MATH 631 NOTES, FALL 2018

Notes from Math 631, *Algebraic Geometry I*, taught at the University of Michigan, Fall 2018. Notes written by the students: Anna Brosowsky, Ryan Capouellez, Jack Harrison Carlisle, Shelby Cox, Karthik Ganapathy, Sameer Kailasa, Sayantan Khan, Michael Mueller, William C Newman, Khoa Dang Nguyen, Swaraj Sridhar Pande, Yuping Ruan, Eric Winsor, Yueqiao Wu, Jingchuan Xiao, Hua Xu, Jit Wu Yap, Hang Yin and Bradley Zykoski and edited by Prof. David E Speyer. Comments are welcome!

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September 5: Preview of algebraic geometry. Algebraic geometry relates algebraic properties of polynomial equations to geometric properties of their solution set.

The first theorem of algebraic geometry is the fundamental theorem of algebra:

Theorem (Fundamental Theorem of Algebra, misstated). Let $f(z) = f_d z^d + f_{d-1}z^{d-1} + \cdots + f_0$. Then there are d points in $\{z : f(z) = 0\}$.

We have related an algebraic property of the polynomial f – its degree – to a geometric property – the cardinality – of its zero set. If "cardinality" doesn't sound geometric to you, you can say that I computed $|\pi_0|$ or $\dim H^0$.

Of course, there are some caveats to the above:

- We need to say what field we are taking solutions in – it should be algebraically closed.
- We need to require that $f_d \neq 0$.
- We need to count with multiplicity.

Each of these caveats represents a more general issue that we'll see throughout the subject of algebraic geometry (namely, the need to work in algebraically closed fields, the need to take projective completions, and the need to keep track of nilpotents). Because of caveats like this, algebraic geometry has a reputation as a technical subject. However, I hope to convince you that algebraic geometry is fundamentally not technical – the essence of this result is that the number of solutions equals the degree.

Algebraic geometry is a field that has reinvented itself several times. What version of algebraic geometry are we studying?

Before the twentieth century, algebraic geometry meant studying the solutions of polynomial equations, in $\mathbb{C}^n$ or $\mathbb{R}^n$, using all the tools of analysis, differential geometry and algebraic topology. This is still an important, active, subject, but it is not what we are doing.

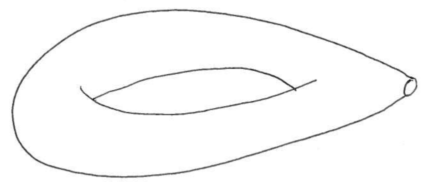
In the twentieth century, the major project of algebraic geometry was to redevelop the tools of analysis, differential geometry and algebraic topology in a purely algebraic way, so they can be used in any algebraically closed field. Major names here are Zariski and Weil in the first half of the twentieth century, followed by Grothendieck and Serre in the sixties. Our textbook by Shafarevich, takes this as its goal, but from a perspective early in the project. We will take a similar perspective this term, but will try to prepare you next term to read Hartshorne's book, which is closer to the Grothendieck perspective.

This project is still ongoing – work on stacks, derived algebraic geometry or $\mathbb{A}^1$-homotopy theory are all still seeking new foundations. However, I want to emphasize that there are many good problems in algebraic geometry which can be understood at the basic level of Shafarevich! You don't need to spend years on foundations to read and do interesting research!

So, why should we try to rebuild geometric tools in a purely algebraic way? I'll give three answers: The one which originally drew me to algebraic geometry, the one which historically captured the interest of the mathematical community, and what I think is the best answer now.

What originally drew me to algebraic geometry: There are no space filling curves. There are no functions which don't equal their Taylor series. Every function is given by a polynomial which you can write down. If you compare the difficulty of writing down, say a 3-manifold, to that of writing down an algebraic variety, you'll see that an algebraic variety is just a finite list of polynomials. Compared to analysis and differential geometry, I loved (and still love!) the idea of a subject where the fundamental objects are well behaved and can be written down using a finite amount of data.

What drew the mathematical community to this project was work of Weil. Here is an example of the sort of thing Weil was studying: Consider the equation $y^2 = x^3 - x - 1$. In $\mathbb{C}^2$, the solutions of these equations form a genus one surface with one puncture:



Weil was considering this equation (and many others) not over $\mathbb{C}$, but over the finite fields $\mathbb{F}_{p^k}$. It is a good idea to add in one more solution, corresponding to the missing puncture. With this correction, the number of solutions over $\mathbb{F}_{3^k}$ is

$$1,\ 7,\ 28,\ 91,\ 271,\ 784,\ 2269,\ 6643,\ 19684,\ 58807\cdots$$

and turns out to be given by

$$3^k - \left(\frac{3+\sqrt{-3}}{2}\right)^k - \left(\frac{3-\sqrt{-3}}{2}\right)^k + 1.$$

More generally, for any prime p, there are complex numbers α_p and $\overline{\alpha}_p$, such that $\alpha_p\overline{\alpha}_p = p$, such that the number of solutions over $\mathbb{F}_{p^k}$ is

$$p^k - \alpha_p^k - \overline{\alpha_p}^k + 1.$$

Weil realized that this formula can be thought of as

$$\det(A^k - \mathrm{Id})$$

where A is a 2×2 matrix with eigenvalues α_p and $\overline{\alpha}_p$. (In the $p = 3$ example, we could take $A = \left[\begin{smallmatrix} 2 & 1 \\ -1 & 1 \end{smallmatrix}\right]$.)

Moreover, Weil gave an insightful way to think of this. The map Frob : $(x, y) \mapsto (x^p, y^p)$ is a permutation of the $\overline{\mathbb{F}}_p$ solutions of this equation, and the $\mathbb{F}_{p^k}$ solutions are the fixed points of Frob^k. Now, let's go back to the complex case. The complex solutions (with the puncture filled in) look topologically like $\mathbb{R}^2/\mathbb{Z}^2$. An endomorphism of $\mathbb{R}^2/\mathbb{Z}^2$ looks like multiplication by a 2×2 integer matrix. And the number of fixed points of multiplication by A^k is $\det(A^k - \mathrm{Id})$!

Thus, Weil's computations suggest that the curve $y^2 = x^3 - x - 1$ in some sense is of genus 1, and the map $(x, y) \mapsto (x^p, y^p)$ in some senselooks like multiplication by a 2×2 matrix of determinant p. This suggests a need to develop the language of algebraic topology to work over fields like $\overline{\mathbb{F}_p}$.

The best reason to redevelop geometry in purely algebraic language, in my opinion, is to gain a new understanding of geometry. Just as learning French can teach you how English works, I found that learning algebraic geometry gives a new, clarifying perspective on the differential geometry and topology I supposedly already knew.

September 7: Basic definitions, slicing and projecting. Let k be an algebraically closed field. For a subset S of $k[x_1, \ldots, x_n]$, we define

$$Z(S) = \{(a_1, \ldots, a_n) \in k^n : f(a) = 0\ \forall f \in S\}.$$

For a subset X of k^n, we define

$$I(X) = \{(a_1, \ldots, a_n) \in k^n : f(a) = 0 \ \forall a \in X\}.$$

We verified that

Proposition. The maps Z and I are inclusion reversing correspondences between subsets of $k[x_1, \ldots, x_n]$ and subsets of k^n.

Proposition. We have $Z(I(X)) \supseteq X$ and $I(Z(S)) \supseteq S$.

Proposition. We have $Z \circ I \circ Z = Z$ and $I \circ Z \circ I = I$.

Thus, $Z \circ I$ and $I \circ Z$ are inverses between the image of I and the image of Z.

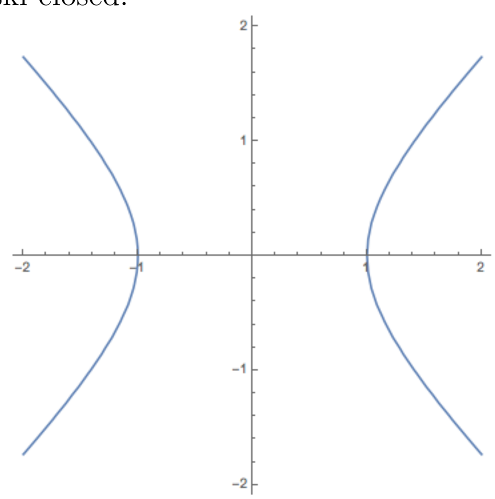
A set $X \subseteq k^n$ is called ***Zariski closed*** if $X = Z(S)$ for some S. In other words, if $X = Z(I(X))$. In general, for $X \subseteq k^n$, we put $\overline{X} = Z(I(X))$ and call $\overline{X}$ the ***Zariski closure*** of X. You will check on the problem set that the Zariski closed sets are the closed sets of a topology and $\overline{X}$ is the closure of X.

We could make a definition that a subset S of $k[x_1, \ldots, x_n]$ is "geometrically closed" if $S = I(Z(S))$. However, in a week, we will in fact prove the Nullstellansatz, which says that $S = I(Z(S))$ if and only if S is a radical ideal.

In the meantime, we discussed two important ways to reduce the number of variables.

Proposition (Slicing)**.** Let $X \subset k^{n+1}$ be Zariski closed, with $X = Z(S)$. Then $X' := X \cap \{x_{n+1} = 0\}$ is Zariski closed, with $X' = Z(S \cup \{x_{n+1}\})$.

Let $\pi : k^{n+1} \to k^n$ be the projection onto the first n coordinates. If $X \subset k^{n+1}$ is Zariski closed, then $\pi(X)$ need not be Zariski closed. Consider $X = \{x_1 x_2 = 1\}$. Then $\pi(X) = \{x_1 \neq 0\}$ which is not Zariski closed.



Proposition (Projection)**.** Let $X \subset k^{n+1}$ be Zariski closed, with $I = I(X)$. Then $I(\pi(X))$ is $I \cap k[x_1, \ldots, x_n]$, so $Z(I \cap k[x_1, \ldots, x_n]) = \overline{\pi(X)}$.

A confusing point that was not explained well in class: This proposition started with a variety X and set $I = I(X)$. If we start with I an ideal I and put $X = Z(I)$, it is not clear that $Z(I \cap k[x_1, \ldots, x_n]) = \overline{\pi(X)}$. To see this, note that the situation is different when k is not algebraically closed. Indeed, consider the ideal $I = \langle x^2 + y^2 + 1 \rangle$ in $\mathbb{R}[x, y]$. The zero set of I, in $\mathbb{R}^2$, is $\emptyset$, so $\overline{\pi(\emptyset)} = \overline{\emptyset} = \emptyset$. But $I \cap \mathbb{R}[x] = (0)$, and $Z((0)) = \mathbb{R}$.

For algebraically closed fields, this issue does not happen, but we will only be able to conclude this after we know the Nullstellansatz.

September 12: Nakayama's lemma; finite maps are closed. Before we start our main material, a piece of vocabulary which has occurred on the problem sets but not yet in class: For k an algebraically closed field and A a k-algebra, we made the preliminary definition $\mathrm{MaxSpec}(A) = \mathrm{Hom}_{k-\mathrm{alg}}(A, k)$. Given a map $\phi : A \to B$ of k-algebras, the induced map on MaxSpec's, $\phi^* : \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$, sends $\beta : B \to A$ to $\beta \circ \phi : A \to k$. The problem set gives you a good opportunity to get used to how this constructions tuns algebra into geometry.

Let $X \subset \mathbb{A}^{n+1}$ be Zariski closed and let π be the projection onto the first n coordinates. We have seen that $\pi(X)$ need not be Zariski closed. We would like conditions under which $\pi(X)$ *is* closed. Let's expand this algebraically and see what it means. Let $I = I(X)$. It will be convenient to put $R = k[x_1, \ldots, x_n]$, to view the coordinate ring of $\mathbb{A}^{n+1}$ as $R[y]$.

We would like some condition under which we have the implication: If a lies in $\overline{\pi}(X)$, then there exists $(a, y) \in X$. Taking the contrapositive, we would like that, if $X \cap \{x_1 = a_1, \ldots, x_n = a_n\} = \emptyset$, then $a \notin \overline{\pi(X)}$.

Now, $X \cap \{x_1 = a_1, \ldots, x_n = a_n\} = Z(I + \mathfrak{m}_a)$ where $\mathfrak{m}_a = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. So (using that k is algebraically closed) the condition that $X \cap \{x_1 = a_1, \ldots, x_n = a_n\} = \emptyset$ is equivalent to $I + \mathfrak{m}_a R[y] = (1)$. The desired conclusion that $a \notin \overline{\pi(X)}$ translates into asking that there is some $f \in I \cap R$ such that $f(a) \neq 0$. So we want that, under some hypothesis, the condition $I + \mathfrak{m}_a R[y] = (1)$ implies $\exists f \in R \cap I$ with $f \notin \mathfrak{m}_a$.

This conclusion sounds nicer in terms of the ring $S = R[y]/I$. We want to know that, if $\mathfrak{m}_a S = S$, then there exists some $f \in R$, with $f = 0$ in S, and $f \notin \mathfrak{m}_a$.

The missing condition is that S is finitely generated as an R module. It turns out that the ring structure of S is a distraction, we only need its structure as an R module. Renaming $\mathfrak{m}_a$ to I and S to M, what we need is:

Theorem (Nakayama's Lemma, version 1)**.** Let R be a commutative ring, let I be an ideal of R and let M be a **finitely generated** R-module. Suppose that $\mathfrak{m}M = M$. Then there is some $f \in R$ with $f \equiv 1 \bmod I$ and $fM = 0$.

Proof. Let $g_1, g_2, \ldots, g_N$ generate M as an R-module. Since $IM = M$, for each j, there are $h_{ij} \in I$ such that

$$g_j = \sum_i h_{ij} g_i.$$

Organizing the h_{ij} into a matrix H and the g_j into a vector $\vec{g}$, we have

$$(\mathrm{Id}_N - H)\vec{g} = 0.$$

Left multiplying by the adjugate of $\mathrm{Id}_N - H$, we deduce that $\det(\mathrm{Id}_N - H)\vec{g} = \vec{0}$. Let f be the element $\det(\mathrm{Id}_N - H)$ of R. Then $f\vec{g} = 0$, meaning that $fg_j = 0$ for each j, and thus $fM = 0$. But $H \equiv 0 \bmod I$, so $f = \det(\mathrm{Id}_N - H) \equiv \det \mathrm{Id}_N = 1 \bmod I$ as desired. ☐

To summarize the geometric conclusion:

Theorem. Let $X \subset k[x_1, \ldots, x_n, y]$ be Zariski closed with ideal I, and suppose that $k[\boldsymbol{x}, y]/I$ is finitely generated as a $k[\boldsymbol{x}]$-module. Then $\pi(X) = Z(I \cap k[\boldsymbol{x}])$. In particular, $\pi(X)$ is Zariski closed.

We note that, in our example of a non-Zariski closed projection, the ring $k[x, y]/(xy - 1) \cong k[x, x^{-1}]$ is not finitely generated as a $k[x]$-module.

So, when is $R[y]/I$ a finitely generated R-module?

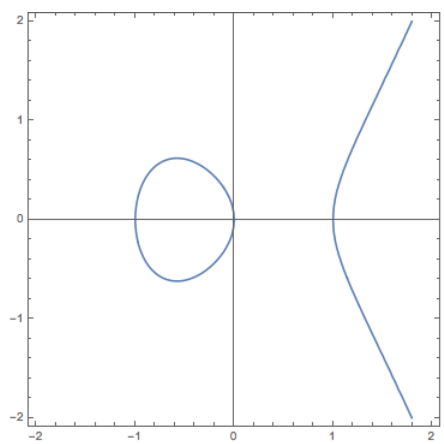
Lemma. The quotient ring $R[y]/I$ is finitely generated as an R-module if and only if I contains a polynomial of the form $y^d + r_{d-1}y^{d-1} + \cdots + r_1 y + r_0$.

Proof. In one direction, if $y^d + r_{d-1}y^{d-1} + \cdots + r_1 y + r_0 \in I$, then $R[y]/I$ is spanned by y^{d-1}, y^{d-2}, $\ldots$, y, 1. The reverse direction is left to homework. $\square$

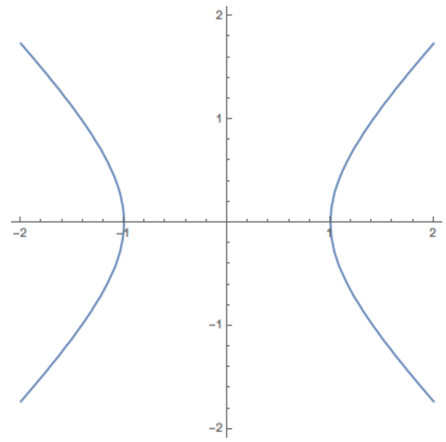
So we have the geometric conclusion:

Theorem. Let $g \in k[\boldsymbol{x}, y]$ be a polynomial of the form $y^d + g_{d-1}(\boldsymbol{x})y^{d-1} + \cdots g_1(\boldsymbol{x})y + g_0(\boldsymbol{x})$. Then $\pi : Z(g) \to \mathbb{A}^n$ is a closed map. For any ideal I containing g, we have $\pi(X) = Z(I \cap k[\boldsymbol{x}])$.

