradiction (indeed, on a proper, geometrically irreducible variety, there are no effective divisors with effective inverses).

Now, assume that some higher cohomology group of L does not vanish. Let $i \in \mathbf{Z}_{\geq 0}$ be minimal such that $H^i(A, L)$ is nonzero. Consider the composition

$$A \xrightarrow{(\mathrm{id}, e)} A \times A \xrightarrow{m} A.$$

This composition is the identity map, and hence the identity map on $H^i(A, L)$ factors through a map to the cohomology $H^i(A \times A, m^* L)$ on the product. Thus, it suffices then to show that $H^i(A \times A, m^* L) = 0$. However, there is an isomorphism $m^*(L) \simeq \mathrm{pr}_1^*(L) \otimes \mathrm{pr}_2^*(L)$ by Lemma 12.5, so we can apply the Künneth formula in order to compute $H^i(A \times A, m^* L)$. The minimality of i (and the $i = 0$ case!) immediately implies that $H^i(A \times A, m^* L)$ vanishes. □

Proposition 12.11. *If $L \in \mathrm{Pic}(A)$ is ample, then $\phi_L : A(k) \to \mathrm{Pic}^0(A)$ is surjective with kernel $K(L)(k)$.*

Recall that, if $L \in \mathrm{Pic}(A)$ is ample, the subgroup $K(L)(k)$ of $A(k)$ is finite; thus, Proposition 12.11 is asserting that $\mathrm{Pic}^0(A)$ is very large!

Proof. We showed in Corollary 9.7 that $K(L)(k)$ is the kernel of ϕ_L, so it remains to show that ϕ_L is surjective. Assume there exists a line bundle $M \in \mathrm{Pic}^0(A)$ such that M does not lie in the image of ϕ_L. Consider the line bundle

$$K \coloneqq \Lambda(L) \otimes \mathrm{pr}_1^*(M^{-1}) \simeq m^*L \otimes \mathrm{pr}_2^*(L^{-1}) \otimes \mathrm{pr}_1^*(L^{-1} \otimes M^{-1})$$

on $A \times A$. For $x \in A(k)$, the restrictions of K to $A \times \{x\}$ and $\{x\} \times A$ are

$$K|_{A\times\{x\}} \simeq t_x^*(L) \otimes L^{-1} \otimes M^{-1} \text{ and } K|_{\{x\}\times A} \simeq t_x^*(L) \otimes L^{-1}.$$

Since M does not lie in the image of ϕ_L, $K|_{A\times\{x\}}$ defines a nontrivial degree-zero line bundle on A, for any $x \in A(k)$. In particular, the restriction of K to any fibre of pr_2 has no cohomology by Lemma 12.10, and hence $R^i(\mathrm{pr}_2)_*(K) = 0$ for all $i \geq 0$. It follows from the Leray spectral sequence for $\mathrm{pr}_2\colon A \times A \to A$ that $H^i(A \times A, K) = 0$ for all $i \geq 0$.

On the other hand, we have $\mathrm{Supp}(R^i(\mathrm{pr}_1)_*(K)) \subset K(L)$ for all $i \geq 0$. Since L is ample, $K(L)$ is finite by Corollary 9.9, and so the Leray spectral sequence for pr_1 gives an isomorphism

$$H^0(A, R^i(\mathrm{pr}_1)_*(K)) \simeq H^i(A \times A, K) = 0$$

for all $i \geq 0$. We must conclude that $R^i(\mathrm{pr}_1)_*(K) = 0$ for all $i \geq 0$, and hence $H^i(A, K|_{\{x\}\times A}) = 0$ for all $i \geq 0$. Taking $x = e$ gives $H^i(A, \mathcal{O}_A) = 0$ for all $i \geq 0$, a contradiction. □

13. October 19th

13.1. **The Dual Abelian Variety (Continued).** Last time, we began discussing the degree-zero line bundles on an abelian variety, with the goal of defining the dual abelian variety à la Mumford. If A is an abelian variety over an algebraically closed field k and L is a line bundle on A, recall that $L \in \mathrm{Pic}^0(A)$, iff $\phi_L = 0$ iff $K(L) = A$ (and in this case we say that L has *degree-zero*). By Corollary 9.7, a line bundle L on A lies in $\mathrm{Pic}^0(A)$ precisely when it is "translation-invariant".

The last class concluded with a proposition explaining that all line bundles in $\mathrm{Pic}^0(A)$ can be expressed in terms of a single ample line bundle on A. This proposition is repeated below.

Proposition 13.1. *If $L \in \mathrm{Pic}(A)$ is ample, then $A(k) \to \mathrm{Pic}^0(A)$ is surjective with kernel equal to $K(L)(k)$.*

One should think of Proposition 13.1 as roughly saying that $\mathrm{Pic}^0(A)$ looks like the k-points of some abelian variety, namely one whose k-points are the k-points of A modulo the finite subgroup $K(L)(k)$.

Remark 13.2. It follows immediately from Proposition 13.1 that, for each fixed $n \in \mathbf{Z}_{>0}$, the set of n-torsion line bundles on A is finite.

Proof. Let $M \in \mathrm{Pic}^0(A)$ and assume that M does not lie in the image of ϕ_L. Consider the line bundle

$$K \coloneqq \Lambda(L) \otimes \mathrm{pr}_1^*(M^{-1}) \in \mathrm{Pic}(A \times A).$$

In order to arrive at a contradiction, we analyze the higher pushforwards of K along the two projections to compute the cohomology of K in two different ways. Observe that for any $x \in A(k)$, we have

$$K|_{A\times\{x\}} \simeq t_x^*(L) \otimes L^{-1} \otimes M^{-1} \text{ and } K|_{\{x\}\times A} \simeq t_x^*(L) \otimes L^{-1}.$$

As M does not in the image of ϕ_L, none of the fibres $K|_{A\times\{x\}}$ are trivial, and thus its cohomology always vanishes by Lemma 12.10; in particular, $R^i(\mathrm{pr}_2) * (K) = 0$ for all $i \in \mathbf{Z}_{\geq 0}$. The Leray spectral sequence allows one to conclude that higher cohomology of K also vanishes, i.e. $H^i(A \times A, K) = 0$ for all $i > 0$.

Now, consider the first projection: if $x \in A(k)$, then $K|_{\{x\}\times A}$ is trivial iff $x \in K(L)(k)$, so $R^i(\mathrm{pr}_1)_*(K)$ has support contained in $K(L)$. As L is ample, $K(L)$ is finite by Corollary 9.9, but any coherent sheaf that is supported on a finite subscheme has no higher cohomology. It follows from the Leray spectral sequence that there is an isomorphism

$$H^i(A \times A, K) \simeq \bigoplus_{x \in K(L)} \left(R^i(\mathrm{pr}_1)_*(K)\right)_x$$

for all $i \in \mathbf{Z}_{\geq 0}$. We conclude that $R_i(\mathrm{pr}_1)_*(K) = 0$ for all $i \in \mathbf{Z}_{\geq 0}$, and hence Thus, $H^i(A, K|_{\{x\}\times A}) = 0$ for all $i \in \mathbf{Z}_{\geq 0}$ and $x \in A(k)$. However, if $x = e$, then $K|_{\{x\}\times A} \simeq \mathcal{O}_A$, which has $H^0(A, \mathcal{O}_A) \neq 0$, a contradiction. □

The trick appearing in the proof of Proposition 13.1 (namely, using different pushforwards to argue that cohomology vanishes) will often be used in the construction of the abelian variety.

The upshot of Proposition 13.1 is that we have constructed the k-points of the dual abelian variety in a "sort of geometric" manner. In order to complete the construction, we must first discuss the general theory of quotients of schemes by the actions of finite groups. This will allows us to place a scheme structure on $A/K(L)$ and then port this to a scheme structure on $\mathrm{Pic}^0(A)$. This description is problematic, in that appears to depend on the choice of the ample line bundle L on A; however, we will eventually describe the functor of points of the aforementioned scheme structure on $\mathrm{Pic}^0(A)$, from which it will follow that the construction is independent of this choice.

13.2. **Group Schemes Are Smooth In Characteristic Zero.** We will first show that group schemes are reduced (and hence smooth) in characteristic zero. While this result is not strictly necessary for the construction of the dual abelian variety, it is psychologically comforting to know that a finite group scheme in characteristic zero is reduced, and so it is just a finite group.

Theorem 13.3. [Cartier] *If k is a field of characteristic zero and G is a group scheme locally of finite type over k, then G is reduced; in particular, G is smooth over k.*

The second assertion follows from the first by the observations that a reduced scheme over a field of characteristic zero is geometrically reduced, and that a geometrically reduced group scheme over a field is smooth (indeed, one has a dense open subset that is smooth over the base field, and one can use the group structure to cover the group scheme by smooth opens). The original proof of Theorem 13.3 is due to Carter [Car62] and it is exposited nicely[29] in [Mum66, §25], and there is a (very short) proof due to Oort [Oor66]. We instead give a proof of the smoothness of such a group scheme following [Sta17, Tag 0BF6].

The proof proceeds via a sequence of lemmas, the first of which uses the characteristic zero assumption in a crucial way.

Lemma 13.4. *Let R be a noetherian local ring and let $k \to R$ be a map of $\mathbf{Q}$-algebras. If there exists $f \in R$ such that the R-linear map $df : R \to \Omega^1_{R/k}$ is a direct summand, then f is a nonzero divisor.*

The differential $df : R \to \Omega^1_{R/k}$ appearing in Lemma 13.4 is, more explicitly, the R-linear map sending $1 \mapsto df$, i.e. if $x \in R$, then $df(x) \coloneqq x \cdot df$.

Remark 13.5. Lemma 13.4 fails in characteristic $p > 0$: if $R = k[x]/(x^p)$ and $f = x$, then $\Omega^1_{R/k} \simeq Rdx$ (in particular, df is an isomorphism), but f is a zero divisor in R.

Proof. As $df : R \to \Omega^1_{R/k}$ is a direct summand, there is an R-linear section $s : \Omega^1_{R/k} \to R$ of df; s gives rise to a (R-linear) derivation $\theta : R \to R$ given by $g \mapsto s(dg)$. Notice that for any $a \in R$, we have a decomposition

$$d(a) = \theta(a)df + c(a)$$

for some $c(a) \in \ker(s)$. By construction, $\theta(f) = 1$; in particular, for any $n \in \mathbf{Z}_{>0}$, $\theta(f^n) = nf^{n-1}\theta(f) = nf^{n-1}$ by the derivation property. Assume there is $g \in R$ such that $fg = 0$, then we must show that $g = 0$. To see this, observe that

$$0 = \theta(fg) = \theta(f)g + f\theta(g) = g + f\theta(g)$$

[29] In fact, the proof of Cartier shows more generally that any locally noetherian group scheme over a field of characteristic zero is reduced, as pointed out here: https://mathoverflow.net/questions/22553/are-group-schemes-in-char-0-reduced-yes. It turns out that even the local noetherianity hypothesis is not necessary: Perrin shows in [Per75] that every group scheme over a field of characteristic zero is reduced.

and hence $g \in (f)$. Inductively, assume that $g = f^n h$ for some $h \in R$. As $fg = 0$, we have $f^{n+1}h = 0$, and so
$$0 = \theta(f^{n+1}h) = \theta(f^{n+1})h + f^{n+1}\theta(h) = (n+1)f^n h + f^{n+1}\theta(h),$$
i.e. $g \in (f^{n+1})$. Here, we have used that $n + 1 \neq 0$ since R has characteristic zero. Therefore, $g \in \bigcap_{n\geq 1}(f^n)$, and hence $g = 0$ by Krull's intersection theorem. □

Lemma 13.6. *If k is a field of characteristic zero, and R is a finite type k-algebra such that $\Omega^1_{R/k}$ is projective, then R is smooth over k.*

The slogan of Lemma 13.6 is that smoothness can be detected by the local freeness of differentials in characteristic zero (though this is completely false in positive characteristic by the same example as in Remark 13.5).

The key point in the proof of Lemma 13.6 is to use that fact that a ring over a field is regular iff it is smooth over the base field.

Proof. By base changing to the algebraic closure, we may assume that k is algebraically closed. After replacing R with a localization, we may assume that k is a noetherian local ring with residue field k. Thus, $\Omega^1_{R/k}$ is free (of rank n, say). The goal is to show that R is regular of dimension n.

If $\mathfrak{m} \subseteq R$ is the maximal ideal, then there is an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\simeq} \Omega^1_{R/k} \otimes_R k$ given by $g \mapsto dg$ (note that it is at this point that we use the fact that the residue field of R is k); in particular, $\mathfrak{m}/\mathfrak{m}^2$ is an n-dimensional vector space over k. If $f \in \mathfrak{m}$ is nonzero in $\mathfrak{m}/\mathfrak{m}^2$, then df is nonzero in $\Omega^1_{R/k} \otimes_R k$, and hence $R \xrightarrow{\cdot df} \Omega^1_{R/k}$ is a direct summand. By Lemma 13.4, f is a nonzero divisor. Now, $R/(f)$ has module of differentials that is free of rank $n - 1$, so one can inductively show that R is regular of dimension n. □

The smoothness assertion in Theorem 13.3 can now be deduced from the previous two lemmas.

Corollary 13.7. *If k is a field of characteristic zero, and G is a group scheme locally of finite type over k, then the local ring $\mathcal{O}_{G,e}$ is regular; in particular, G is smooth over k.*

Proof. If $\pi: G \to \mathrm{Spec}(k)$ is the structure map of G, then there is an isomorphism $\Omega^1_{G/k} \simeq \pi^*(e^*\Omega^1_{G/k})$ by Proposition 2.6. Applying Lemma 13.6 to the stalk of $\Omega^1_{G/k}$ at the identity e gives that the local ring $\mathcal{O}_{G,e}$ of G at the identity is regular. □

13.3. **Quotients by Finite Group Schemes.** Let k be a field, let G be a finite group scheme over k, and let X be a scheme of finite type over k. There is an obvious notion of a G-action on X (either in terms of the functor points or in terms of diagrams). Denote the action map of G on X by $\mathrm{act}: G \times X \to X$.

There is a scheme-theoretic version of the usual notion of a free group action (i.e. one with trivial stabilizers).

Definition 13.8. A G-action on X is *free* if the map $(\mathrm{act}, \mathrm{pr}_2): G \times X \to X \times X$ is a closed immersion.

Remark 13.9. If X is separated over k, then a G-action on X is free iff the map $(\mathrm{act}, \mathrm{pr}_2)$ is a monomorphism of presheaves (that is, for any k-scheme T, the induced map $(G \times X)(T) \to (X \times X)(T)$ on T-valued points is injective). This is because G is a finite (and hence proper) k-scheme, so $(\mathrm{act}, \mathrm{pr}_2)$ is a proper map, and a proper map is a closed immersion iff it is a monomorphism by [Sta17, Tag 04XV].

Example 13.10. If A is an abelian variety over k and if $G \subseteq A$ is a finite subgroup scheme, then translation gives a free G-action on A.

We would like to discuss not only quotients of schemes by finite group actions, but how one can carry a sheaf through this process. To that end, we introduce the following notation: for any k-scheme S and any S-valued point $g \in G(S)$, write $a_g: X_S \to X_S$ for the induced action map, where $X_S := X \times_k S$. By virtue of the transitivity property of the group action, we have the relation
$$a_h \circ a_g = a_{h\cdot g}$$
for $g, h \in G(S)$.

Definition 13.11. A *G-equivariant quasicoherent sheaf* on X is the data of a quasicoherent sheaf $\mathcal{F}$ on X, along with an isomorphism

$$\lambda_g \colon a_g^*(\mathcal{F}_S) \xrightarrow{\simeq} \mathcal{F}_S$$

for all $g \in G(S)$, such that the following transitivity condition is satisfied: for any $g, h \in G(S)$, we have a commutative diagram

$$\begin{tikzcd} a_{h\cdot g}^*(\mathcal{F}_S) \arrow[r, "\lambda_{h\cdot g}"] \arrow[d, equal] & \mathcal{F}_S \\ a_g^*(a_h^*(\mathcal{F}_S)) \arrow[r, "a_g^*(\lambda_h)"] & a_g^*(\mathcal{F}_S) \arrow[u, "\lambda_g"'] \end{tikzcd}$$

That is, $\lambda_{h\cdot g} = \lambda_g \circ a_g^*(\lambda_h)$. Here, $\mathcal{F}_S$ is the pullback of $\mathcal{F}$ along $X_S \to S$.

If the group G acts trivially on X, then a G-equivariant sheaf on X is the same a sheaf on X that is equipped with a G-action. Thus, a G-equivariant sheaf is really a notion of a G-action on a sheaf that is compatible with a G-action on the scheme.

Now, we can make precise the notion of a quotient X/G of a scheme X by the action of a finite group scheme G. If X is affine, then one simply want the quotient X/G to be the spectrum of the ring of G-invariants; in general, we must place certain hypotheses on X so that we may reduce to this case in the construction of the quotient.

Theorem 13.12. *Let X be a separated scheme of finite type over k, and let G be a finite group scheme over k equipped with a free action on X. Assume that any finite subset of X is contained in an affine open subset of X.*[30] *Then, there exists a universal G-invariant*[31] *map $\pi \colon X \to X/G$ satisfying the following:*

(1) The map $\pi \colon X \to X/G$ is a G-torsor for the fppf topology; that is, π is finite, flat, surjective, and the map $(\mathrm{act}, \mathrm{pr}_2) \colon G \times X \to X \times_{\pi, X/G} X$ is an isomorphism.

(2) The degree of π is $\deg(\pi) = \mathrm{rank}(G) \coloneqq \dim_k(\mathcal{O}_G(G))$.

(3) If $f \coloneqq \pi \circ \mathrm{act} \colon X \times G \to X/G$ is the standard map, then

$$\mathcal{O}_{X/G} \longrightarrow \pi_* \mathcal{O}_X \underset{\mathrm{pr}_2^*}{\overset{\mathrm{act}^*}{\rightrightarrows}} f_* \mathcal{O}_{X \times G}$$

is an equalizer diagram[32].

(4) If k is algebraically closed, then π induces a bijection $X(k)/G(k) \xrightarrow{\simeq} (X/G)(k)$ and a homeomorphism $|X|/|G| \xrightarrow{\simeq} |X/G|$ on topological spaces.

(5) The pullback π^* gives an equivalence of categories[33]

$$\mathrm{QCoh}(X/G) \xrightarrow{\simeq} \{G\text{-equivariant quasicoherent sheaves on } X\}.$$

(6) There is a norm map[34] $\mathrm{Nm} \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X/G)$ *with the property that the composition*

$$\mathrm{Pic}(X/G) \xrightarrow{\pi^*} \mathrm{Pic}(X) \xrightarrow{\mathrm{Nm}} \mathrm{Pic}(X/G)$$

is the multiplication-by-$\mathrm{rank}(G)$ *map* $\mathcal{L} \mapsto \mathcal{L}^{\otimes \mathrm{rank}(G)}$.

[30]This condition is satisfied e.g. if X is quasi-projective over k: indeed, in a projective space, one can always find a hypersurface that avoids a finite sets of points, and thus the complement of an ample divisor contains this finite set. If the group G is no longer assumed to be finite, one must replace this condition with the following: any G-orbit is contained in an affine open subset.

[31]If X is a scheme with a G-action, a map $\varphi \colon X \to Y$ is *G-invariant* if there is an equality $\varphi \circ \mathrm{act} = \varphi \circ \mathrm{pr}_2$ of maps $G \times X \to Y$.

[32]In the special case when $X = \mathrm{Spec}(A)$, then (3) asserts that $X/G = \mathrm{Spec}(B)$, where B is the kernel of the map $A \to \prod_{g \in G} A$ given by $a \mapsto (g(a) - a)_{g \in G}$; that is, B is the subring of G-invariants of A.

[33]This is a special case of faithfully flat descent.

[34] In the affine case, the norm map can be constructed explicitly: if $A \to B$ is a finite flat ring extension, then any $b \in B$ acts on B via the left multiplication map $m_b \colon B \to B$. As B is a finite projective A-module, $\det(m_b) \in A$. Setting $\mathrm{Nm}(b) \coloneqq \det(m_b)$ gives a group homomorphism $\mathrm{Nm} \colon B^* \to A^*$, which globalizes to a map on Picard groups. For details, see [Sta17, Tag 0BCX].

In the condition (1), the assertion that $(\mathrm{act}, \mathrm{pr}_2)$ be an isomorphism can be rephrased as demanding that the commutative diagram

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{act}"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & G \end{tikzcd}$$

is Cartesian. Said differently, the fibres of π "look like copies of G". From this description, we immediately see (2), as well as the fact that X/G is affine whenever X is.

The conclusion of (4) is false without the hypothesis that k is algebraically closed: e.g. if k is a field such that $k^2 \subsetneq k$, then the squaring map $\mathbf{G}_{m,k} \to \mathbf{G}_{m,k}$ given by $z \mapsto z^2$ is not surjective on k-points, but it can be identified with the quotient map for the $\mathbf{Z}/2$-action on $\mathbf{G}_{m,k}$ given by $z \mapsto -z$.

Example 13.13. If A is an abelian variety and $G = A[n]$ is the n-torsion subgroup of A (viewed as acting on A by translation), (this is a subgroup of A so it acts on A by translation), then $A/G \simeq A$ via $[n]\colon A \to A$.

We will not prove Theorem 13.12, but refer the reader to [MFK94, §0-1], as well as [BLR90, §6] for a general discussion of (fppf) descent and torsors. Note that properties (2-6) in Theorem 13.12 follow from (1) since we assume that G is finite. Next time, we will use Theorem 13.12 to show that Pic^0 exists as a variety.

14. October 24th

Last time, we stated a general theorem about quotients of schemes by the actions of finite groups. The goal of this lecture (and the next) is to use this general theorem to construct the dual abelian variety.

14.1. **Construction of the Dual Abelian Variety.** Let A be an abelian variety over a field k (we do not assume that k is algebraically closed). The goal is to attach a "canonical" abelian variety structure to the set $\mathrm{Pic}^0(A)$ (by using the isomorphism $(A/K(L))(k) \simeq \mathrm{Pic}^0(A)$ established in Proposition 13.1 in the case of an algebraically closed ground field).

The first step is to ignore the description above, and instead write what the functor of points of the dual abelian variety must be, and show a representability result.

Theorem 14.1. *Consider the category $\mathcal{C}_A$ of triples (S, L, ι), where*

(1) S is a k-scheme;
(2) $F \in \mathrm{Pic}(S \times A)$ is a line bundle such that $F|_{\{s\}\times A} \in \mathrm{Pic}^0(A)$ for all geometric points $s \in A$;
(3) $\iota\colon F|_{S\times\{e\}} \simeq \mathcal{O}_S$ is an isomorphism (called a rigidification *of L).*

Then, the category $\mathcal{C}_A$ has a final object, denoted $(A^t, \mathcal{P}, \iota_{\mathrm{univ}})$; that is, for any k-scheme S, there is a natural bijection

$$\mathrm{Hom}_k(S, A^t) \xrightarrow{\simeq} \{(F, \iota)\colon (S, F, \iota) \in \mathcal{C}_A\}/\simeq$$

given by pulling back $(\mathcal{P}, \iota_{\mathrm{univ}})$ along $S \to A^t$.

Including the rigidification ι in the definition of the category $\mathcal{C}_A$ forces any automorphism of an object in $\mathcal{C}_A$ to be the identity (i.e. the category $\mathcal{C}_A$ is *discrete*). If one wishes to remove this added datum, one must use stacks instead.

To determine the abelian variety structure on A^t, it remains to specify the identity element by Corollary 2.5. Under the bijection described in Theorem 14.1, there is a map $\mathrm{Spec}(k) \to A^t$ corresponding the structure sheaf $\mathcal{O}_A$ on A with the standard isomorphism of it with itself. The identity e of A^t is precisely this map.

The strategy of the proof of Theorem 14.1 (following [Mum08, §13]) is the following:

(1) choose an ample line bundle $L \in \mathrm{Pic}(A)$ and define $A^t := A/K(L)$;
(2) construct the universal line bundle $(\mathcal{P}, \iota_{\mathrm{univ}})$, called the *Poincaré bundle*;
(3) check that $(A^t, \mathcal{P}, \iota_{\mathrm{univ}})$ satisfies the universal property.

Unsurprisingly, (1) and (2) are easy, and (3) is the difficult part; note that, once (3) is known, it follows a posteriori that A^t is independent of the choice of ample line bundle L. The modern proof of Theorem 14.1 is very different: it uses Artin's representability theorem [Sta17, Tag 07SZ, Tag 07Y3], but this requires the machinery of stacks.

Let us proceed with the proof of Theorem 14.1. Fix an ample line bundle L on A. Form the Mumford bundle $\Lambda(L) \in \mathrm{Pic}(A \times A)$ and the finite subgroup scheme $K(L) \subseteq A$, and set $A^t \coloneqq A/K(L)$, where the quotient is taken in the sense of Theorem 13.12.

To construct the Poincaré bundle $\mathcal{P}$ on A^t, the idea is to descend the Mumford bundle $\Lambda(L)$ from $A \times A$ (here, we thinking of $\mathcal{P}$ as a line bundle on $A^t \times A = (A \times A)/(K(L) \times \{0\})$, so "descent" is in the sense of Theorem 13.12(5)).