Geometrically, the difference between a monic polynomial $y^2 - x^3 + x$, and a nonmonic polynomial $xy - 1$, is that the zero locus of a monic polynomial does not have vertical asymptotes.



$y^2 = x^3 - x$

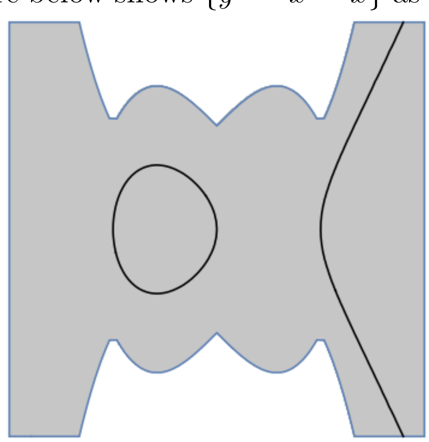


$xy = 1$

A remark on motivation in the classical geometry case: It is also true, over $k = \mathbb{R}$ or $\mathbb{C}$, that if $g \in k[\boldsymbol{x}, y]$ is monic in y, then $\pi : Z(g) \to k^n$ is closed in the classical topology on k^n. Proof: If $h(y) = y^d + h_{d-1}y^{d-1} + \cdots + h_0$ is a polynomial in $k[y]$, and $h(r) = 0$, then $|r| \leq 1 + \max(|h_j|)$. (Exercise!) So

$$Z(g) \subseteq \{(\boldsymbol{x}, y) : |y| \leq 1 + \max(|g_{d-1}(\boldsymbol{x})|, \ldots, |g_1(\boldsymbol{x})|), |g_0(\boldsymbol{x})|).$$

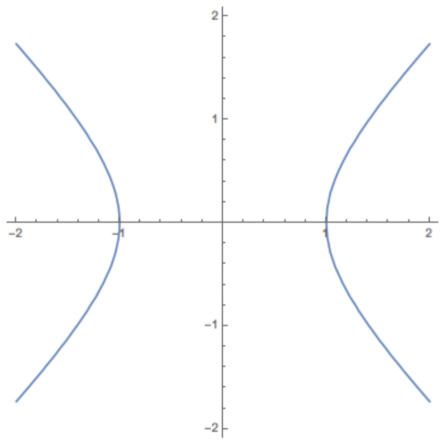
The right hand side is proper over k^n, and $Z(g)$ is closed in it, so $Z(g) \to k^n$ is proper and, in particular, closed. The figure below shows $\{y^2 = x^3 - x\}$ as a subset of $\{|y| \leq |x^3 - x| + 1\}$:



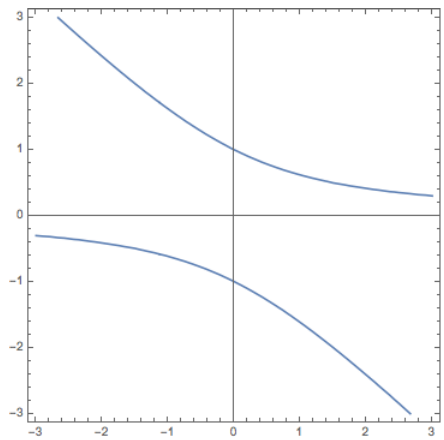
September 14: Proof of the Nullstellansatz. Today, we prove the Nullstellansatz! We first want:

Lemma (Noether's normalization lemma, first version)**.** Let $g(x_1, \ldots, x_n, y)$ be a nonzero polynomial with coefficients in an infinite field k. Then there exist $c_1, \ldots, c_n \in k$ such that $g(x_1 + c_1 y, x_2 + c_2 y, \ldots, x_n + c_n y, y)$ is monic as a polynomial in y.

For example, $xy = 1$ is not finite over the x-line, but $(x + cy)y = 1$ is finite over the x-line for $c \neq 0$. Geometrically, this means that we can shear $Z(g)$ so that make sure it has no vertical asymptotes.



$xy = 1$



$(x + y)y = 1$

Proof. Write $g(\boldsymbol{x}, y) = g_d(\boldsymbol{x}, y) + g_{d-1}(\boldsymbol{x}, y) + \cdots + g_0(\boldsymbol{x}, y)$ where g_j is homogenous of total degree j and $g_d \neq 0$. Then $g(x_1 + c_1 y, \ldots, x_n + c_n y, y) = g_d(c_1, c_2, \ldots, c_n, 1)y^d +$ (lower order terms in y). Since g_d is a nonzero homogenous polynomial, the polynomial $g_d(t_1, \ldots, t_n, 1)$ is not zero. Since k is infinite, we can find some specific $(c_1, \ldots, c_n) \in k^n$ where $g_d(c_1, c_2, \ldots, c_n, 1) \neq 0$. $\square$

We now prove the Weak Nullstellansatz:

Theorem (Weak Nullstellansatz)**.** Let k be an algebraically closed field and let I be an ideal of $k[\boldsymbol{x}]$. If $Z(I) = \emptyset$ then $I = (1)$.

Proof. We will be showing the contrapositive: If $I \neq (1)$, then $Z(I) \neq \emptyset$ or, in other words, $I \supseteq \mathfrak{m}_a$ for some $a \in k^n$.

Our proof is by induction on n. For the base case, $n = 1$, since $k[x]$ is a PID we have $I = \langle g(x) \rangle$ for some g and, since $I \neq (1)$, the polynomial g has positive degree. Then g has a root a, by the definition of being algebraically closed, and $\langle g \rangle \subseteq \mathfrak{m}_a$.

We now turn to the inductive case; assume the result is known for $k[x_1, \ldots, x_n]$ and let I be an ideal of $k[x_1, \ldots, x_n, y]$. If $I = (0)$, the result is clearly true. If not, let $g(x_1, \ldots, x_n, y)$ be a nonzero polynomial in I. By Noether's normalization lemma, we may make a change of variables such that g is monic in y and thus $k[\boldsymbol{x}, y]/I$ is finite as a $k[\boldsymbol{x}]$-module.

Put $J = I \cap k[x_1, \ldots, x_n]$. Since $I \neq (1)$, we also have $J \neq (1)$ so, by induction, there is some $a \in Z(J) \subseteq k^n$. By yesterday's result, we can lift $(a_1, \ldots, a_n)$ to some $(a_1, \ldots, a_n, a_{n+1}) \in Z(I) \subset k^{n+1}$. $\square$

We can now prove the Strong Nullstellansatz, using a method called Rabinowitsch's trick:

Theorem (Strong Nullstellansatz)**.** Let k be an algebraically closed field and let I be an ideal of $k[\boldsymbol{x}]$. Suppose that h is 0 on all of $Z(I)$. Then $h \in \sqrt{I}$.

Taking $h = 1$ yields the Weak Nullstellansatz. We will now show that the Weak Nullstellansatz implies the Strong:

Proof. We consider the zero set of I in one dimension higher. Since h is 0 on $Z(I)$, the polynomial $1-h(\boldsymbol{x})y$ is nowhere vanishing on $Z(I) \subset \mathbb{A}^{n+1}$. So By the Weak Nullstellansatz, we deduce that $1-h(\boldsymbol{x})y$ is a unit in $k[\boldsymbol{x}, y]/I = (k[\boldsymbol{x}]/I)[y]$. By the homework, this implies that h is nilpotent in $k[\boldsymbol{x}]/I$. $\square$

September 17: Affine varieties, regular functions, and regular maps. In what follows we will set up a correspondence between geometric objects and algebraic ones. We begin by defining our spaces, and an appropriate notion of maps between them.

Definition. An ***affine variety*** X is a Zariski closed subset of $\mathbb{A}^m$.

Definition. Given an affine variety $X \subseteq \mathbb{A}^m$, a function $\varphi : X \to k$ is called ***regular*** if φ is the restriction of some polynomial f in $k[x_1, \dots, x_m]$ to X. A map $\varphi : X \to \mathbb{A}^n$ is called ***regular*** if each of its coordinate functions[1] is regular.

Definition. Given affine varieties $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$, a ***regular map*** from X to Y is a function $f : X \to Y$ such that the composition

$$X \xrightarrow{f} Y \hookrightarrow \mathbb{A}^n$$

is regular, in the sense of the previous definition.

Given an affine variety $X \subseteq \mathbb{A}^m$, we can consider the ring of regular functions on X, which we will denote by $\mathcal{O}_X$. This gives us a method by which to associate a ring to an affine variety. Moreover, given any regular map $\varphi : X \to Y$, we obtain the "pullback" map $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X$ which acts on the regular function $g : Y \to k$ by

$$\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X$$
$$\varphi^*(g : Y \to k) = (g \circ \varphi : X \to k)$$

This construction defines a contravariant functor from the category of affine varieties to the category of finitely generated k-algebras with no nilpotents.[2]

Let's construct a (contravariant) functor in the other direction. Recall that $\operatorname{MaxSpec} A := \operatorname{Hom}_{k-alg}(A, k)$. Since A is finite generated, we can choose generators $x_1, \dots, x_n$ for A and write $A = k[x_1, \dots, x_n]/I$. A homomorphism $A \to k$ is determined by the images of the x_i, so by a point $(a_1, \dots, a_n) \in \mathbb{A}^n$. But such a homomorphism only exists if $f(a) = 0$ for all $f \in I$. In other words, once we choose generators, $\operatorname{MaxSpec} A$ is in canonical bijection with $Z(I)$.

Suppose B and A are finitely generated k-algebras without nilpotents, and $\psi : B \to A$ is a k-algebra homomorphism. Then this induces a map $\psi^* : \operatorname{MaxSpec}(A) \to \operatorname{MaxSpec}(B)$ given by $(h : A \to k) \mapsto (h \circ \psi : B \to k)$.

[1]The coordinate functions are the maps $\pi_i \circ \varphi : X \to k$, where π_i is the projection of X onto the line $k \cong \{(x_1, \dots, x_m) \in \mathbb{A}^m : x_j = 0 \text{ for all } j \neq i\}$.

[2]The former category has regular maps of affine varieties as its arrows, and the latter category has k-algebra homomorphisms as its arrows.

If we let **AffVar** denote the category of affine varieties and **FGAlg** denote the category of finitely generated k-algebras with no nilpotents, we have:

Theorem. *The contravariant functor* $\mathbf{AffVar}^{op} \to \mathbf{FGAlg}$ *taking a regular map* $\varphi : X \to Y$ *to its pullback* $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_Y$*, defines an equivalence of categories.*

This theorem suggests that in some sense all of the information about an algebraic variety X is contained in its coordinate ring $\mathcal{O}_X$.

Moving on, we recall that we have developed a notion of nice maps between algebraic varieties, namely regular maps. These play the role that smooth maps play in the category of smooth manifolds. When working with a smooth manifold M, one also has a notion of when a map $f : M \to \mathbb{R}$ is smooth *at some point* $x \in M$. We will soon state the appropriate notion of regularity of a map $f : X \to k$ at some point $\mathbf{x} \in X$. In fact, we define such a notion for a function on *any* subset of $\mathbb{A}^n$:

Definition. *Let* X *be any subset of* $\mathbb{A}^n$*. A function* $f : X \to k$ *is* **regular at** $\mathbf{x} \in X$ *if there exist* $g, h \in k[x_1, \dots, x_n]$*, with* $h(\mathbf{x}) \neq 0$*, such that*

$$f = \frac{g}{h}$$

on a Zariski open neighborhood of $\mathbf{x}$*.*

Continuing our analogy with manifold theory, we recall that a map $f : M \to \mathbb{R}$ is smooth if and only if it is smooth at every point $x \in M$. The analogous fact for regular maps is stated below, and we will cover the proof in class soon:

September 19: Regularity, Connected Components and Idempotents. We start with a proof of the theorem mentioned last time.

Theorem. *Let* X *be a Zariski closed subset of* $\mathbb{A}^n$*. A function* $f : X \to k$ *is regular if and only if* f *is regular at every* $\mathbf{x} \in X$*.*

Proof. Suppose that $f : X \to k$ is regular. Then, we can choose $g = f$, and $h = 1$ so that we have $f = \frac{g}{h}$ on all of X, which is a neighborhood of every point $\mathbf{x} \in X$. Thus, f is regular at every point.

Now suppose that $f : X \to k$ is regular at every point $x \in X$. We can find an open neighborhood $V_{\mathbf{x}}$, and rational functions $g_{\mathbf{x}}, h_{\mathbf{x}} \in k[x_1, \dots, x_n]$, with $h_{\mathbf{x}}(\mathbf{y}) \neq 0$, $\forall \mathbf{y} \in V_{\mathbf{x}}$, and $f(\mathbf{y}) = \frac{g_{\mathbf{x}}(\mathbf{y})}{h_{\mathbf{x}}(\mathbf{y})}$, or $h_{\mathbf{x}}(\mathbf{y}) f(\mathbf{y}) = g_{\mathbf{x}}(\mathbf{y})$, $\forall \mathbf{y} \in V_{\mathbf{x}}$.

Note that $V_{\mathbf{x}} \subset X$ is open in X implies that $X \setminus V_{\mathbf{x}}$ is closed in X (which is closed in $\mathbb{A}^n$), and so $X \setminus V_{\mathbf{x}}$ is a closed subset of $\mathbb{A}^n$ and is thus an affine variety. Now, since $X \setminus V_{\mathbf{x}}$ is closed and $\mathbf{x}$ is not in $X \setminus V_{\mathbf{x}}$, we have some polynomial $p \in I(X \setminus V_{\mathbf{x}})$ such that $p(\mathbf{x}) \neq 0$. Now we can take $V'_{\mathbf{x}} = V_{\mathbf{x}} \cap \{\mathbf{y} \in X | p(\mathbf{y}) \neq 0\}$, $g'_{\mathbf{x}} = p * g_{\mathbf{x}}$, and $h'_{\mathbf{x}} = p * h_{\mathbf{x}}$ so that we have $h'_{\mathbf{x}}(\mathbf{y}) f(\mathbf{y}) = g'_{\mathbf{x}}(\mathbf{y})$, $\forall \mathbf{y} \in X$.

Let $J = (\{h'_{\mathbf{x}} | x \in X\})$, the ideal generated by the $h'_{\mathbf{x}}$s. Note that for each $\mathbf{x} \in X$, we have $h'(\mathbf{x}) \neq 0$, so, by invoking the Nullstellansatz, the ideal $I(X) + J = (1)$. Thus we can write

$$1 = q(\mathbf{y}) + \sum a_i(\mathbf{y}) * h'_i(\mathbf{y})$$

for $\mathbf{y} \in \mathbb{A}^n$, where $q \in I(X)$, $a_i \in k[x_1, \dots, x_n]$, and $h'_i \in J$. Now for $\mathbf{y} \in X$, we have $1 = \sum a_i(\mathbf{y}) * h'_i(\mathbf{y})$, and multiplying by f on both sides we get $f(y) = \sum a_i(\mathbf{y}) * g'_i(\mathbf{y})$, for $\mathbf{y} \in \mathbb{A}^n$, so f is a polynomial restricted to X. ∎

It is important to note that the requirement that X was Zariski closed (as apposed to being an open subset of a zariski closed set) is necessary. For example, the function $f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ defined by $f(y) = 1/y$ is regular at every point $y \neq 0$, but it is not a polynomial.

It is also important to note that not every regular function on an open subset of a zariski closed set is given by a quotient of polynomials. For example, let $X = Z(x_1x_2 - x_3x_4) \subset \mathbb{A}^n$, and $U = X \setminus Z(\{x_2, x_3\}$, and define $f(x_1, x_2, x_3, x_4) = \frac{x_1}{x_3}$ if $x_3 \neq 0$, and $f(x_1, x_2, x_3, x_4) = \frac{x_2}{x_4}$ if $x_4 \neq 0$; there is no single expression $\frac{g}{h}$ for this f with h nonzero on all of U.