Lemma 14.2. *The Mumford bundle $\Lambda(L)$ is $K(L)$-equivariant; more precisely, a $K(L)$-equivariant structure on $\Lambda(L)$ is uniquely determined after fixing an isomorphism $L|_{\{e\}} \simeq k$.*

Proof. For any k-scheme T and any $x \in A(T)$, notice that

$$t^*_{(x,e)}(\Lambda(L)_T) = t^*_{(x,e)}(\Lambda(L_T)) = t^*_{(x,e)}\left(m_T^*(L_T) \otimes \mathrm{pr}_1^*(L_T^{-1}) \otimes \mathrm{pr}_2^{-1}(L_T)\right) = m_T^* t_x^*(L_T) \otimes \mathrm{pr}_1^*\, t_x^*(L_T^{-1}) \otimes \mathrm{pr}_2^*(L_T^{-1}).$$

Now, if $x \in K(L)(T)$, then we can write

$$t_x^*(L_T) \simeq L_{T^{-1}} \otimes M_0$$

for some line bundle M_0 that is pulled back from T. Substituting this into the above expression for $t^*_{(x,e)}(\Lambda(L)_T)$ yields

$$t^*_{(x,e)}(\Lambda(L_T)) = (m_T^*(L_T) \otimes m_T^*(M_0)) \otimes (\mathrm{pr}_1^*(L_T^{-1}) \otimes \mathrm{pr}_1^*(M_0^{-1})) \otimes \mathrm{pr}_2^*(L_T^{-1}). \tag{14.1}$$

The two maps $\mathrm{pr}_T \circ m_T \colon (A \times_k A)_T \to T$ and $\mathrm{pr}_T \circ \mathrm{pr}_1 \colon (A \times_k A)_T \to T$ coincide (even better, in the world of T-schemes, all maps to T coincide), and hence $m_T^*(M_0) \simeq \mathrm{pr}_1^*(M_0)$. Thus, (14.1) simplifies to give an isomorphism

$$t^*_{(x,e)}(\Lambda(L)) \simeq \Lambda(L).$$

In the above, we were not precise enough: we must fix an isomorphism $t^*_{(x,e)}(\Lambda(L)) \simeq \Lambda(L)$ (a priori, there is a $\mathbf{G}_m$ worth of choices). To do so, we can use the rigidification.

It suffices to fix this isomorphism after pullback along the map $i \colon A_T \hookrightarrow (A \times_k A)_T$ given by inclusion into the first factor, i.e. $x \mapsto (x, e)$ on points (this follows because the global sections of A_T and $(A \times_k A)_T$ coincide; this is carefully discussed in [Mum08, §13]). The composition $t_{(x,e)} \circ i \colon A_T \to A_T$ is equal to $i \circ t_x$, so we can write

$$i^*(t^*_{(x,e)}(\Lambda(L_T))) = t_x^* i^*(\Lambda(L)) \simeq t_x^*\left(\underline{V}\right),$$

where $V = L|_{\{e\}}$ is the fibre of L at origin e, and $\underline{V}$ denotes the constant line bundle with fibre equal to V. Similarly, $i^*(\Lambda(L_T)) \simeq \underline{V}$. Therefore, it suffices to fix an isomorphism between $t_x^*(\underline{V})$ and $\underline{V}$, but the rigidification gives a canonical choice for this isomorphism. □

The upshot of Lemma 14.2 is that the canonical $K(L)$-equivariant structure on $\Lambda(L)$ descends to give a line bundle $\mathcal{P}$ on $A^t \times A$ by Theorem 13.12(5).

To construct the rigidification ι_{univ}, we must specify an isomorphism $\mathcal{P}|_{A^t \times \{e\}} \simeq \mathcal{O}_{A^t}$. By descent, it suffices to give an isomorphism $\Lambda(L)|_{A \times \{e\}} \simeq \mathcal{O}_A$ that is compatible with the $K(L)$-equivariant structure on $\Lambda(L)$. Observe that

$$\Lambda(L)|_{A \times \{e\}} = (m^*(L) \otimes \mathrm{pr}_1^*(L^{-1}) \otimes \mathrm{pr}_2^*(L^{-1}))|_{A \times \{e\}} = L \otimes L^{-1} \otimes \underline{L^{-1}|_{\{e\}}} \simeq \underline{L^{-1}|_{\{e\}}}, \tag{14.2}$$

where recall that $\underline{L^{-1}|_{\{e\}}}$ denotes the constant line bundle with fibre $L^{-1}|_{\{e\}}$. Thus, a choice of a trivialization $L|_{\{e\}} \simeq k$, combined with (14.2), gives an isomorphism $\Lambda(L)|_{A \times \{e\}} \simeq \mathcal{O}_A$. One must check that this is compatible with the $K(L)$-equivariant structure, but this is left as an exercise.

We have constructed the triple $(A^t, \mathcal{P}, \iota_{\text{univ}})$, and now we must show that it has the correct universal property, i.e. given $(S, F, \iota) \in \mathcal{C}_A$, we must show that there is a unique map $\alpha\colon S \to A^t$ such that (F, ι) is pulled back from $(\mathcal{P}, \iota_{\text{univ}})$ along $(\alpha, \text{id})\colon S \times A \to A^t \times A$. For simplicity, we will ignore the rigidifications.

The map α is constructed by first building its graph. Consider the line bundle $M := \text{pr}_{13}^*(F^{-1}) \otimes \text{pr}_{23}^*(\mathcal{P})$ on $S \times A^t \times A$, and the maximal closed subscheme $\Gamma_S \subseteq S \times A^t$ such that M is pulled back from Γ_S.

Claim 14.3. The composition $\Gamma_S \subseteq S \times A^t \xrightarrow{\text{pr}_1} S$ is an isomorphism; in particular, the composition

$$\alpha\colon S \simeq \Gamma_S \subseteq S \times A^t \xrightarrow{\text{pr}_2} A^t$$

is the map in the universal property above.

The proof of the claim will be broken up into a sequence of seven steps.

Step 1: Preliminary reductions: the formation of Γ_S is compatible with base change, so we may first assume that k is algebraically closed. It suffices to check that $\Gamma_S \to S$ is an isomorphism after base change to every Artin local ring $S = \text{Spec}(B)$ for an Artin local ring B with residue field k. In particular, S has only one point, i.e. $|S| = \{s\}$.

By hypothesis, we have $F|_{\{s\}\times A} \in \text{Pic}^0(A)$, i.e. $F|_{\{s\}\times A} = \mathcal{P}|_{\{b\}\times A}$ for some $b \in A^t(k)$ (using the fact that $A^t(K) \to \text{Pic}^0(A)$ is surjective). Replacing M by $M \otimes \text{pr}_3^*(\mathcal{P}^{-1}|_{\{b\}\times A})$ leaves Γ_S is unchanged (since the additional factor is a pullback from the third factor). Thus, we may assume that $F|_{\{s\}\times A} \simeq \mathcal{O}_A$.

Step 2: Freeness of the cohomology of M: The cohomology groups $H^i(S\times A^t\times A, M)$ are naturally B-modules, and we claim that they are all free.

Observe that $M|_{\{s\}\times A^t\times\{a\}}$ is a degree-zero line bundle on A^t for any base point $a \in A(k)$. Indeed, this holds for $a = e$ by assumption, and hence it holds everywhere by Lemma 12.9.

Claim 14.4. There exist only finitely-many $a \in A(k)$ for which $M|_{\{s\}\times A^t\times\{a\}}$ is trivial.

Proof. Consider the quotient map $\pi\colon A \to A^t = A/K(L)$, and the pullback

$$\pi^*\left(M|_{\{s\}\times A^t\times\{a\}}\right) = \pi^*\left(\mathcal{P}|_{A\times\{a\}}\right) = \Lambda(L)|_{A\times\{a\}} \simeq t_a^*(L) \otimes L^{-1}.$$

In particular, if $M|_{\{s\}\times A^t\times\{a\}}$ is trivial, then $a \in K(L)$; however, $K(L)$ is finite, so there are only finitely-many such a's. □

The upshot of Claim 14.4 is that the sheaf $\bigoplus_{i\in\mathbf{Z}} R^i\,\text{pr}_{13,*}(M)$ has finite support (as the support has only finitely many k-points). Now we can play the same game as in the proof of Proposition 13.1: as this sheaf has finite support, the Leray spectral sequence for pr_{13} degenerates to give an isomorphism

$$H^i(S \times A^t \times A, M) \simeq H^0(S \times A, R^i\,\text{pr}_{13,*}(M)).$$

By looking at a small enough neighbourhood of the support of $\bigoplus_{i\in\mathbf{Z}} R^i\,\text{pr}_{13,*}(M)$, we may assume that F is trivial, and hence the projection formula gives isomorphisms

$$H^0(S \times A, R^i\,\text{pr}_{13,*}(M)) \simeq H^0(S \times A, R^i\,\text{pr}_{13,*}(\text{pr}_{23}^*\,\mathcal{P})) \simeq B \otimes_k H^i(A^t \times A, \mathcal{P}).$$

In particular, $H^i(S \times A^t \times A, M)$ is a free B-module.

Next time, we will complete the proof of Claim 14.3.

15. October 26th

We continue with the construction of the dual abelian variety, which began last class.

15.1. **Construction of the Dual Abelian Variety (Continued).** Let A be an abelian variety of dimension g over k, and let L be an ample line bundle. Consider the quotient $A^t \coloneqq A/K(L)$ in the sense of Theorem 13.12. If one identifies $A^t \times A \simeq (A \times A)/(K(L) \times \{e\})$, then the Poincaré bundle $\mathcal{P}$ is the line bundle on $A^t \times A$ that is descended from $\Lambda(L)$ (this exists by Lemma 14.2).

Consider a triple (S, F, ι), where S is a k-scheme, $F \in \mathrm{Pic}(S \times A)$ is a line bundle such that $F|_{\{s\} \times A} \in \mathrm{Pic}^0(A)$ for all geometric points s of S, and $\iota\colon F|_{S \times \{e\}} \simeq \mathcal{O}_S$ is a trivialization of the fibre of F above the origin. Set $M \coloneqq \mathrm{pr}_{23}^*(\mathcal{P}) \otimes \mathrm{pr}_{13}^*(F^{-1})$ on $S \times A^t \times A$.

Let $\Gamma_S \subseteq S \times A^t$ be the maximal closed subscheme such that M is pulled back from Γ_S along pr_2. Following the strategy outlined in the previous class (as in Claim 14.3), it remains to show that $\Gamma_S \xrightarrow{\simeq} S$ via pr_1. The proof of this claim proceeds in six steps, the first two of which were discussed last time, and we briefly recall below:

Step 1: Preliminary reductions: using that the formation of Γ_S commutes with base change, we may assume that k is algebraically closed and $S = \mathrm{Spec}(B)$ for some Artin local ring B with residue field k. Let $s \in S$ be the closed point. In addition, after possibly tensoring M with a line bundle pulled back from A, we may assume that $F|_{\{s\} \times A}$ is trivial. Broadly, the goal is to analyze the cohomology of M in order to say something about Γ_S.

Step 2: Freeness of the cohomology of M: we saw last time that the sheaf $\bigoplus_{i \in \mathbf{Z}} R^i \mathrm{pr}_{13,*}(M)$ is supported on the finite k-scheme $S \times K(L)$. The Leray spectral sequence always degenerates for a coherent sheaf with finite support, so there is a non-canonical isomorphism

$$H^i(S \times A^t \times A, M) \simeq H^0(S \times A^t, R^i(\mathrm{pr}_{13})_*(M)),$$

with the non-canonicity coming from a choice of isomorphism between F and the trivial line bundle on a sufficiently small neighbourhood of s. Now $H^0(S \times A^t, R^i \mathrm{pr}_{13,*}(M))$ is a free B-module, and hence $H^i(S \times A^t \times A, M)$ is as well.

Step 3: Vanishing of most of the cohomology of M via pr_{12}: if $a \in A^t(k)$ is such that $M|_{\{s\} \times \{a\} \times A}$ is non-trivial, then $R^i \mathrm{pr}_{12,*}(M)$ vanishes in a small neighbourhood of $(s, a) \in S \times A^t$, by semicontinuity. As $M|_{\{s\} \times \{a\} \times A} = \mathcal{P}|_{\{a\} \times A}$, it follows that $\mathrm{Supp}(R^i \mathrm{pr}_{12,*}(M)) \subseteq S \times \{e\}$.

Set $R \coloneqq B \otimes_k \mathcal{O}_{A^t,e}$. The sheaf $R^i \mathrm{pr}_{12,*}(M)$ is coherent and supported only at the point (s, e), so it can be thought of as an Artinian module over R. Consider the perfect complex

$$K^\bullet = \left(K^0 \longrightarrow K^1 \longrightarrow \ldots \longrightarrow K^g\right)$$

over R that "universally computes the cohomology" of the higher direct images $R^i \mathrm{pr}_{12,*}(M)$, in the sense of Proposition 5.3. This perfect complex $K^\bullet$ has the following special feature.

Lemma 15.1. [Mum08, p. 118] *If $\mathcal{O}$ is a regular local ring of dimension g and $M^\bullet = \left(M^0 \to M^1 \to \ldots \to M^g\right)$ is a perfect complex of $\mathcal{O}$-modules such that $H^i(M^\bullet)$ is Artinian for all i, then $H^i(M) = 0$ for all $i < g$.*

Applying Lemma 15.1 to $K^\bullet$, now viewed as a complex of $\mathcal{O}_{A^t,e}$-modules (instead of just as an R-module), implies that $H^i(K^\bullet) = 0$ for all $i < g$. However, by construction of the complex $K^\bullet$, we have $R^i \mathrm{pr}_{12,*}(M) = H^i(K^\bullet) = 0$ for $i < g$.

Consider the only remaining pushforward $N \coloneqq R^g \mathrm{pr}_{12,*}(M)$, which is an Artinian R-module. We can play the same game as in Step 2: the Leray spectral sequence for pr_{12} gives isomorphisms

$$H^i(S \times A^t \times A, M) \simeq H^0(S \times A^t, R^i(\mathrm{pr}_{12})_* M)$$

since all $R^i \mathrm{pr}_{12,*}(M)$ have finite support. As all higher direct images vanish except possibly in top degree, we see that

$$H^i(S \times A^t \times A, M) \simeq \begin{cases} 0 & i < g \\ N & i = g \end{cases} \tag{15.1}$$

The cohomology groups in (15.1) are free B-modules by Step 2, so N is free over $B \subseteq R$.

Step 4: $\Gamma_S \to S$ is a homeomorphism: As S is a one-point topological space, it suffices to show that Γ_S consists of a single point. The fibre of M above any point of Γ_S is trivial, so has non-vanishing cohomology and hence Γ_S is contained in the support of $\bigoplus_{i\in\mathbf{Z}} R^i \operatorname{pr}_{12,*}(M)$, which is precisely $\{(s,e)\}$ by Step 3. It remains to show that (s,e) actually lies in Γ_S, but this is easy to check: both $\operatorname{pr}_{23}^*(\mathcal{P})|_{\{s\}\times\{e\}\times A}$ and $\operatorname{pr}_{13}^*(F^{-1})|_{\{s\}\times\{e\}\times A}$ are trivial, and hence $M|_{\{s\}\times\{e\}\times A}$ is also trivial, i.e. $(s,e)\in\Gamma_S$.

Therefore, $|\Gamma_S| = \{(s,e)\}$ (and it is certainly the case that the one-point space $\{(s,e)\}$ maps homeomorphically to the one-point set $\{e\}$ under the projection map).

But we are algebraic geometers in the 21st century! We are not satisfied doing this at the level of points! We want it at the level of schemes! With this in mind...

Step 5: Finding an explicit scheme structure on Γ_S: we must return to our proof of the existence of Γ_S, as in Proposition 6.1. Let $K^\bullet$ be the perfect complex from Step 3, set

$$K^{\bullet,\vee} = (K^{g,\vee} \to K^{g-1,\vee} \to \ldots \to K^{0,\vee})$$

to be the R-linear dual of $K^\bullet$, and set $Q \coloneqq \operatorname{coker}\left(d^\vee \colon K^{1,\vee} \to K^{0,\vee}\right)$. The universal property asserts that

$$\operatorname{Hom}_R(Q,T) \simeq H^0(\operatorname{Spec}(T) \times_{S\times A^t} (S\times A^t\times A), M) \tag{15.2}$$

for any R-algebra T. Repeating the argument of Proposition 6.1, we can apply (15.2) to $T = k$ (thought of as the residue field of R) to get

$$\operatorname{Hom}_R(Q,k) \simeq H^0(\{s\}\times\{e\}\times A, M|_{\{s\}\times\{e\}\times A}) \simeq H^0(A,\mathcal{O}_A) \simeq k.$$

Now, as $\operatorname{Hom}_R(Q,k) \simeq \operatorname{Hom}_K(Q\otimes_R k, k)$ is 1-dimensional over k, Nakayama's lemma implies that Q is cyclic, so there is a non-canonical isomorphism $Q \simeq R/I$. One can check that $\Gamma_S = \operatorname{Spec}(R/I) \subseteq \operatorname{Spec}(R)$.

Step 6: $\Gamma_S \to S$ is an isomorphism: we must show that the composition $B \to R = B\otimes_k \mathcal{O}_{A^t,e} \to R/I$ is an isomorphism.

For injectivity, first note that $H^i(K^{\bullet,\vee})$ is an Artinian module for all $i\in\mathbf{Z}$, since the same is true for $K^\bullet$. Moreover, applying Lemma 15.1 again gives that $0 \to K^{\bullet,\vee} \to Q \to 0$ is a resolution. As $I\cdot Q = 0$, we have $I\cdot H^i(G(K^{\bullet,\vee})) = 0$ for any R-linear functor G. In particular, taking $G = \operatorname{Hom}_R(-,R)$, we get $I\cdot H^i(K) = 0$ for all $i\in\mathbf{Z}$. The cohomology $H^g(K^\bullet) = N$ is a free B-module and $I\cdot N = 0$, it follows that $I\cap B = \{0\}$. One can also check that N is nonzero, so $B \to R/I$ is injective.

For surjectivity, Nakayama's lemma implies that it suffices to work modulo the maximal ideal $\mathfrak{m}_B$. Recall that formation of Γ_S commutes with base change, so we have reduced to the case $S = \operatorname{Spec}(k)$. In this case, $\operatorname{Spec}(R/I) \hookrightarrow A^t$ is the maximal closed subscheme such that $\mathcal{P}$ is pulled back from $\operatorname{Spec}(R/I)$. Consider the Cartesian diagram

$$\begin{tikzcd} A\times A \arrow[r, "(\pi,\mathrm{id})"] \arrow[d, "\mathrm{pr}_1"'] & A^t\times A \arrow[d, "\mathrm{pr}_1"] \\ A \arrow[r, "\pi"] & A^t \end{tikzcd}$$

where $\pi\colon A\to A^t$ is the quotient map. Then, $\pi^{-1}(\operatorname{Spec}(R/I))$ is the maximal closed subscheme of A such that $(\pi,\mathrm{id})^*(\mathcal{P})$ is pulled back from below; however, $(\pi,\mathrm{id})^*(\mathcal{P}) = \Lambda(L)$, and hence $\pi^{-1}(\operatorname{Spec}(R/I)) = K(L)$. Now, π is a finite flat map of degree $\operatorname{rank}(K(L))$, which is also the length of $K(L)$. It follows that the length of R/I is 1, and hence $R/I \simeq k$. This completes the proof of the claim, and hence of the construction of the dual abelian variety.

Remark 15.2. There are now many different ways to construct the Picard scheme, e.g. using Artin's representability theorem. The proof presented above, following [Mum08, §13], is very typical for moduli problems, e.g. the construction of the moduli space of curves proceeds similarly.

The construction of the dual abelian variety

Corollary 15.3. *If A is an abelian variety of dimension g over k, then*

$$H^i(A^t \times A, \mathcal{P}) = \begin{cases} 0 & i < g \\ k & i = g \end{cases}$$

Proof. The computation for $i < g$ is immediate from Step 3 by taking $S = \mathrm{Spec}(k)$. For $i = g$, let $Q \simeq R/I$ be as in Step 5, then $Q = H^g(K^{\bullet,\vee})$ by definition, so $K^{\bullet,\vee}$ is in fact a resolution of k. Over the regular local ring $\mathcal{O}_{A^t,e}$, any two resolutions are homotopy equivalent to one another; in particular, the Koszul resolution $M^\bullet$ of k is homotopy equivalent to $K^{\bullet,\vee}$. The Koszul complex is self-dual, so $K^\bullet$ is homotopy equivalent to $M^\bullet$. It follows that $H^i(A^t \times A, \mathcal{P}) = H^g(K^\bullet) = k$. $\square$

Corollary 15.4. *If A is an abelian variety of dimension g over k, then $H^1(A, \mathcal{O}_A)$ has rank g and $H^i(A, \mathcal{O}_A) = \Lambda^i H^1(A, \mathcal{O}_A)$ has rank $\binom{g}{i}$.*

If one is more careful with the proof of Corollary 15.4, one can show that the cohomology algebra of A is exactly the exterior algebra on $H^1(A, \mathcal{O}_A)$.

Proof. The proof of Corollary 15.3 gives a homotopy equivalence between $K^\bullet$ and the Koszul complex over R, where $K^\bullet$ is the perfect complex that universally computes the cohomology of $R^i \mathrm{pr}_{1,*}(\mathcal{P})$. For the Koszul complex over R, the i-th term is $\Lambda^i R^{\oplus g}$, and the differentials are zero modulo the maximal ideal of R. Taking fibres over the identity e (algebraically, tensor with the residue field k) $- \otimes_R k$), $R^i \mathrm{pr}_{1,*}(\mathcal{P})|_{\{e\}}$ gives $H^i(A, \mathcal{O}_A)$, and the fibre of the Koszul complex gives

$$0 \longrightarrow \Lambda^g k^g \xrightarrow{0} \dots \xrightarrow{0} \Lambda^2 k^g \xrightarrow{0} k^g \xrightarrow{0} k \longrightarrow 0,$$

from which the result follows. $\square$

15.2. **Duality.** The goals of this section are to explain that $A \mapsto A^t$ is contravariantly functorial in the abelian variety, and that there is a canonical isomorphism $A \xrightarrow{\simeq} (A^t)^t$.

Let us discuss the functoriality: given a homomorphism $f\colon A \to B$ of abelian varieties over k, then we must construct a map $B^t \to A^t$ on the dual abelian varieties. Consider the map

$$g = (\mathrm{id}, f)\colon B^t \times A \to B^t \times B$$

and the pullback $g^*(\mathcal{P}_B) \in \mathrm{Pic}(B^t \times A)$. From the universal property of the dual abelian variety, to construct a map $B^t \to A^t$, we must check that $g^*(\mathcal{P}_B)$ comes with a trivialization at the origin of A, and all fibres of pr_1 lie in $\mathrm{Pic}^0(A)$.

For any geometric point $b \in B^t(k)$, we have

$$g^*(\mathcal{P}_B)|_{\{b\} \times A} \simeq f^*(\mathcal{P}_B|_{\{b\} \times B}),$$

and $\mathcal{P}_B|_{\{b\} \times B}$ is a degree-zero line bundle on B, and it is the trivial line bundle when $b = e$. It follows that $g^*(\mathcal{P}_B)|_{\{b\} \times A}$ is trivial if $b = e$, so all fibres must lie in $\mathrm{Pic}^0(A)$ by Lemma 12.9.

In order to get a map $B^t \to A^t$, we it remains to find a trivialization of $g^*(\mathcal{P}_B)|_{B^t \times \{e_A\}}$. Since f is a homomorphism of abelian varieties, it sends the origin to the origin, so $g^*(\mathcal{P}_B)|_{B^t \times \{e_A\}}$ is canonically isomorphic to $\mathcal{P}_B|_{B^t \times \{e_B\}}$, and this comes with a preferred trivialization.

Thus, by the universal property of A^t, there is a map $f^t\colon B^t \to A^t$ such that

$$(f^t, \mathrm{id})^*(\mathcal{P}_A) \simeq (\mathrm{id}, f)^*(\mathcal{P}_B). \tag{15.3}$$

The map f^t is often called the *dual isogeny* of f, the defining feature of which is the isomorphism (15.3) between the pullbacks of Poincaré bundles.

16. October 31st

Last time, we completed the construction of the dual abelian variety. If A is an abelian variety of dimension g over k, then we showed the following:

(1) the functor $(\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Sets})$ given by
$$S \mapsto \left\{(F,\iota)\colon \begin{matrix} F \in \mathrm{Pic}(S\times A) \text{ such that } F|_{\{s\}\times A} \in \mathrm{Pic}^0(A_{k(s)}) \text{ for all geometric points } s \text{ of } S, \\ \text{and a trivialization } i\colon F|_{S\times\{e\}} \simeq \mathcal{O}_S \end{matrix}\right\} / \simeq$$
is representable by a k-scheme A^t, the Poincaré bundle $\mathcal{P}_A$, and the universal trivialization ι_{univ}.

(2) The cohomology of the Poincaré bundle is given by
$$H^i(A^t\times A, \mathcal{P}_A) = \begin{cases} 0 & i<g \\ k & i=g \end{cases},$$
In fact, we proved the stronger statement that
$$R^i\,\mathrm{pr}_{1,*}(\mathcal{P}_A) = \begin{cases} 0 & i<g \\ k(e) & i=g \end{cases},$$
where $k(e)$ is the skyscraper sheaf at $e \in A^t(k)$.

(3) [Functoriality] If $f\colon A\to B$ is a homomorphism of abelian varieties, then there is a homomorphism $f^t\colon B^t\to A^t$ of abelian varieties characterized by
$$(f^t,\mathrm{id})^*(\mathcal{P}_A) \simeq (\mathrm{id},f)^*(\mathcal{P}_B)$$
on $B^t\times A$. One can check the formation of f^t is compatible with composition, so $A\mapsto A^t$ is a functor.

Exercise 16.1. For an elliptic curve E over k, show that $E^t\simeq E$ and $\mathcal{P}_E$ is identified with
$$\mathcal{O}_{E\times E}\left(\Delta - \mathrm{pr}_1^*(e) - \mathrm{pr}_2^*(e)\right)$$
under this isomorphism, where $\Delta\colon E\hookrightarrow E\times E$ is the diagonal.

16.1. **Duality (Continued).**

Theorem 16.2. [Biduality] *There is a canonical isomorphism* $\alpha\colon A \xrightarrow{\simeq} (A^t)^t$.

Proof. To construct the map α, we can use the Poincaré bundle $\mathcal{P}_A\in\mathrm{Pic}(A\times A^t)$ and the universal property of A^t. First, we must show that $\mathcal{P}_A|_{\{x\}\times A^t}\in\mathrm{Pic}^0(A^t)$ for any geometric point x of A. This holds at $x=e$ using the trivialization ι_{univ}, and hence it holds everywhere by Lemma 12.9.

In addition, we must give an isomorphism $\mathcal{P}_A|_{A\times\{e\}}\simeq\mathcal{O}_A$. Such an isomorphism exists because the map $A^t(k)\simeq\mathrm{Pic}^0(A)$is given by $x\mapsto\mathcal{P}_A|_{A\times\{x\}}$, and we can fix a particular isomorphism by requiring it to agree with $\iota_{\mathrm{univ}}|_{\{e\}\times\{e\}}$ over $\{e\}\times\{e\}\subseteq A\times\{e\}$ (this is sufficient since $\{e\}\times\{e\}$ and $A\times\{e\}$ have the same global sections).

Therefore, using the universal property of A^t, we get a map $\alpha\colon A\to(A^t)^t$ that induces a natural isomorphism
$$\pi^*(\mathcal{P}_{A^t})\simeq\mathcal{P}_A, \tag{16.1}$$
where $\pi=(\alpha,\mathrm{id})\colon A\times A^t\to(A^t)^t\times A^t$. To check that α is an isomorphism, we will show that α is finite (hence surjective, since A and A^t have the same dimension), flat (by miracle flatness), and of degree 1. Then, we use the fact that a finite, flat map of degree 1 is an isomorphism.

It is left as an exercise that α is finite (one uses the description $A\times A\to A^t\times A=(A\times A)/(K(L)\times\{e\})$ and argue as in the construction of the dual abelian variety). It remains to show that $\deg(\alpha)=1$, for which we require the following general lemma.

Lemma 16.3. *Let X be a proper variety over k, and let G be a finite group scheme acting freely on X. If $\mathcal{F}$ is a coherent sheaf on X/G, then*
$$\chi(X/G, \mathcal{F}) \cdot \deg(\pi) = \chi(X, \pi^* \mathcal{F})$$
where $\pi\colon X \to X/G$ is the quotient map.

The proof of Lemma 16.3 is similar to Proposition 11.6; see [Mum08, §12] for details.

Applying Lemma 16.3 to the map $\pi = (\alpha, \mathrm{id})$ with $G = \ker(\alpha)$, we see that
$$\deg(\pi) \cdot (-1)^g = \deg(\pi) \cdot \chi((A^t)^t \times A^t, \mathcal{P}_{A^t}) = \chi(A \times A^t, \pi^* \mathcal{P}_{A^t}) = \chi(A \times A^t, \mathcal{P}_A) = (-1)^g,$$
where the third equality follows from (16.1). It follows that $\deg(\pi) = 1$. $\square$

16.2. **Fourier–Mukai Transforms.** The goal is to now develop categorical relationships between an abelian variety and its dual, which are provided by certain integral transforms on the derived category. The immediate goal is to prove Fourier–Mukai equivalence. As applications, we aim to discuss Atiyah's classification of vector bundles on an elliptic curve. First, let us fix some notation and recall the basic tools in the study of derived categories.

Let k be a field. All schemes are of finite type over k, and let Sch_k denote the category of such schemes. We write equality for natural equivalence.

Notation 16.4. Let $X \in \mathrm{Sch}_k$.

(1) The *unbounded derived category* $\mathrm{D}(X) = \mathrm{D}_{\mathrm{qc}}(X) = \mathrm{D}_{\mathrm{qc}}(X, \mathcal{O}_X)$ of X is the full subcategory of $\mathrm{D}(X, \mathcal{O}_X)$ given by
$$\left\{K \in \mathrm{D}(X, \mathcal{O}_X)\colon H^i(K) \in \mathrm{QCoh}(X) \text{ for all } i \in \mathbf{Z}\right\}.$$
If X has affine diagonal (e.g. if X is separated), then $\mathrm{D}(X) = \mathrm{D}(\mathrm{QCoh}(X))$, but this is not always helpful for our purposes, as injective resolutions in $\mathrm{QCoh}(X)$ are complicated. Similarly, the *bounded derived category* $\mathrm{D}^b(X)$ is the full subcategory of $\mathrm{D}(X)$ consisting of complexes whose cohomology vanishes outside of a finite range. Recall that if $X = \mathrm{Spec}(R)$, then $\mathrm{D}(X) = \mathrm{D}(\mathrm{Mod}_R)$.

(2) Consider the full subcategory
$$\mathrm{D}^b_{\mathrm{coh}}(X) \coloneqq \{K \in \mathrm{D}(X)\colon H^i(K) \in \mathrm{Coh}(X) \text{ and } H^i(K) = 0 \text{ for all } |i| \gg 0\}.$$
Under certain hypotheses, $\mathrm{D}^b_{\mathrm{coh}}(X) = \mathrm{D}^b(\mathrm{Coh}(X))$, but this is again not always useful, since $\mathrm{Coh}(X)$ does not have enough injectives or projectives in general.

(3) The derived category $\mathrm{D}(X)$ has a symmetric monoidal structure, denoted by $(M, N) \mapsto M \otimes^L N$. Often, we will write $M \otimes N$ for $M \otimes^L N$, though it is important to remember that we are using the *derived* tensor product.

The main tools that are needed to work with derived categories are summarized below (and these were discussed in more detail in §7 and §8).

Proposition 16.5. *Let $f\colon X \to Y$ be a morphism in Sch_k.*

(1) There is an exact functor $Rf_*\colon \mathrm{D}(X) \to \mathrm{D}(Y)$ which is lax-monoidal, meaning that there is a natural map
$$Rf_*(M) \otimes Rf_*(N) \longrightarrow Rf_*(M \otimes N),$$
which is not an isomorphism in general. Moreover, if f is proper, then Rf_* preserves $\mathrm{D}^b_{\mathrm{coh}}$.

(2) There is an exact functor $Lf^*\colon \mathrm{D}(Y) \to \mathrm{D}(X)$ which is symmetric monoidal, meaning that it commutes with the tensor product. Moreover, Lf^* always preserves $\mathrm{D}_{\mathrm{coh}}$, and it preserves D^b and $\mathrm{D}^b_{\mathrm{coh}}$ if f is flat (or more generally, if f has finite Tor dimension).

(3) The functor Rf_* is right-adjoint to Lf^*.

(4) [Projection Formula] *For any $M \in \mathrm{D}(X)$ and $N \in \mathrm{D}(Y)$, there is a natural isomorphism*
$$N \otimes Rf_*(M) \xrightarrow{\simeq} Rf_*\left(Lf^*(N) \otimes M\right).$$

(5) [Base Change] *Given a Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

there is a natural map

$$Lg^* Rf_*(-) \longrightarrow Rf'_* Lg'^*(-) \tag{16.2}$$

of functors $\mathrm{D}(X) \to \mathrm{D}(Y')$*. In addition, (16.2) is an isomorphism if either* f *or* g *is flat (or more generally, if* f *and* g *are Tor independent).*

Mukai gave a construction of functors between derived categories using a categorification of the integral transforms that appear in Fourier analysis.

Construction 16.6. [Mukai] Given $X, Y \in \mathrm{Sch}_k$ and $K \in \mathrm{D}(X \times_k Y)$, there is a functor $\phi_K \colon \mathrm{D}(X) \to \mathrm{D}(Y)$ given by

$$N \mapsto R\,\mathrm{pr}_{2,*}\left(L\,\mathrm{pr}_2^*(N) \otimes K\right),$$

and a functor $\psi_K \colon \mathrm{D}(Y) \to \mathrm{D}(X)$ given by

$$M \mapsto R\,\mathrm{pr}_{1,*}\left(L\,\mathrm{pr}_2^*(M) \otimes K\right).$$

Functors of the form ϕ_K or ψ_K are often called *integral transforms*, and the complex K is called the *kernel* of ϕ_K or of ψ_K.

Almost all functors between derived categories that one encounters are integral transforms.

Example 16.7.

(1) If $X \in \mathrm{Sch}_k$ is separated, then $\mathrm{id} = \phi_K$ for $K = R\Delta_* \mathcal{O}_X \in \mathrm{D}(X \times_k X)$, where $\Delta \colon X \to X \times_k X$ is the diagonal. Indeed,

$$N \mapsto R\,\mathrm{pr}_{2,*}\left(L\,\mathrm{pr}_1^*(N) \otimes R\Delta_*\mathcal{O}_X\right) \simeq R\,\mathrm{pr}_{2,*}\left(R\Delta_*(N)\right) = R(\mathrm{pr}_2 \circ \Delta)_*(N) = N.$$

(2) If $f \colon X \to Y$ in Sch_k, then $Rf_* = \phi_K$ and $Lf^* = \psi_K$ for $K = Ri_*\mathcal{O}_\Gamma$, where $i \colon \Gamma \hookrightarrow X \times_k Y$ is the graph of f. Indeed, we have

$$N \mapsto \phi_K(N) = R\,\mathrm{pr}_{2,*}(L\,\mathrm{pr}_1^*(N) \otimes Ri_*\mathcal{O}_\Gamma) \simeq R\,\mathrm{pr}_{2,*}(Ri_*N) \simeq Rf_*(N).$$

Exercise 16.8. For $X \in \mathrm{Sch}_k$ and $M \in \mathrm{D}(X)$, find the kernel for the functor $N \mapsto N \otimes M$.

Remark 16.9. Orlov showed that every fully faithful functor arises as an integral transform, but recently there have been examples of functors between derived categories that are not integral transforms.

Proposition 16.10. [Convolution] *Let* $X, Y, Z \in \mathrm{Sch}_k$*. Given* $K \in \mathrm{D}(X \times Y)$ *and* $L \in \mathrm{D}(Y \times Z)$*, set*

$$K * L := R\,\mathrm{pr}_{13,*}\left(L\,\mathrm{pr}_{12}^*(K) \otimes L\,\mathrm{pr}_{23}^*(L)\right) \in \mathrm{D}(X \times Z).$$

Then, $\phi_L \circ \phi_K \simeq \phi_{K*L}$ *(and similarly for* ψ*).*

Next time, we will discuss the proof of Proposition 16.10 and the Fourier–Mukai equivalence.

17. November 2nd

The plan is to prove the Fourier–Mukai equivalence and its basic corollaries, and then discuss two applications: Atiyah's classification of vector bundles on an elliptic curve, and (possibly) Hacon's work on generic vanishing theorems using the Fourier–Mukai machinery.

17.1. **Fourier–Mukai Transforms (Continued).** We adopt the notation of Notation 16.4, except we will omit the L and R from the notation of pullback and pushforward for simplicity. Given two schemes $X, Y \in \mathrm{Sch}_k$, there is a functor $\mathrm{D}(X \times_k Y) \longrightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathrm{D}(X), \mathrm{D}(Y))$, given by

$$K \mapsto \phi_K := \mathrm{pr}_{2,*}\left(\mathrm{pr}_1^*(-) \otimes K\right).$$

Similarly, there is a functor $\mathrm{D}(X \times_k Y) \longrightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathrm{D}(Y), \mathrm{D}(X))$ given by $K \mapsto \psi_K$.

Proposition 17.1. [Convolution] *Let $X, Y, Z \in \mathrm{Sch}_k$. Given $K \in \mathrm{D}(X \times_k Y)$ and $L \in \mathrm{D}(Y \times_k Z)$, set*

$$K * L := \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*(K) \otimes \mathrm{pr}_{23}^*(L)) \in \mathrm{D}(X \times_k Z).$$

Then, $\phi_L \circ \phi_K = \phi_{K*L}$ (and similarly for ψ).

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & & X \times Y \times Z \arrow[dl, "\mathrm{p}_{XY}"'] \arrow[dr, "\mathrm{p}_{YZ}"] \arrow[ddll, bend right, "\mathrm{p}_X"'] \arrow[ddrr, bend left, "\mathrm{p}_Z"] & & \\
 & X \times Y \arrow[dl, "\mathrm{pr}_X"] \arrow[dr, "\mathrm{pr}_Y"] & & Y \times Z \arrow[dl, "\mathrm{pr}_Y"'] \arrow[dr, "\mathrm{pr}_Z"'] & \\
X & & Y & & Z
\end{tikzcd}$$

Moreover, the middle diamond is Cartesian. If $M \in \mathrm{D}(X)$, then

$$\begin{aligned}
(\phi_L \circ \phi_K)(M) &= \phi_L(\mathrm{pr}_{Y,*}(\mathrm{pr}_X^*(M) \otimes K)) \\
&= \mathrm{pr}_{Z,*}(\mathrm{pr}_Y^*(\mathrm{pr}_{Y,*}(\mathrm{pr}_X^*(M) \otimes K)) \otimes L) \\
&\simeq \mathrm{pr}_{Z,*}(\mathrm{p}_{YZ,*}(\mathrm{p}_{XY}^*(\mathrm{pr}_X^*(M) \otimes K)) \otimes L) \\
&= \mathrm{pr}_{Z,*}(\mathrm{p}_{YZ,*}(\mathrm{p}_X^* M \otimes \mathrm{p}_{XY}^* K \otimes \mathrm{p}_{YZ}^* L)) \\
&= \mathrm{p}_{Z,*}(\mathrm{p}_X^* M \otimes \mathrm{p}_{XY}^* K \otimes \mathrm{p}_{YZ}^* L),
\end{aligned}$$

where the first isomorphism follows from flat base change for the middle diamond, and the second isomorphism follows from the projection formula for p_{YZ}. Now, if $p_{XZ} \colon X \times Y \times Z \to X \times Z$ denotes the projection onto the first and third factors, we have shown that

$$(\phi_L \circ \phi_K)(M) = \mathrm{pr}_{Z,*}(\mathrm{pr}_X^* M \otimes (K * L)) = \phi_{K*L}(M).$$

The proof for ψ is similar. ☐

The upshot of Proposition 17.1 is that there is a diagram

$$\begin{tikzcd}
\mathrm{D}(X \times Y) \times \mathrm{D}(Y \times Z) \arrow[r, "(K,L) \mapsto K*L"] \arrow[d, "(\phi,\phi)"] & \mathrm{D}(X \times Z) \arrow[d, "\phi"] \\
\mathrm{Fun}\left(\mathrm{D}(X), \mathrm{D}(Y)\right) \times \mathrm{Fun}\left(\mathrm{D}(Y), \mathrm{D}(X)\right) \arrow[r] & \mathrm{Fun}\left(\mathrm{D}(X), \mathrm{D}(Z)\right)
\end{tikzcd}$$

that commutes, up to canonical isomorphism. There is an analogous diagram for the ψ's.

17.2. **The Fourier–Mukai Equivalence.** Let A be an abelian variety of dimension g over k, and let A^t be the dual abelian variety. Let $\mathcal{P}_A \in \mathrm{Pic}(A \times A^t)$ be the Poincaré bundle of A (notice that we have swapped the factors A and A^t). There is also the Poincaré bundle $\mathcal{P}_{A^t} \in \mathrm{Pic}(A^t \times A)$ (this is an abuse of notation: $\mathcal{P}_{A^t}$ is a line bundle on $A^t \times (A^t)^t$, which is canonically isomorphic to $A^t \times A$ by biduality). Furthermore, there are preferred and compatible trivializations at the origin: $\mathcal{P}_A|_{A \times \{e\}} \simeq \mathcal{O}_A$ and $\mathcal{P}_A|_{\{e\} \times A^t} \simeq \mathcal{O}_{A^t}$.

Theorem 17.2. [Mukai] *The integral transform $\phi_{\mathcal{P}_A}$ is an equivalence of triangulated categories In fact,*

$$\phi_{\mathcal{P}_{A^t}} \circ \phi_{\mathcal{P}_A} \simeq [-1]^*[-g].$$

It follows immediately from Theorem 17.2 that $\phi_{\mathcal{P}_A}$ preserves $\mathrm{D}^b_{\mathrm{coh}}$, as well.
The proof of Theorem 17.2 requires the following lemma.

Lemma 17.3. *If $\mu\colon A \times A^t \times A \to A \times A^t$ is the morphism given on points by $(a, b, c) \mapsto (m(a, c), b)$, then the line bundle*

$$\mu^*(\mathcal{P}_A^{-1}) \otimes \mathrm{pr}_{12}^*(\mathcal{P}_A) \otimes \mathrm{pr}_{23}^*(\mathcal{P}_A)$$

on $A \times A^t \times A$ is trivial.

Remark 17.4. Lemma 17.3 is a "families" version of the statement that $m^*L \simeq \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^L$ for $L \in \mathrm{Pic}^0(A)$ (as in Lemma 12.5).

Proof. Let Q be the line bundle in the statement. The idea is to apply the theorem of the cube (Corollary 6.8) to Q, and to do so we must verify that certain restrictions are trivial. One such calculation is

$$Q|_{\{e\}\times A^t \times A} = \mathcal{P}_{A^t}^{-1} \otimes \mathrm{pr}_{23}^*(\mathcal{P}_A|_{\{e\}\times A^t}) \otimes \mathcal{P}_{A^t} \simeq \mathcal{O}_{A^t}.$$

The others restrictions are computed similarly. □

Proof of Theorem 17.2. In order to show the natural equivalence $\phi_{\mathcal{P}_{A^t}} \circ \phi_{\mathcal{P}_A} \simeq [-1]^*[-g]$, we can write $\phi_{\mathcal{P}_{A^t}} \circ \phi_{\mathcal{P}_A} \simeq \phi_{\mathcal{P}_A * \mathcal{P}_{A^t}}$ by Proposition 17.1, so it suffices to identify $\mathcal{P}_A * \mathcal{P}_{A^t}$ with $\mathcal{O}_\Gamma[-g]$, where $\Gamma \subseteq A \times A$ denotes the graph of $[-1]$ (here, we are using Example 16.7).

To that end, write

$$\mathcal{P}_A * \mathcal{P}_{A^t} = \mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*(\mathcal{P}_A) \otimes \mathrm{pr}_{23}^*(\mathcal{P}_{A^t})) = \mathrm{pr}_{13,*}(\mu^*(\mathcal{P}_A)),$$

where $\mu\colon A \times A^t \times A \to A \times A^t$ is as in Lemma 17.3. Consider the Cartesian square

$$\begin{tikzcd} A \times A^t \times A \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_{13}"'] & A \times A^t \arrow[d, "\mathrm{pr}_1"] \\ A \times A \arrow[r, "m"] & A \end{tikzcd}$$

Thus, $\mathcal{P}_A * \mathcal{P}_{A^t} \simeq m^* \mathrm{pr}_{1,*}(\mathcal{P}_A)$ via flat base change. However, we know that $\mathrm{pr}_{1,*}(\mathcal{P}_A) \simeq k(e)[-g]$ by Corollary 15.3, where $k(e)$ is the skyscraper sheaf at the origin. There is another Cartesian square

$$\begin{tikzcd} \Gamma \arrow[r, hook, "i"] \arrow[d, "m"'] & A \times A \arrow[d, "m"] \\ \{e\} \arrow[r, hook, "i"] & A \end{tikzcd}$$

and flat base change for this square gives isomorphisms

$$m^* \mathrm{pr}_{1,*}(\mathcal{P}_A) \simeq m^*(k(e)[-g]) \simeq \mathcal{O}_\Gamma[-g],$$

which completes the proof of the equivalence. □

The upshot of Theorem 17.2 is that there is an equivalence $\mathrm{D}^b_{\mathrm{coh}}(A) \xrightarrow{\simeq} \mathrm{D}^b_{\mathrm{coh}}(A^t)$, which can be used to transport objects of interest on A to objects on A^t that are (hopefully) easier to understand. The meta-theorem is that global features on A become local features on A^t under this equivalence; as a concrete example, vector bundles on A are mapped to skyscraper sheaves on A^t.

17.3. **Properties of the Fourier–Mukai Equivalence.** For notational simplicity, assume throughout that the field k is algebraically closed. Let A be an abelian variety of dimension g over k. Set $\phi_A \coloneqq \phi_{\mathcal{P}_A}$.

Property 17.5. If the degree-zero line bundle $M_x \in \mathrm{Pic}^0(A)$ corresponds to the point $x \in A^t(k)$, then there is a canonical isomorphism

$$\phi_A(F \otimes M_x) \simeq t_x^* \phi_A(F)$$

for any $F \in \mathrm{D}(A)$.

Proof. If we write $\phi_A(F \otimes M_x) = \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(F) \otimes \mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A)$ and

$$t_x^*(\phi_A(F)) = t_x^* \,\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(F) \otimes \mathcal{P}_A) = \mathrm{pr}_{2,*}(t_{(e,x)}^* \,\mathrm{pr}_1^*(F) \otimes t_{(e,x)}^*(\mathcal{P}_A)) = \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(F) \otimes t_{(e,x)}^*(\mathcal{P}_A)),$$

then it suffices to show that $\mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A \simeq t_{(e,x)}^*(\mathcal{P}_A)$ as line bundles on $A \times A^t$. To do so, we can use the seesaw principle (Theorem 6.3).

First, we claim that both $\mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A$ and $t_{(e,x)}^* \mathcal{P}_A$ restrict to the line bundle $M_x \otimes M_y$ on the fibre $A \times \{y\}$, for any $y \in A^t(k)$. This is immediate for $\mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A$. On the other hand, we have

$$\left(t_{(e,x)}^*(\mathcal{P}_A)\right)|_{A \times \{y\}} \simeq \mathcal{P}_A|_{A \times \{t_x(y)\}} = \mathcal{P}_A|_{A \times \{x+y\}} \simeq M_{x+y}$$

and the claim then follows from the fact the map $A^t(k) \simeq \mathrm{Pic}^0(A)$, given by $x \mapsto M_x$, is a group homomorphism.

Now, it is enough to check that both $\mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A$ and $t_{(e,x)}^*(\mathcal{P}_A)$ restrict to the same line bundle on $\{e\} \times A^t$. Note that $(\mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A)|_{\{e\} \times A^t}$ is trivial, since both factors are trivial. Furthermore,

$$\left(t_{(e,x)}^*(\mathcal{P}_A)\right)|_{\{e\} \times A^t} \simeq t_x^*(\mathcal{P}_A|_{\{e\} \times A^t})$$

and this is certainly trivial. In particular, the restrictions of $\mathrm{pr}_1^*(M_x) \otimes \mathcal{P}_A$ and $t_{(e,x)}^*(\mathcal{P}_A)$ to $\{e\} \times A^t$ coincide. $\square$

Property 17.5 can be used to compute some common examples of the Fourier–Mukai transform.

Example 17.6. If M_x is as in Property 17.5, then $\phi_A(M_x) \simeq k(-x)[-g]$, where $k(-x)$ denotes the skyscraper sheaf at $-x$. Indeed, set $F = \mathcal{O}_A$ in Property 17.5 to get

$$\phi_A(M_x) \simeq t_x^* \phi_A(\mathcal{O}_A) = t_x^*(\mathrm{pr}_{2,*}(\mathcal{P}_A)) = t_x^*(k(e)[-g]) = k(-x)[-g].$$

Combining (the proof of) Property 17.5 with biduality gives another similar formula, described below.

Property 17.7. If a point $a \in A(k)$ corresponds to the degree-zero line bundle $N_a \in \mathrm{Pic}^0(A^t)$ under the biduality isomorphism, then there is a canonical isomorphism

$$\phi_A(t_a^* F) \simeq N_{-a} \otimes \phi_A(F)$$

for $F \in \mathrm{D}(A)$.

Under the Fourier–Mukai equivalence, tensor products are exchanged with a convolution-type operation (note that this is not the same convolution that appears in Proposition 17.1, but an operation reminiscient of the convolution of two functions on a locally compact group that one encounters in harmonic analysis). This is made precise below.

Property 17.8. If $M, N \in \mathrm{D}(A)$, then there is a canonical isomorphism

$$\phi_A(M) \otimes \phi_A(N) \simeq \phi_A(M *_A N),$$

where $M *_A N \coloneqq m_*(\mathrm{pr}_1^*(M) \otimes \mathrm{pr}_2^*(N)) \in \mathrm{D}(A)$.

Proof. Consider the commutative diagram

$$
\begin{tikzcd}
& A \times A \times A^t \arrow[rr, "\mu=(m,\mathrm{id})"] \arrow[d, "\mathrm{pr}_{12}"] \arrow[ddl, bend right, "\mathrm{pr}_1"'] \arrow[ddr, bend left, "\mathrm{pr}_2"] \arrow[rrr, bend left, "\mathrm{pr}_3"] & & A \times A^t \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & A^t \\
& A \times A \arrow[rr, "m"'] \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & & A & \\
A & & A & &
\end{tikzcd}
$$

with Cartesian square $\mathrm{pr}_1 \circ \mu = m \circ \mathrm{pr}_{12}$. Thus, by flat base change and the projection formula, we find that

$$
\begin{aligned}
\phi_A(M *_A N) &= \mathrm{pr}_{2,*}(\mathcal{P}_A \otimes \mathrm{pr}_1^*(m_*(\mathrm{p}_1^*(M) \otimes \mathrm{p}_2^*(N)))) \\
&\simeq \mathrm{pr}_{2,*}(\mathcal{P}_A \otimes \mu_* \,\mathrm{pr}_{12}^*(\mathrm{p}_1^*(M) \otimes \mathrm{p}_2^*(N))) \\
&\simeq \mathrm{pr}_{2,*}\, \mu_*(\mu^*(\mathcal{P}_A) \otimes \mathrm{pr}_1^*(M) \otimes \mathrm{pr}_2^*(N)) \\
&\simeq \mathrm{pr}_{2,*}\, \mu_*(\mathrm{pr}_{13}^*(\mathcal{P}_A) \otimes \mathrm{pr}_{23}^*(\mathcal{P}_A) \otimes \mathrm{pr}_1^*(M) \otimes \mathrm{pr}_2^*(N)) \\
&= \mathrm{pr}_{3,*}((\mathrm{pr}_{13}^*(\mathcal{P}_A) \otimes \mathrm{pr}_1^*(M)) \otimes (\mathrm{pr}_{23}^*(\mathcal{P}_A) \otimes \mathrm{pr}_3^*(N))).
\end{aligned}
$$

The third isomorphism is a consequence of Lemma 17.3. Now, observe that $\mathrm{pr}_3 \colon A \times A \times A^t \to A^t$ is the self-fibre product of $\mathrm{pr}_2 \colon A \times A^t \to A^t$, and so Künneth implies that

$$\phi_A(M *_A N) \simeq \mathrm{pr}_{2,*}(\mathcal{P}_A \otimes \mathrm{pr}_1^*(M)) \otimes \mathrm{pr}_{1,*}(\mathcal{P}_A \otimes \mathrm{pr}_2^*(N)),$$

which is precisely $\phi_A(M) \otimes \phi_A(N)$, as required. $\square$

18. November 7th

18.1. **Properties of the Fourier–Mukai Equivalence (Continued).** Let A be an abelian variety of dimension g over an algebraically closed field k. Last time, we discussed various properties of the Fourier–Mukai transform $\phi_A : \mathrm{D}(A) \to \mathrm{D}(A^t)$; in particular, we showed (up to sign) that

- For a point $x \in A^t(k)$, the functor ϕ_A exchanges t_x^* with $- \otimes M_x$, where $M_x \in \mathrm{Pic}^0(A)$ is the degree-zero line bundle corresponding to the point x.
- There is a natural isomorphism $\phi_A(- *_A -) = \phi_A(-) \otimes \phi_A(-)$ of bifunctors on $\mathrm{D}(A)$.