We now turn our attention to the notion of connectedness of affine varieties. Recall that a topological space X is said to be disconnected if we can find $X_1, X_2 \subset X$ such that $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$, and X_1, $X_2 \neq \emptyset$. A space is connected if it is not disconnected.[3]

Assuming that an affine variety $X \subset \mathbb{A}^n$ is disconnected, we can find find $X_1, X_2 \subset X$ as above, and define $f(\mathbf{x}) = 0$, if $\mathbf{x} \in X_1$, and $f(\mathbf{x}) = 1$, if $\mathbf{x} \in X_2$. Note that this function is regular at every $\mathbf{x} \in X$. By our result above, it must be given by a polynomial in $\mathcal{O}_X$. Also note that our f is idempotent, meaning $f^2 = f$.

Now suppose we are given an affine variety X, and a idempotent element, f, of $\mathcal{O}_X$, with $f \neq 0, 1$ (such an idempotent is called nontrivial). Then we can define $X_1 = f^{-1}(\{0\})$, and $X_2 = f^{-1}(\{1\})$, and check that these have the properties $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$, and X_1, $X_2 \neq \emptyset$, using the fact that we must have either $f(\mathbf{y}) = 0$ or $f(\mathbf{y}) = 1$. Thus we have proved

Theorem. *An affine variety X is connected $\iff$ its coordinate ring $\mathcal{O}_X$ contains no nontrivial idempotent elements.*

In fact, we have proved slightly more: we have given a bijection between the (ordered) pairs of subspaces that disconnect X and nontrivial idempotent of $\mathcal{O}_x$.

Now, a useful lemma from algebra says that

Lemma. A ring contains nontrivial idempotents $\iff$ it is the direct sum of two nontrivial rings.

Combining this with our result above, we get that

Theorem. *An affine variety X is connected $\iff$ its coordinate ring is not the direct sum of two nontrivial rings.*

September 21: Irreducible Components. We state Hilbert's Basis theorem, which we proved in the 2nd problem set:

Theorem (Hilbert's Basis Theorem)**.** Finitely generated k-algebras are noetherian rings.

Theorem (Hilbert's Basis theorem, Restatement 1)**.** Every ideal in the polynomial ring $k[x_1, \dots x_n]$ is finitely generated.[4]

One implication of the above restatement is that the zero set of any ideal can be realized as the zero set of *finitely* many polynomials.

[3]In Professor Speyer's opinion, the empty set is neither connected nor disconnected, just as 1 is neither prime nor composite. But not everyone will agree on this point.

[4]Even though the initial proofs of the theorem weren't constructive, now we can explicitly construct generators of a given ideal in the polynomial ring. See *Gröbner Basis.*

Theorem (Hilbert's Basis theorem, Restatement 2)**.** $\nexists$ an infinite chain $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_m \subsetneq \ldots$ of ideals in $k[x_1, \ldots, x_n]$

Using the algebro-geometric dictionary, we obtain:

Corollary. *$\nexists$ an infinite chain $X_1 \supsetneq X_2 \supsetneq \ldots \supsetneq X_m \supsetneq \ldots$ of Zariski closed subsets in $\mathbb{A}^n$.*

The above corollary illustrates the fact that the Zariski topology behaves differently from the classical topology. Instead of working with connected components, we will develop a new way of decomposing subsets of $\mathbb{A}^n$ which takes this into account.

Definition. A topological space X is ***reducible*** if $X = X_1 \cup X_2$ where X_1 and X_2 are *proper* closed subsets of X.

Definition. A topological space is ***irreducible*** if it is nonempty and not reducible.

In the previous class, we saw that X is connected if and only if its ring of regular functions is not a direct sum. We have a similar algebraic description for when X is irreducible.

Lemma. Let X be a Zariski closed subset of $\mathbb{A}^n$ and let A be the ring of regular functions on X. Then, X is reducible if and only if A is an integral domain.

Proof. Let f_1, f_2 be nonzero elements in A such that $f_1 f_2 = 0$. Let $X_j = Z(f_j)$. X_j is Zariski closed by the definition of the Zariski topology. Furthermore, X_j is proper since f_j is a nonzero element, and hence doesn't vanish on all of X. Furthermore, since $f_1 f_2 = 0$, $X = X_1 \cup X_2$, which means that X is reducible.

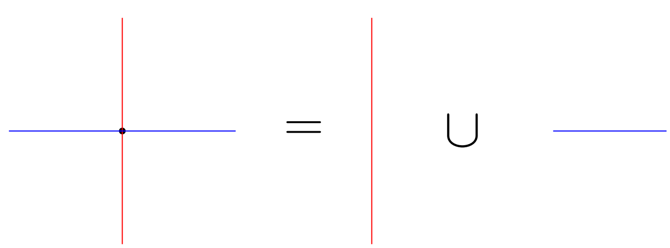
Now, suppose X is reducible. We obtain a decomposition of $X = X_1 \cup X_2$, where X_1 and X_2 are proper closed subsets. Now, let $f_1 \in I(X_1)$ and $f_2 \in I(X_2)$ be nonzero elements. Then $f_1 f_2$ vanishes on X as X is the union of X_1 and X_2. Hence, A is not an integral domain. $\square$

The above lemma should reinforce the idea that irreducible components are nicer to work with than connected components - coordinate rings of connected components needn't even be integral domains!

Now, we show that any variety can be decomposed into irreducible subsets.

Theorem. Let $X \subseteq \mathbb{A}^n$ be a Zariski closed. There are irreducible varieties $X_1, X_2, \ldots X_N$ such that $X = \bigcup_{i=1}^{N} X_i$.

Here is an example:



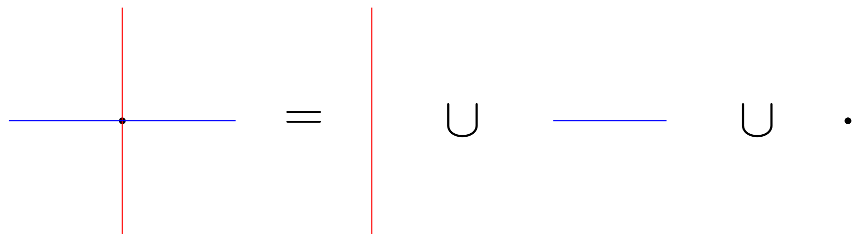
Proof. Recursively build a tree with vertices labeled by varieties. We label the root with X. If a vertex v is labeled by Y and Y is reducible with $Y = Y_1 \cup Y_2$, then we place two children below v, labeled by Y_1 and Y_2. If v is labeled by an irreducible variety, then make it a leaf.

If the tree is finite, then X is the union of the irreducible labels of the leaves, as desired.

If the tree is infinite, then it has an infinite path. This corresponds to a chain of varieties $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$, a contradiction. $\square$

The result about decomposing topological spaces into connected components also has a uniqueness clause; can we expect something similar for the above decomposition?

On the face of it, no.



However, notice that the problem arised when we threw in irreducible subvarieties which are contained in bigger irreducible subsets of X. We can prevent this by defining:

Definition. Let X be a Zariski closed subset of $\mathbb{A}^n$. $Y \subseteq X$ is an ***irreducible component*** of X if

- Y is irreducible,
- Y is closed in X, and
- $\nexists\, Y'$ irreducible and closed in X such that $Y \subsetneq Y'$.

Looking back at the above example, the single point was not an irreducible component of $Z(xy)$.

Theorem (Irreducible Decomposition)**.** Let X be Zariski-closed in $\mathbb{A}^n$. Then,

(1) If $X = \bigcup_{i=1}^N X_i$, with X_i irreducible, and $Z \subseteq X$ is irreducible, then Z is contained in one of the X_i.
(2) If $X = \bigcup_{i=1}^N X_i$, with X_i irreducible, then each irreducible component is equal to one of the X_i.
(3) X has finitely many irreducible components.
(4) X is the union of its irreducible components.

Since irreducible components of X are the maximal irreducible closed subvarieties of X, they correspond to *minimal primes* in the coordinate ring of X.

Proof. To prove (1), note that $Z = \bigcup_i (Z \cap X_i)$. Since Z is irreducible and $Z \cap X_i$ is closed in Z, this means that one of the $Z \cap X_i$ equals Z, so, for that i, we have $Z \subseteq X_i$.

For (2), let Y be an irreducible component of X. By (1), we know that Y is contained in some X_i. But, by the definition of being an irreducible component, this implies that $Y = X_i$.

For (3), we have just shown that all the irreducible components occur in the finite list X_1, X_2, $\ldots$, X_N, so there are finitely many.

We finally come to (4). Choose a decomposition $X = \bigcup_{i=1}^N X_i$ into irreducible subvarieties where N is minimal. Suppose, for the sake of contradiction that one of the X_i is not an irreducible component; without loss of generality let it be X_N. So $X_N \subsetneq X'$ for some irreducible X'. Using (1), we have $X' \subseteq X_j$ for some j, and this j must not be N. So $X_N \subsetneq X' \subseteq X_j$ and thus $\bigcup_{i=1}^N X_i = \bigcup_{i=1}^{N-1} X_i$, contradicting minimality.

□

September 24: Projective spaces. We'll now start to see projective varieties in projective spaces. To start with, we settle some notations: Let k denote a algebraic closed field, V denote a finite dimensional k-vector space, and $\mathbb{P}(V) = (V - \{0\})/k^*$ the projective space. Write $\mathbb{P}^n = \mathbb{P}(k^{\oplus(n+1)})$. We'll use $(z_1, \cdots, z_{n+1})$ to denote the coordinates on k^{n+1}, and $[z_1 : z_2 : \cdots : z_{n+1}]$ to denote homogeneous coordinates on $\mathbb{P}^n$.

The first observation is that inside $\mathbb{P}^n$, there sits a copy of $\mathbb{A}^n$, via the inclusion map

$$i : \mathbb{A}^n \to \mathbb{P}^n, (z_1, \cdots, z_n) \mapsto [z_1 : z_2 : \cdots : z_n : 1].$$

We then have a decomposition $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1} = \{z_{n+1} \neq 0\} \cup \{z_{n+1} = 0\}$. Similarly, if $V = H \oplus k$, where H is a hyperplane, we have $\mathbb{P}(V) = H \cup \mathbb{P}(H) = \{[\mathbf{h} : 1]\} \cup \{[\mathbf{h} : 0]\}$.

The reason why we're considering the projective space is to try to draw an analogy to the fact in manifold theory that every compact manifold can be embedded in some $\mathbb{R}^n$. However, there are no positive dimensional subvarieties of $\mathbb{A}^n$ which deserve to be called compact. (Literally speaking, $\mathbb{A}^n$ is compact in the Zariski topology, but we will see soon that this is misleading.) $\mathbb{P}^n$ does deserve to be called compact, as we will soon see.

In this course we will see:

- Affine varieties: Closed subsets of $\mathbb{A}^n$.
- Quasi-affine varieties: Open subsets of affine varieties.
- Projective varieties: Closed subsets of $\mathbb{P}^n$.
- Quasi-projective varieties: Open subsets of projective varieties.

Figure 1 shows their relations.

We won't deal with any notion of variety more abstract than a quasi-projective variety in this term. More general abstract notions of variety could make a great final project, though!

There are three ways to talk about projective spaces:

- Work in $V - \{0\}$ and do dilation invariant things.
- Work in homogeneous coordinates: If $g \in k[x_1, \cdots, x_{n+1}]$ is a homogeneous polynomial, then $Z(g)$ is a well-defined subset of $\mathbb{P}^n$.
- Work locally in an affine chart, i.e., split $V = H \oplus k$ and think of $H \subseteq \mathbb{P}(V)$. For example, we can cover $\mathbb{P}^2$ with homogeneous coordinates $[x_1 : x_2 : x_3]$ using three charts $\{x_i \neq 0\}, i = 1, 2, 3$.

Example. Let's look at a curve in different coordinate charts. Consider the curve $x_1^2 + x_2^2 = x_3^2$ in $\mathbb{P}^2$. On chart $\{x_3 \neq 0\}$, the equation becomes $(\frac{x_1}{x_3})^2 + (\frac{x_2}{x_3})^2 = 1$, and this is a circle. On chart $\{x_1 \neq 0\}$, the equation is $1 + (\frac{x_2}{x_1})^2 = (\frac{x_3}{x_1})^2$, which illustrates a hyperbola.

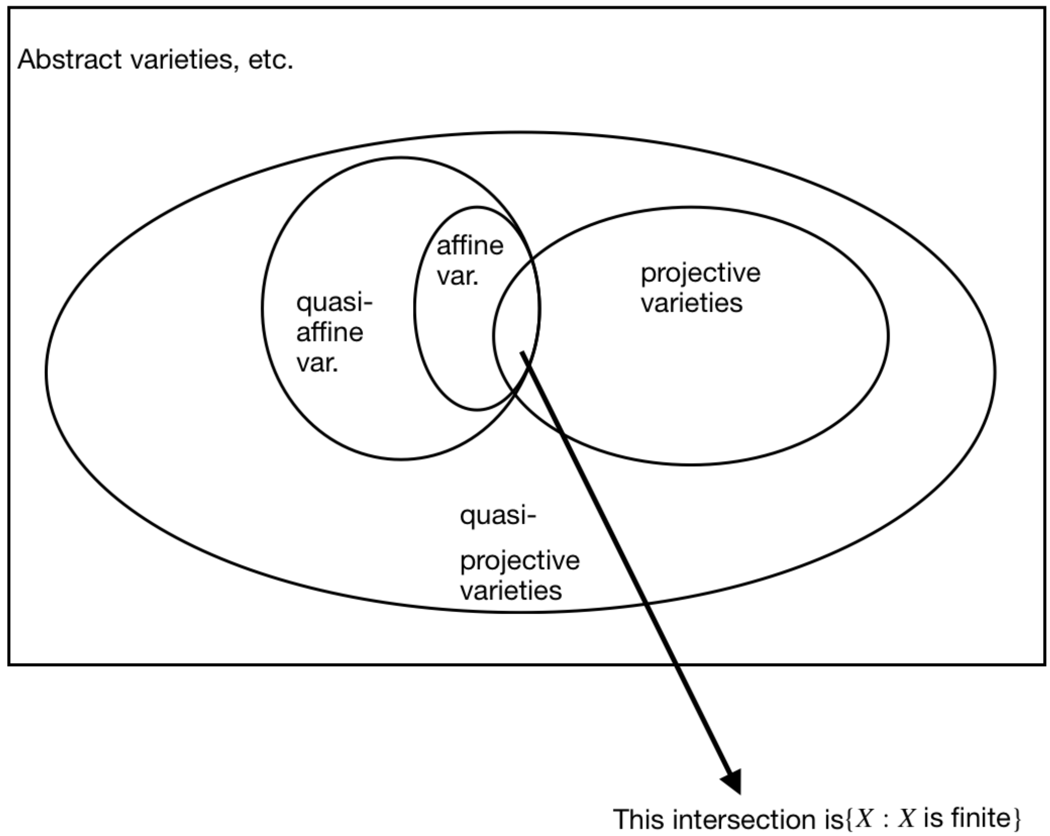
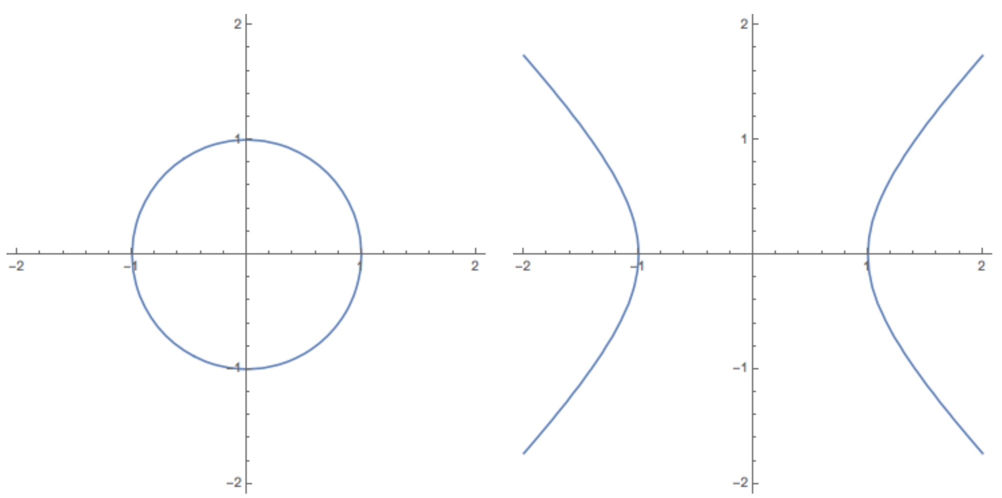


FIGURE 1. Various classes of varieties



$$(\tfrac{x_1}{x_3})^2 + (\tfrac{x_2}{x_3})^2 = 1 \qquad\qquad 1 + (\tfrac{x_2}{x_1})^2 = (\tfrac{x_3}{x_1})^2.$$

Corresponding to the three ways of talking about projective spaces, we have three ways of describing the topology on $\mathbb{P}^n$:

Definition. A set X is **closed** in $\mathbb{P}(V)$ if one of the following holds:

- $\pi^{-1}(X)$ is closed in $V - \{0\}$, or equivalently, $\pi^{-1}(X) \cup \{0\}$ is closed in V, where $\pi : V - \{0\} \to \mathbb{P}(V)$ is the projection map;
- $X = \bigcap_{g \in S} Z(g)$, where S is a set of homogeneous polynomials in $k[x_1, \cdots, x_{n+1}]$.
- $X \cap H$ is closed in every affine chart H, or equivalently, $X \cap \{x_j \neq 0\}$ is closed in $\{x_j \neq 0\} \cong \mathbb{A}^n, \forall j$.

We also have three ways to define a regular function on $\mathbb{P}^n$:

Definition. Let $X \subset \mathbb{P}^n$, and $x \in X$. $f : X \to k$ is a function. We say f is **regular at x** if one of the following holds:

- $f \circ \pi$ is regular on $\pi^{-1}(X)$ at $\tilde{x}$, where $\tilde{x} \in V - \{0\}$, and $\pi(\tilde{x}) = x$.
- $f = \frac{g}{h}$ on an open neighborhood of $x \in X$, where g, h are homogeneous polynomials of the same degree, and $h(x) \neq 0$.
- $f|_H$ is regular at x for every affine chart H containing x, or equivalently, $f|_H$ is regular at x for an affine chart H containing x.

September 26: Pause to look at a homework problem. Today we looked at various ways of solving the tricky homework question of splitting a variety into irreducible pieces. The variety in question is $X = Z(wy - x^2, xz - y^2)$. We want to think geometrically; what are the solutions?

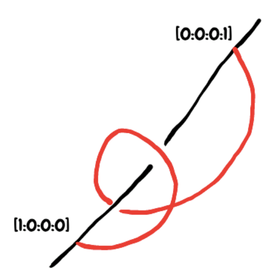
(1) Suppose $x = 0$, then $y = 0$ so the solutions are of the form
$$(w, 0, 0, z) \qquad \text{and} \qquad \mathbb{A}^2 \cong X_1 := \{x = y = 0\} \subset \mathbb{A}^4.$$

(2) Suppose $x \neq 0$, then $wy = x^2$ so $w, y \neq 0$ and $w = \frac{x^2}{y}$, $z = \frac{y^2}{x}$. Thus the solutions are of the form
$$\left(\frac{x^2}{y}, x, y, \frac{y^2}{x}\right)$$
which is a geometric progression!

There are now two modes of thought on how to proceed for defining this second component of the variety:
$$X_2' := \{\text{geometric progressions}\} \qquad \text{or} \qquad X_2'' := Z(wy - x^2, xz - y^2, wz - xy).$$

These end up being the same set, but the proofs proceed differently.

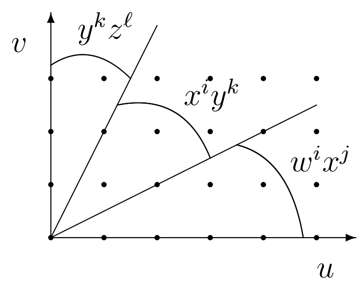
For visualization purposes, its easiest to draw the relation between these sets projectively.



Method 1: From the geometric progression perspective, a sequence (w, x, y, z) is a geometric progression if and only if it is of the form (u^3, u^2v, uv^2, v^3). So let's define $\varphi : \mathbb{A}^2 \to \mathbb{A}^4$ via $(u, v) \mapsto u^3, u^2v, uv^2, v^3)$. We'll see on the homework that if X and Y are topological spaces, $\phi : X \to Y$ is a continuous surjection, and X is irreducible, then Y is irreducible. Thus this image is X_2' and is irreducible.

Method 2: Try to prove that $R := k[w, x, y, z]/\langle wy - x^2, xz - y^2, wz - xy\rangle$ is a domain. (This actually would also prove that the ideal is radical, but luckily that is true). We could show that $R \cong k[u^3, u^2v, uv^2, v^3] \subset k[u, v]$. This map is clearly onto, but what about the kernel? Suppose $g(u^3, u^2v, uv^2, v^3) \neq 0$, we'll reduce with respect to a Gröbner basis. Using lex order with $w > z > y > x$, then $\underline{wy} - x^2, \underline{xz} - y^2, \underline{wz} - xy$ is already a Gröbner basis, and so we can keep doing replacements with these generators to decrease the w and z degrees. Thus we can write $g \equiv \sum g_{ijk\ell} w^i x^j y^k z^\ell$ where either $i = k = 0$, $i = \ell = 0$, or $j = k = 0$. We can graph the possible exponents of monomials $u^a v^b$, and from the picture we can see that there is no cancellation between the terms contributed by $w^i x^j$, by $x^i y^k$, and $y^k z\ell$. So g must actually be zero, and this is an isomorphism.

$$w^i x^j \mapsto u^{3i+2j} v^j$$
$$x^j y^k \mapsto u^{2j+k} v^{j+2k}$$
$$y^k z^\ell \mapsto u^k v^{2k+3\ell}$$



Method 3: Someone in class proposed to look at the map $\mathbb{A}^4 \to \mathbb{A}^4$ where $(a, b, c, d) \mapsto (ac, ad, bc, bd)$, which we can restrict to a map $Z(ad^2 - bc^2) \to X_2''$. Some algebra has to be checked, but this probably works.

Method 4: Let's prove $X_2 = Z(\langle wy - x^2, xz - y^2, wz - xy\rangle)$ is irreducible. We see that $X_2 \cap \{w \neq 0\}$ implies that

$$p := \frac{x}{w} \qquad q := \frac{y}{w} = \frac{x^2}{w^2} \qquad r := \frac{z}{w} = \frac{x^3}{w^3}$$

so that $q = p^2$, $r = p^3$. The intersection of X_2 with $\{w \neq 0\}$ is thus clearly irreducible. Put $U = X_2 \cap \{w \neq 0\}$.

(This paragraph, added by Professor Speyer, is what he would have said if we were enough on the ball, and he still feels like it is a lot longer than it should be.) Let $X_2 = \bigcup Y_i$ is the decomposition into irreducible components. So $X_2 \cap U = \bigcup (Y_i \cap U)$ so we have $Y_i \cap U = U$ for some Y_i, let's say Y_1. We claim each irreducible component Y_j other than Y_1 must be contained in $\{w = 0\}$. To see this, suppose for the sake of contradiction that $Y_j \cap U$ is nonempty. Then $Y_j \cap U$ is dense in Y_j, since Y_j is irreducible. But $Y_j \cap U$ would lie in $Y \cap U = Y_1 \cap U \subset Y_1$, so a dense subset of Y_j would lie in Y_1, and thus $Y_j \subseteq Y_1$, a contradiction. We thus see that any other irreducible component of X_2 must be contained in $\{w = 0\}$.

But $X_2 \cap \{w = 0\}$ is easily checked to be the z-axis, and the z-axis is easily checked to be in the Zariski closure of $X_2 \cap U$.

September 28: Topology and Regular Functions on Projective Spaces. There are three ways of thinking about almost anything in projective space – by coning and working in affine space, by working with homogenous polynomials, and by working in affine charts. This class was devoted to proving the equivalence of these three ways through group work.

We write $z_1, \ldots, z_{n+1}$ for the homogenous coordinate on $\mathbb{P}^n$ and π for the map $\mathbb{A}^{n+1} \to \mathbb{P}^n$.

Theorem. (Closed Sets in $\mathbb{P}^n$) The following are equivalent:

(1) $\pi^{-1}(X)$ is closed in $\mathbb{A}^{n+1} \setminus \{0\}$.
(2) $X = \cap_{g \in S} Z(g)$, where S is a set of homogeneous polynomials in $k[z_1, \ldots, z_{n+1}]$.
(3) X is closed in $\{z_j \neq 0\} \cong \mathbb{A}^n$ for $1 \leq j \leq n+1$.

Proof. (1) $\implies$ (2): If $\pi^{-1}(X)$ is closed in $\mathbb{A}^{n+1} \setminus \{0\}$, let $I = I(\pi^{-1}(X) \cup 0)$. It is enough to show I is a homogeneous ideal. Let $f \in I$ and let $f = f_0 + f_1 + \cdots + f_d$ be the decompostion of f into homogeneous parts. Then $f(\lambda x) = f_0(x) + \lambda f_1(x) + \cdots + \lambda^d f_d(x)$. So, if $x \in \pi^{-1}(X) \setminus \{0\}$ then $\sum \lambda^j f_j(x) = 0$ for all nonzero $\lambda \in k$, so $f_0(x) = f_1(x) = \cdots = f_d(x) = 0$ and the f_j are in I as desired.

(2) $\implies$ (3): If S is a set of homogeneous polynomials such that $Z(S) = X$, then, $X \cap \{z_i \neq 0\} = Z(\{g(z_1, \ldots, z_{i-1}, 1, \ldots, z_n) | g \in S\})$. In particular, $X \cap \{z_i \neq 0\}$ is closed in $\{z_i \neq 0\} \cong \mathbb{A}^n$.

(3) $\implies$ (1): Let $X \cap \{z_i \neq 0\}$ be given by $Z(\{f_j\}) \subset \mathbb{A}^n$. Now, $\mathbb{A}^{n+1} \setminus \{0\}$ is covered by $U_i = \pi^{-1}(\{z_i \neq 0\})$, $i = 1, \ldots, n+1$. Therefore, to show that $\pi^{-1}(X)$ is closed in $\mathbb{A}^{n+1} \setminus \{0\}$, it suffices to show that $\pi^{-1}(X) \cap U_i$ is closed in $\mathbb{A}^{n+1} \setminus \{0\}$. But, $\pi^{-1}(X) \cap U_i = \pi^{-1}(X \cap \{z_i \neq 0\}) = Z(\{f_j\}) \cap U_i \subset \mathbb{A}^{n+1} \setminus \{0\}$ is closed.

□

Theorem. (Regular Functions on $\mathbb{P}^n$) Let $X \subset \mathbb{P}^n$ and $x \in X$ and let $f : X \to k$. Then, the following are equivalent:

(1) The function $f \circ \pi$ is regular at $\tilde{x}$ where $\tilde{x} \in \pi^{-1}(x)$.
(2) There are homogeneous polynomials g, h, $h(x) \neq 0$, degree g = degree h, such that, $f = \frac{g}{h}$ on an open neighbourhood of x.
(3) f is regular when restricted to $\{z_j \neq 0\}$ where j is chosen such that $x_j \neq 0$.

Proof. (1) $\implies$ (3): If $f \circ \pi$ is regular at $\tilde{x}$ where $\tilde{x} \in \pi^{-1}(x)$, then, in a neighbourhood U of $\tilde{x}$, $f \circ \pi = \frac{g}{h}$ for some g, $h \in k[z_1, \ldots, z_{n+1}]$ and $h(y) \neq 0$ on U. Then, if $x_j \neq 0$, then choosing a neighbourhood V of x in $\{z_j \neq 0\}$ such that $V \subset \pi(U)$, we have, $f = \frac{g(z_0, \ldots, \tilde{x_j}, \ldots, z_n)}{h(z_0, \ldots, \tilde{x_j}, \ldots, z_n)}$ on V where $\tilde{x} = (\tilde{x_1}, \ldots, \tilde{x_{n+1}}) \in \mathbb{A}^{n+1}$. Therefore, f is regular when restricted to $\{z_j \neq 0\}$.

(3) $\implies$ (2): If f is regular at x when restricted to $\{z_j \neq 0\}$, then in a neighborhood of x, we have, $f([z_0 : \cdots : z_{n+1}]) = \frac{g(\frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j})}{h(\frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j})}$. Then, in the same neighborhood, we have $f([z_0 : \cdots : z_{n+1}]) = \frac{z_j^N g(\frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j})}{z_j^N h(\frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j})}$ where $N >$ degree g, degree h so that $z_j^N g(\frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j})$ and $z_j^N h(\frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j})$ are homogeneous polynomials of the same degree.

(2) $\implies$ (1): If on a neighbourhood U of x, we have $f = \frac{g}{h}$ for f, g homogeneous polynomials of same degree, then on $\pi^{-1}(U)$, $f \circ \pi([z_0 : \cdots : z_{n+1}]) = \frac{g(z_1, \ldots, z_{n+1})}{h(z_1, \ldots, z_{n+1})}$. Therefore, $f \circ \pi$ is regular at $\tilde{x}$. □

October 1 : Products. Summary: We talk about products of quasi-projective varieties, and show that they exist, and actually are quasi-projective varieties themselves.

In the case of quasi-affine varieties, the product of varieties sitting inside $\mathbb{A}^m$ and $\mathbb{A}^n$ are actually varieties sitting inside $\mathbb{A}^{m+n}$. However, we say in one of the problem sets that the Zariski topology on the product $X \times Y$ of affine varieties X and Y is not the same as the product topology on $X \times Y$ (unlike the categories of topological spaces, or smooth manifolds).

The regular functions on $X \times Y$ are just polynomials in $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$, where the $\{x_i\}$ are coordinate functions on $\mathbb{A}^m$, and $\{y_j\}$ are coordinate functions on $\mathbb{A}^n$. If we want to describe the ring of regular functions on $X \times Y$ in more algebraic terms, we have the following description.

$$\mathcal{O}_{X\times Y} \cong \mathcal{O}_X \otimes_k \mathcal{O}_Y$$

Proposition. *For any affine variety Z, and maps $f_X : Z \to X$ and $f_Y : Z \to Y$, there exists a unique map $f_{X\times Y} : Z \to X \times Y$, which make the following diagram commute.*

$$\begin{tikzcd}
& Z \arrow[d, dashed, "\exists! f_{X\times Y}"] \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] & \\
& X \times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
X & & Y
\end{tikzcd}$$

Proof. There can clearly exist at most one such map, i.e. $f_{X\times Y} = (f_X, f_Y)$, since regular functions are also set functions. The only thing we need to verify is that this is actually a regular map, but that follows by checking on each coordinate. □

When dealing with projective varieties though, products get a little harder. It's not even clear what $\mathbb{P}^m \times \mathbb{P}^n$ is (it's certainly not $\mathbb{P}^{m+n}$). But here's a more fundamental question: what is the topology we want on $\mathbb{P}^m \times \mathbb{P}^n$, and what are the functions we want to call regular on $\mathbb{P}^m \times \mathbb{P}^n$? The answer to the first question is that a subset U of $\mathbb{P}^m \times \mathbb{P}^n$ is open if $U \cap (\mathbb{A}^m \times \mathbb{A}^n)$ is open for all affine open sets in $\mathbb{P}^m \times \mathbb{P}^n$. In a similar spirit, we call a function $f : \mathbb{P}^m \times \mathbb{P}^m \to k$ regular if the restriction to each affine open chart as before gives a regular function. Now we know that the product of $\mathbb{P}^m \times \mathbb{P}^n$ looks like locally: it locally looks like an affine variety. We still don't know whether this a projective variety or not.

The Segre embedding answers our question, by realizing $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ as a closed subset of $\mathbb{P}^{mn-1}$. As the name suggests, it's an injective map μ from $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ to $\mathbb{P}^{mn-1}$.

$$\mu : ([x_1 : \cdots : x_m], [y_1 : \cdots , y_n]) \mapsto [x_1 y_1 : \cdots : x_m y_n]$$

A basis independent way of writing the same map is the following.

$$\mu : ([v], [w]) \mapsto [v \otimes w]$$

We want to show that the map μ is an embedding, i.e. it's injective, its image is closed, and the inverse map from the image is also regular. To show all these results, the following lemma will be useful.

Lemma. *If we restrict μ to the chart where $x_m \neq 0$ and $y_n \neq 0$, then we get a map from $\mathbb{A}^{m-1} \times \mathbb{A}^{n-1}$ to $\mathbb{A}^{mn-1}$ which has a regular right inverse σ.*

Proof. Restricting to the given coordinate charts, and normalizing the coordinates so that $x_m = 1$, and $y_n = 1$, the map μ is given by the following formula.

$$\mu([x_1 : \cdots : x_{m-1} : 1], [y_1 : \cdots : y_{n-1} : 1]) = \begin{pmatrix} x_1y_1 & x_2y_1 & \cdots & y_1 \\ x_1y_2 & x_2y_2 & \cdots & y_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1y_{n-1} & x_2y_{n-1} & \cdots & y_{n-1} \\ x_1 & x_2 & \cdots & 1 \end{pmatrix}$$

From this formula, it's easy to see what the right inverse will be: simply the projection onto the last rows and columns. That also tells us why the inverse is regular. □

Now we'll use this lemma to get the properties we want from μ.