Lemma 18.1. *Let $x \in A^t(k)$ and let $M_x \in \mathrm{Pic}^0(A)$ be the corresponding degree-zero line bundle. If $F \in \mathrm{D}(A)$, then there is canonical isomorphism*

$$R\Gamma(A, F \otimes M_x) \simeq \phi_A(F)|_{\{x\}} \in \mathrm{D}(k). \tag{18.1}$$

Similarly, if $G \in \mathrm{D}(A^+)$, then there is a canonical isomorphism

$$R\Gamma(A, \phi_{A^t}(G) \otimes M_x) \simeq G[-g]|_{\{-x\}} \in \mathrm{D}(k). \tag{18.2}$$

Remark 18.2. Taking $F = \mathcal{O}_A$ and $x = e$ in (18.1) gives a canonical isomorphism

$$R\Gamma(A, \mathcal{O}_A) \simeq \phi_A(\mathcal{O}_A)|_{\{e\}} \simeq \mathrm{pr}_{2,*}(\mathcal{P}_A)|_{\{e\}} \simeq k(e)[-g]|_{\{x\}}.$$

It may not be immediately obvious that the above formula makes sense: the left-hand side is known to be an exterior algebra on the first cohomology of $\mathcal{O}_A$, whereas the right-hand side appears to simply be a skyscraper sheaf. To reconcile this, remember that the right-hand side is calculated in the derived category, and so it is equal to

$$k(e)[-g] \otimes k(e) = \mathrm{Tor}_*^{\mathcal{O}_{A^t,e}}(k(e), k(e))[-g].$$

Now, $\mathrm{Tor}_*^{\mathcal{O}_{A^t,e}}(k(e), k(e))$ is itself an exterior algebra (since $\mathcal{O}_{A^t,e}$ is a regular local ring): one can check that it is equal to $\Lambda^* \mathrm{Tor}_1^{\mathcal{O}_{A^t,e}}(k(e), k(e))$ by using the Koszul complex.

Remark 18.3. Taking $G \in \mathrm{Pic}(A^t)$ and $x = e$ in (18.2) gives a canonical isomorphism

$$R\Gamma(A, \phi_{A^t}(G)) \simeq G[-g]|_{\{e\}}.$$

The right-hand side is (non-canonically) isomorphic to $k(e)[-g]$, so taking the Euler characteristic of the above formula yields

$$\chi(A, \mathcal{O}_{A^t}(G)) = (-1)^g.$$

Proof of Lemma 18.1. The second assertion follows from the first by taking $F = \phi_{A^t}(G)$; thus, it suffices to prove (18.1). Consider the Cartesian diagram

$$\begin{tikzcd} A \times \{x\} \arrow[r] \arrow[d] & A \times A^t \arrow[d, "\mathrm{pr}_2"] \\ \{x\} \arrow[r] & A^t \end{tikzcd}$$

Expanding the definition of $\phi_A(F)|_{\{x\}}$ and applying the base change formula for the above Cartesian diagram gives

$$\phi_A(F)|_{\{x\}} = \mathrm{pr}_{2,*}\left(\mathrm{pr}_1^*(F) \otimes \mathcal{P}_A\right)|_{\{x\}} \simeq R\Gamma\left(A, (\mathrm{pr}_1^*(F) \otimes \mathcal{P}_A)|_{A \times \{x\}}\right) \simeq R\Gamma(A, F \otimes M_x),$$

which completes the proof. ☐

Exercise 18.4. If $f : A \to B$ is a homomorphism of abelian varieties, what happens to the Fourier transforms ϕ_A and ϕ_B under the functors f_* and f^*? More precisely, show that there is a canonical isomorphism

$$\phi_B \circ f_* \simeq (f^t)^* \circ \phi_A,$$

where $f^t \colon B^t \to A^t$ is the dual map to f.

18.2. **Homogeneous Vector Bundles.** Let A be an abelian variety of dimension g over a field k, and assume for notational simplicity that k is algebraically closed. As before, set $\phi_A := \phi_{\mathcal{P}_A}$.

Definition 18.5. An object $E \in \mathrm{D}(A)$ is *homogeneous* if for any $x \in A(k)$, there exists an isomorphism

$$t_x^*(E) \simeq E.$$

The homogeneous objects of the derived category $\mathrm{D}(A)$ ought to be thought of as a generalization of the class of degree-zero line bundles. Note that we do not make any compatibility assumptions on the isomorphisms appearing in Definition 18.5.

Theorem 18.6. *The functor $\phi_A[g] \colon \mathrm{D}(A) \to \mathrm{D}(A^t)$ restricts to an equivalence of categories*

$$\{\textit{homogeneous vector bundles on } A\} \xrightarrow{\simeq} \left\{\textit{coherent sheaves on } A^t \textit{ with finite support}\right\}.$$

Moreover, under this equivalence, the rank of a homogeneous vector bundle $\mathcal{E}$ on A coincides with the length of the coherent sheaf $\phi_A(\mathcal{E})[g]$.

One assertion, implicit in the statement of Theorem 18.6, is that homogeneous vector bundles on A form an abelian category (because it is an abelian subcategory of $\mathrm{Coh}(A)$) and it is closed under taking extensions.

A crucial tool in the proof of Theorem 18.6 is the following criterion for an object of the derived category of A to have finite support (in the sense that each cohomology sheaf has finite support).

Lemma 18.7. *If $G \in \mathrm{D}^b_{\mathrm{coh}}(A)$ satisfies $G \otimes L \simeq G$ for all $L \in \mathrm{Pic}^0(A)$, then G has finite support.*

In fact, the converse to the lemma is true (and the proof is immediate).

Proof. The operation of tensoring by a line bundle is flat, so it is enough to show the statement in each degree separately. Thus, we may assume that G is a coherent sheaf on A placed in degree 0.

If $\mathrm{Supp}(G)$ is not zero dimensional, then it must contain an irreducible curve $C \subseteq \mathrm{Supp}(G)$. Consider the normalization $f : \widetilde{C} \to C \subseteq A$ of C. By assumption, $G|_C$ has nonzero rank at every point of C, and hence $f^*(G)$ has nonzero rank. In particular,

$$\overline{G} := f^*(G)/\text{torsion}$$

is a nonzero vector bundle on $\widetilde{C}$. The hypothesis on G ensures that $f^*(G) \otimes f^*L \simeq f^*G$ for all $L \in \mathrm{Pic}^0(A)$, and quotienting by the torsion on both sides gives that $\overline{G} \otimes f^*(L) \simeq \overline{G}$ for all $L \in \mathrm{Pic}^0(A)$. Now, both $\overline{G} \otimes f^*(L)$ and $\overline{G}$ are vector bundles, so taking we may take determinants to obtain an isomorphism

$$\det\left(\overline{G}\right) \otimes (f^*(L))^{\mathrm{rank}(\overline{G})} \simeq \det\left(\overline{G}\right)$$

for all $L \in \mathrm{Pic}^0(A)$. Since the group $\mathrm{Pic}^0(A)$ is divisible and $\det\left(\overline{G}\right)$ is a line bundle, it follows that $f^*(L) = \mathcal{O}_{\widetilde{C}}$ for all $L \in \mathrm{Pic}^0(A)$.

Now, consider the map $\pi = (f, \mathrm{id}) : \widetilde{C} \times A^t \to A \times A^t$. Having the equality $f^*(L) = \mathcal{O}_{\widetilde{C}}$ for all $L \in \mathrm{Pic}^0(A)$ is equivalent to the assertion that $\pi^*(\mathcal{P}_A)$ is trivial along the fibres of pr_2. Thus, by the seesaw theorem, we must have that $\pi^*(\mathcal{P}_A)$ is isomorphic to $\mathrm{pr}_2^*(M)$ for some line bundle M on A^t. Using the isomorphism $\mathcal{P}_A \simeq \mathcal{P}_{A^t}$, it follows that the map $C \to (A^t)^t$ corresponding to $\mathcal{P}_{A^t}$ is the constant map with image equal to the point corresponding to M; unwinding definitions and using biduality, one can check that this is the same map as $f : C \to A$. Thus, f is constant, a contradiction. $\square$

Proof of Theorem 18.6. Let N be a coherent sheaf on A^t with finite support. The Fourier–Mukai transform ϕ_A is an equivalence by Theorem 17.2, so we may write $N = \phi_A(M)$ for some $M \in \mathrm{D}^b_{\mathrm{coh}}(A)$. Any such N must be an iterated extension of copies of various skyscraper sheaves $k(x) = \phi_A(M_{-x})[g]$ for various $x \in A^t$. Thus, we have (by duality) that M is a coherent sheaf concentrated in a single degree and an iterated extension of sheaves of the form M_{-x} (this calculation is expanded upon in Example 18.8). Thus, $M[-g]$ is a vector bundle on A. The homogeneity of $M[-g]$ follows immediately from the invariance of N under all the endofunctors $- \otimes L$ for $L \in \mathrm{Pic}^0(A)$ and Property 17.7. Thus, we have constructed a functor

$$\{\text{coherent sheaves on } A^t \text{ with finite support}\} \xrightarrow{\phi_A^{-1}[-g]} \{\text{homogeneous vector bundles on } A\}. \tag{18.3}$$

Conversely, Let E be a homogeneous vector bundle on A. It follows immediately from Property 17.7 that $\phi_A(E)$ is invariant under the endofunctor $- \otimes L$ for all $L \in \mathrm{Pic}^0(A^t)$, and hence $\phi_A(E)$ is a complex with finite support by Lemma 18.7. If N lived in more than one degree (i.e. if N has at least two distinct nonzero cohomology sheaves), then so would the vector bundle E, a contradiction. Thus, N lives in a single degree (in fact, in degree g), and hence $\phi_A(E)[g]$ is actually a coherent sheaf with finite support. $\square$

Example 18.8. Given an extension $k(x) \to N \to k(x)$ in $\mathrm{D}(A^t)$, it may be written as

$$\phi_A(M_{-x})[g] \to \phi_A(M) \to \phi_A(M_{-x})[g]$$

for some $M \in \mathrm{D}(A)$. The duality of ϕ_A gives an exact triangle

$$M_{-x}[g] \to M \to M_{-x}[g]$$

in $\mathrm{D}(A)$; thus, $\mathcal{H}^{-g}(M)$ is a vector bundle and $\mathcal{H}^i(M) = 0$ for $i \neq g$. It follows that $M = \mathcal{H}^{-g}(M)[g]$.

18.3. **Unipotent Vector Bundles.** Let A be an abelian variety of dimension g over an algebraically closed field k.

Definition 18.9. A vector bundle on A is *unipotent* if it arises as an iterated extension of copies of $\mathcal{O}_A$.

Recall that there is an isomorphism $H^1(A, \mathcal{O}_A) \simeq \mathrm{Ext}^1(\mathcal{O}_A, \mathcal{O}_A)$ (see e.g. [Har77, III, Proposition 6.3]) and $H^1(A, \mathcal{O}_A)$ is a k-vector space of dimension g; in particular, there are many nontrivial unipotent vector bundles on A.

Example 18.10. If E is an elliptic curve over k, then $H^1(E, \mathcal{O}_E) = 1$. Thus, there is exactly one nontrivial unipotent vector bundle of rank 2 on E.

Theorem 18.11. *The functor $\phi_A[g]$ restricts to an equivalence of categories*

$$\{\textit{unipotent vector bundles on } A\} \xrightarrow{\simeq} \left\{\textit{coherent sheaves on } A^t \textit{ supported at } e\right\}.$$

Proof. Restrict the equivalence of categories in Theorem 18.6 to the abelian subcategory generated by $\mathcal{O}_A$ on one side (i.e. to the thick subcategory of homogeneous vector bundles generated by $\mathcal{O}_A$), and to the the abelian subcategory generated by the skyscraper sheaf at the origin on the other. As $\phi_A(\mathcal{O}_A)[g] = k(e)$, the result follows. $\square$

19. November 9th

19.1. **Cohomology of Ample Line Bundles.** Let A be an abelian variety of dimension g over a field k. Assume for simplicity that k is algebraically closed. Recall that for a line bundle $L \in \operatorname{Pic} A$, there is a map $\phi_L : A \to A^t$ given by

$$x \mapsto t_x^*(L) \otimes L^{-1}$$

with kernel denoted by $K(L)$. Consider the map $\alpha = (\phi_L, \mathrm{id}) : A \times A \to A^t \times A$, and note that $\alpha^*(\mathcal{P}_A) = \Lambda(L)$; this can be seen e.g. by comparing fibres on one factor, using the seesaw theorem, and the trivialization of $\mathcal{P}_A$ at a point. We have seen that, if L is ample, the map ϕ_L is a finite surjective morphism, and hence $K(L)$ is finite. Today, we will discuss what line bundles L give rise to the finitude of $K(L)$.

Definition 19.1. We say $L \in \operatorname{Pic}(A)$ is *non-degenerate* if $K(L)$ is finite.

It is clear that the non-degeneracy of $L \in \operatorname{Pic}(A)$ is equivalent to the morphism ϕ_L being either finite or surjective. The first main result about non-degenerate line bundles that we will show is the following:

Theorem 19.2. *Let $L \in \operatorname{Pic}(A)$ be non-degenerate.*

(1) There is a vector bundle E on A^t and an integer $0 \le i(L) \le g$ such that

$$\phi_A(L) = E[-i(L)].$$

In particular, $R\Gamma(A, L) \simeq E[-i(L)]|_{\{e\}}$, and

$$H^i(A, L) = \begin{cases} E|_{\{e\}} & i = i(L) \\ 0 & \textit{otherwise} \end{cases}$$

(2) There is an equality $\left(\dim H^{i(L)}(A, L)\right)^2 = \operatorname{rank}(K(L)) = \deg(\phi_L)$; in particular, $H^{i(L)}(A, L) \neq 0$.

Note that Theorem 19.2(1) immediately implies that very ample line bundles L on A have no higher cohomology, since they must have global sections, and thus $i(L) = 0$. The integer $i(L)$ appearing in Theorem 19.2 is called the *index* of the line bundle L.

The proof of Theorem 19.2 rests on the following sequence of lemmas.

Lemma 19.3. *If $L \in \operatorname{Pic} A$, then*

$$\phi_L^*(\phi_A(L)) = R\Gamma(A, L) \otimes L^{-1}. \tag{19.1}$$

In particular, each cohomology sheaf $\mathcal{H}^i\left(\phi_L^*(\phi_A(L))\right)$ is a vector bundle on A, and $\mathcal{H}^i(\phi_A(L))$ is a vector bundle on A^t.

Proof. That the cohomology sheaves are vector bundles follows from the faithful flatness of ϕ_L, so it suffices to show (19.1). Consider the map $\alpha = (\mathrm{id}, \phi_L) : A \times A \to A \times A^t$, which satisfies $\alpha^*(\mathcal{P}_A) \simeq \Lambda(L)$. By the definition of ϕ_A, we can write

$$\phi_L^* \phi_A(L) = \phi_L^*((\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(L) \otimes \mathcal{P}_A)). \tag{19.2}$$

Consider the diagram

$$\begin{tikzcd} A \arrow[d, equal] & A \times A \arrow[l, "\mathrm{p}_1"'] \arrow[r, "\mathrm{p}_2"] \arrow[d, "\alpha"] & A \arrow[d, "\phi_L"] \\ A & A \times A^t \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & A^t \end{tikzcd}$$

The right-hand square is Cartesian, and the left-hand square is commutative. Using flat base change for the right-hand square, the expression (19.2) becomes

$$\phi_L^*(\phi_A(L)) = \mathrm{p}_{2,*}\, \alpha^*(\mathrm{pr}_1^*(L) \otimes \mathcal{P}_A) = \mathrm{p}_{2,*}(\mathrm{p}_1^*(L) \otimes \Lambda(L)). \tag{19.3}$$

Recall that $\Lambda(L) \coloneqq m^*(L) \otimes \mathrm{p}_1^*(L^{-1}) \otimes \mathrm{p}_2(L^{-1})$, and thus (19.3) becomes

$$\mathrm{p}_{2,*}(m^*(L) \otimes \mathrm{p}_2^*(L^{-1})) = \mathrm{p}_{2,*}(m^*(L)) \otimes L^{-1}$$

by the projection formula. Now, observe that the commutative square

$$\begin{tikzcd} A \times A \arrow[r, "m"] \arrow[d, "\mathrm{p}_2"'] & A \arrow[d, "f"] \\ A \arrow[r, "d"] & \mathrm{Spec}(k) \end{tikzcd}$$

is Cartesian, so applying flat base change once again yields the isomorphism

$$\phi_L^*\phi_A(L) \simeq f^*(f_*L) \otimes L^{-1} = R\Gamma(A, L) \otimes L^{-1},$$

which completes the proof. □

Lemma 19.4. *If $L \in \mathrm{Pic}(A)$ is non-degenerate, then $\chi(A, L)^2 = \mathrm{rank}(K(L))$.*

Proof. By Lemma 16.3, we have

$$\chi(A, \phi_L^*\phi_A(L)) = \mathrm{rank}\, K(L)\chi(A^t, \phi_A(L))$$

since $\deg(\phi_L) = \mathrm{rank}\, K(L)$. Note that Remark 18.3 implies that $\chi(A^t, \phi_A(L)) = (-1)^g$, and hence we have $\chi(A, \phi_L^*\phi_A(L)) = \mathrm{rank}(K(L)) \cdot (-1)^g$. By Lemma 19.3, we get that

$$\chi(A, \phi_L^*\phi_A(L)) = \chi(A, R\Gamma(A, L) \otimes_k L^{-1}).$$

Now, $R\Gamma(A, R\Gamma(A, L) \otimes L^{-1}) = R\Gamma(A, L) \otimes R\Gamma(A, L^{-1})$ by the projection formula. Using that χ is a ring homomorphism, we find that

$$\chi(A, R\Gamma(A, L) \otimes L^{-1}) = \chi(R\Gamma(A, L) \otimes R\Gamma(A, L^{-1})) = \chi(A, L) \cdot \chi(A, L^{-1}).$$

The canonical bundle on A is trivial, so Serre duality then gives that

$$\chi(A, R\Gamma(A, L) \otimes L^{-1}) = \chi(A, L)^2(-1)^g.$$

Thus, we have computed $\chi(A, \phi_L^*\phi_A(L))$ in two different ways, which give the desired equality. □

Lemma 19.5. *If $L \in \mathrm{Pic}(A)$ and $i\colon K(L) \hookrightarrow A$ is the inclusion, then*

$$L *_A [-1]^*(L^{-1}) \simeq i_*\left(L|_{K(L)}\right)[-g].$$

In particular, if L is non-degenerate, then $L *_A [-1]^*(L^{-1})$ is finitely-supported.

Proof. Expanding the definition of $L *_A [-1]^*(L^{-1})$, we must show that

$$m_*(\mathrm{p}_1^*\, L \otimes \mathrm{p}_2^*[-1]^* L^{-1}) = i_*(L|_{K(L)})[-g] \tag{19.4}$$

Consider the diagram

$$\begin{tikzcd} A \times A \arrow[r, "\eta"] \arrow[dr, "m"'] & A \times A \arrow[d, "\mathrm{p}_1"] \\ & A \end{tikzcd}$$

where η is the map given by $\eta(a, b) := (m(a, b), -b)$. Note that $\eta^2 = \mathrm{id}$, and thus $\eta_* = \eta^*$. It follows that the left-hand side of (19.4) becomes

$$\mathrm{p}_{1,*}\left(\eta^* \mathrm{p}_1^*(L) \otimes \eta^* \mathrm{p}_2^*[-1]^*(L^{-1})\right) = \mathrm{p}_{1,*}\left(m^*(L) \otimes \mathrm{p}_2^*(L^{-1})\right) = \mathrm{p}_{1,*}\left(\Lambda(L) \otimes \mathrm{p}_1^*(L)\right) = \mathrm{p}_{1,*}(\Lambda(L)) \otimes L,$$

where the final equality follows from the projection formula. Consider now the Cartesian square

$$\begin{tikzcd} A \times A \arrow[r, "\alpha"] \arrow[d, "\mathrm{p}_1"'] & A^t \times A \arrow[d, "\mathrm{pr}_1"] \\ A \arrow[r, "\phi_L"] & A^t \end{tikzcd}$$

We know that $\alpha^*(\mathcal{P}_A) = \Lambda(L)$, so flat base change for the above Cartesian square gives isomorphisms

$$\mathrm{p}_{1,*}(\Lambda(L)) \otimes L = \mathrm{p}_{1,*}(\alpha^*(\mathcal{P}_A)) \otimes L \simeq \phi_L^*\left(k(e)[-g]\right) \otimes L \simeq i_*\mathcal{O}_{K(L)}[-g] \otimes L \simeq i_*(L|_{K(L)})[-g],$$

which completes the proof. $\square$

Proof of Theorem 19.2. We must show that, if $L \in \mathrm{Pic}(A)$ is non-degenerate, then there is a vector bundle E on A and an integer $0 \leq i(L) \leq g$ such that $\phi_A(L) \simeq E[-i(L)]$. This statement, when coupled with Lemma 19.4, immediately gives the rest of the theorem.

Since L is non-degenerate, the map ϕ_L is faithfully flat; Both local freeness and being concentrated in a single degree can be detected after faithfully flat pullback, so it suffices to show that $\phi_L^*\phi_A(L) \simeq E'[-i(L)]$ for some vector bundle E' on A. Note that Lemma 19.3 implies that $\phi_L^*\phi_A(L) = R\Gamma(A, L) \otimes L^{-1}$. Thus, it suffices to show that $R\Gamma(A, L)$ lives in a single degree.

By Lemma 19.5, we have $L *_A [-1]^*(L^{-1}) = i_*(L|_{K(L)})$, where $i\colon K(L) \hookrightarrow A$ is the inclusion. Using that ϕ_A exchanges tensor products and convolution, we find that

$$\phi_A(L) \otimes \phi_A([-1]^*(L^{-1})) \simeq \phi_A(i_*(L|_{K(L)})[-g]). \tag{19.5}$$

As L is non-degenerate, the subscheme $K(L)$ is finite, and so the right-hand side of (19.5) is finitely-supported. Now, Theorem 18.6 asserts that ϕ_A (up to shift) gives a bijection between the finitely-supported coherent sheaves on A^t and the homogeneous vector bundles on A; thus, the right-hand side of (19.5) is of the form $E''[-g]$ for some homogeneous vector bundle E'' on A. Applying ϕ_L^* to $\phi_A(i_*(L|_{K(L)})[-g])$ gives

$$\left(R\Gamma(A, L) \otimes L^{-1}\right) \otimes \left(R\Gamma(A, [-1]^*L^{-1}) \otimes [-1]^*(L)\right) \simeq E'''[g]$$

for some E''' a vector bundle on A. Reorganizing terms, we have

$$R\Gamma(A, L) \otimes R\Gamma(A, [-1]^*(L^{-1})) \simeq E''''[-g].$$

Both factors in the above tensor product have but a single nonzero cohomology group, so the result follows. $\square$

The upshot of Theorem 19.2 is that $H^i(A, L) = 0$ for all $i > 0$ whenever $L \in \mathrm{Pic}(A)$ is very ample. Now, our goal is to replace the assumption "very ample" by "ample". More precisely, the goal is to show that $i(L) = i(L^{\otimes n})$, from which the desired cohomology vanishing follows. It is difficult to deal with the tensor powers $L^{\otimes n}$ directly, so instead we proceed by showing that the index is invariant under the operation of tensoring with a degree-zero line bundle, from which the invariance of the index will be deduced.

Lemma 19.6. *If $L, L' \in \mathrm{Pic}(A)$ are algebraically equivalent and non-degenerate, then $i(L) = i(L')$.*

Proof. By hypothesis, there exists a smooth connected curve C and line bundle $\mathcal{L}$ on $C \times A$ such that

$$\begin{cases} \mathcal{L}|_{\{c_0\}\times A} \simeq L \\ \mathcal{L}|_{\{c_1\}\times A} \simeq L' \end{cases}$$

for some $c_0, c_1 \in C$. The line bundle $M := \mathcal{L} \otimes \mathrm{pr}_2^*(L^{-1})$ satisfies $M|_{\{c_0\}\times A} \simeq \mathcal{O}_A$; thus, $M|_{\{c\}\times A} \in \mathrm{Pic}^0(A)$ for all $c \in C$ by Lemma 12.9. Therefore, we obtain a map $g : C \to A^t$ with the property that $\alpha^*(\mathcal{P}_A) = M$, where α denotes the map $\alpha := (g, \mathrm{id}) : C \times A \to A^t \times A$. As $\phi_L : A \to A^t$ is surjective and L is non-degenerate, it follows that the tensor product $L \otimes (L')^{-1} \in \mathrm{Pic}^0(A)$ can be written as $\phi_L(x)$ for some $x \in A$. However, $\phi_L(x) = t_x^*(L) \otimes L^{-1}$, so we may cancel the L^{-1} factor to obtain $L' = t_x^*(L)$. Thus, the pullback t_x^* induces an isomorphism $H^*(A, L) \simeq H^*(A, L')$, and the result follows. □

20. November 14th

20.1. **Cohomology of Ample Line Bundles (Continued).** Let A be an abelian variety of dimension g over an algebraically closed field k. Let $L \in \mathrm{Pic}(A)$ be a non-degenerate line bundle on A, i.e. the map $\phi_L \colon A \to A^t$ is finite. Last time, we showed that there is a vector bundle E on A and an integer $0 \le i(L) \le g$, called the *index* of L, such that

$$\phi_A(L) = E[-i(L)] \in \mathrm{D}^b_{\mathrm{coh}}(A^t).$$

Moreover, the index can also be computed from the cohomology of L: we saw that

$$R\Gamma(A, L) = H^{i(L)}(A, L)[-i(L)].$$

It is clear that a very ample line bundle has index zero (since it has nonzero global sections). Last time, we proved that $i(L) = i(M)$ whenever L, M are algebraically equivalent non-degenerate line bundles on A. Our goal is to use this fact to show that ample line bundles also have index zero.