Corollary. *The map μ must be injective.*

This follows because any map that has a right inverse must be injective.

Corollary. *The image of μ is a closed set.*

Proof. It suffices to check the intersection of the image with each affine chart is closed. Let's check the affine chart $z_{mn} \neq 0$. On this open set, the image is the image of μ when restricted to the open sets of the lemma. Now we use the fact that σ is the right inverse to μ. That means $\sigma^{-1}(\mathbb{A}^{m-1} \times \mathbb{A}^{n-1})$ is exactly the image of μ. But since σ is a regular map, the pre-image of a closed set is closed, which gives us the result. □

Corollary. *The map from $\mu(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$ to $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ is regular.*

We already know that the inverse map locally is regular, thanks to the lemma. But that's all we need, since to prove regularity, it suffices to check locally.

Now what we're interested in knowing is what the image of $\mathbb{P}^1 \times \mathbb{P}^1$ looks like when it's sitting inside $\mathbb{P}^3$. To make visualization simpler, we'll assume we're working over the field $\mathbb{C}$. The map from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^3$ is given by $([x_1 : x_2], [y_1, y_2]) \mapsto [x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2]$. The image is the zero set of the polynomial $z_1z_4 - z_2z_3$. We can change coordinates to make this polynomial easier to visualize. We pick new coordinates $[w_1 : w_2 : w_3 : w_4]$, where $z_1 = w_1 + iw_2$, $z_4 = w_1 - iw_2$, $z_2 = w_3 + iw_4$, and $z_3 = w_3 - iw_4$. In these new coordinates, our polynomial becomes $w_1^2 + w_2^2 = w_3^2 + w_4^2$. We now restrict to the set where $w_4 \neq 0$, and we normalize w_4 to be 1. That makes the polynomial $w_1^2 + w_2^2 = w_3^2 + 1$, in the affine chart isomorphic to $\mathbb{C}^3$. Complex three space is too high dimensional to visualize, so we just look at the real part of this variety. We get something that looks like Figure 2. Notice that this is covered with two families of lines. One is lines of the form $\mathbb{P}^1 \times \{\text{point}\}$, and the other is lines of the form $\{\text{point}\} \times \mathbb{P}^1$.

October 3 : Projective maps are closed. Today we discussed the following important theorem.

Theorem. *Let B be a quasi-projective variety and let X be closed in $B \times \mathbb{P}^n$. Let $\pi : B \times \mathbb{P}^n \to B$ denote the projection map. Then $\pi(X)$ is closed.*

The proof will be given on Friday and we first talked about some applications and the significance of it.

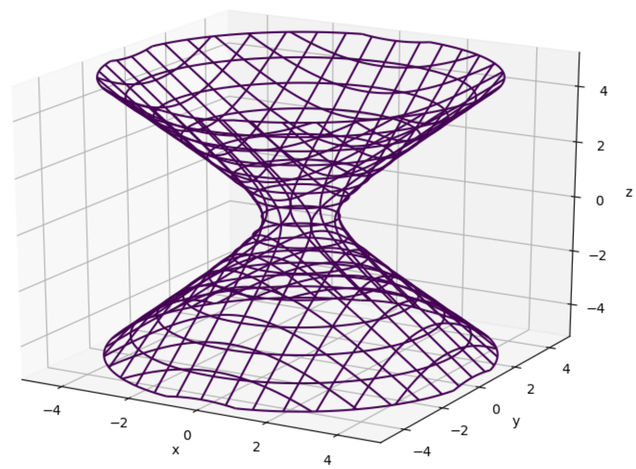


FIGURE 2. An affine piece of $\mathbb{P}^1 \times \mathbb{P}^1$, Segre embedded in $\mathbb{P}^3$

Take B to be $\mathbb{A}^{(m+1)+(n+1)}$ with coordinates $(f_0, \ldots, f_m, g_0, \ldots, g_n)$ and $\mathbb{P}^n = \mathbb{P}^2$ with co-ordinates $[x : y]$. Then we can look at the set

$$V = Z(f_0 x^m + f_1 x^{m-1} y + \cdots + f_m y^m, g_0 x^n + g_1 x^{n-1} y + \cdots + g_n y^n)$$

which is closed in $B \times \mathbb{P}^n$. Hence by our theorem, its projection onto $\mathbb{A}^{(m+1)+(n+1)}$ is closed. If some point, say $(f_0, \ldots, f_m, g_0, \ldots, g_n)$ is in the projection, then it implies that the two polynomials $f_0 x^m + \cdots + f_m y^m$ and $g_0 x^n + \cdots + g_n y^n$ have a common zero and vice versa. Now since it is closed in $\mathbb{A}^{(m+1)+(n+1)}$, it implies that given two homogeneous polynomials f, g in variables x, y and of degree m, n, there exists polynomial equations in the coefficients that determine whether they have a common zero. In fact, the relevant subvariety of $\mathbb{A}^{m+n+2}$ is cut out by a single hypersurface, known as the ***resultant***.

Similarly, one can ask if any number of polynomials in any number of variables have a common root in projective space.

A particularly interesting case is to ask when f, $(\partial f)/(\partial x_1)$, $(\partial f)/(\partial x_2)$, $\ldots$, $(\partial f)/(\partial x_m)$, have a common root – in other words, when $Z(f)$ is singular.

The theorem also implies that we can think of $\mathbb{P}^n$ as a compact set. The following proposition helps us to see why.

Proposition. *Let X be a topological space. Then X is compact if and only if for any other space B, the projection of any closed subset of $B \times X$ into B is closed.*

This is true for arbitrary topological spaces; see Martín Escardó, "Intersections of compactly many open sets are open". At the moment, the best source I can give for this document is Escardo's webpage. See also the discussion at Mathoverflow. We'll make our lives easy by just proving the result for metric spaces.

Proof. First, suppose that X is compact. Let (b_n) be any sequence in $\pi(X)$ with a limit point b. Let b_n to (b_n, x_n) in X. As X is sequentially compact, there is a convergent subsequence $x_{n_k} \to x$. Then (b_{n_k}, x_{n_k}) converges to (b, x) and so b is in the projection, implying that the projection is closed.

Conversely, assume that X has this property but is not compact. Then there exists a sequence (x_n) with no convergent subsequence. Now let $B = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots, 0\}$ and consider the subset $\{(\frac{1}{n}, x_n) \mid n \in \mathbb{N}\}$ of $B \times X$. Then this is closed as the (x_n) have no convergent subsequence. But its projection is just $\{\frac{1}{1}, \ldots, \frac{1}{n}, \ldots\}$ which has a limit point, 0, which is not in the projection. Hence the projection is not closed — a contradiction. □

This proposition also sorts of explain why the projection of the hyperbola $\{xy = 1\}$ in $\mathbb{A}^2$ to $\mathbb{A}^1$ is not closed. The points with x-coordinate approaching 0 have the y-coordinates' escaping to infinity and thus have no convergent subsequence. Hence we are unable to obtain any point with x-coordinate 0, although we can get any point with x-coordinate around it.

Another way $\mathbb{P}^n$ behaves like compact sets is with the following property.

Proposition. *Let X be a compact connected complex manifold and $f : X \to \mathbb{C}$ a holomorphic function. Then f must be constant.*

Proof. If f were not constant, then by connectedness and the open mapping theorem, its image has to be open. But by compactness, it is also compact in $\mathbb{C}$ which cannot be true as there are no open compact non-empty set in $\mathbb{C}$. □

Proposition. *Let X be a closed connected subvariety of $\mathbb{P}^n$ and $f : X \to k$ a regular function. Then f is constant.*

Proof. We may view f as a regular function from $X \to \mathbb{A}^1$ and then as $\mathbb{A}^1$ injects into $\mathbb{P}^1$, we get a regular function $f : X \to \mathbb{P}^1$. Now consider the graph of f, $\Gamma(f)$, which is a subset of $\mathbb{P}^n \times \mathbb{P}^1$. By a homework problem, we know that $\Gamma(f)$ is closed and so its projection to $\mathbb{P}^1$ is closed, which is just the image of f. But the point $\{\infty\}$ is not in it where we view $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ and so the only possible closed sets are finite sets of points. But as the image is connected, the only possibility is the set having exactly one point and so f is constant. □

October 5 : Proof that projective maps are closed. Today we prove the "projective varieties behave like compact things" theorem from last time.

Theorem. Let B be a quasiprojective variety, and let $X \subset B \times \mathbb{P}^n$ be Zariski closed. If $\pi : B \times \mathbb{P}^n \to B$ is the projection onto first coordinate, then $\pi(X) \subset B$ is Zariski closed in B.

We first note that it will suffice to prove this in the case where B is an *affine* variety. Indeed, if $B = \bigcup_\alpha V_\alpha$ where each $V_\alpha \subset B$ is an open set isomorphic to an affine variety, then if $\pi|_{V_\alpha \times \mathbb{P}^n}(X \cap (V_\alpha \times \mathbb{P}^n)) = \pi(X) \cap V_\alpha$ is closed in V_α for all α, it follows that $\pi(X)$ is closed in B (since closedness is a local property, i.e. can be checked on an open cover). Now, we actually can cover B by affine varieties re: the following lemma.

Lemma. Any quasiprojective variety permits a cover by open sets that are isomorphic to affine varieties.

Proof. Suppose $B \subset \mathbb{P}^n$ is a quasiprojective variety. Since $\mathbb{P}^n$ is covered by the standard affine charts $\mathbb{A}^n_{z_i \neq 0}$, we have a cover of B by quasiaffine varieties $B \cap \mathbb{A}^n_{z_i \neq 0}$. So, it suffices to prove any quasiaffine variety is covered by affine varieties. In general, let $V \subset \mathbb{A}^n$ be quasiaffine, and let $X = \overline{V}$ be an affine closed set. Then $Y := X \setminus V$ is closed in X, hence it is the zero set of some $f_1, \cdots, f_n \in \mathcal{O}(X)$. It follows that $V = \bigcup_{i=1}^n X \cap \{f_i \neq 0\}$. Each set $D_X(f_i) := X \cap \{f_i \neq 0\}$ is called a *distinguished open set*, and by a homework problem, each $D_X(f)$ for $f \in \mathcal{O}(X)$ is isomorphic to an affine variety. □

Back to the proof: moving forward, let us assume B is affine and denote by $\mathcal{O}(B)$ the ring of regular functions on B. Again by a homework problem, we know that any Zariski closed subset $X \subset B \times \mathbb{P}^n$ is of the form $X = Z(I)$ where $I \subseteq \mathcal{O}(B)[x_0, \cdots, x_n]$ is a homogeneous ideal. We will study the ring $S(X) := \mathcal{O}(B)[x_0, \cdots, x_n]/I$, the "homogeneous coordinate ring" of X. This ring does not consist of regular functions on X, but its homogeneous ideals are still in correspondence with the closed subsets of X.

In particular, since $\pi : X \to \pi(X)$ is continuous, $\pi^{-1}(b) \cap X$ is closed in X. The corresponding ideal in $S(X)$ is $\mathfrak{m}_b S(X) := \mathfrak{m}_b[x_0, \cdots, x_n]/I$ where $\mathfrak{m}_b \subset \mathcal{O}(B)$ is the maximal ideal of functions vanishing at b; we mean precisely that $\pi^{-1}(b) \cap X = Z(\mathfrak{m}_b S(X))$. By the "projective Nullstellensatz," it follows that $\pi^{-1}(b) \cap X$ is empty if and only if $\mathfrak{m}_b S(X) = S(X)$ or $\mathfrak{m}_b S(X) \supset \langle x_0, \cdots, x_n \rangle^d$ for some $d \geq 0$. Equivalently, $\pi^{-1}(b) \cap X$ is empty if and only if $(S(X)/\mathfrak{m}_b S(X))_d = 0$ for some $d \geq 0$, where $(S(X)/\mathfrak{m}_b S(X))_d$ denotes the d-graded piece of the quotient ring.

To show $\pi(X)$ is closed in B, we should show its complement is open, i.e. that the set of $b \in B$ with $\pi^{-1}(b) \cap X$ empty is open. By the above, we know that if

$$U_d := \{b \in B \,:\, (S(X)/\mathfrak{m}_b S(X))_d = 0\}$$

then $\pi(X)^c = \bigcup_{d \geq 0} U_d$. Thus, it will suffice to show each U_d is open. Here is where the sorcery of Nakayama's Lemma comes into play.

Lemma (Nakayama Statement 2)**.** Suppose R is a ring, M is a finitely generated R-module, and $I \subset R$ is an ideal. Then $IM = M$ if and only if there is some $r \in R$ such that $r \equiv 1 \pmod{I}$ and $rM = 0$.

Proof. The only if direction is the hard one, which you use the standard determinant trick to show. Conversely, if there is $r \in R$ such that $r \equiv 1 \pmod{I}$ and $rM = 0$, then we can write $1 = r + i$ for some $i \in I$, hence $M = (r + i)M = iM \subset IM$ so $IM = M$. □

To apply Nakayama, we think of $S(X)_d$ as a finitely generated $\mathcal{O}(B)$-module. If $\xi \in U_d$, then $0 = (S(X)/\mathfrak{m}_\xi S(X))_d = S(X)_d/\mathfrak{m}_\xi S(X)_d \implies \mathfrak{m}_\xi S(X)_d = S(X)_d$, so the hypothesis of Nakayama holds with $I = \mathfrak{m}_\xi$. Thus, there is some $f \in \mathcal{O}(B)$ such that $f \equiv 1 \pmod{m_\xi}$ and $fS(X)_d = 0$.

Now, note that for any $\tau \in D_B(f)$, since $f(\tau) \neq 0$, there is some $c \in k$ such that $\tilde{f} = cf \equiv 1 \pmod{\mathfrak{m}_\tau}$. Since $\tilde{f}S(X)_d = 0$, the converse of Nakayama's lemma shows $(S(X)/\mathfrak{m}_\tau S(X))_d = 0$; hence $\xi \in D_B(f) \subset U_d$, which shows U_d is open. This completes the proof. □

October 8 : Finite maps. A map of commutative algebras $A \to B$ is called ***finite*** if B is a finitely generated A-module with respect to this map. We also call the corresponding map of affine varieties $\operatorname{MaxSpec} B \to \operatorname{MaxSpec} A$ finite.

Proposition. *The composition of finite maps is finite:*

$$\begin{tikzcd} A \arrow[r, "finite"] \arrow[rr, bend right, dashed, "finite"'] & B \arrow[r, "finite"] & C \end{tikzcd}$$

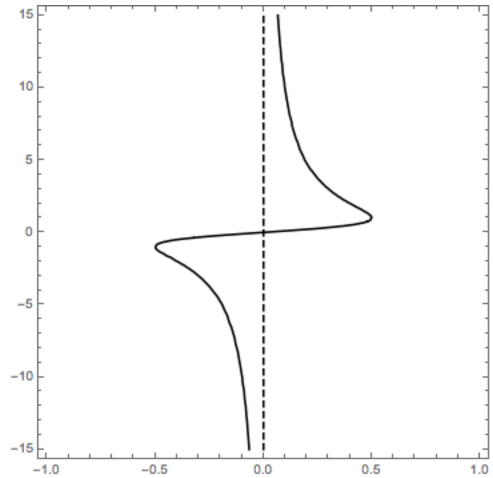


FIGURE 3. The curve $xy^2 - y + x = 0$ and its vertical asymtote

Proof. Let $c_1, \ldots, c_n \in C$ be generators for C as a B-module, and let $b_1, \ldots, b_m \in B$ be generators for B as an A-module. Then $\{b_i c_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ generates C as an A-module, since each $\gamma \in C$ is of the form $\sum_j \beta_j c_j$ for some $\beta_1, \ldots, \beta_n \in B$ and each $\beta_j = \sum_i \alpha_{ij} b_i$ for some $\alpha_{1j}, \ldots, \alpha_{mj} \in A$, so that

$$\gamma = \sum_{j=1}^{n} \beta_j c_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \alpha_{ij} b_i \right) c_j = \sum_{i,j} \alpha_{ij} (b_i c_j).$$

This shows that C is a finitely generated A-module. $\square$

Note that if $A \to B$ is finite, so is $A \otimes_k C \to B \otimes_k C$; if $b_1, \ldots, b_n$ generate B as an A-module, then $b_1 \otimes 1, \ldots, b_n \otimes 1$ generate $B \otimes_k C$ as an $A \otimes_k C$-module. Geometrically, this corresponds to the fact that if $Y \to X$ is finite then $Y \times Z \to X \times Z$ is also finite.

From our proof of the Nullstellensatz, we know that finite maps are closed. A finite map $Y \to X$ is also ***universally closed***, i.e. for every Z, the map $Y \times Z \to X \times Z$ is closed. This follows from the fact that $Y \times Z \to X \times Z$ is finite, as mentioned above.

Not every closed map is universally closed. For example, the curve $C = Z(xy^2 - y + x)$ has a vertical asymptote at $x = 0$. The projection of C onto the x-axis is a closed map, because the image of the whole curve is $\mathbb{A}^1$ (the point $(0, 0)$ maps down to the origin) and the image of any finite set is clearly finite. However, C is not universally closed. Let $g(x, y) = xy - 1$ and let $X \subset C \times \mathbb{A}^1$ be the graph of Γ. Then the projection of X onto $\mathbb{A}^1$ is the range of C, and we see that g is nowhere 0 on C, so the projection of X omits the point 0 and is not closed.

In the context of topological spaces, a map $Y \to *$ is universally closed if and only if the projection $Y \times Z \to Z$ is closed for all Z, which is equivalent to compactness of Y by a previous proposition. More generally, a map $f : X \to Y$ of topological spaces is universally closed if and only if it is proper, meaning that, for $K \subseteq Y$ compact, the preimage $f^{-1}(K)$ is compact.

Proposition. *Finite maps of affine varieties have finite fibers. That is, if $X = \operatorname{MaxSpec}(A)$, $Y = \operatorname{MaxSpec}(B)$, and $f : Y \to X$ is finite with $x \in X$, then $f^{-1}(x)$ is finite.*

Proof. Let $\varphi : A \to B$ be the corresponding algebra map, and let $\mathfrak{m}_x \subset A$ be the maximal ideal corresponding to $x \in X$. Then $Z(\varphi(\mathfrak{m}_x)) \subset Y$ corresponds to the set of maximal ideals in B containing $\varphi(\mathfrak{m}_x)$. If $f(y) = x$, then $\varphi^{-1}(\mathfrak{m}_y) = \mathfrak{m}_x$, so

$$\varphi(\mathfrak{m}_x) = \varphi(\varphi^{-1}(\mathfrak{m}_y)) \subset \mathfrak{m}_y \implies \mathfrak{m}_y \in Z(\varphi(\mathfrak{m}_x)).$$

Conversely if $\mathfrak{m}_y \in Z(\varphi(\mathfrak{m}_x))$, then

$$\varphi(\mathfrak{m}_x) \subset \mathfrak{m}_y \implies \mathfrak{m}_x \subset \varphi^{-1}(\varphi(\mathfrak{m}_x)) \subset \varphi^{-1}(\mathfrak{m}_y)$$

so by maximality, $\mathfrak{m}_x = \varphi^{-1}(\mathfrak{m}_y)$ and thus $f(y) = x$. This shows that $Z(\varphi(\mathfrak{m}_x))$ corresponds to $f^{-1}(x)$, so the regular functions on $f^{-1}(x)$ correspond to $B/I(f^{-1}(x)) = B/\sqrt{\mathfrak{m}_x B}$.

Now consider the sequence

$$B \twoheadrightarrow B/\mathfrak{m}_x B \twoheadrightarrow B/\sqrt{\mathfrak{m}_x B}$$

Note that $B/\mathfrak{m}_x B$ is a finitely generated $A/\mathfrak{m}_x$-module, i.e. a finite-dimensional vector space over $k \cong A/\mathfrak{m}_x$, since if $b_1, \dots, b_n$ generate B as an A-module then $b_1/\mathfrak{m}_x, \dots, b_n/\mathfrak{m}_x$ generate $B/\mathfrak{m}_x B$ as an $A/\mathfrak{m}_x$-module. Since $B/\sqrt{\mathfrak{m}_x B}$ is a quotient of $B/\mathfrak{m}_x B$, it is also a finite-dimensional vector space over k: say $\dim_k B/\sqrt{\mathfrak{m}_x B} = d$.

Suppose $|f^{-1}(x)| = \infty$. Let $m \in \mathbb{N}$ and choose any finite set $S \subset f^{-1}(x)$ with $|S| = m$. Then S is an affine variety and $B/\sqrt{\mathfrak{m}_x B} \twoheadrightarrow \mathcal{O}_S$ is a surjection, and $\mathcal{O}_S \cong \prod_{i=1}^m k$ so it follows that $\dim_k \mathcal{O}_S = m$ and then $\dim_k B/\sqrt{\mathfrak{m}_x B} \geq m$. This then implies $\dim_k B/\sqrt{\mathfrak{m}_x B} = \infty$, contradicting our earlier statement, so $f^{-1}(x)$ is finite. (In fact $|f^{-1}(x)| = d$, since $f^{-1}(x) \cong \prod_{i=1}^{|f^{-1}(x)|} k$ implies $\dim_k B/\sqrt{\mathfrak{m}_x B} = |f^{-1}(x)|$.) $\square$

Theorem. *A map $Y \to X$ of (affine) varieties is finite if and only if it has finite fibers and is universally closed.*

This theorem seems to be hard, for unclear reasons. It appears as Theorem 29.6.2 in Ravi Vakil's The Rising Sea – Foundations of Algebraic Geometry and the proof invokes some serious machinery, such as Zariski's Main Theorem. Prof. Speyer would like to know if anyone knows a simple proof.

We now want to define finite maps between non-affine varieties. We need the following theorem/definitions:

Theorem. *Let $Y \xrightarrow{f} X$ be a regular map of quasiprojective varieties. The following are equivalent:*

- *For all affine $U \subset X$, $f^{-1}(U)$ is affine and $f^{-1}(U) \to U$ is finite;*
- *There exists an affine cover $\{U_i\}$ of X such that $f^{-1}(U_i)$ is affine and $f^{-1}(U_i) \to U_i$ is finite.*

If these conditions hold, we call f **finite**.

Theorem. *Let $Y \xrightarrow{f} X$ be a regular map of quasiprojective varieties. The following are equivalent:*

- *For all affine $U \subset X$, $f^{-1}(U)$ is affine;*
- *There exists an affine cover $\{U_i\}$ of X such that $f^{-1}(U_i)$ is affine.*

If these conditions hold, we call f **affine**.

These theorems seem somewhat hard. A reference for the former is Proposition 8.2.1 in Milne's Algebraic Geometry . For the latter, see Proposition 7.3.4 in Vakil. Shafarevich, to

Professor Speyer's annoyance, only proves the weaker statement that, if Y and X are affine, and X has an affine cover U_i such that $f^{-1}(U_i)$ is affine and $f^{-1}(U_i) \to U_i$ is finite, then $\mathcal{O}_Y$ is a finite $\mathcal{O}_X$ module. (Theorem 5 in Chapter I.5.3.)

Here's one easy case: if U and V are affine, $f : V \to U$ is a regular map and h is regular on U, then

$$f^{-1}(D(h)) = D(f^*h).$$

Also, if $\mathcal{O}_V$ is a finitely generated $\mathcal{O}_U$-module, then $(f^*h)^{-1}\mathcal{O}_V$ is a finitely generated $f^{-1}\mathcal{O}_U$-module.

Remark. At this point someone asked whether every affine open subset of an affine variety is a hypersurface complement. Using local or sheaf cohomology, one can show the following result:

Proposition. *Let Y be irreducible, $X \subset Y$ closed, Y affine, with $Y - X$ affine. Then X has pure codimension 1.*

Another question is whether every affine open subset is a distinguished open subset, which is false. For example, let Y be an elliptic curve in $\mathbb{A}^2$, like $y^2 = x(x-1)(x-3)$. Let $X = \{p\}$, where p is not torsion in the group law on Y. Then $Y - X$ is affine, but is not a distinguished open.

It is still true in this more general context of quasiprojective varieties that

- Finite maps are universally closed
- Finite maps have finite fibers

because both of these statements are local on the target. Also, a composition of finite maps is finite; again, this is checkable on a cover of the target.

Finally, we remark on Noether's normalization lemma:

Lemma (Noether's Normalization Lemma (v1))**.** *Let $f \in k[x_1, \ldots, x_n]$ where k is an infinite field, with $f \neq 0$. Then there exists a linear change of coordinates on $\mathbb{A}^n$ such that*

$$f = cx_n^d + (\textit{lower order terms in } x_n)$$

where $c \in k$, $c \neq 0$, and $d = \deg(f)$. In such a coordinate system, $Z(f) \to \mathbb{A}^{n-1}$ is finite.

Embed $\mathbb{A}^n$ in $\mathbb{P}^n$ via $(x_1, \ldots, x_n) \mapsto [x_1 : x_2 : \cdots : x_n : 1]$ and consider $Z(\tilde{f})$ in $\mathbb{P}^n$, where $\tilde{f} \in k[x_1, \ldots, x_n, x_{n+1}]$ is the homogenization of f. The condition

$$f = cx_n^d + (\text{lower order in } x_n)$$

means that $Z(f) \not\ni [0 : \cdots : 0 : 1 : 0]$, since each of the terms with lower order in x_n have a term x_i $(i \neq n)$ and therefore vanish.

Note that the map $\mathbb{P}^n - \{[0 : \cdots : 0 : 1 : 0]\} \to \mathbb{P}^{n-1}$ deleting the n^{th} coordinate is regular, so if $Z(\tilde{f}) \not\ni [0 : \cdots : 0 : 1]$ (e.g. if the condition above holds), we have a regular map $Z(\tilde{f}) \to \mathbb{P}^{n-1}$ which is closed; see Figure 4 and Figure 5.

October 10: An important lemma. Let X and Y be irreducible affine varieties, $f : Y \to X$ a regular map with dense image. Let $X = \operatorname{MaxSpec} A$ and $Y = \operatorname{MaxSpec} B$. The aim of today, which was constructed as a sequence of problems, was to show that there is a nonempty open subset U of X contained in the image of Y.

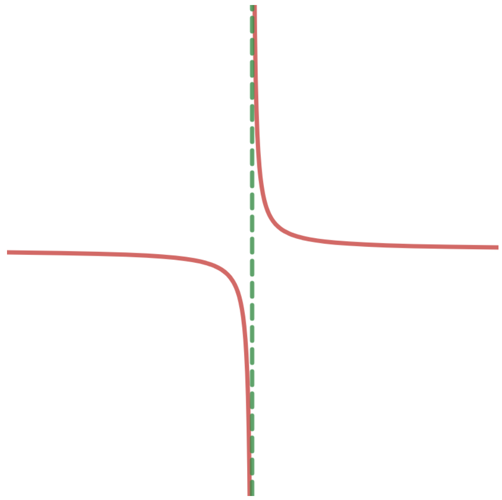


FIGURE 4. The projection of the hyperbola onto the horizontal axis is not closed and therefore not finite.

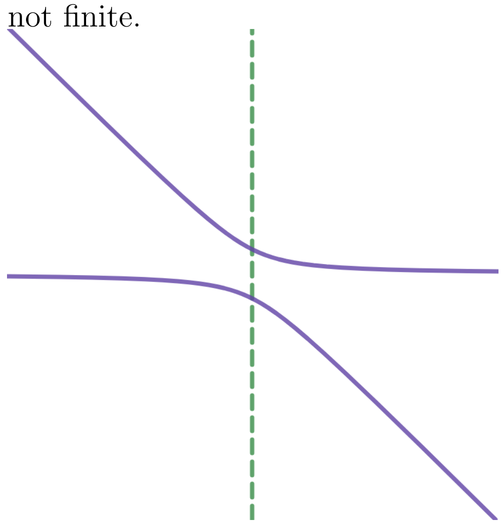


FIGURE 5. The projection of this skewed hyperbola onto the horizontal axis is finite.

Note that this is a way in which regular maps are nicer than maps of manifolds: Take Y to be $\mathbb{R}$ and X to be the torus $\mathbb{R}^2/\mathbb{Z}^2$. Let $f : Y \to X$ be the map $f(y) = (y, \sqrt{2}y) \bmod \mathbb{Z}^2$. Then $f(Y)$ is dense in X, but contains no nonempty open set.

Problem. Show that A and B are domains and A injects into B.

Proof. Since $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are irreducible, the ideals $I(X) \subset k[x_1, \ldots, x_n]$ and $I(Y) \subset k[y_1, \ldots, y_m]$ are prime, so the ring of regular functions $A = \mathcal{O}_X = k[x_1, \ldots, x_n]/I(X)$ and $B = \mathcal{O}_Y = k[y_1, \ldots, y_m]/I(Y)$ are domains.

The regular map $f : Y \to X$ induces a ring homomorphism $f^* : A \to B$ by $p \mapsto p \circ f$. Since $f(Y)$ is dense in X, $X = \overline{f(Y)} = Z(I(f(Y)))$, hence $I(X) = I(Z(I(f(Y)))) = I(f(Y))$. This means regular functions on X vanishing on $f(Y)$ vanish everywhere on X. Therefore if $f^*(p) = p \circ f = 0$, then p vanishes on the image of f, hence vanishes on X, i.e. $p = 0$ in $\mathcal{O}_X$. This shows injectivity of $f^* : A \to B$. $\square$

Put $K = \operatorname{Frac} A$ and $L = \operatorname{Frac} B$. Let $y_1, \ldots, y_r \in B$ be a transcendence basis for L over K, so we have $A \subset A[y_1, \ldots, y_r] \subset B$ and every element in B is algebraic over $K(y_1, \ldots, y_r)$. Geometrically we can factor f as

$$Y \xrightarrow{g} X \times \mathbb{A}^r \xrightarrow{h} X$$

Let $z_1, z_2, \ldots, z_s$ generate B as a k-algebra. Let z_i satisfy the polynomial $z_i^{N_i} + \sum_{j=0}^{N_i-1} a_{ij} z_i^j = 0$ where $a_{ij} \in \operatorname{Frac}(A[y_1, \ldots, y_r])$. Write $a_{ij} = p_{ij}/q_{ij}$ with p_{ij} and $q_{ij} \in A[y_1, \ldots, y_r]$, and put $Q = \prod_{i=1}^{s} \prod_{j=0}^{N_i-1} q_{ij}$.

Problem. Show that $Q^{-1}B$ is finite over $Q^{-1}A[y_1, \ldots, y_r]$.

Proof. We'll show that the images of monomials $\prod_i z_i^{e_i}$ for $0 \leq e_i < N_i$ in $Q^{-1}B$ generate $Q^{-1}B$ as a $Q^{-1}A[y_1, \ldots, y_r]$-module. Since each z_i satisfies $z_i^{N_i} = \sum_{j=0}^{N_i-1} \frac{p_{ij}}{q_{ij}} z_i^j$, the image of z_i in $Q^{-1}B$ satisfies

$$\begin{aligned}
\frac{{z_i}^{N_i}}{1} &= -\frac{\sum_{j=0}^{N_i-1} \frac{p_{ij}}{q_{ij}} z_i^j}{1} \\
&= -\frac{Q(\sum_{j=0}^{N_i-1} \frac{p_{ij}}{q_{ij}} z_i^j)}{Q} \\
&= -\frac{\sum_{j=0}^{N_i-1} t_{ij} z_i^j}{Q}
\end{aligned}$$

where $t_{ij} \in A[y_1, \ldots, y_r]$. Therefore the images of $z_i^{N_i}$ and hence z_i^N for any $N \geq N_i$ in $Q^{-1}B$ is generated by the images of $z_i, z_i^2, \ldots, z_i^{N_i-1}$ over $Q^{-1}A[y_1, \ldots, y_r]$.

Let $\frac{b}{Q^k}$ be any element in $Q^{-1}B$, $b \in B$. Since $z_1, \ldots, z_s$ generate B as a k-algebra, $b = \sum_I \alpha_I z^I$ for some $\alpha_I \in k$, where $I = (i_1, \ldots, i_s)$ is a multi-index and $z^I = z_1^{i_1} \ldots z_s^{i_s}$. By what we showed earlier, we can replace each z_i^N by an $A[y_1, \ldots, y_r]$-linear combination of $z_i, z_i^2, \ldots, z_i^{N_i-1}$ and possibly changing the exponent k of Q. In other words, we may assume that (1) $i_j \leq N_i - 1$ for all $j = 1, 2, \ldots, s$ and multi-index I, and (2) $\alpha_I \in A[y_1, \ldots, y_r]$. This shows that $Q^{-1}B$ is finitely generated over $Q^{-1}A[y_1, \ldots, y_r]$ by images of monomials as stated. $\square$

Problem. Show that $g(Y)$ contains the distinguished open $D(Q)$.