Lemma 20.1. *If $f \colon A \to B$ is a finite surjective morphism of abelian varieties and $L \in \mathrm{Pic}(B)$ is a non-degenerate line bundle on B, then $i(L) = i(f^*(L))$.*

Proof. We must first show that $f^*(L)$ is non-degenerate. In fact, we claim that $\phi_{f^*(L)} = f^t \circ \phi_L \circ f$, from which it follows that $f^*(L)$ is non-degenerate, since f^t, ϕ_L, and f are all finite and surjective. To see the claim, notice that

$$\phi_{f^*(L)}(x) = t_x^*(f^*(L)) \otimes f^*(L^{-1}) = f*(t_{f(x)}^*(L)) \otimes f^*(L^{-1}) = f^*(t_{f(x)}^*(L) \otimes L^{-1}) = f^*(\phi_L(f(x))) = (f^t \circ \phi_L \circ f)(x)$$

for any scheme-theoretic point x of B. It remains to compute the index. Recall that $f_*^t(\phi_B(L)) = \phi_A(f^*(L))$. As L is non-degenerate, f^t is finite flat and so $f_*^t(\phi_B(L)) = f_*^t(E[-i(L)]) = E'[-i(L)]$ for some vector bundle E' on A. The result follows. □

Lemma 20.2. *For any $n \in \mathbf{Z}_{>0}$ and any non-degenerate line bundle $L \in \mathrm{Pic}(A)$, we have $i(L) = i(L^n)$.*

Proof. We will prove only the case $n = m^2$ (we may reduce to this case using Zarhin's trick). Apply Lemma 20.1 to the multiplication-by-m map $[m] \colon A \to A$ to get that $i(L) = i([m]^*(L))$. However, there exists $N \in \mathrm{Pic}^0(A)$ such that $[m]^*(L) = L^{\otimes m^2} \otimes N$. Thus, $[m]^*(L)$ and $L^{\otimes m^2}$ are algebraically equivalent, so

$$i(L^{\otimes n}) = i(L^{\otimes m^2}) = i([m]^*(L)) = i(L)$$

by Lemma 19.6. □

Corollary 20.3. *If $L \in \mathrm{Pic}(A)$ is an ample line bundle on A, then $i(L) = 0$ and $\Big(\dim H^0(A, L)\Big)^2 = \deg(\phi_L)$.*

Proof. Choose $m \gg 0$ such that $L^{\otimes m^2}$ is very ample, in which case $i(L) = i(L^{\otimes m^2}) = 0$ by Lemma 20.2. The second assertion is immediate, since $\deg(\phi_L) = \mathrm{rank}(K(L))$. □

Definition 20.4. A *principally polarized abelian variety* (ppav) is a pair $\left(A, \varphi \colon A \xrightarrow{\simeq} A^t\right)$, where A is an abelian variety over a field k, and φ is a homomorphism of abelian varieties that (after base change to $\overline{k}$) is of the form ϕ_L for some ample line bundle $L \in \operatorname{Pic}(A)$.

20.2. **Stable Vector Bundles on Curves.** Let C be a smooth, projective, geometrically connected curve over k. If E is a vector bundle on C, write $r(E) := \operatorname{rank}(E) \geq 0$ and $d(E) := \deg(\det(E)) \in \mathbf{Z}$. Both r and d are additive in short exact sequences, so they give functions

$$K_0(C) \xrightarrow{(r,d)} \mathbf{Z}^{\oplus 2}.$$

Thus, we can define r and d on any object in the bounded coherent derived category.

Definition 20.5. For any nonzero vector bundle E on C, define the *slope* of E as

$$\mu(E) := \frac{d(E)}{r(E)} \in \mathbf{Q}.$$

Furthermore, we say:

(1) E is *semistable* if for any nonzero quotient bundle Q of E, we have $\mu(Q) \geq \mu(E)$;
(2) E is *stable* if for any nonzero, nontrivial quotient bundle Q of E, we have $\mu(Q) > \mu(E)$.

Lemma 20.6. *If $0 \to E_1 \to E \to E_2 \to 0$ is a short exact sequence of vector bundles on C, then*

$$\min\{\mu(E_1), \mu(E_2)\} \leq \mu(E) \leq \max\{\mu(E_1), \mu(E_2)\}.$$

Proof. Set $d_i = d(E_i)$ and $r_i = r(E_i)$; they satisfy $r_i \geq 0$ and $\frac{d_1}{r_1} \leq \frac{d_2}{r_2}$. Observe that

$$\mu(E) = \frac{d_1 + d_2}{r_1 + r_2} \leq \frac{d_2 \frac{r_1}{r_2} + d_2}{r_1 + r_2} \leq \frac{d_2\left(\frac{r_1 + r_2}{r_2}\right)}{r_1 + r_2} = \frac{d_2}{r_2} = \mu(E_2).$$

The other inequality is similar. □

Lemma 20.7. *Fix a nonzero vector bundle E on C. Then, E is semistable iff for any nonzero, nontrivial subsheaf $F \subseteq E$, we have $\mu(F) \leq \mu(E)$ (and similarly for stable vector bundles).*

Proof. If E is semistable, fix a nonzero subsheaf $F \subseteq E$. Let $\overline{F}$ denote the saturation of F in E. Then, $E/\overline{F}$ is a vector bundle on C. To see that $\mu(F) \leq \mu(\overline{F})$, use the fact that $\overline{F}/F$ is torsion, so it has rank zero and positive degree, so $\operatorname{rank}(F) = \operatorname{rank}(\overline{F})$ and $\deg(\overline{F}) \geq \deg(F)$. To see that $\mu(\overline{F}) \leq \mu(E)$, use that $Q := E/\overline{F}$ is a vector bundle and Lemma 20.6 gives that

$$\min\left\{\mu(Q), \mu(\overline{F})\right\} \leq \mu(E) \leq \max\left\{\mu(\overline{F}), \mu(Q)\right\}.$$

Refining Lemma 20.6 for the boundary case of equality gives the result. The converse is proved similarly, and it is left as an exercise. □

Lemma 20.8. *If E, F are semistable vector bundles on C such that $\mu(E) > \mu(F)$, then* $\operatorname{Hom}(E, F) = 0$.

Proof. If $f \colon E \to F$ is a nonzero map of vector bundles, let $Q \subseteq F$ denote the image. Then, the semistability of F implies that $\mu(Q) \leq \mu(F)$ by Lemma 20.7. Similarly, the semistability of E implies that $\mu(Q) \geq \mu(E)$, a contradiction. □

Lemma 20.9. *If $0 \to E_1 \to E \to E_2 \to 0$ is a short exact sequence of vector bundles on C all of which have the same slope μ, then E is semistable iff both E_1 and E_2 are semistable.*

Proof. If E is semistable and $F \subseteq E_1$ is a nontrivial subsheaf, then $\mu(F) \leq \mu(E) = \mu = \mu(E_1)$, so E_1 is semistable. Similarly, one can see that E_2 is semistable using the quotient formulation.

Conversely, if both E_1 and E_2 are semistable, take a nonzero, nontrivial subsheaf $F \subseteq E$ and we must show that $\mu(F) \leq \mu(E)$. Consider the short exact sequence

$$0 \to F \cap E_1 \to F \to F/(F \cap E_1) \to 0,$$

then we have $\mu(F) \leq \max\{\mu(F \cap E_1), \mu(F/(F \cap E_1))\}$. Now, $F \cap E_1 \subseteq E_1$ and $F/(F \cap E_1) \subseteq E_2$, so:

- if $F \cap E_1 \neq 0$, then $\mu(F \cap E_1) \leq \mu(E_1) = \mu$ by the semistability of E_1;
- if $F/(F \cap E_1) \neq 0$, then $\mu(F/(F \cap E_1)) \leq \mu(E_2) = \mu$ by the semistability of E_2.

Thus, if both $F \cap E_1$ and $F/(F \cap E_1)$ are nonzero, then $\mu(F) = \mu$. If one is zero, a similar argument applies. □

Example 20.10.

(1) Every line bundle is stable.
(2) Extensions of line bundles of the same slope are semistable (e.g. if $L_1, \ldots, L_n$ are semistable with $\mu(L_i) = \mu(L_j)$ for all i, j, then $\bigoplus_{i=1}^n L_i$ is semistable).
(3) If $C = \mathbf{P}^1$, the vector bundle $\bigoplus_{i=1}^n \mathcal{O}_C(a)$ is semistable with slope a for all $n \geq 1$ and $a \in \mathbf{Z}$, and all semistable vector bundles of slope a are of this form (to see this, use Grothendieck's theorem). Moreover, there do not exist semistable vector bundles on $\mathbf{P}^1$ with slope in $\mathbf{Q} \backslash \mathbf{Z}$. In fact, there is an equivalence of categories
$$\mathrm{Vect}_k \xrightarrow{\simeq} \mathrm{Vect}(\mathbf{P}^1_k)_{\substack{\text{semistable}\\ \text{of slope } a}}$$
given by $V \mapsto V \otimes \mathcal{O}(a)$.

Lemma 20.11. *Stable vector bundles are simple: if E is a stable vector bundle on C, then $End(E)$ is a division ring.*

Proof. Take a nonzero endomorphism $f \in \mathrm{End}(E)$ with image $Q \coloneqq \mathrm{im}(f) \subseteq E$. If f is not surjective, then $\mu(Q) < \mu(E)$; however, there is a surjection $E \twoheadrightarrow Q$, so $\mu(Q) \geq \mu(E)$, a contradiction. Thus, f is an isomorphism, since any surjective endomorphism of a coherent sheaf is an isomorphism. □

Example 20.12. Suppose the k-curve C has genus zero, but $C(k) = \emptyset$. Then, $H^1(C, K_C) \simeq \mathrm{Ext}^1_C(\mathcal{O}_C, K_C) = k$. Choose a non-split extension

$$0 \to K_C \to E \to \mathcal{O}_C \to 0,$$

then this extension remains non-slit after base changing to $\overline{k}$, hence $E \otimes \overline{k} \simeq \mathcal{O}(-1)^{\oplus 2}$, because $C_{\overline{k}} \simeq \mathbf{P}^1_{\overline{k}}$. We claim that E is stable and $\mathrm{End}(E) \simeq \mathbf{H}$ is some quaternion algebra.

Indeed, suppose there is a nonzero line subbundle $L \subseteq E$, then we must show that $\mu(L) < \mu(E) = -1$. Note that $L \otimes \overline{k} \simeq \mathcal{O}(-i)$, and we want that $i > 1$. From the fact that $\mathcal{O}(-i) = L \otimes \overline{k} \subseteq E \otimes \overline{k} = \mathcal{O}(-1)^{\oplus 2}$, we have $i \geq 1$. If $i = 1$, then $L^{-1} \otimes \overline{k} \simeq \mathcal{O}(1)$, hence $H^0(C, L^{-1}) = k^{\oplus 2}$. Now, any nonzero global section $s \in H^0(C, L^{-1})$ gives a subscheme $Z(s) \subseteq C$ that is a k-point, which contradicts the assumption that $C(k) = \emptyset$.

21. November 16th

21.1. **Stable Vector Bundles on Curves (Continued).** Last time, we defined the notion of a semistable vector bundle on a smooth projective curve C over a field k. Today, we will explore further properties of such vector bundles.

Theorem 21.1. [Harder–Narasimhan Filtration] *If E is a vector bundle on C, then there is a unique filtration $0 = E_0 \subseteq \ldots \subseteq E_\ell = E$ such that for all $i = 1, \ldots, \ell$,*

- *the quotient $Q_i \coloneqq E_i/E_{i-1}$ is semistable;*
- $\mu(Q_{i-1}) > \mu(Q_i)$.

The subsheaf E_1 appearing in Theorem 21.1 is often called the *maximal destabilizing subsheaf* of E; if E is semistable, then $E_1 = E$.

Proof. We proceed by induction on the rank of E. To construct E_1, we will take a certain subsheaf with maximal slope, but first we must show that there is a bound on the slopes of subsheaves of E.

Choose an inclusion $E \subset L^{\oplus n}$ for a sufficiently ample line bundle L on C. Since L is semistable, $L^{\oplus n}$ is as well; thus, for any subsheaf $F \subset E$ we have $\mu(F) \leq \mu(L^{\oplus n})$. A priori, a bounded set of numbers need not achieve the maximum (the slopes are rational numbers, and so they could have irrational supremum). However, note that since the rank of E is fixed, the denominators of $\mu(F)$ are bounded; thus, there actually exists a subsheaf of maximal slope $\mu = \mu_{\max}(E)$. Moreover, note that any subsheaf achieving this slope is semistable.

Now, if F and G are subsheaves of E with $\mu(F) = \mu(G) = \mu$ (in particular, F and G are semistable), then $\mu(F + G) = \mu$, since $F + G$ is a quotient of the semistable sheaf $F \oplus G$. By semistability, we have that $\mu(F + G) \geq \mu(F \oplus G)$, and then the maximality of μ implies that $\mu(F + G) = \mu$. Thus, there exists a maximal subsheaf E_1 with maximal slope.

In fact, the quotient E/E_1 is a vector bundle: if not, the saturation E_1^{sat} must have slope strictly large than μ, which contradicts the maximality of E_1 and of μ.

By induction, we choose a Harder–Narasimhan filtration for E/E_1, and take preimages in E to get a filtration as in the statement. All we must show is that the $\mu(E_i)$'s actually decrease in i. By induction, it suffices to check this just for the leftmost step of the filtration, i.e. $\mu(E_1) > \mu(E_2/E_1)$. If $\mu(E_1) \leq \mu(E_2/E_1)$, then the short exact sequence

$$0 \to E_1 \to E_2 \to E_2/E_1 \to 0$$

implies that $\mu(E_2) \geq \mu(E_1) = \mu$. By the maximality of μ, we must have $\mu(E_2) = \mu(E_1)$. In particular, the maximality of E_1 implies that $E_1 = E_2$, so $E_2/E_1 = 0$, i.e. E_1 is already semistable.

The uniqueness of the Harder–Narasimhan filtration is left as an exercise. □

Corollary 21.2. *If C has genus 1 and F is an indecomposable vector bundle on C, then F is semistable.*

Proof. It suffices to show that the Harder–Narasimhan filtration of any bundle E on C is split, since then any bundle is the direct sum of semistable bundles, and so the result follows from the indecomposability of F.

The idea is simple: let $E_0 \subset \ldots \subset E_k = E$ be the Harder–Narasimhan filtration, with quotients $Q_i = E_i/E_{i-1}$. We proceed by induction on the length ℓ of the filtration, with the $\ell = 1$ case being trivial. By induction on ℓ, we assume all but the last term of the filtration is split, i.e. $E_{\ell-1} = \bigoplus_{i=1}^{\ell-1} E_i/E_{i-1}$. It remains to split the short exact sequence

$$0 \to E_{\ell-1} \to E_\ell \to E_k/E_{\ell-1} \to 0.$$

The obstruction to the splitting is an element of

$$\bigoplus_{i=1}^{\ell-1} \mathrm{Ext}^1(E_\ell/E_{\ell-1}, E_i/E_{i+1}).$$

By Serre duality, we have that

$$\mathrm{Ext}^1(E_\ell/E_{\ell-1}, E_i/E_{i+1}) = \mathrm{Hom}(E_i/E_{i-1}, E_\ell/E_{\ell-1}) \tag{21.1}$$

since all twists by the canonical bundle disappear (since the canonical bundle is trivial). Finally, the Hom-group appearing in (21.1) is zero since $\mu(E_i/E_{i-1}) > \mu(E_\ell/E_{\ell-1})$ for $i < \ell$. □

One consequence of Corollary 21.2 is that, on elliptic curves, the study of vector bundles is the same as the study of semistable vector bundles.

Example 21.3. Let C be a smooth projective curve. If $f : D \to C$ is a finite étale morphism, then $f_*\mathcal{O}_D$ is semistable vector bundle of slope zero.

It suffices to show the statement after pullback to a finite flat cover of C, because degree zero line bundles pull back to degree zero (and similarly for negative degree line bundles). Thus, we may pull back along a suitable finite flat cover $C' \to C$ to ensure that $D' = D \times_C C' \simeq \coprod C'$ as curves over C'. In this case, the statement is clear, since $f_*\mathcal{O}_{D'} \simeq \bigoplus \mathcal{O}_{C'}$.

21.2. **Atiyah's Theorem on Vector Bundles on Elliptic Curves.** In this section, we discuss Atiyah's classification of vector bundles on an elliptic curve, following [Ati57]. Let E be an elliptic curve over k. Use a principal polarization to fix an isomorphism $E \simeq E^t$. The identity point $e \in E(k)$ defines a line bundle $L = \mathcal{O}(e)$, and the corresponding map $\phi_L : E \to E^t$ has degree $\sqrt{\dim H^0(E, L)}$; by the Riemann–Roch theorem, $\sqrt{\dim H^0(E, L)} = 1$, so ϕ_L is an isomorphism. In the sequel, we use ϕ_L to identify E and E^t.

Theorem 21.4. *Fix $\mu \in \mathbf{Q}$. There is an equivalence of categories*

$$T\colon \left\{ \begin{matrix} \text{semistable vector bundles} \\ \text{on } E \text{ of slope } \mu \end{matrix} \right\} \xrightarrow{\simeq} \left\{ \begin{matrix} \text{torsion coherent} \\ \text{sheaves on } E \end{matrix} \right\}.$$

In particular, the category of torsion coherent sheaves on E is independent of μ, and there exist semistable vector bundles on E of any slope. Moreover, if F is a semistable vector bundle on E, then

$$\operatorname{length}(T(F)) = \gcd(\operatorname{rank}(F), \deg(F)).$$

In order for it to form a reasonable category, we include the zero vector bundle in the category of semistable vector bundles of a fixed slope.

We have already seen an example where the existence of vector bundles with a specified slope is completely false: on $\mathbf{P}^1$, we saw that a semistable vector bundle must have integer slope.

Prior to proving Theorem 21.4, we require a sequence of lemmas.

Lemma 21.5. *For any $F \in \mathrm{D}^b_{\mathrm{Coh}}(E)$, we have $\deg(\phi_E(F)) = -\operatorname{rank}(F)$ and $\operatorname{rank}(\phi_E(F)) = \deg(F)$; in particular,*

$$\mu(\phi_E(F)) = -\mu(F)^{-1}.$$

Recall that the rank and degree functions are additive, so they descend to functions on the derived category.

Proof. As E is an elliptic curve, the Riemann–Roch theorem implies that $\chi(E, F) = \deg(F)$ and $\operatorname{rank}(F|_{\{e\}}) = \operatorname{rank}(F)$. However, we know that $R\Gamma(E, \phi_E(F)) = F[-1]|_{\{e\}}$ and $R\Gamma(E, F) \simeq \phi_E(F)|_{\{e\}}$. Taking Euler characteristics of these equalities give the desired equalities. □

Lemma 21.6. *If F is a vector bundle on E and $\mu(F) < 0$, then $\phi_E(F[1])$ is a semistable vector bundle with slope $-\mu(F)^{-1}$.*

The condition that $\mu(F) < 0$ in Lemma 21.6 is necessary: $\mu(F) = 0$ if and only if F has degree zero, in which case $\phi_E(F)$ is a skyscraper sheaf and not a vector bundle!

Proof. Granted the semistability of $\phi_E(F)$, the slope calculation is immediate from Lemma 21.5. We may assume $k = \overline{k}$ and that F is indecomposable. Since ϕ_E is an equivalence, $\phi_E(F[1])$ is also indecomposable as an object of $\mathrm{D}^b_{\mathrm{coh}}(E^t)$. . As F has negative slope, for any degree zero line bundles $L \in \operatorname{Pic}^0(E)$, we have $\operatorname{Hom}(L, F) = 0$; furthermore, $\operatorname{Hom}(L, F) = H^0(E, F \otimes L^{-1})$, so the twist of F by any degree zero line bundle has vanishing zeroth cohomology. It follows that

$$\phi_E(F)|_{\{x_L\}} \simeq R\Gamma(E, F \otimes L) \tag{21.2}$$

where $x_L \in E^t$ is the point corresponding to the line bundle L. The right-hand side of (21.2) is concentrated in degree 1, so by a semicontinuity and base change-type argument, it follows that $\phi_E(F[1])$ is a vector bundle. As $\phi_E(F[1])$ is indecomposable, it is semistable by Corollary 21.2. □

Lemma 21.7. *Fix $\mu \in \mathbf{Q}$. There is an equivalence between the category of vector bundles of degree μ, and the category of vector bundles of degree zero, obtained by iterating the operations*

$$F \mapsto \phi_E(F[1]) \tag{21.3}$$

and

$$F \mapsto F \otimes \mathcal{O}(e) \tag{21.4}$$

and their inverse.

Proof. If $\mu = 0$, there is nothing to show. If $\mu < 0$, then we may pass from slope μ to $-\mu^{-1}$ using (21.3), and from slope μ to $\mu + 1$ by (21.4). Recall that $\mathrm{SL}_2(\mathbf{Z})$ acts transitively on $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ by fractional linear transformation. In addition, $\mathrm{SL}_2(\mathbf{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrices S and T exactly correspond to the action of (21.3) and (21.4) on the slopes, so the result follows immediately from the transitivity of the $\mathrm{SL}_2(\mathbf{Z})$ action. $\square$

Proof of Theorem 21.4. We may assume that k is algebraically closed. The first goal is to prove, by induction on $\operatorname{rank}(F)$, that $\phi_E(F)[1]$ is a torsion coherent sheaf on E. The case $\operatorname{rank}(F) = 1$ has already been discussed.

In general, if there exists a nonzero map $L \to F$ with $L \in \operatorname{Pic}^0(E)$, then F/L is a vector bundle of slope zero; thus, we proceed by induction. More precisely, we must show that if $L \to F$ is a nonzero map, then L is saturated and so F/L is a vector bundle bundle. If $L \subseteq F$ was not saturated, then the saturation $L' \subseteq F$ would give a subbundle with $\mu(L') = \deg(L') > \deg L = 0$, violating the semistability of F.

If there are no such maps, then $H^0(F \otimes L) = 0$ for all $L \in \operatorname{Pic}^0(E)$, and then by Lemma 21.6 we have that $\phi_E(F)[1]$ is a vector bundle. Thus, we have that $\operatorname{rank}(\phi_E(F)[1]) = \pm\chi(E, F) = \deg(F) = 0$, so $\phi_E(F)[1] = 0$. It follows that $F = 0$.

Conversely, as the torsion coherent sheaves are generated under extensions by skyscraper sheaves of points, this functor is surjective. Finally, the length of $\phi_E(F)[1]$ is $\chi(E, \phi_E(F)[1]) = \operatorname{rank}(F)$. $\square$

Corollary 21.8. *If k is algebraically closed and F is an indecomposable vector bundle on E of degree d and rank r, then the following are equivalent:*

(1) F is stable;
(2) $\gcd(d, r) = 1$;
(3) F is simple as a sheaf, i.e. $\operatorname{Hom}(F, F) = k$;
(4) F is simple as an object of the category of semistable vector bundles of slope μ.

In particular, there are stable vector bundles on E of arbitrary rational slope.

The proof Corollary 21.8 is straightforward from Theorem 21.4, and it is left as an exercise.

22. November 21st

For the rest of the semester, we will be discussing special topics. The first topic is Hacon's work on generic vanishing theorems.

22.1. **Generic Vanishing Theorems.** This story was originally developped by Green–Lazarsfeld [GL87, GL91] in the late 80's-early 90's, advanced further by Hacon [Hac04] in the early 00's, and studied more recently by Popa–Schnell [PS13].

Let X be a smooth, proper variety over $\mathbf{C}$, and fix a rational point $x \in X(\mathbf{C})$. The rational point x is used to define the Albanese map $a\colon X \to \operatorname{Alb}(X)$, where $\operatorname{Alb}(X)$ is the universal abelian variety receiving a map from X sending x to the identity. (For example, if X is a curve, then $\operatorname{Alb}(X) = \operatorname{Jac}(X)$). The construction of a and $\operatorname{Alb}(X)$ will be discussed later.

We will work towards the generic vanishing theorem below, following Hacon's treatment [Hac04].

Theorem 22.1. [Green–Lazarsfeld] *For $L \in \underline{\operatorname{Pic}}^0(X)$ general and $i < \dim(a(X))$, then $H^i(X, L) = 0$ and $H^{\dim(X)-i}(X, \omega_X \otimes L) = 0$.*

Remark 22.2. We record two easy corollaries of Theorem 22.1.

(1) If X has *maximal Albanese dimension* (that is, the map $a\colon X \to \mathrm{Alb}(X)$ is generically finite) and $L \in \underline{\mathrm{Pic}}^0(X)$ is general, then $H^i(X, L) = 0$ for all $i < \dim(X)$ and $H^i(X, \omega_X \otimes L) = 0$ for $i > 0$.

(2)
$$(-1)^{\dim(X)}\chi(X, \mathcal{O}_X) = (-1)^{\dim(X)}\chi(X, \omega_X) \geq 0$$
To get this, the Picard variety is connected, so twist $\mathcal{O}_X$ or ω_X by a sufficiently general L, and they're connected by a flat family and Euler characteristic doesn't change.

When dealing with the Albanese map, the general philosophy is the following: the larger the Albanese dimension, the more of the variety that can be "seen" using abelian varieties. Here, the Albanese dimension refers to the dimension of the image of the Albanese map.

There are two main inputs into the proof of Theorem 22.1:

(1) purely algebraic input: Fourier–Mukai machinery, as developped by Hacon in [Hac04];
(2) Hodge-theoretic input: Kollàr's vanishing theorem (a Kodaira-type vanishing theorem for morphisms), which we will state but not prove.

22.2. **Picard & Albanese Schemes.** Let k be a field of characteristic zero, let X be a smooth, proper scheme over k, and let $x \in X(k)$ be a basepoint. In this section, we will describe, but not construct, the Picard scheme $\underline{\mathrm{Pic}}(X)$ and the Albanese scheme $\mathrm{Alb}(X)$.

The Picard scheme $\underline{\mathrm{Pic}}(X)$ of X will be a scheme parametrizing line bundles on X. We define it first as the functor $\underline{\mathrm{Pic}}(X)\colon (\mathrm{Sch}/k) \to (\mathrm{Ab})$ given by
$$T \mapsto \big(L \in \mathrm{Pic}(X \times T), \iota\colon L|_{\{x\}\times T} \simeq \mathcal{O}_T\big) / \simeq,$$
where the abelian group structure on this set is the obvious one, i.e. tensor together the two line bundles. The category of such pairs (L, ι) has no nontrivial automorphisms (i.e. it is *discrete*) because we have fixed the trivialization ι of L at the basepoint x. In addition, the k-points of this functor give the Picard group: $\underline{\mathrm{Pic}}(X)(k) = \mathrm{Pic}(X)$ (in particular, one does need to record the trivialization ι for k-points).

Theorem 22.3. [Grothendieck] *The functor $\underline{\mathrm{Pic}}(X)$ is a representable by a locally finitely-presented group scheme over k, and the identity component $\underline{\mathrm{Pic}}^0(X)$ is an abelian variety over k.*

For a proof of Theorem 22.3, see [FGI+05, Part 5]. The only assertion in Theorem 22.3 that uses the characteristic zero assumption is the smoothness of the connected component $\underline{\mathrm{Pic}}^0(X)$. It is possible that the Picard scheme is non-reduced in positive characteristic: Igusa [Igu55] gave an example of a surface over a field of characteristic 2 with this property.

The *Poincaré bundle* $\mathcal{P}_X \in \mathrm{Pic}(X \times \underline{\mathrm{Pic}}(X))$ is the line bundle corresponding to the identity map from $\underline{\mathrm{Pic}}(X)$ to itself; it comes equipped with a *universal trivialization* $\iota^{\mathrm{univ}}\colon \mathcal{P}_X|_{\{x\}\times\underline{\mathrm{Pic}}(X)} \simeq \mathcal{O}_{\underline{\mathrm{Pic}}(X)}$.

Remark 22.4. The construction $(X, x) \mapsto \underline{\mathrm{Pic}}(X)$ is contravariantly functorial (as expected, since line bundles pull back).

Example 22.5. If X is a smooth, proper curve of genus g, then $\underline{\mathrm{Pic}}(X) \simeq \mathrm{Jac}(X) \times \mathbf{Z}$. In fact, if $k = \mathbf{C}$, then one can explicitly construct $\underline{\mathrm{Pic}}^0(X)$ as a complex-analytic space: it follows from the exponential exact sequence that
$$\underline{\mathrm{Pic}}^0(X)^{\mathrm{an}} = H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})/H^1(X^{\mathrm{an}}, \mathbf{Z}).$$

The complex-analytic realization of $\underline{\mathrm{Pic}}^0(X)$ for curves in Example 22.5 suggests the following:

Proposition 22.6. *For any smooth, proper k-scheme X, the abelian variety $\underline{\mathrm{Pic}}^0(X)$ has dimension $h^1(X, \mathcal{O}_X)$.*

Proof. It suffices to show that the tangent space $T_e\underline{\mathrm{Pic}}^0(X)$ at the identity is canonically isomorphic to $H^1(X, \mathcal{O}_X)$. Recall that there is an identification
$$T_e\underline{\mathrm{Pic}}^0(X) = \ker\big(\underline{\mathrm{Pic}}(X)(k[\epsilon]) \to \underline{\mathrm{Pic}}(X)(k)\big)$$

as groups, and moreover $\underline{\mathrm{Pic}}(X)(k[\epsilon]) = \mathrm{Pic}(X[\epsilon])$, where $X[\epsilon] \coloneqq X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$. Similarly, $\underline{\mathrm{Pic}}(X)(k) = \mathrm{Pic}(X)$. Thus, $\ker(\mathrm{Pic}(X[\epsilon]) \to \mathrm{Pic}(X))$ is computed via the following "poor man's version" of the exponential exact sequence:

$$1 \to 1 + \epsilon\mathcal{O}_{X[\epsilon]} \to \mathcal{O}^*_{X[\epsilon]} \to \mathcal{O}^*_X \to 1. \tag{22.1}$$

This is a short exact sequence of sheaves on X, where we identify the topological space of $X[\epsilon]$ with the topological space of X. Furthermore, there is an isomorphism $1 + \epsilon\mathcal{O}_{X[\epsilon]} \simeq \mathcal{O}_X$ of sheaves on X, given by $1 + \epsilon\alpha \mapsto \alpha$ on local sections (one thinks of this isomorphism and its inverse as "log" and "exp"). The long exact sequence in cohomology associated to (22.1) gives

$$H^0(X, \mathcal{O}^*_{X[\epsilon]}) \longrightarrow H^0(X, \mathcal{O}^*_X) \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{\exp} \underbrace{H^1(X, \mathcal{O}^*_{X[\epsilon]})}_{\simeq \mathrm{Pic}(X[\epsilon])} \to \underbrace{H^1(X, \mathcal{O}^*_X)}_{\simeq \mathrm{Pic}(X)}.$$

However, $H^0(X, \mathcal{O}^*_X) \simeq k^*$ because X is smooth and proper, so the map $H^0(X, \mathcal{O}^*_{X[\epsilon]}) \longrightarrow H^0(X, \mathcal{O}^*_X)$ must be surjective, and hence exp is injective, as required. $\square$

Now, we will discuss the Albanese scheme of X, which (in our setting) is the universal abelian variety receiving a map from X. It can be realized as the dual abelian variety of $\underline{\mathrm{Pic}}^0(X)$. More precisely, the pair $(\mathcal{P}_X, \iota^{\mathrm{univ}})$ defines a map $X \to \underline{\mathrm{Pic}}^0(X)^t$, because there exists a unique trivialization of $\mathcal{P}_X|_{X\times\{e\}}$ compatible with ι^{univ} over $\mathcal{P}_X|_{\{x\}\times\{e\}}$. Informally, the map $X \to \underline{\mathrm{Pic}}^0(X)^t$ is described as follows: given $y \in X(k)$, map it to

$$\mathcal{P}_X|_{\{y\}\times\underline{\mathrm{Pic}}^0(X)} \in \mathrm{Pic}^0(\underline{\mathrm{Pic}}^0(X)).$$

Definition 22.7. The *Albanese scheme* of X is $\mathrm{Alb}(X) \coloneqq \underline{\mathrm{Pic}}^0(X)^t$, and the preceding map $a\colon X \to \mathrm{Alb}(X)$ is the *Albanese map.*

Claim 22.8. The Albanese map $a\colon X \to \mathrm{Alb}(X)$ is the universal map $f\colon X \to B$ such that B is an abelian variety and $f(x) = e$.

Proof. It is clear that $a(x) = e$. Given such a map $f\colon X \to B$, it induces a pullback map $f^*\colon \underline{\mathrm{Pic}}^0(B) \to \underline{\mathrm{Pic}}^0(X)$, which one can dualize to get a map $\mathrm{Alb}(X) \to B$. It is an exercise in unwinding the definitions to verify that the composition $X \to \mathrm{Alb}(X) \to B$ coincides with f. $\square$

The Albanese map $a\colon X \to \mathrm{Alb}(X)$ induces an isomorphism $\underline{\mathrm{Pic}}^0(\mathrm{Alb}(X)) \xrightarrow{\simeq} \underline{\mathrm{Pic}}^0(X)$. The upshot of this is the following: every degree zero line bundle on X comes from $\mathrm{Alb}(X)$ via pullback (here, a degree zero line bundle on X, by definition, means that it comes from a point of $\underline{\mathrm{Pic}}^0(X)$).

Remark 22.9. One can think of the Albanese scheme construction as an algebro-geometric analogue of the Postnikov towers from algebraic topology.

22.3. **Hacon's Theorem.** In this section, we will develop the purely algebraic machinery required to prove Theorem 22.1.

Notation 22.10. Let A be an abelian variety over any field k.

1. Let $\phi_A\colon \mathrm{D}(A) \xrightarrow{\simeq} \mathrm{D}(A^t)$ denote the Fourier–Mukai transform, defined by the Poincaré bundle $\mathcal{P}_A$.
2. [Verdier[35] Duality] For any smooth, proper k-scheme X of dimension n, consider the *Verdier duality functor* $\mathbf{D}_X(-) \coloneqq \underline{\mathrm{RHom}}(-, \omega_X[n])$ as an endofunctor of the derived category $\mathrm{D}(X)$. We record some basic facts and examples below.
   (a) The functor $\mathbf{D}_X$ is an equivalence on $\mathrm{D}^b_{\mathrm{coh}}(X)$ (in fact, it is an involution, i.e. $\mathbf{D}_X^2 = \mathrm{id}$) and, when $X = \mathrm{Spec}(k)$, $\mathbf{D}_X$ just gives duality of finite-dimensional vector spaces.

[35] In this context, it may be more appropriate to refer to this as "Grothendieck duality".

(b) If $f\colon X \to Y$ is a proper map between smooth, proper k-schemes, then $Rf_* \circ \mathbf{D}_X \simeq \mathbf{D}_Y \circ Rf_*$ as functors on $\mathrm{D}^b_{\mathrm{coh}}(X)$. In addition, by taking $Y = \mathrm{Spec}(k)$, we get that

$$R\Gamma(X, \underline{\mathrm{RHom}}(F, \omega_X[n])) \simeq (R\Gamma(X, F))^\vee$$

which is precisely the statement of (the derived category version of) Serre duality (here, $\vee$ denotes the linear dual of a vector space). In particular, this implies that $H^i(X, F)$ is dual to $\mathrm{Ext}^{n-i}(F, \omega_X)$ for any coherent sheaf F on X, which is the more familiar version of Serre duality.

(3) For an ample line bundle L on A^t, set $E(L) \coloneqq \phi_{A^t}(L)$. By Theorem 19.2, $E(L)$ is a vector bundle on A, viewed as a complex by placing it in degree zero. Recall that

$$\phi_L^*(E(L)) \simeq H^0(A^t, L) \otimes L^{-1}.$$

In particular, $\phi_L^*(E(L))$ is an anti-ample vector bundle on A^t.

Theorem 22.11. [Hac04, Theorem 1.2] *If $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$, then the following are equivalent:*

(1) for a sufficiently ample line bundle L on A^t, $H^i(A, F \otimes E(L)^\vee) = 0$ for all $i \neq 0$;
(2) for a sufficiently ample line bundle L on A^t, $H^i(A^t, \phi_A(\mathbf{D}_A(F)) \otimes L) = 0$ for all $i \neq 0$;
(3) the complex $\phi_A(\mathbf{D}_A(F))$ is concentrated in degree zero.

Proof. The statement (1) is equivalent to the assertion that $\mathbf{D}_k\left(R\Gamma(A, F \otimes E(L)^\vee)\right)$ is concentrated in degree zero. By Verdier duality, this can be written as

$$\mathbf{D}_k\left(R\Gamma(A, F \otimes E(L)^\vee)\right) \simeq R\Gamma(A, \mathbf{D}_A(F \otimes E(L)^\vee)) \simeq R\Gamma(A, \mathbf{D}_A(F) \otimes E(L)), \tag{22.2}$$

where the second isomorphism is immediate from the definition of $\mathbf{D}_A$. Furthermore, (22.2) is isomorphic to

$$R\Gamma(A, \mathbf{D}_A(F) \otimes \mathrm{pr}_{1,*}(\mathcal{P} \otimes \mathrm{pr}_2^*(L))), \tag{22.3}$$

where pr_1 and pr_2 denote the usual projections from $A \times A^t$. Now, by the projection formula, (22.3) is isomorphic to

$$\begin{aligned} R\Gamma(A, \mathrm{pr}_{1,*}(\mathrm{pr}_1^*(\mathbf{D}_A(F) \otimes \mathcal{P} \otimes \mathrm{pr}_2^*(L))) &\simeq R\Gamma(A \times A^t, \mathrm{pr}_1^*(\mathbf{D}_A(F)) \otimes \mathcal{P} \otimes \mathrm{pr}_2^*(L)) \\ &\simeq R\Gamma(A^t, \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\mathbf{D}_A(F)) \otimes \mathcal{P}) \otimes L), \end{aligned}$$

and this last expression is exactly $\phi_A(\mathbf{D}_A(F))$. The assertion that the complex $\phi_A(\mathbf{D}_A(F))$ is concentrated in degree zero is equivalent to (2). Thus, we have shown the equivalence between (1) and (2).

The equivalence between (2) and (3) follows the the general lemma below. (2) iff (3) follows from the following much more general statement:

Lemma 22.12. *If X is a projective k-scheme and $M \in \mathrm{D}^b_{\mathrm{coh}}(X)$, then M is concentrated in degree zero iff $R\Gamma(X, M \otimes L)$ is concentrated in degree zero for all sufficiently ample line bundles L on X.*

To obtain the equivalence between (2) and (3) from Lemma 22.12, take $M = \phi_A(\mathbf{D}_A(F))$.

Proof. If M is concentrated in degree zero, then M is a coherent sheaf, so this is precisely the statement of Serre's vanishing theorem. Conversely, given $M \in \mathrm{D}^b_{\mathrm{coh}}(X)$, consider standard hypercohomology spectral sequence

$$E_2^{p,q}\colon H^p(X, \mathcal{H}^q(M) \otimes L) \Longrightarrow H^{p+q}(X, M \otimes L), \tag{22.4}$$

which allows one to compute the cohomology of M in terms of its cohomology sheaves. For a sufficiently ample line bundle L on X, we have

(i) $H^n(X, M \otimes L) = 0$ for all $n > 0$, by assumption;
(ii) $H^p(X, \mathcal{H}^q(M) \otimes L) = 0$ for all $p > 0$ and all $q \in \mathbf{Z}$, by Serre's vanishing theorem.

By (ii), the spectral sequence (22.4) is completely degenerate, and so there is an isomorphism

$$H^0(X, \mathcal{H}^q(M) \otimes L) \simeq H^q(X, M \otimes L)$$

and (i) implies that $H^0(X, \mathcal{H}^q(M) \otimes L) = 0$ for all $q \neq 0$. However, this cannot occur unless $\mathcal{H}^q(M) = 0$ for all $q \neq 0$; therefore, M is concentrated in degree zero. □

This completes the proof of Theorem 22.11 □

23. November 28th

We will continue today our discussion of Hacon's proof of a generic vanishing theorem. The setup is as follows: let X be a smooth,proper k-scheme, where k is a field of characteristic zero. Assume for simplicity that k is algebraically closed. Last time, we introduced the Albanese map $a\colon X \to \mathrm{Alb}(X)$, where $\mathrm{Alb}(X) = \underline{\mathrm{Pic}}^0(X)^t$. Write $A := \mathrm{Alb}(X)$ and $A^t = \underline{\mathrm{Pic}}^0(X)$, and let g denote the dimension of A.

The generic vanishing theorem to be proven is as follows:

Theorem 23.1. *If $M \in A^t(k)$ is general, then $H^i(X, \omega_X \otimes M) = 0$ for all $i > 0$.*

In fact, we will show a stronger statement than Theorem 23.1 that gives some control on the loci of $M \in A^t(k)$ for which the desired vanishing holds.

One easy corollary of Theorem 23.1 is the following: $\chi(X, \omega_X) = \dim H^0(X, \omega_X \otimes M) \geq 0$ for general $M \in A^t(k)$ general (here, we use that the Euler characteristic is invariant in flat families, and moreover that there are flat families in $\underline{\mathrm{Pic}}^0(X)$ connecting M and $\mathcal{O}_X$).

23.1. **Hacon Complexes & Hacon Sheaves.** Last time, we showed an abstract statement Theorem 22.11, due to Hacon, that asserted certain properties of a complex in the derived category of an abelian variety are equivalent. We repeat a simplified version of this result below.

Theorem 23.2. *If $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$, then the following are equivalent:*

(1) for sufficiently ample line bundle L on A^t, $R\Gamma(A, F \otimes E(L)^\vee)$ lives in degree zero;
(2) the complex $\phi_A(\mathbf{D}_A(F))$ lives in degree zero.

In the statement of Theorem 23.2, recall that $E(L) = \phi_{A^t}(L)$ is the Fourier transform and it is known to be a vector bundle placed in degree zero, because the line bundle L is ample, hence non-degenerate. Moreover, a complex "lives in degree zero" if its homology sheaves are zero outside of degree zero.

Definition 23.3. A complex $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$ is a *Hacon complex* if it satisfies one of the equivalent conditions of Theorem 23.2. Furthermore, if F is a coherent sheaf placed in degree zero, then we say F is a *Hacon sheaf*[36].

The key properties of Hacon complexes that will be needed are isolated in the following theorem.

Theorem 23.4. *Let F be a Hacon complex.*

(1) The cohomology support locus[37]

$$S^i(A^t, F) := \{M \in A^t(k)\colon H^i(A, F \otimes M) \neq 0\}$$

is closed in A^t, and each irreducible component of $S^i(A^t, F)$ has codimension $\geq i$. In particular, a general point of A^t does not lie in any of the S^i's.

(2) If F is a Hacon sheaf, then there is a sequence of inclusions

$$S^g(A^t, F) \subseteq S^{g-1}(A^t, F) \subseteq \ldots \subseteq S^0(A^t, F).$$

(3) If F is a Hacon sheaf, then $\chi(A, F) \geq 0$.

[36]In the literature, Hacon sheaves are also known as GV sheaves (where 'GV' is short for 'generic vanishing').

[37]It is in defining the cohomology support loci that it is convenient to assume that the field k is algebraically closed. Without this assumption on k, one can specify the T-valued points of $S^i(A^t, F)$ for any k-scheme T, but we will ignore this.

The key tools of the proof are commutative algebraic, but we first require some compatibility between the Fourier transform and the Verdier duality functor.

Lemma 23.5. *If $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$, then*

$$\mathbf{D}_{A^t}\left(\phi_A(F)\right) \simeq [-1]^*\phi_A\left(\mathbf{D}_A(F)\right)[g].$$

The proof is left as an exercise (the idea being that the Verdier duality function is simply $\mathrm{RHom}(-,\mathcal{O}_A)$ since the canonical sheaf is trivial on an abelian variety, and $\mathrm{RHom}(-,\mathcal{O}_A)$ commutes with arbitrary pullbacks and pushforwards).

From Lemma 23.5, we can deduce the following alternate description of Hacon complexes. We adopt the following notation: set $(-)^\vee := \underline{\mathrm{RHom}}(-,\mathcal{O}_A) = \mathbf{D}_A(-)[-g]$.

Proposition 23.6. *If $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$, then the following are equivalent:*

(1) F is a Hacon complex;
(2) $\phi_A(F)^\vee$ lives in degree zero;
(3) $\phi_A(F) \simeq E^\vee$ for some coherent sheaf $E \in \mathrm{Coh}(A^t) \subseteq \mathrm{D}^b_{\mathrm{coh}}(A^t)$.

The upshot of Proposition 23.6 is that the Fourier transform ϕ_A gives an equivalence between the subcategory of Hacon complexes in $\mathrm{D}^b_{\mathrm{coh}}(A)$ and the subcategory $\mathrm{Coh}(A^t) \subseteq \mathrm{D}^b_{\mathrm{coh}}(A^t)$.

Proof. The condition (1) is (by definition) equivalent to the assertion that $\phi_A(\mathbf{D}_A(F))$ is a coherent sheaf placed in degree zero. By Lemma 23.5, this occurs precisely when $\mathbf{D}_{A^t}(\phi_A(F))$ is a coherent sheaf placed in homological degree g. Using the fact that $\mathbf{D}_A(-) = (-)^\vee[g]$, this is in turn equivalent to $(\phi_A(F))^\vee$ being a coherent sheaf placed in degree zero, which is precisely the condition (2). The equivalence between (2) and (3) is formal since $(-)^\vee$ is a duality. ☐

Lemma 23.7. *If R is a regular ring and M is a finitely-generated R-module, then $\mathrm{Ext}^i_R(M,R)$ is supported in codimension $\geq i$; that is, $\mathrm{Ext}^i_R(M,R)$ vanishes after localizing at a prime ideal of height $< i$.*

Proof. Pick a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ of height $< i$. By the noetherianity of R, we have

$$\mathrm{Ext}^i_R(M,R) \otimes_R R_\mathfrak{p} = \mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p})$$

and this vanishes because the global dimension of $R_\mathfrak{p}$ is $\leq \dim(R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) < i$, since R is regular. ☐

Remark 23.8. The ideas of Lemma 23.7 can be pushed further: under the autoequivalence

$$\mathrm{RHom}(-,R)\colon\ \mathrm{D}(R) \xrightarrow{\simeq} \mathrm{D}(R),$$

the image of the subcategory $\mathrm{Mod}^{\mathrm{fg}}_R \subseteq \mathrm{D}(R)$ is the subcategory of perverse R-modules.

Corollary 23.9. *If $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$ is a Hacon complex, then $\mathrm{Supp}\left(\mathcal{H}^i(\phi_A(F))\right) \subseteq A^t$ has codimension $\geq i$.*

Proof. Since F is Hacon sheaf, we can write $\phi_A(F) = E^\vee$ for some $E \in \mathrm{Coh}(A^t)$. Locally, the cohomology sheaf $\mathcal{H}^i(E^\vee)$ is an Ext group as in Lemma 23.7, so applying the lemma completes the proof. ☐

Proof of Theorem 23.4. To prove (1), take a Hacon complex $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$ and we must show that the cohomology support locus $S^i(A^t, F)$ is a closed subset of A^t of codimension $\geq i$. The closedness follows from the semicontinuity theorem, and the bound on the codimension can be deduced from the following commutative algebra lemma.

Lemma 23.10. *If R is a noetherian ring and $M \in \mathrm{D}^b_{\mathrm{coh}}(R)$, then the locus*

$$S^i(R,M) := \{x \in \mathrm{Spec}(R)\colon H^i(M \otimes k(x)) \neq 0\}$$

is closed in $\mathrm{Spec}(R)$, *and* $S^i(R,M) \subseteq \displaystyle\bigcup_{j \geq i} \mathrm{Supp}\left(H^j(M)\right)$.

In fact, Lemma 23.10 holds more generally over any commutative ring and for perfect complexes. Now, granted Lemma 23.10, (locally) taking $M = \phi_A(F)$ immediately gives (1).

Proof of Lemma 23.10. The closedness of $S^i(R, M)$ follows from the semicontinuity theorem. Furthermore, if the desired containment does not hold, then there exists $x \in S^i(R, M)$ such that

$$H^j(M) \otimes_R R_x = 0$$

for all $j \geq i$, and hence $M \otimes_R R_x \in \mathrm{D}^{<i}$ (here, $\mathrm{D}^{<i}$ denotes the subcategory of $\mathrm{D}^b_{\mathrm{coh}}(R)$ consisting of those complexes whose cohomology lives in degree strictly less than i). Now,

$$M \otimes_R k(x) = (M \otimes_R R_x) \otimes_{R_x} k(x) \in \mathrm{D}^{<i},$$

so we have $H^i(M \otimes k(x)) = 0$; however, $x \in S^i(R, M)$, so $H^i(M \otimes k(x)) \neq 0$, a contradiction. □

For the assertion (2), $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$ is a Hacon sheaf, so all cohomology of its twists by degree zero line bundles vanishes in degree larger than g. It remains to show that $S^i(A^t, F) \subseteq S^{i-1}(A^t, F)$ for $i > 0$, or equivalently: if $x \in A^t(k)$ corresponds to the degree zero line bundle M on A and it is such that $H^i(F \otimes M) \neq 0$, then we must show that $H^{i-1}(F \otimes M) \neq 0$. As before, write $\phi_A(F) = E^\vee$ for some coherent sheaf $E \in \mathrm{Coh}(A^t)$. Then, we have that

$$E^\vee \otimes_{\mathcal{O}_{A^t}} k(x) = \phi_A(F) \otimes_{\mathcal{O}_{A^t}} \kappa(x) \simeq R\Gamma(A, F \otimes M). \tag{23.1}$$

Note that $E^\vee = \underline{\mathrm{RHom}}(E, \mathcal{O}_{A^t})$, so (23.1) can be recast as

$$\underline{\mathrm{RHom}}(E, k(x)) \simeq \underline{\mathrm{RHom}}(E, \mathcal{O}_{A^t}) \otimes k(x) \simeq R\Gamma(A, F \otimes M), \tag{23.2}$$

where the first isomorphism follows by replacing the coherent sheaf with a finite free resolution (which exists since A is smooth), and the assertion is obvious for a free module.

From the description (23.2), the required non-vanishing is immediately deduced from the following commutative algebra lemma.

Lemma 23.11. *Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring, and let M be a finitely-generated R-module. If $\mathrm{Ext}^i_R(M, k) \neq 0$ for some $i > 0$, then $\mathrm{Ext}^{i-1}_R(M, k) \neq 0$.*

Proof. Let $K^\bullet \to M$ be a minimal free resolution (that is, each K^i is a finite free R-module, and all differentials in the complex $K^\bullet$ are zero mod $\mathfrak{m}$). It has the property that $\mathrm{rank}(K^i) = \dim_\kappa \mathrm{Ext}^i_R(M, \kappa)$; thus, if $\mathrm{Ext}^j_R(M, \kappa) = 0$ for some $j \in \mathbf{Z}_{\geq 0}$, it follows that $K^j = 0$. By minimality, one then has $K^\ell = 0$ for all $\ell \geq j$. In particular, $\mathrm{Ext}^{i-1}_R(M, \kappa)$ cannot be zero. □

Finally, it remains to show (3): if $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$ is a Hacon sheaf, then $\chi(A, F) \geq 0$. By (1), $\bigcup_{i>0} S^i(A^t, F)$ is a proper closed subset of A^t. Therefore, for any degree zero line bundle $M \in A^t \setminus \bigcup_{i>0} S^i(A^t, F)$, we have

$$R\Gamma(A, F \otimes M) = H^0(A, F \otimes M),$$

and hence $\chi(A, F) = \chi(A, F \otimes M) = \dim H^0(A, F \times M) \geq 0$. □

The proof of the generic vanishing theorem relies on constructing interesting geometric examples of Hacon sheaves. Recall that $a \colon X \to A := \mathrm{Alb}(X)$ denotes the Albanese map and $A^t := \underline{\mathrm{Pic}}^0(X)$ is the identity component of the Picard scheme.