Proof. We have $f^{-1}(D(Q)) = D(f^*Q)$ essentially by definition. The map $D(f^*Q) \to D(Q)$ corresponds to the map of algebras $Q^{-1}A \to Q^{-1}B$. So $f(D(f^*Q)) = f(Y) \cap D(Q)$ is closed in $D(Q)$. But also, $f(Y)$ is dense in X and $D(Q)$ is open in X, so $f(Y) \cap D(Q)$ is dense in X. Combining these two facts, $f(Y) \cap D(Q) = D(Q)$, as desired. $\square$

Problem. Show that, for any nonzero $Q \in A[y_1, \ldots, y_r]$, the projection $\pi(D(Q))$ contains a nonempty open subset of X.

Proof. Write $Q = \sum a_{i_1 \cdots i_r} y_1^{i_1} \cdots y_r^{i_r}$, where the $a_{i_1 \cdots i_r}$ are in A. Let x be any point of X. As long as any of the $a_i(x)$ are nonzero, the polynomial $\sum a_{i_1 \cdots i_r}(x) y_1^{i_1} \cdots y_r^{i_r}$ is not identically zero as a function of the y_j. So, as long as any $a_i(x)$ is nonzero, we have $x \in \pi(D(Q))$. We have shown that $\pi(D(Q)) = \bigcup_{i_1 \cdots i_r} D(a_{i_1 \cdots i_r})$. $\square$

October 12 : Chevalley's Theorem. The goal of this day was to prove Chevalley's theorem, which shows that the images of regular maps cannot be too terrible. We proceeded by a series of problems:

Theorem (Chevalley)**.** If Y is constructible in $\mathbb{A}^n$ and $f : \mathbb{A}^n \to \mathbb{A}^m$ is regular, then $f(Y)$ is constructible.

Before we prove Chevalley's theorem, we first introduce the concept of constructible subsets.

Definition (Constructible Subsets)**.** Let T be a topological space. A subset X of T is called ***constructible*** if it can be built from finitely many open and closed sets using the operations of union, intersection and complement.

Example. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the map $(x, y) \to (x, xy)$. We have $f(\mathbb{A}^2) = Z(x)^c \cup Z(y)$, which is construcible.

Problem. Let C be a constructible subset of a topological space T. Show that we can write C in the form $\bigcup_{i=1}^m \bigcap_{j=1}^{n_i} X_{ij}$, where each X_{ij} is either open or closed.

Proof. Since any open or closed subset is in this form and any constructible subset is obtained by finitely many union, intersection or complement operations on sets in this form, it suffice to show that if C_1, C_2 can be written in this form, then so can $C_1 \cup C_2$, $C_1 \cap C_2$ and C_1^c. Write $C_1 = \bigcup_{i=1}^M \bigcap_{j=1}^{m_i} X_{ij}$, $C_2 = \bigcup_{l=1}^N \bigcap_{k=1}^{n_l} Y_{kl}$, where X_{ij}, Y_{kl} are either open or closed subsets of T, we have

$$C_1 \cup C_2 = \left(\bigcup_{i=1}^{M} \bigcap_{j=1}^{m_i} X_{ij} \right) \bigcup \left(\bigcup_{l=1}^{N} \bigcap_{k=1}^{n_l} Y_{kl} \right);$$

$$C_1 \cap C_2 = \bigcup_{1 \le i \le M,\ 1 \le l \le N} \left(\bigcap_{1 \le j \le m_i,\ 1 \le k \le n_l} X_{ij} \cap Y_{kl} \right);$$

$$C_1^c = \bigcap_{i=1}^{M} \bigcup_{j=1}^{m_i} X_{ij}^c = \bigcup_{1 \le j_i \le m_i} \bigcap_{i=1}^{M} X_{ij_i}^c.$$

Therefore, any constructible subset of T can be written in the form $\bigcup_{i=1}^m \bigcap_{j=1}^{m_i} X_{ij}$, where each X_{ij} is either open or closed. □

Problem. Show further that we can write C in the form $C = \bigcup_{i=1}^m (K_i \cap U_i)$ where each K_i is closed and each U_i is open.

Proof. This follows if we write

$$\bigcap_{j=1}^{m_i} X_{ij} = \left(\bigcap_{1 \le j \le m_i,\ X_{ij} \text{ is open}} X_{ij} \right) \bigcap \left(\bigcap_{1 \le j \le m_i,\ X_{ij} \text{ is closed}} X_{ij} \right).$$

□

Problem. Show that every constructible set is a union of affine varieties.

Proof. Write $U_i^c = Z(f_1, ..., f_{t_i})$, then $U_i = \bigcup_{j=1}^{t_i} D(f_j)$, where $D(f_j) := \{x | f_j(x) \neq 0\}$ are distinguished open subsets. Since $D(f_j) \cap K_i$ are affine varieties, we have every constructible set is a finite union of affine varieties. □

According to our discussion above, we only need to work with affine varieties since any constructible set is a finite union of affine varieties. Let Y be an affine variety and $f : Y \to \mathbb{A}^m$ a regular map. Let $Y = \bigcup_{i=1}^r Y_r$ be the decomposition of Y into irreducible components. Since $f(Y) = \bigcup_{i=1}^r f(Y_r)$, if all the $f(Y_i)$ are constructible, then $f(Y)$ is constructible.

Problem. Let Y be an irreducible affine variety and $f : Y \to \mathbb{A}^m$ a regular map. Let $X = \overline{f(Y)}$. By the lemma proved last time there is a non-empty U open in X such that $U \subset f(Y)$. Put $Y' = Y - f^{-1}(U)$. Show that if $f(Y')$ is constructible, then $f(Y)$ is constructible.

Proof. This follows from the fact that $f(Y') \cup U = f(Y)$. □

Problem. Let Y be an affine variety and $f : Y \to \mathbb{A}^m$ a regular map. Show that $f(Y)$ is constructible.

Proof. We prove by contradiction. If $f(Y)$ is not constructible, we can construct a infinite descending chain of closed subsets of Y as follows:

Let Y_1 be one of its irreducible components such that $f(Y_1)$ is not constructible. (If such Y_1 does not exist then the image of every irreducible component of Y is constructible, which implies that $f(Y)$ is constructible and hence contradicts with our assumption.) We construct irreducible closed sets Y_i such that $f(Y_i)$ are not constructible, inductively. Apply the lemma we proved last time, there exsits an open subset $U_i \subset \overline{f(Y_i)}$ such that $U_i \subset f(Y_i)$. Let $Y_i' = Y_i - f^{-1}(U_i)$, by regularity of f and the previous problem, we have Y_i' is closed and $f(Y_i')$ is not constuctible. Then we define Y_{i+1} to be one of the irreducible components of Y_i' such that $f(Y_{i+1})$ is not constuctible.

Assume $Y \subset \mathbb{A}^n$. From our definition of $\{Y_i\}_{i\in\mathbb{Z}_+}$, $Y_{i+1} \subsetneq Y_i$, which implies $I(Y_i) \subsetneq I(Y_{i+1}) \subset k[x_1, ..., x_n]$. This contradicts to the Hilbert Basis theorem, that is, $k[x_1, ..., x_n]$ is Noetherian. □

Chevalley's theorem follows from our last problem and the fact that any constructible set $Y = \bigcup_{i=1}^m Y_i$ where Y_i is affine and $f(Y) = \bigcup_{i=1}^m f(Y_i)$.

October 17: Noether normalization, start of dimension theory. Suppose $X \subseteq \mathbb{A}^n$ is Zariski closed.

Lemma. *(Noether Normalization Lemma, first version) If $X \neq \mathbb{A}^n$ $(n > 0)$, then there is a linear map $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ such that $\pi : X \to \mathbb{A}^{n-1}$ is finite.*

So $\pi(X) \subseteq \mathbb{A}^{n-1}$ is Zariski closed. If $\pi(X) \neq \mathbb{A}^{n-1}$, we can repeat this argument to see that there is a linear map $\pi' : \mathbb{A}^{n-1} \to \mathbb{A}^{n-2}$ such that $\pi' : \pi(X) \to \mathbb{A}^{n-2}$ is finite. Continuing in this manner:

Lemma. *(Noether Normalization Lemma, second version) If $X \neq \emptyset$, then there exists a nonnegative integer d and $\pi : \mathbb{A}^n \to \mathbb{A}^d$ such that $\pi|_X : X \to \mathbb{A}^d$ is finite and surjective.*

Correspondingly, suppose V is a finite dimensional k-vector space, $X \subseteq \mathbb{P}(V)$, $X \neq \emptyset$, then there is a surjective linear map $\pi : V \to W$ such that $X \neq \mathbb{P}(V) - \mathbb{P}(\operatorname{Ker}\pi)$ and $\pi : X \to \mathbb{P}(W)$ is finite and surjective.

We would consider: could we have an affine variety X such that the maps $X \to \mathbb{A}^{d_1}$ and $X \to \mathbb{A}^{d_2}$ are both finite and surjective? The answer is NO. To see why, we remember the notion of transcendence degree:

Let L/K be a field extension. An s-tuple of elements $(y_1, ..., y_s)$ in L are called

- *algebraically independent* if there is no polynomial relation among $(y_1, ..., y_s)$ with coefficients in K;
- *algebraically spanning* if $\forall z \in L$, z is algebraic over $K[y_1, ..., y_s] \subseteq L$;
- *a transcendence basis* if $(y_1, ..., y_s)$ is both algebraically independent and algebraically spanning.

Conceptually, these notions act like the ones in linear algebra:

- All transcendence basis have the same size, which is called the *transcendence degree of* L/K;
- Any algebraic independent set can be extended to a transcendence basis;
- Any algebraic spanning set contains a transcendence basis.

Remark. If you've never seen transcendence degree before, you might like to look at Problem 5, Problem Set 9 at Professor Speyers 594 webpage. I actually wrote solutions!

Remark. For those who know the terminology, transcendence bases form a matroid. (Further details not given in class:) A matroid of this form is called *algebraic*. Not all matroids are algebraic, see Ingleton and Main, *Non-Algebraic Matroids exist*, (Bull. of the London Math. Soc., 1975).

Remark. (Remark not made in class:) If K has characteristic zero, then there is a finite dimensional L vector space, called $\Omega^1_{L/K}$, and a map $d : L \to \Omega^1_{L/K}$, such that $(y_1, \ldots, y_s)$ are (algebraically independent/algebraically spanning) if and only if $(dy_1, \ldots, dy_s)$ are (linearly independent/spanning). We will learn about this in a few weeks. For K of characteristic p, the vector space $\Omega^1_{L/K}$ still exists and has many other good properties, but this property does not hold and there is no replacement vector space V and map $d : L \to V$ to fix this. See Lindström, *The non-Pappus matroid is algebraic*, (Ars. Combin. 1983).

Let $X = \operatorname{MaxSpec} R$ be irreducible and affine. Let $\pi : X \to \mathbb{A}^d$ be finite and surjective, we have $k[y_1, ..., y_d]$ is injective since π has dense image. So $k[y_1, ..., y_d] \subseteq R$ and $k(y_1, ..., y_d) \subseteq Frac(R)$ is a finite field extension. So the transcendence degree of $Frac(R/k)$ is d.

Lemma. *(On the problem set) If X is irreducible and affine, $U \subseteq X$ is nonempty, affine and open, then $Frac(\mathcal{O}_U) = Frac(\mathcal{O}_X)$.*

Corollary. *If X is an irreducible quasiprojective variety, U, V are affine, open, nonempty subsets, then $Frac(\mathcal{O}_U) \cong Frac(\mathcal{O}_V)$. We call it $Frac\ X$.*

For irreducible X, we will define the dimension of X to be the transcendence degree of $Frac\ X/k$.

In a reducible space, the different components have may have different dimensions. For example, $Z(xz, yz) = Z(z) \cup Z(x, y) \subset \mathbb{A}^3$ is the union of a 2-plane and a line. If $X = Y_1 \cup ... \cup Y_r$ where Y_i's are irreducible components of X, set $\dim X = \max\ \dim Y_j$. We say that X is *pure dimensional* if $\dim Y_j = d$ for all j. So the example is not pure dimensional.

We have the following easy consequences:

- If $X \subseteq Y$, $X \neq \emptyset$, then $\dim X \leqslant \dim Y$;
- Finite surjective maps preserve dimension;
- If $f : X \to Y$ has dense image, $Y \neq \emptyset$, then $\dim X > \dim Y$;
- $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$; if $f \in k[x_1, ..., x_n]$, $x \notin k$, then $Z(f) \subset \mathbb{A}^n$ has dimension $n-1$ (in fact this is pure dimension since $k[x_1, ..., x_n]$ is a UFD);

- If $f = f_1 ... f_r$ is an irreducible factorization then $Z(f) = \bigcup_i Z(f_i)$ is an irreducible decomposition.

The following are not straightforward, we will work on them through the next classes:

- If $X \subsetneqq Y$, $X \neq \emptyset$, Y is irreducible, then $\dim X < \dim Y$, so any chain $X_0 \subsetneqq X_1 \subsetneqq ... \subsetneqq X_l \subseteq Y$ of irreducible subvarieties of Y has $l \leqslant \dim Y$;
- If X, Y are irreducible, $X \subseteq Y$, then there exists an irreducible chain $X = Z_0 \subsetneqq Z_1 \subsetneqq ... \subsetneqq Z_l = Y$, $l = \dim Y - \dim X$;
- If X is irreducible, $f \in \mathcal{O}_U$, $Z(f) \neq \emptyset$ or X, then $Z(f)$ is of pure dimension $\dim X - 1$;
- We want a result that roughly says that, if $f : X \to Y$ is surjective, then most fibers of f have dimension $\dim X - \dim Y$.

October 19: Lemmas about polynomials over UFDs. We are going to go through a bunch of commutative algebra lemmas about the behavior of polynomials over UFDs, which will be useful at several points in the course. Our immediate payoff will be the following lemma from Shafarevich I.6.2:

Lemma. If A is a UFD, f, $g \in A$ are relatively prime, $A \subseteq B$, B is a domain and finite A-module, and $h \in B$, then if $f \mid gh$, there's $k \in \mathbb{N}$ so that $f \mid h^k$.

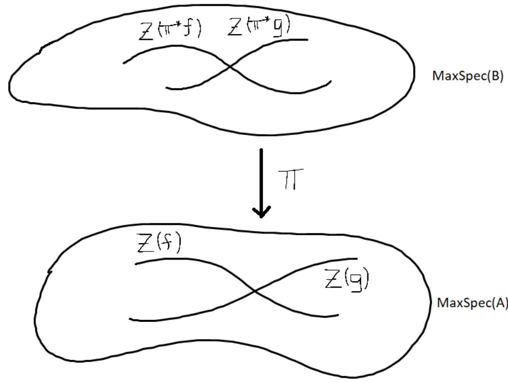
WARNING: In the end, Shafarevich's proof turned out to be much harder to flesh out than it should have been, and Professor Speyer has found a route he likes better. This has the effect that this particular lemma is no longer crucial. However, many of the other lemmas proved this day are still useful and relevant. In particular, the lemma proved this day which turns out to be most useful is that, if $A \subset B$ are domains, with A a UFD and B finite over A, and $\theta \in B$, then the minimal polynomial of θ over $\mathrm{Frac}(A)$ has coefficients in A.

We make some remarks:

Remark. This lemma essentially says that if f, g are relatively prime in A, then they act relatively prime in the larger ring, B. Also note that the lemma is easy to prove if A is a PID, since we have $x, y \in A$ so that $fx + gy = 1$. Multiplying both sides by h, we get $fxh + gyh = h$. Since f divides fxh and gyh ($f \mid gh$ by assumption), $f \mid h$ as well (and we don't even need a larger power of h).

Remark. Karthik noted that the conclusion of the lemma is similar to the ideal (f) being primary in B. Certainly if (f) is primary, then the conclusion of the lemma holds. However, the hypotheses of the theorem don't force (f) to be primary, or even force $\sqrt{(f)}$ to be prime, so it is unclear what to do with this.