Theorem 23.12. *For any $k \in \mathbf{Z}_{\geq 0}$, $R^k a_*(\omega_X)$ is a Hacon sheaf.*

The key input into the proof of Theorem 23.12 is Kollár's vanishing theorem; this is the part of the proof that seriously requires the characteristic zero assumption. After proving Theorem 23.12, we will see next time how to deduce the generic vanishing theorem.

Proof. Fix an ample line bundle L on A^t. Then, $E(L) := \phi_{A^t}(L)$ a vector bundle on A with the following two properties:

- $\phi_L \colon A^t \to A$ is finite étale;
- $\phi_L^*(E(L)) \simeq H^0(A, L) \otimes L^{-1}$.

The goal is to show that $H^i(A, R^k a_*(\omega_X) \otimes E(L)^\vee) = 0$ for all $i \neq 0$, provided L is sufficiently ample.

As ϕ_L is finite and surjective, the map $\mathcal{O}_A \to (\phi_L)_*(\mathcal{O}_{A^t})$ is a direct summand (indeed, as we are working in characteristic zero, we can use the map $\frac{1}{\deg(\phi_L)}\mathrm{trace}(-)$ as a splitting). Therefore, it suffices to show the vanishing of

$$H^i(A, R^k a_*(\omega_X) \otimes E(L)^\vee \otimes (\phi_L)_*(\mathcal{O}_{A^t})) = H^i(A^t, \phi_L^*(R^k a_*(\omega_X)) \otimes \phi_L^*(E(L))^\vee),$$

for all $i \neq 0$, where the isomorphism follows from the projection formula. Now, as $\phi_L^*(E(L))$ is a direct sum of copies of L^{-1}, it suffices to show the vanishing of

$$H^i(A^t, \phi_L^*(R^k a_*(\omega_X)) \otimes L)$$

for all $i \neq 0$. To that end, consider the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "b"'] & X \arrow[d, "a"] \\ A^t \arrow[r, "\phi_L"] & A \end{tikzcd}$$

As ϕ_L is finite étale, we have $\phi_L^*(R^k a_*(\omega_X)) \simeq R^k b_*(\omega_{X'})$. Thus, it suffices to show that

$$H^i(A^t, R^k b_*(\omega_{X'}) \otimes L) = 0$$

for all $i > 0$, and this follows from Kollár vanishing (which we will explain next time). $\square$

24. November 30th

Today we will complete our discussion of Hacon's proof of a generic vanishing theorem, and then we will introduce the Jacobians of curves, with the aim of proving the Torelli theorem following Beilinson–Polishchuk [BP01].

24.1. **Hacon Complexes & Hacon Sheaves (Continued).** For an abelian variety A over an algebraically closed field k, we introduced last time the notion of a Hacon sheaf $F \in \mathrm{D}^b_{\mathrm{coh}}(A)$ (one characterization was that $\phi_A(F) = E^\vee$ for some $E \in \mathrm{Coh}(A^t)$), and we proved various properties about the associated cohomology support loci $S^i(A^t, F) := \{M \in A^t(k) \colon H^i(A, F \otimes M) \neq 0\}$. In particular, we showed that $S^i(A^t, F) \subseteq A^t$ has codimension $\geq i$, and there is a sequence of inclusions

$$S^g(A^t, F) \subseteq S^{g-1}(A^t, F) \subseteq \ldots \subseteq S^0(A^t, F).$$

Furthermore, we were discussing a construction that produces interesting geometric examples of Hacon sheaves, detailed in the theorem below.

Theorem 24.1. *Assume the field k has characteristic zero. If X is a smooth, projective variety over k and $a \colon X \to A = \mathrm{Alb}(X)$ is the Albanese map, then $R^k a_*(\omega_X)$ is a Hacon sheaf for all $k \in \mathbf{Z}_{\geq 0}$.*

Proof. Last time, we reduced the proof to following assertion: L is an ample line bundle on A^t, consider the pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "b"'] & X \arrow[d, "a"] \\ A^t \arrow[r, "\phi_L"] & A \end{tikzcd}$$

where both horizontal maps are finite étale; then, we must show that

$$H^i(A^t, R^k b * (\omega_{X'}) \otimes L) = 0 \tag{24.1}$$

for all $i > 0$. This statement looks similar to Kodaira vanishing, and it is a "relative version" of Kodaira vanishing that will be applied to deduce this, due to Kollár [Kol86a, Kol86b].

Theorem 24.2. [Kol86a, Theorem 2.1] *If $f\colon X \to Y$ is a surjective map of projective varieties over $\mathbf{C}$ and X is smooth, then*

(a) for any ample line bundle L on Y, we have $H^i(Y, R^k f_*\omega_X \otimes L) = 0$ for all $i > 0$;
(b) each $R^j f_*\omega_X$ is torsion-free; in particular, $R^j f_*\omega_X = 0$ if $j > \dim(X) - \dim(Y)$;
(c) there is decomposition $Rf_*\omega_X = \bigoplus_j R^j f_*\omega_X[-j]$ in the derived category $\mathrm{D}(Y)$.

The proof of Kollár's vanishing theorem relies on certain Hodge-theoretic results; more precisely, the decomposition in (c) is obtained by combining the BBD Decomposition theorem [BlBD82] with Saito's theory of mixed Hodge modules [Sai86]. Both (a) and (b) can be deduced from (c). We will not discuss any further the proof of Kollár's theorem.

Now, applying Kollár's vanishing theorem for $X' \to Y := b(X') \subseteq A^t$ precisely gives the vanishing (24.1) (this can be done since the sheaves $R^k b_*(\omega_{X'})$ are only supported on $b(X')$). □

24.2. **Proof of a Generic Vanishing Theorem.** We will apply Theorem 24.1, along with the properties of Hacon sheaves, to prove the generic vanishing theorem of Green–Lazarsfeld [GL87, GL91].

Corollary 24.3. [Green–Lazarsfeld] *Let X be a smooth, projective variety over k, $a\colon X \to A = \mathrm{Alb}(X)$ be the Albanese map, $A^t = \underline{\mathrm{Pic}}^0(X)$, and*

$$S^i(\omega_X) := \{M \in A^t(k)\colon H^i(X, \omega_X \otimes M) \neq 0\}.$$

Then, each cohomology support locus $S^i(\omega_X) \subseteq A^t$ is closed of codimension $\geq i - (\dim(X) - \dim(a(X)))$.

In particular, if $a\colon X \to \mathrm{Alb}(X)$ is generically finite, then $\dim(X) = \dim(a(X))$, and so $S^i(\omega_X)$ has codimension $\geq i$. Furthermore, if $M \in A^t(k)$ is general, then

$$\chi(X, \omega_X) = \chi(X, \omega_X \otimes M) = h^0(X, \omega_X \otimes M) \geq 0.$$

Proof. Combining the projection formula and the fact that $a\colon X \to A$ induces an isomorphism $\underline{\mathrm{Pic}}^0(A) \simeq \underline{\mathrm{Pic}}^0(X)$ (by construction), we find that $S^i(\omega_X) = S^i(A^t, Ra_*(\omega_X))$. For any $M \in \mathrm{Pic}^0(A)$, there is the hypercohomology spectral sequence

$$E_2^{p,q}\colon H^p(A, R^q a_*(\omega_X) \otimes M) \Longrightarrow H^{p+q}(A, Ra_*(\omega_X) \otimes M)$$

The existence of the spectral sequence implies that

$$S^n(A^t, Ra_*(\omega_X)) \subseteq \bigcup_p S^p(A^t, R^{n-p} a_*(\omega_X))$$

for any $n \in \mathbf{Z}_{\geq 0}$. Now, each $R^{n-p}a_*(\omega_X)$ is a Hacon sheaf by Theorem 24.1, so $S^p(A^t, R^{n-p}a_*(\omega_X))$ has codimension $\geq p$. Therefore, suffices to show that if $S^p(A^t, R^{n-p}a_*(\omega_X))$ is non-empty, then

$$p \geq n - (\dim(X) - \dim(a(X))).$$

Equivalently, it suffices to show that if $R^{n-p}a_*(\omega_X) \neq 0$, then $n - p \leq \dim(X) - \dim(a(X))$. This is precisely part (b) of Kollár's vanishing theorem (Theorem 24.2). □

24.3. **Jacobians.** For simplicity, fix an algebraically closed field k of characteristic zero. Let C be a smooth, projective curve over k of genus $g \geq 1$. The *Jacobian* $\mathrm{Jac}(C) \coloneqq \underline{\mathrm{Pic}}^0(C)$ of C is the identity component of the Picard scheme $\underline{\mathrm{Pic}}(C)$ of C.

The aim of of the rest of the course is to discuss the proof of the Torelli theorem (stated below), following Beilinson–Polishchuk [BP01].

Theorem 24.4. [Torelli Theorem] *The Jacobian* $\mathrm{Jac}(C)$ *comes equipped with a canonical principal polarization, denoted by* Θ*, and the pair* $(\mathrm{Jac}(C), \Theta)$ *determines* C *up to (non-unique) isomorphism.*

In more fancy terms, the Torelli theorem says that the map $\mathcal{M}_g \to \mathcal{A}_g$ between the moduli space of curves of genus g and the moduli space of principally polarized abelian varieties of dimension g, given by $C \mapsto (\mathrm{Jac}(C), \Theta)$, is injective on points. The non-uniqueness of the isomorphism in the statement of the Torelli theorem is an avatar of the fact that $\mathcal{M}_g \to \mathcal{A}_g$ is not a closed immersion - it has non-reduced fibres along the hyperelliptic locus.

24.4. **Symmetric Powers & Jacobians.** Fix an integer $r \in \mathbf{Z}_{\geq 0}$. Let C^r denote the product of r-many copies of C; C^r carries a natural S_r-action given by permuting the factors, where S_r denotes the symmetric group on r letters.

Theorem 24.5. *The quotient* $\mathrm{Sym}^r(C) \coloneqq C^r/S_r$ *exists in the category of* k*-schemes. Moreover,* $\mathrm{Sym}^r(C)$ *is a smooth, projective variety and there is a canonical identification*

$$\mathrm{Sym}^r(C)(k) = \mathrm{Div}_{\mathrm{eff}}(C)_r,$$

where $\mathrm{Div}_{\mathrm{eff}}(C)_r$ *denotes the set of effective divisors on* C *of degree* r*.*

Sketch. For any quasi-projective k-variety X with an action of a finite group G, the quotient X/G exists as a k-scheme. The construction is as follows:

- if $X = \mathrm{Spec}(R)$, then $X/G = \mathrm{Spec}(R^G)$, where $R^G \subseteq R$ denotes the ring of G-invariants;
- in general, X has a cover $\{U_i\}$ by G-stable open affines, so set $V_i = U_i/G$, and glue the schemes $\{V_i\}$ together to build the quotient X/G.

In particular, $\mathrm{Sym}^r(C)$ exists as a k-scheme and it is normal (because a ring of invariants is integrally closed); in addition, $\mathrm{Sym}^r(C)$ is projective, since the quotient map $C^r \to \mathrm{Sym}^r(C)$ is finite and surjective.

The above discussion did not require C to be a curve, but now we will show that $\mathrm{Sym}^r(C)$ is smooth over k, for which the curve assumption is crucial. It suffices to show that all completed local rings of $\mathrm{Sym}^r(C)$ are power series rings in some number of variables. For notational simplicity, we consider the point of $\mathrm{Sym}^r(C)$ which is the "worst offender", and is constructed as follows: fix $x \in C(k)$ and consider the S_r-invariant point $y = (x, \dots, x) \in C^r(k)$, which has image $z \in \mathrm{Sym}^r(C)(k)$. We claim that $\widehat{\mathcal{O}}_{\mathrm{Sym}^r(C),z}$ is a formal power series ring.

As C is smooth at x and C is a curve, one obtains an isomorphism $\widehat{\mathcal{O}}_{C,x} \simeq k[[t]]$ by choosing a local coordinate t at x. Thus, $\widehat{\mathcal{O}}_{C^r,y} \simeq k[[t_1, \dots, t_r]]$, and one can check that the induced S_r-action on $\widehat{\mathcal{O}}_{C^r,y}$ simply permutes the variables $t_1, \dots, t_r$. It follows that

$$\widehat{\mathcal{O}}_{\mathrm{Sym}^r(C),z} = \left(\widehat{\mathcal{O}}_{C^r,y}\right)^{S_r} = k[[t_1, \dots, t_r]]^{S_r},$$

and it is a fundamental result in the theory of symmetric functions that the ring $k[[t_1, \dots, t_r]]^{S_r}$ is a power series ring in the elementary symmetric functions in r variables. Therefore, we have shown that $\mathrm{Sym}^r(C)$ is smooth at z.

It remains to show that the set $\mathrm{Sym}^r(C)(k)$ of k-points can be identified with the set of effective degree-r divisors on C. By general properties of quotients and the assumption that k is algebraically closed, we have $\mathrm{Sym}^r(C)(k) = C^r(k)/S_r$. There is a map $\psi \colon C^r(k) \to \mathrm{Div}_{\mathrm{eff}}(C)_r$ given by

$$(x_1, \dots, x_r) \mapsto \sum_{i=1}^r [x_i],$$

where $[x_i]$ denotes the prime divisor on C associated to $x_i \in C(k)$. The map ψ is clearly S_r-invariant, so it descends to a map $C^r(k)/S_r \to \mathrm{Div}_{\mathrm{eff}}(C)_r$, and this is clearly bijective. $\square$

Remark 24.6. The identification of $\mathrm{Sym}^r(C)(k)$ with the effective divisors on C of degree r appearing in Theorem 24.5 admits the following generalization: for any k-scheme T,

$$\mathrm{Hom}_k(T, \mathrm{Sym}^r(C)) = \{\text{relative effective Cartier divisors on } C \times T \text{ of degree } r\}. \tag{24.2}$$

That is, the right-hand side of (24.2) consists of Cartier divisors $Z \subseteq C \times T$ such that the induced map $Z \to T$ is finite flat of degree r. Equivalently, this condition says that each fibre of $Z \to T$ is a degree r divisor on the corresponding fibre of $C \times T \to T$. For an explanation of this more general statement, see [Mil86, Theorem 3.13].

The goal now is to relate the schemes $\mathrm{Sym}^r(C)$ and $\underline{\mathrm{Pic}}(C)$. There are two maps that allow one to do so:

(1) Consider the map $C^r \to \underline{\mathrm{Pic}}(C)$ given on k-points by

$$(x_1, \ldots, x_r) \mapsto \mathcal{O}_C([x_1] + \ldots + [x_r]).$$

This map is S_r-invariant, so it descends to a map $\sigma_r \colon \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)$ (on k-points, σ_r takes an effective degree r divisor D to the line bundle $\mathcal{O}_C(D)$).

(2) Taking degrees yields a map $\deg\colon \underline{\mathrm{Pic}}(C) \to \mathbf{Z}$ of k-groups schemes, where $\mathbf{Z}$ is thought of as the discrete k-group scheme consisting of the disjoint union of $\mathbf{Z}$-copies of $\mathrm{Spec}(k)$ and the group structure is defined by the usual one on $\mathbf{Z}$.

Set $\underline{\mathrm{Pic}}(C)_r := \deg^{-1}(r) \subseteq \underline{\mathrm{Pic}}(C)$, which is both an open and closed subscheme of $\underline{\mathrm{Pic}}(C)$. The image of σ_r lands in the subscheme $\underline{\mathrm{Pic}}(C)_r$, so we will write $\sigma_r \colon \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)_r$.

Theorem 24.7. *For $r \in \mathbf{Z}_{\geq 0}$, consider the map $\sigma_r \colon \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)_r$ defined above.*

(a) Each fibre of σ_r is (set-theoretically) a projective space of some dimension (and possibly empty);
(b) For $r > g-1$, σ_r is surjective; in particular, the fibres of σ_r are geometrically connected[38].
(c) For $r > 2g-2$, σ_r is a projective bundle.
(d) For $r \leq g$, the map σ_r is birational onto its image $W^r := \sigma_r(\mathrm{Sym}^r(C)) \subseteq \underline{\mathrm{Pic}}(C)_r$; in particular, W^{g-1} is a divisor on $\underline{\mathrm{Pic}}(C)_{g-1}$
(e) The Jacobian $\mathrm{Jac}(C) := \underline{\mathrm{Pic}}^0(C)$ coincides with $\underline{\mathrm{Pic}}(C)_0$.

Next class, we will discuss the proof of Theorem 24.7 and continue to move towards a proof of the Torelli theorem.

25. December 5th

Last time, we began discussing the relationships between symmetric powers of curves and their Jacobians, which we continue today.

25.1. **Symmetric Powers & Jacobians (Continued).**

The setup is as follows: C is a smooth proper curve of genus g over an algebraically closed field k of characteristic zero. For $r \in \mathbf{Z}_{>0}$, let $\mathrm{Sym}^r(C) := C^r/S_r$ be the r-th symmetric power of C, and let $\underline{\mathrm{Pic}}(C)_r$ denote the degree r component of $\underline{\mathrm{Pic}}(C)$.

Last time, we constructed a map $\sigma_r \colon \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)_r$. The moduli-theoretic interpretation of σ_r is as follows: $\mathrm{Sym}^r(C)$ classifices relative effective Cartier divisors on C of degree r, and $\underline{\mathrm{Pic}}(C)_r$ parametrizes degree r line bundles, and the map σ_r is given by $D \mapsto \mathcal{O}(D)$. The key properties of the map σ_r are summarized in the theorem below.

Theorem 25.1. *Fix $r \in \mathbf{Z}_{>0}$.*

(1) Each fibre of σ_r is a projective space.

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[38]By convention, a connected scheme is non-empty.

(2) For $r > g-1$, σ_r is surjective; in particular, the fibres are geometrically connected, as they are non-empty projective spaces.
(3) For $r > 2g-2$, σ_r is a projective space bundle.
(4) For $r \leq g$, the map σ_r is birational onto its image $W^r \coloneqq \sigma_r(\mathrm{Sym}^r(C)) \subseteq \underline{\mathrm{Pic}}(C)_r$
(5) The group $\underline{\mathrm{Pic}}(C)_0$ is connected, and therefore $\mathrm{Jac}(C) = \underline{\mathrm{Pic}}(C)_0$.

Note that, by combining (2) and (4) above, we find that $W^{g-1} = \underline{\mathrm{Pic}}(C)_{g-1}$.

Proof. For (1), if $L \in \underline{\mathrm{Pic}}(C)_r$, then we have $\sigma_r^{-1}(L) = \mathbf{P}\left(H^0(C, L)\right)$, i.e. $\sigma_r^{-1}(L)$ consists of all effective divisors D on C such that $\mathcal{O}_C(D) \simeq L$ (this is usually called the *linear system* $|L|$ of L).

For (2), we must show that $H^0(C, L) \neq 0$ for any $L \in \underline{\mathrm{Pic}}(C)_r$ whenever $r \geq g-1$. This is an easy application of the Riemann–Roch theorem, which implies that $\chi(C, L) = r + 1 - g > 0$, and the basic inequality $\chi(C, L) \leq h^0(C, L)$ on a curve.

For (3), if $r > 2g-2$ and $L \in \underline{\mathrm{Pic}}(C)_r$, then the Riemann–Roch theorem says that

$$h^0(C, L) - h^0(C, K_C \otimes L^{-1}) = r + 1 - g,$$

but now $\deg(K_C \otimes L^{-1}) < 0$, and so $h^0(C, K_C \otimes L^{-1}) = 0$. Thus, $h^0(L) = r + 1 - g$, and hence all fibres of σ_r are projective spaces of dimension $r - g$.

By miracle flatness, σ_r is a smooth and proper morphism of relative dimension $r-g$ (this alone does not mean that σ_r is a projective bundle, e.g. Brauer–Severi varieties). We must construct a vector bundle $\mathcal{E}$ on $\underline{\mathrm{Pic}}(C)_r$ such that $\mathrm{Sym}^r(C) \simeq \mathbf{P}(\mathcal{E})$ over $\underline{\mathrm{Pic}}(C)_r$. Consider the Poincaré bundle $\mathcal{P}$ on $C \times \underline{\mathrm{Pic}}(C)_r$ (recall that $\mathcal{P}$ is such that $\mathcal{P}|_{C \times \{L\}} \simeq L$ for $L \in \underline{\mathrm{Pic}}(C)_r$), and so the obvious guess is to set

$$\mathcal{E} \coloneqq \mathrm{pr}_{2,*}(\mathcal{P}) \in \mathrm{Coh}(\underline{\mathrm{Pic}}(C)_r).$$

One must first show that $\mathcal{E}$ is a vector bundle, and that the fibres compute the correct spaces of global sections. These claims fall out of cohomology base change: note that $H^1(C, L) = H^0(C, K_C \otimes L^{-1})^\vee = 0$, so the semicontinuity theorem, $\mathcal{E}$ is a vector bundle on $\underline{\mathrm{Pic}}(C)_r$. Moreover, the formation of $\mathcal{E}$ commutes with base change, so $\mathcal{E}|_{\{L\}} \simeq H^0(C, L)$ for any $L \in \underline{\mathrm{Pic}}(C)_r$. Thus, we have maps

$$\begin{tikzcd}
\mathbf{P}(\mathcal{E}) \arrow[dr] & & \mathrm{Sym}^r(C) \arrow[dl, "\sigma_r"] \\
& \underline{\mathrm{Pic}}(C)_r &
\end{tikzcd}$$

and we must construct an isomorphism between $\mathbf{P}(\mathcal{E})$ and $\mathrm{Sym}^r(C)$ so that the resulting diagram commutes. To produce this map, we can use the universal property of $\mathrm{Sym}^r(C)$, and this is an isomorphism on fibres (by construction) so it must be an isomorphism.

For (4), the image W^r of σ_r is irreducible, because it is the image of an irreducible variety. Now, we have a proper surjective map $\mathrm{Sym}^r(C) \to W^r$ and we must show that is is birational; since both the source and target are irreducible, it suffices to exhibit a single point of W^r where the fibre of σ_r is a single point, i.e. a single point $x \in W^r$ such that the fibre $\sigma_r^{-1}(x)$ is *scheme-theoretically* a single point. Fix $x \in \mathrm{Pic}(C)_r$. By (1), the fibre $\sigma_r^{-1}(x)$ is reduced (indeed, it is a projective space).

Said differently, we must find a degree r line bundle L on C such that $h^0(C, L) = 1$. To find such a line bundle, we use the following lemma.

Lemma 25.2. *For $0 \leq r \leq g$, there exists a non-empty open set $U \subseteq C^r$ such that*

$$h^0\left(C, \mathcal{O}_C\left(\sum_{i=1}^r [x_i]\right)\right) = 1$$

for any $(x_1, \ldots, x_r) \in U$.

The assertion (4) follows immediately from Lemma 25.2.

One might think that Lemma 25.2 says that a generic line bundle of degree r on C has a 1-dimensional space of global sections, but this is assuming that the map σ_r is birational! This is only a statement about generic *effective* line bundles of degree r.

Proof of Lemma 25.2. Consider the following assertion:

$$\text{If } D \text{ is an effective divisor on } C \text{ with } h^1(D) > 0, \text{ then there exists a non-empty open subset } V \subseteq C \text{ such that } h^1(D+[x]) = h^1(D) - 1 \text{ for all } x \in V. \tag{$*$}$$

In the setting above, we know that adding a point $[x]$ to D will cause h^1 to drop by at most 1, and $(*)$ asserts that h^1 does indeed drop for a generic choice of point x.

The proof of $(*)$ is left as an exercise (briefly, one must just guarantee that x avoids D in order for h^1 to drop).

Applying $(*)$ to the trivial divisor D_0 (i.e. D_0 satisfies $\mathcal{O}_C(D_0) = \mathcal{O}_C$), we learn that $h^1(D_0) = h^1(\mathcal{O}_C) = g > 0$. By induction (on the degree of the divisor) and applying $(*)$, we get that there exists an open subset $U \subseteq C^r$ such that

$$h^1\left(D_0 + [x_1] + \ldots + [x_r]\right) = g - r$$

for any $(x_1, \ldots, x_r) \in U$. Now, one can use Riemann–Roch to conclude: one has

$$h^0\left(\sum_{i=1}^r [x_i]\right) - h^1\left(\sum_{i=1}^r [x_i]\right) = r + 1 - g,$$

and hence $h^0\left(\sum_{i=1}^r [x_i]\right) = 1$. □

The upshot of Lemma 25.2 is, as discussed before, that $\sigma_r \colon \mathrm{Sym}^r(C) \to W^r$ is birational for $r \le g$.

For (5), proving that $\underline{\mathrm{Pic}}(C)_0$ is connected is equivalent to proving $\underline{\mathrm{Pic}}(C)_r$ is connected for $r \gg 0$ (using the group structure). However, the map $\sigma_r \colon \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)_r$ is surjective for $r > g - 1$, so the connectedness of $\mathrm{Sym}^r(C)$ implies the connectedness of $\underline{\mathrm{Pic}}(C)_r$. □

25.2. **Jacobian & the Albanese.** Still on route to the Torelli theorem, the goal is now to show that $\mathrm{Jac}(C)$ is isomorphic to its dual, $\mathrm{Jac}(C)^t \coloneqq \mathrm{Alb}(C)$, via the universal property of $\mathrm{Alb}(C)$.

Construction 25.3. For $r, s \in \mathbf{Z}_{\ge 0}$, there is an obvious map $C^r \times C^s \to C^{r+s}$, given by concatenating tuples of points. The group $S_r \times S_s$ acts on $C^r \times C^s$, the group S_{r+s} acts on C^{r+s}on the right-side, and there is an obvious map $S_r \times S_s \to S_{r+s}$ of symmetric groups that intertwines the actions. Therefore, on quotients, one gets a map

$$\mathrm{Sym}^r(C) \times \mathrm{Sym}^s(C) \to \mathrm{Sym}^{r+s}(C). \tag{25.1}$$

In the moduli-theoretic interpretation of the symmetric powers, the above map takes a degree r divisor and a degree s divisor and outputs their sum, which is a degree $(r+s)$ divisor.

Definition 25.4. The *infinite symmetric product* of C is

$$\mathrm{Sym}(C) \coloneqq \bigcup_{r \ge 0} \mathrm{Sym}^r(C),$$

which is a scheme via the above decomposition as a disjoint union of schemes.

The maps (25.1) make $\mathrm{Sym}(C)$ into a commutative monoid scheme. Consider the map

$$C \xrightarrow{\simeq} \mathrm{Sym}^1(C) \subseteq \mathrm{Sym}(C) \tag{25.2}$$

obtained by composing the obvious isomorphism $C \simeq \mathrm{Sym}^1(C)$ with the natural inclusion $\mathrm{Sym}^1(C) \subseteq \mathrm{Sym}(C)$.

Lemma 25.5. *The map (25.2) is the universal map from C into a commutative monoid scheme.*

Proof. If $f\colon C \to M$ is a map from C into a commutative monoid scheme M, the composition

$$C^r \xrightarrow{f^r} M^r \xrightarrow{\text{sum}} M$$

is S_r-equivariant, where the sum map is given from the monoid law on M (and the S_r-equivariance follows from the commutativity of the monoid law). Thus, one obtains a map $f_r\colon \operatorname{Sym}^r(C) \to M$ for all $r \in \mathbf{Z}_{>0}$, and hence a map $F\colon \operatorname{Sym}(C) \to M$ of commutative monoid schemes. One can verify that postcomposing (25.2) with F gives exactly the map f. □

Consider the map

$$\sigma \coloneqq \coprod_{r\geq 0} \sigma_r\colon \operatorname{Sym}(C) = \coprod_{r\geq 0} \operatorname{Sym}^r(C) \to \underline{\operatorname{Pic}}(C) = \coprod_{r\in\mathbf{Z}} \underline{\operatorname{Pic}}(C)_r. \tag{25.3}$$

If one thinks of $\operatorname{Sym}(C)$ as parametrizing relative effective Cartier divisors (of unspecified degree) on C, then the map σ this is precisely the map $D \mapsto \mathcal{O}(D)$. It is easy to check that σ is a map of commutative monoid schemes, because adding divisors is compatible with tensoring line bundles.

Lemma 25.6. *The map $C \to \underline{\operatorname{Pic}}(C)$ given by*

$$x \mapsto \mathcal{O}_C\left([x]\right)$$

is the universal map from C into a commutative group scheme that is locally finitely presented and with the identity component being an abelian variety.

In fact, the map $C \to \underline{\operatorname{Pic}}(C)$ is universal for maps from C into all commutative group schemes, but we leave the added assumptions in the statement of Lemma 25.6 to simplify the proof (and because we will only apply the lemma with these assumptions).

Proof. If $f\colon C \to G$ is any map as above, then Lemma 25.5 implies that there is a map $F\colon \operatorname{Sym}(C) \to G$. Via restriction, one obtains a map

$$F_{>2g-2}\colon \coprod_{r>2g-2} \operatorname{Sym}^r(C) \longrightarrow G.$$

Similarly restricting σ_r gives a map

$$\coprod_{r>2g-2} \sigma_r\colon \coprod_{r>2g-2} \operatorname{Sym}^r(C) \to \coprod_{r>2g-2} \underline{\operatorname{Pic}}(C)_r.$$

The assumptions on the group scheme G mean that G has a discrete set of connected components, and each component is an abelian variety.

Now, we have previously shown the following result (though we formulated it differently):

Lemma 25.7. *Let A be an abelian variety over k. If $f\colon X \to Y$ is a proper surjective map of normal varieties with fibres covered by rational curves and $g\colon X \to A$ is any map, then g factors over f.*

Applying Lemma 25.7 to each connected component of G, we see that $F_{>2g-2}$ factors as

$$\coprod_{r>2g-2} \operatorname{Sym}^r(C) \xrightarrow{\coprod_{r>2g-2}\sigma_r} \coprod_{r>2g-2} \underline{\operatorname{Pic}}(C)_r \xrightarrow{H_{>2g-2}} G.$$

Now, one can use the group structure on G to extend $H_{>2g-2}$ to a map from all of $\underline{\operatorname{Pic}}(C)$ to G. The uniqueness of the factorization is left as an exercise. □

Thus, Lemma 25.5 and Lemma 25.6 gives a map $C \to \operatorname{Sym}(C)$ and its group completion $\sigma\colon \operatorname{Sym}(C) \to \underline{\operatorname{Pic}}(C)$, each with a universal property.

Corollary 25.8. *Fix a point $P \in C(k)$. Consider the* Abel-Jacobi map $\mathrm{AJ}\colon C \to \mathrm{Jac}(C) \subseteq \underline{\mathrm{Pic}}(C)$*, given by*

$$x \mapsto \mathcal{O}_C\left([x] - [P]\right).$$

Then, AJ *is the Albanese map for (C, P).*

Proof. Given a map $f\colon C \to A$ to an abelian variety A such that $f(P) = e$, we must show that f factors over AJ in a unique fashion. By Lemma 25.6, f factors $\underline{\mathrm{Pic}}(C)$ in a unique way, so one gets a map $F\colon \underline{\mathrm{Pic}}(C) \to A$, and hence a map $F_0\colon\ \mathrm{Jac}(C) = \underline{\mathrm{Pic}}^0(C) \to A$ by restriction. This map F has the property that $F \circ \sigma_1 = f$, so $F\left(\mathcal{O}_C([x])\right) = f(x)$ for any $x \in C$. However, observe that for any $x \in C$,

$$\begin{aligned} F\left(\mathcal{O}_C([x])\right) &= F\left(\mathcal{O}_C([x] - [P]) \otimes \mathcal{O}_C([P])\right) \\ &= F_0(\mathcal{O}_C([x] - [P])) + F(\mathcal{O}_C([P])) \\ &= (F_0 \circ \mathrm{AJ})(x) + e \\ &= (F_0 \circ \mathrm{AJ})(x). \end{aligned}$$

Therefore, f factors as $F_0 \circ \mathrm{AJ}$, and it is left as an exercise to check that this factorization is unique. ☐

The upshot of Corollary 25.8 is that, after choosing a basepoint, one obtains an isomorphism

$$\mathrm{Jac}(C) \xrightarrow{\simeq} \mathrm{Alb}(C) = \mathrm{Jac}(C)^t. \tag{25.4}$$

One can then ask whether this isomorphism arises as a principal polarization.

Fact 25.9. The isomorphism (25.4) has the form ϕ_L for some $L \in \mathrm{Pic}(\mathrm{Jac}(C))$, and it corresponds to the divisor $W^{g-1} \subseteq \underline{\mathrm{Pic}}(C)_{g-1}$ if one identifies $\underline{\mathrm{Pic}}(C)_0 \xrightarrow{\simeq} \underline{\mathrm{Pic}}(C)_{g-1}$ via some choice of degree $(g-1)$ line bundle.

Note that, since $K(L)$ is a finite scheme and L is effective, it follows that W^{g-1} is ample.

25.3. **Determinants.** The construction of the determinant of a vector bundle extends to a general construction on the bounded derived category, called the determinant of cohomology. The fundamental fact, that we invoke without proof, is the following:

Fact 25.10. If X is a smooth k-scheme, there is a determinant map

$$\det\colon\ \mathrm{D}^b_{\mathrm{coh}}(X) \to \mathrm{Pic}(X)$$

that extends the usual determinant on the subcategory $\mathrm{Vect}(X)$ of vector bundles. Moreover, it has the following property: if $K \to L \to M$ is an exact triangle in $\mathrm{D}^b_{\mathrm{coh}}(X)$, then there is a canonical isomorphism

$$\det(L) \simeq \det(K) \otimes \det(M).$$

The construction of the determinant is nontrivial to set up carefully, though the idea is clear: resolve a coherent sheaf by vector bundles, take determinants, tensor together, and then one must show that the resulting complex is independent of choice of resolution (which is the hard part). See the paper [KM76] for the details.

Next time, we will use the determinant of cohomology to define the theta divisor on the Jacobian.

26. December 7th

Today we will finish discussing the Torelli theorem, but we will not be able to prove it, simply set up the statement and the tools so that one can go ahead and read the proof of [BP01].

The setup was the following: let C be a smooth proper curve of genus g over an algebraically closed field k of characteristic zero, and we constructed maps

$$C \longrightarrow \mathrm{Sym}(C) \longrightarrow \underline{\mathrm{Pic}}(C),$$

where $C \to \mathrm{Sym}(C)$ is universal for maps from C into commutative monoid schemes, and the composition $C \to \underline{\mathrm{Pic}}(C)$ is universal for maps from C into commutative group schemes. We have also produced isomorphisms

$\mathrm{Jac}(C) = \underline{\mathrm{Pic}}^0(C) \simeq \underline{\mathrm{Pic}}^0(C)^t = \mathrm{Alb}(C)$, but this was very specific to the case of curves, since we are identifying points with divisors.

Furthermore, there is the theta divisor $W^{g-1} \subseteq \underline{\mathrm{Pic}}(C)$, which consists of the set of degree $g-1$ line bundles on C with a section (i.e. that are effective).

The next goal is to describe these isomorphisms $\mathrm{Jac}(C) \simeq \mathrm{Alb}(C)$, or more precisely to realize them as ϕ_L for some line bundle L (i.e. realize them as principal polarizations).

Finally, we introduced the determinant of cohomology: if X is a smooth k-scheme, there is a map

$$\det\colon \mathrm{D}^b_{\mathrm{coh}}(X) \to \mathrm{Pic}(X)$$

extending the usual determinant on the subcategory of vector bundles (in brief, resolve the terms of the complex by vector bundles, take determinants, multiply these together, and the nontrivial claim is that different choices of resolutions yield quasi-isomorphic complexes). There is an analogous construction for perfect complexes on any scheme, which we will not discuss here.[39]

26.1. **Towards the Torelli Theorem.** The theta divisor can be recovered via the determinant of cohomology, for which we need the following key, general proposition that allows us to construct line bundles and sections with certain properties.

Proposition 26.1. *Let S be a connected, smooth k-scheme, and let $f\colon X \to S$ be a proper, smooth family of curves. Let $M \in \mathrm{Vect}(X)$ be a vector bundle on X such that $Rf_*(M)$ is generically acyclic; that is, the cohomology of M on the generic fibre of f is zero in all degrees, or equivalently there is an open subset of S over which M has trivial cohomology. If $L \coloneqq \det(Rf_*(M)) \in \mathrm{Pic}(S)$, then the line bundle L^{-1} comes equipped with a canonical section Θ such that*

$$Z(\Theta) = S^i(S, Rf_*(M)) \text{ for } i = 0, 1 \tag{26.1}$$

where recall that $S^i(S, Rf_*(M)) \coloneqq \{s \in S\colon H^i(X_s, M|_{X_s}) \neq 0\}$.

Note that the left-hand side of (26.1) does not depend on the index i.

Proof. We will first show that $S^0(S, Rf_*(M)) = S^1(S, Rf_*(M))$. The set $S^0(S, Rf_*(M))$ is the locus of $s \in S$ where $H^0(X_s, M|_{X_s}) \neq 0$, and $S^1(S, Rf_*(M))$ is the locus of $s \in S$ where $H^1(X_s, M|_{X_s}) \neq 0$. By the assumption that M is generically acyclic, both cohomology groups vanish for general $s \in S$; in particular, $\chi(X_s, M_S) = 0$ for $s \in S$ general. By the invariance of Euler characteristics in flat families, it follows that $\chi(X_s, M_s) = 0$ for all $s \in S$, since S is connected. It immediately follows that the two sets coincide, and they are also equal to the locus

$$\{s \in S\colon R\Gamma(X_s, M|_{X_s}) \neq 0\},$$

because if the complex is nonzero, then at least one of the two cohomology groups must be nonzero.

To construct the section Θ, we will make the following choices (and one can subsequently show that the resulting divisor is independent of such choices): choose an effective Cartier divisor $D \subseteq X$ that is finite flat over S, and sufficiently "positive" on the fibres (meaning that the degree on the fibres of f is sufficiently positive). The positivity of D will be made precise below. There is a short exact sequence

$$0 \to M \to M(D) \to M(D)|_D \to 0$$

and if D is sufficiently positive so that $M(D)$ has no higher cohomology, then there is a long exact sequence of the form

$$0 \to R^0 f_*(M) \to R^0 f_*(M(D)) \to R^0 f_*(M(D)|_D) \to R^1 f_*(M) \to 0. \tag{26.2}$$

Such a divisor D exists: by the semicontinuity theorem, it is enough to check the vanishing of cohomology on the fibres of f, and this can easily be arranged.

[39]See this discussion on MathOverflow: https://mathoverflow.net/questions/7124/determinant-of-a-perfect-complex.

As the map $D \to S$ is finite flat, $R^0 f_*(M(D)|_D)$ is a vector bundle on S. In addition, if D is sufficiently positive, we may assume that $R^0 f_*(M(D))$ is a vector bundle on S (again by cohomology and base change). Thus, by the long exact sequence (26.2), the complex $Rf_*(M)$ is computed by

$$Rf_*(M) \simeq \left(R^0 f_*(M(D)) \xrightarrow{\delta} R^0 f_*(M(D)|_D)\right),$$

where the above is a quasi-isomorphism of complexes, and the term $R^0 f_*(M(D))$ of the right-hand complex lives is placed in degree zero.

As $Rf_*(M)$ is generically acyclic, both vector bundles $R^0 f_* * (M(D))$ and $R^0 f_*(M(D)|_D)$ have the same rank. Therefore, we can write

$$L := \det(Rf_*(M)) \simeq \det(R^0 f_*(M(D))) \otimes \left(\det(R^0 f_*(M(D)|_D))\right)^{-1}$$

and set $\Theta := \det(\delta) \in H^0(S, L^{-1})$. Now, the zero locus $Z(\Theta)$ of Θ is by definition

$$\begin{aligned} Z(\Theta) &= \{s \in S \colon \Theta \otimes \kappa(s) = 0\} \\ &= \{s \in S \colon \delta \otimes \kappa(s) \text{ is not invertible}\} \\ &= \{s \in S \colon R\Gamma(X_s, M|_{X_s}) = (Rf_* M) \otimes \kappa(s) \text{ has nonzero } H^0 \text{ and } H^1\} \end{aligned}$$

and this set is precisely the locus of $s \in S$ where $R\Gamma(X_s, M|_{X_s})$ is nonzero in the derived category of X_s, which, by our previous discussion, is equal to both $S^0(S, Rf_*(M))$ and $S^1(S, Rf_*(M))$. □

We can apply this key proposition to our original setting involving curves and certain associated abelian varieties.

Proposition 26.2. *Let A be an abelian variety and let $a \colon C \to A$ be a non-constant (hence, finite) map. We write coherent sheaves on C as coherent sheaves on A, i.e. we identity them with their pushforward along a. For $L \in \mathrm{Pic}(C)$, write $d(L) := \det(\phi_A(L)) \in \mathrm{Pic}(A^t)$.*

Then, for any $L \in \mathrm{Pic}(C)$, the map $\phi_{d(L)} \colon A^t \to (A^t)^t = A$ given by

$$x \mapsto t_x^*(d(L)) \otimes d(L)^{-1},$$

is independent of L; in fact, it coincides with the map

$$A^t = \underline{\mathrm{Pic}}^0(A) \xrightarrow{a^*} \underline{\mathrm{Pic}}^0(C) \xrightarrow{a_*} A, \tag{26.3}$$

where the map $a_* \colon \underline{\mathrm{Pic}}^0(C) \to A$ is the map given by the universal property of the Albanese of C.

The map (26.3) is obviously independent of the choice of $L \in \mathrm{Pic}(C)$ (indeed, it depends only on the original map a), so it suffices to prove that the map $d(L)$ coincides with (26.3) in order to show Proposition 26.2.

We will later apply Proposition 26.2 to the Albanese map for the curve C to get an isomorphism between the Jacobian and its dual, which will be precisely the composition that we were originally interested in studying.

Proof. As k is algebraically closed, it suffices to work on k-points (as always, this is only for notational simplicity). Given $x \in A^t(k)$, write $\mathcal{P}_x := \mathcal{P}|_{A \times \{x\}} \in \mathrm{Pic}^0(A)$ for the corresponding line bundle on A. Now, we have

$$\phi_{d(L)}(x) = t_x^*(d(L)) \otimes d(L)^{-1} = t_x^*(\det(\phi_A(L))) \otimes \det(\phi_A(L))^{-1}. \tag{26.4}$$

The formation of determinants is functorial, so (26.4) can be rewritten as

$$\det(t_x^*(\phi_A(L))) \otimes \det(\phi_A(L))^{-1} = \det(\phi_A(L \otimes \mathcal{P}_x)) \otimes \det(\phi_A(L))^{-1} = d(L \otimes \mathcal{P}_x) \otimes d(L)^{-1}.$$

We can now appeal to the following lemma:

Lemma 26.3. *For any divisor D on C, we have an isomorphism*

$$d(L(D)) \simeq d(L) \otimes \mathcal{P}_{\{a_*(D)\} \times A^t}, \tag{26.5}$$

where $a_* \colon \underline{\mathrm{Pic}}(C) \to A$ is the map coming from the universal property of $\underline{\mathrm{Pic}}(C)$ (namely, as the universal commutative group scheme that receives a map from C).

Proof of Lemma 26.3. Both sides of (26.5) are linear in the divisor D, so we may assume that D is a single point, say $D = [P]$. Now, there is a short exact sequence

$$0 \to L \to L(P) \to L(P)|_P \to 0,$$

where $L(P)|_P$ is non-canonically isomorphic to the skyscraper sheaf $k(P)$ at the point P. Thus, one obtains an exact triangle

$$\phi_A(L) \to \phi_A(L(P)) \to \phi_A(k(P)). \tag{26.6}$$

Applying determinants of cohomology to (26.6) yields

$$d(L(P)) = d(L) \otimes \det(\phi_A(k(P))),$$

and $\det(\phi_A(k(P)))$ is the line bundle corresponding to the point P, i.e. $\mathcal{P}|_{\{a^*(P)\}\times A^t}$, which completes the proof of the lemma. □

It is left as an easy exercise to complete the proof of Proposition 26.2, granted Lemma 26.3. □

Corollary 26.4. *Fix a basepoint $P \in C(k)$. Let $a\colon C \to \mathrm{Jac}(C)$ denote the associated Albanese map; explicitly, it is the map $x \mapsto \mathcal{O}_C([x] - [P])$ on $C(k)$.*

(1) The map $\phi = \phi_{d(L)}$ of Proposition 26.2 gives an isomorphism $\mathrm{Jac}(C)^t \xrightarrow{\simeq} \mathrm{Jac}(C)$.
(2) The map ϕ is a principal polarization; more precisely, $d(L)$ is ample for any $L \in \mathrm{Pic}(C)$ of degree $g-1$.
(3) For any choice $L \in \mathrm{Pic}(C)$ of degree $g-1$ line bundle on C, the theta divisor $\Theta \subseteq \mathrm{Jac}(C)$ attached to $d(L)$ coincides with $W^{g-1} \subseteq \underline{\mathrm{Pic}}^{g-1}(C)$, under the standard identification of $\underline{\mathrm{Pic}}^0(C) \xrightarrow{\simeq} \underline{\mathrm{Pic}}^{g-1}(C)$ given by the choice of L.

Proof. The assertion (1) is clear: given a line bundle on an abelian variety, it defines a polarization if the line bundle is ample, and it defines a finite map; for non-degenerate line bundles, ampleness coincides with effectivity.

For (2), it suffices to show that $d(L)$ is effective whenever L has degree $g-1$. Consider Poincaré bundle $\mathcal{P} \in \mathrm{Pic}(C \times \mathrm{Jac}(C))$. As x runs over all points of $\mathrm{Jac}(C)$, the line bundle $\mathcal{P}|_{C\times\{x\}} \otimes \mathrm{p}_1^*(L)$ runs through all points of $\underline{\mathrm{Pic}}^{g-1}(C)$ (because tensoring with the pullback identifies Pic^0 with Pic^{g-1}). Therefore, we have $\phi_A(L) \simeq q_*(\mathcal{P}^{g-1})$, where

$$\begin{tikzcd} & C \times \underline{\mathrm{Pic}}^{g-1}(C) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C & & \underline{\mathrm{Pic}}^{g-1}(C) \end{tikzcd}$$

and $\mathcal{P}^{g-1}$ denotes the universal line bundle on $C \times \underline{\mathrm{Pic}}^{g-1}(C)$; that is, $\mathcal{P}^{g-1} = \mathcal{P} \otimes L$.

Now, we have shown that $d(L) = \det(\phi_A(L)) = \det(q_*(\mathcal{P}^{g-1}))$. The pushforward $q_*(\mathcal{P}^{g-1})$ is generically acyclic, because a generic line bundle of degree $g-1$ has no higher cohomology; thus, by Proposition 26.1, it follows that $\det(q_*(\mathcal{P}^{g-1}))^{-1}$ has a canonical section θ such that

$$Z(\theta) = \left\{x \in \underline{\mathrm{Pic}}^{g-1}(C) \colon H^0\left(C, \mathcal{P}^{g-1}|_{C\times\{x\}}\right) \neq 0\right\} = W^{g-1}$$

Thus, (3) follows. □

The upshot of all of this is that the divisor $W^{g-1} \subseteq \underline{\mathrm{Pic}}^{g-1}(C)$ is ample and it gives a principal polarization, as promised.

26.2. **The Torelli Theorem.** We now present one formulation of the Torelli theorem, following [BP01].

Let A be an abelian variety over k and let $\phi\colon A \xrightarrow{\simeq} A^t$ be a principal polarization, i.e. it is of the form ϕ_L for some ample line bundle L, but L is not uniquely determined. Note that the set of all line bundles $L \in \mathrm{Pic}(A)$ such that $\phi = \phi_L$ is a torsor for $\underline{\mathrm{Pic}}^0(A)$, meaning that if one tensors any such L with a degree zero line bundle, the tensor product remains in this set of line bundles; conversely, given any two such line bundles, their difference is a line bundle of degree zero.

Fix an effective divisor $\Theta \subseteq A$ giving an ample line bundle $L \in \mathrm{Pic}(A)$ such that $\phi_L = \phi$ (if ϕ_L is an isomorphism, then the space of sections of L is 1-dimensional, so any nontrivial element cuts out the same divisor, called the theta divisor; this is Θ). Granted this data, we will try to recover the curve with Jacobian A and there divisor Θ.

Lemma 26.5. *There exists a symmetric theta divisor Θ on A, i.e. one such that $[-1]^*(\Theta) = \Theta$.*

The proof of Lemma 26.5 is not difficult: it suffices to find a line bundle $L \in \mathrm{Pic}(A)$ such that $[-1]^*(L) = L$ and $\phi_L = \phi$; to find such a symmetric line bundle, take any line bundle L such that $\phi_L = \phi$, and symmetrize it in the usual way (using that an abelian variety is 2-divisible).

Now, fix a symmetric theta divisor $\Theta \subseteq A$. We make the following definitions:

(1) Set $Z(A, \phi) \coloneqq \mathrm{coker}(\mathrm{Pic}(A) \to \mathrm{Pic}(\Theta^{\mathrm{ns}}))$, where the superscript ns denotes the non-singular part of the scheme; one must show that $Z(A, \phi)$ depends only on ϕ and not on the choice of Θ.

(2) Set
$$P(A, \phi) \coloneqq \left\{ M \in \mathrm{Pic}(\Theta^{\mathrm{ns}}) \colon \begin{array}{c} 1.\ M \otimes [-1]^*(M) \simeq \omega_{\Theta^{\mathrm{ns}}}, \\ 2.\ M \text{ generates the group } Z(A,\phi) \end{array} \right\}.$$

The theta divisor Θ is almost never smooth, though it is always reduced, so the smooth locus is non-empty. The singularities of the theta divisor are an interesting subject in their own right; for example, there is a theorem of Ein–Lazarsfeld [EL97, Theorem 1] that says it is normal with at worst rational singularities.

The objects defined above, namely $Z(A, \phi)$ and $P(A, \phi)$, depend only on the principally polarized abelian variety (A, ϕ); we have not made any reference to a Jacobian. Thus, we can state one formulation of the Torelli theorem:

Theorem 26.6. [BP01, Theorem 1.3] *Suppose $(A, \phi) = (\mathrm{Jac}(C), \phi)$, where the second $\phi\colon \mathrm{Jac}(C)^t \simeq \mathrm{Jac}(C)$ is the principal polarization constructed in Corollary 26.4. Fix a symmetric theta divisor $\Theta \subseteq \mathrm{Jac}(C)$, and let $j\colon \Theta^{\mathrm{ns}} \hookrightarrow \mathrm{Jac}(C)$ be the inclusion of the nonsingular locus. Then,*

(1) $P(\mathrm{Jac}(C), \phi) \neq \emptyset$

(2) for any line bundle $M \in P(\mathrm{Jac}(C), \phi)$, there exists a coherent sheaf $\mathcal{F}$ on $\mathrm{Jac}(C)$ such that
   - *(a) $\phi_{\mathrm{Jac}(C)}(j_*(M)) \simeq \mathcal{F}[1-g]$;*
   - *(b) the scheme-theoretic support of $\mathcal{F}$ is isomorphic to C.*

In particular, C is determined by $(\mathrm{Jac}(C), \phi)$.

The conclusion of Theorem 26.6 says that one can recover a curve from the data of its Jacobian and a principal polarization. The proof of [BP01, Theorem 1.3] is a few pages long, but it is not too difficult given the preparatory work we have done.

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