Remark. Here is an intuitive argument for the lemma. Making it precise requires us to justify our intuitions about dimension. Let $\pi : \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$ be the map induced by the inclusion. Then $\pi^*(f) \mid \pi^*(g)h$ implies $Z(\pi^*(f)) \subseteq Z(\pi^*(g)h) = Z(\pi^*(g)) \cup Z(h)$. So at every point of $Z(\pi^*(f))$, either $\pi^*(g) = 0$ or $h = 0$, so in particular, $h = 0$ on $Z(\pi^*(f)) - Z(\pi^*(g))$. Now $Z(f)$ should be codimension 1 in X and, since π is finite, the preimage $Z(\pi^*(f))$ should be codimension 1 as well. The condition that f and g are relatively prime means that $Z(f) \cap Z(g)$ should be codimension 2, and likewise for $Z(\pi^* f) \cap Z(\pi^* g)$. So $Z(\pi^*(f)) - Z(\pi^*(g))$ should be dense in $Z(\pi^* f)$ and thus we should have $h = 0$ on $Z(\pi^* f)$. By the Nullstellansatz, this means that $h^k \in (f)$ for some k.



Now, the series of lemmas. For the remainder of this class, A will be a UFD and $K = \mathrm{Frac}(A)$.

Definition. Let $a(x) = \sum_{i=1}^{d} a_i x^i \in A[x]$. We call $a(x)$ **primitive** if $\gcd(a_0, \ldots, a_d) = 1$.

Lemma (Gauss's Lemma)**.** The product of primitive polynomials is primitive.

Proof. Let $a(x) = \sum a_i x^i$, $b(x) = \sum b_j x^j$ be primitive polynomials, and let $c(x) = a(x)b(x) = \sum c_k x^k$ be their product. To show that $c(x)$ is primitive, we must show that for any prime $p \in A$, there is some $c_k \not\equiv 0 \bmod p$, or equivalently, $\overline{c(x)} \neq 0 \in (A/p)[x]$.

Since p is prime, A/p is a domain, so $(A/p)[x]$ is also a domain. Since $a(x)$ and $b(x)$ are primitive, they are both non-zero in $(A/p)[x]$. $(A/p)[x]$ a domain then implies $\overline{c(x)} = \overline{a(x)b(x)} \neq 0 \in (A/p)[x]$. □

Corollary. *If $c(x) \in A[x]$ factors in $K[x]$, then $c(x)$ factors in $A[x]$.*

Proof. Let $c(x) = a(x)b(x)$ with $a(x)$ and $b(x) \in K[x]$. Take $\alpha, \beta \in K$ so that $a(x) = \alpha a_0(x)$ and $b(x) = \beta b_0(x)$ with $a_0(x), b_0(x) \in A[x]$ are both primitive. Then

$$c(x) = (\alpha\beta) a_0(x) b_0(x).$$

If $(\alpha\beta) \notin A$, then there's some prime in the denominator of $(\alpha\beta)$, which cannot divide all the coefficients of $a_0(x)b_0(x)$ since the product $a_0(x)b_0(x)$ is primitive by Gauss's lemma. But $c(x) \in A[x]$ by assumption, so this is impossible. Therefore, $c(x)$ factors in $A[x]$. □

Corollary. *Let $f(x) \in A[x]$ be a monic polynomial. Let $f(x) = \prod g_j(x)$ be a factorization of $f(x)$ into monic irreducible polynomials in $k[x]$. Then all $g_j(x)$ are in $A[x]$.*

Remark. The conclusion of the previous corollary holds as long as A is integrally closed in $\mathrm{Frac}(A)$.

Corollary (Rational Root Theorem)**.** *If $f(x) = \sum f_j x^j \in A[x]$, $p/q \in k$ (written in lowest terms, so that p, q have no common divisors), with $f(p/q) = 0$, then $q \mid f_n$ and $p \mid f_0$.*

Proof. $f(p/q) = 0$ means $f(x) = (x - p/q)g(x)$ in $k[x]$. We can then rewrite the factorization as $(qx - p)\overline{g}(x)$. By the previous corollary, $\overline{g}(x) \in A[x]$. Then $f_n = q\overline{g}_{n-1}$ and $f_0 = p\overline{g}_0$, proving the claim. □

Corollary. *UFD's are integrally closed in their fraction field.*

Proof. We need to show that any element of $k = \mathrm{Frac}(A)$ that satisfies a monic polynomial with coefficients in A is itself an element of A. Let $p/q \in k$ be a root of $x^n + f_{n-1}x^{n-1} + \ldots + f_0$. By the previous corollary, $q \mid 1$ in A, so $1/q \in A$, hence $p/q \in A$. □

Corollary. *If A is a UFD, so is $A[x]$.*

Proof Sketch. First note that if $f(x)$ is irreducible in $A[x]$, then either

(1) $f(x) \in A$ and is irreducible in A.
(2) $f(x)$ is primitive and irreducible in $K[x]$. (If $f(x)$ is not primitive, it factors as $pg(x)$ for $p \in A$. If $f(x)$ is not irreducible in $K[x]$, then it factors in $A[x]$ by the previous lemma.)

Now suppose for a contradiction that

$$\prod p_i \prod f_j(x) = \prod q_k \prod g_l(x)$$

where p_i, $q_k \in A$ are irreducible and $f_j(x)$, $g_l(x)$ are irreducible in $k[x]$ and primitive. Now since $k[x]$ is a UFD, the f_j are a rearrangement of g_l, up to an element of K^*, which since the f_i, g_l are primitive is actually up to an element of A^*.

By Gauss's lemma, the products $\prod f_j$ and $\prod g_l$ are primitive and differ by a unit of A^*. Thus, $\prod p_i$ and $\prod q_k$ differ by a unit of A^* and we can apply unique factorization in A. □

In particular, $\mathbb{Z}[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_n]$ are UFD's.

Now we are ready to prove the main lemma.

Lemma. Let A be a UFD, $f, g \in A$ relatively prime, $A \subseteq B$, B a domain and a (module) finite extension of A, and $h \in B$. If $f \mid gh$, then for some $k \in \mathbb{N}$, $f \mid h^k$.

Proof. Let $u \in B$ so that $gh = fu$. Since $A \subseteq B$ is module finite, u is integral over A, hence satisfies a monic polynomial with coefficients in A. Additionally, as an element of $\mathrm{Frac}(B)$, u has a minimal polynomial over $\mathrm{Frac}(A)$. Because A is a UFD, the minimal polynomial of u divides the monic polynomial given by the integrality of u, so u's minimal monic polynomial is in $A[x]$.

Let the minimal monic polynomial of u be:

$$T^n + a_{n-1}T^{n-1} + \cdots + a_0$$

$h = (f/g)u$, so the minimal polynomial of h is:

$$T^n + \frac{f}{g}a_{n-1}T^{n-1} + \cdots + \frac{f^n}{g^n}a_0$$

Since h is also in B (and thus integral over A), the coefficients $(f/g)^j a_{n-j}$ must also be in A, so $g^j \mid a_{n-1}$ in A because f, g are relatively prime. Therefore,

$$h^n + f\frac{a_{n-1}}{g}h^{n-1} + \cdots + f^n\frac{a_0}{g^n} = 0 \implies h^n = -f\left(\frac{a_{n-1}}{g}h^{n-1} + \cdots + f^{n-1}\frac{a_0}{g^n}\right)$$

So $f \mid h^n$. □

October 22 : Krull's Principal Ideal Theorem – Failed Attempt. The aim of this day was to prove Krull's principal ideal, which comes in both affine and projective versions:

Theorem (Krull's principal ideal theorem, affine version)**.** Let Y be an irreducible affine variety of dimension d and let θ be a polynomial with $Y \not\subseteq Z(\theta)$. Then every irreducible component of $Z(\theta) \cap Y$ has dimension $d-1$.

Theorem (Krull's principal ideal theorem, projective version)**.** Let Y be an irreducible projective variety of dimension d and let θ be a homogenous polynomial with $Y \not\subseteq Z(\theta)$. Then every irreducible component of $Z(\theta) \cap Y$ has dimension $d-1$.

It is easy to reduce the projective version to the affine version and Professor Speyer tried to do the proof purely in the affine world. Shavararevich does a complicated shuffle where he reduces the affine case to the projective case, and then reduces the projective case back to a special case of the affine case. Unfortunately, this shuffle seems to be necessary for Shavarevich's approach (which means that Professor Speyer no longer likes this approach so much).

I am going to omit notes from this day and try a better route the next day.

October 24: Krull's Principal Ideal Theorem – Take Two. The main objective today is to prove Krull's principal ideal theorem. The argument here is drawn from the proof Theorem 3.42 in Milne's notes , Section 3.m. Milne credits it to Tate.

Given a field extension L/K, recall that the *norm* $N_{L/K} : L \to K$ is defined so that $N_{L/K}(\theta)$ is the determinant of the K-linear map $L \to L$ given by $x \mapsto \theta x$. Note that if $T^d + a_{d-1}T^{d-1} + \cdots + a_0$ is the minimal polynomial of θ over K, then $N_{L/K}(\theta) = \pm a_0^{[L:K(\theta)]}$.

We first prove the following lemma.

Lemma. Let A be a UFD, let B be a domain, and let $A \subseteq B$ with B finite over A. Writing $L = \operatorname{Frac} B$, $K = \operatorname{Frac} A$, we have for all $\theta \in B$,

$$N_{L/K}(\theta) \in A \text{ and } \theta | N_{L/K}(\theta) \text{ in } B.$$

Remark. We could weaken the above hypothesis so that A is only integrally closed in $\operatorname{Frac} A$.

Remark. The above lemma does not hold for A a general domain. For example, consider $\mathbb{Z}[\sqrt{8}] \subseteq \mathbb{Z}[\sqrt{2}]$ and $\theta = \sqrt{2}$.

Proof of Lemma. Let $F = K(\theta)$. Then $N_{L/K}(\theta) = N_{F/K}(\theta)$, and so it is enough to show the desired result holds for $N_{F/K}(\theta)$. Let $T^d + a_{d-1}T^{d-1} + \cdots + a_0$ be the minimal polynomial of θ over K. The coefficients a_j all lie in A. In particular,

$$a_0^{[L:F]} = \pm N_{F/K}(\theta) \in A,$$

and

$$a_0 = -(\theta^{d-1} + a_{d-1}\theta^{d-2} + \cdots + a_1)\theta.$$

□

We may now prove Krull's principal ideal theorem.

Theorem (Krull's principal ideal theorem)**.** Let Y be an irreducible quasiprojective variety with $\dim Y = d$, and let θ be a regular function on Y with $\theta \neq 0$. Then every irreducible component Z of $Z(\theta)$ has dimension $d-1$.

Proof. We reduce to the affine case, and then further reduce to a smaller affine neighborhood. Choose a point $p \in Z$ not contained in any other irreducible component of $Z(\theta)$. Choose a *distinguished* open neighborhood U of p such that U does not intersect any other irreducible component of $Z(\theta)$. Since U is a distinguished open, U is affine, and we have $\dim(Z \cap U) = \dim Z$. Furthermore, the zero locus of θ as a function on U is just $Z \cap U$, since $Z(\theta) \cap U = Z \cap U$ by assumption. In summary, we have reduced to the case where $Z(\theta) = Z$, so let us assume this equality from here onward.

Choose a Noether normalization $U \xrightarrow{\pi} \mathbb{A}^d$. Since π is a finite map, its image $\pi(Z)$ is closed in $\mathbb{A}^d$, and $\pi(Z)$ has the same dimension as Z. We want to come up with a function on $\mathbb{A}^d$ that vanishes precisely on $\pi(Z)$.

Let B be such that $U = \operatorname{MaxSpec} B$, and let us write $a = N_{\operatorname{Frac} B/k(x_1,\ldots,x_d)}(\theta)$. By the above lemma, we have $a \in k[x_1, \ldots, x_d]$. Since $k[x_1, \ldots, x_d]$ is a UFD, we can write a prime factorization $a = \prod p_i^{k_i}$. Let $r = \prod p_i$, noting that the principal ideal generated by r is radical, and that $Z(r) = Z(a)$. We will show that $\pi(Z) = Z(a) = Z(r)$. (Note: This will imply that $Z(r)$ is irreducible, so it turns out there is only *one* prime p_i.)

First, we show that $\pi(Z) \subseteq Z(a)$. For this, we must show that $\pi^* a$ vanishes on $Z = Z(\theta)$. But indeed, the above lemma tells us that $\theta | a$ in B, and so $\pi^* a$ does in fact vanish on Z. Hence $\pi(Z) \subseteq Z(a)$.

Now, we show that $Z(r) \subseteq \pi(Z)$. Since $\pi(Z)$ is closed in $\mathbb{A}^d$, it is the zero locus of some collection of polynomials on $\mathbb{A}^d$. Let a' be some such polynomial. We must show that a' vanishes on all of $Z(r)$. Observe that

$$\begin{aligned} a' \text{ vanishes on } \pi(Z) &\iff \pi^* a' \text{ vanishes on } Z \\ &\iff (\pi^* a')^\ell = \theta\beta \text{ for some } \beta \in B,\ \ell \geq 0. \quad \text{(Nullstellensatz)} \end{aligned}$$

Note that the norm map $N = N_{\operatorname{Frac} B/k(x_1,\ldots,x_d)}$ is multiplicative. Applying the norm map to the last of the above equivalent conditions gives the equation

$$\begin{aligned} N((\pi^* a')^\ell) &= N(\theta)N(\beta) \\ (a')^{[\operatorname{Frac} B : k(x_1,\ldots,x_d)]\ell} &= aN(\beta). \end{aligned}$$

In particular, a divides a power of a'. Since B is a UFD, this implies that $r | a'$. Therefore $Z(r) \subseteq Z(a') \subseteq \pi(Z)$, as desired. We conclude that $\pi(Z) = Z(a)$, and hence has dimension $d - 1$. Thus Z also has dimension $d - 1$, and so we are done. □

Remark. Note the significance of a ring's being a UFD to this proof. To get a geometric sense for the UFD condition, we comment that

$$A \text{ is a UFD} \iff \text{Every codimension 1 prime of } A \text{ is principal.}$$

For the remainder of the lecture, we consider some easy variants/corollaries of Krull's principal ideal theorem.

Corollary. *Let Y be of pure dimension d, and let $f_1, \ldots, f_r$ be regular functions on Y. Then every irreducible component of $Z(f_1, \ldots, f_r)$ has dimension $\geq d - r$.*

Proof strategy. Induction on r. □

Corollary. *Let $X, Y \subseteq \mathbb{A}^n$ be of pure dimensions d and e, respectively. Then each irreducible component of $X \cap Y$ has dimension $\geq d + e - n$.*

Proof. We "reduce to the diagonal:"

The intersection $X \cap Y \subseteq \mathbb{A}^n$ is isomorphic to $(X \times Y) \cap \Delta \subseteq \mathbb{A}^{2n}$, where Δ is the diagonal $\{(x, x) \in \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}\}$. Now, $\dim(X \times Y) = d + e$, and Δ is given by n linear equations. Hence $\dim(X \cap Y) = \dim((X \times Y) \cap \Delta) \geq d + e - n$, as desired. $\square$

Corollary. *If $X, Y \subseteq \mathbb{P}^n$ are closed of pure dimensions d and e, respectively, then each irreducible component of $X \cap Y$ has dimension $\geq d + e - n$. Furthermore, if $d + e - n \geq 0$, then $X \cap Y \neq \varnothing$.*

Proof sketch. For the first claim, pass to affine patches.

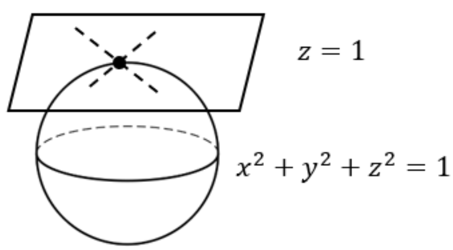
For the second claim, consider the closed subsets $\operatorname{Cone}(X), \operatorname{Cone}(Y) \subseteq \mathbb{A}^{n+1}$. Their intersection has dimension at least $(d+1)+(e+1)-(n+1) \geq 1$, and hence $0 \in \operatorname{Cone}(X) \cap \operatorname{Cone}(Y)$. Therefore there are nonzero points lying in $\operatorname{Cone}(X) \cap \operatorname{Cone}(Y)$. $\square$

October 26: Dimensions of Fibers. From last time, we know that if Y is irreducible of dimension d and f is a nonzero regular function on Y, then $Z(f)$ is pure dimension $d - 1$. We get two corollaries from this.

Corollary. *Suppose Y is pure dimension d, and $f_1, \ldots, f_r$ are regular functions on Y. Then every component of $Z(f_1, \ldots, f_r)$ has dimension $\geq d - 1$.*

Corollary. *Suppose X, Y are closed in $\mathbb{A}^n$ or $\mathbb{P}^n$, X has pure dimension D, and Y has pure dimension e. Then every component of $X \cap Y$ has dimension $\geq d + e - n$.*

These are analogous to results in complex geometry and contrary to the corresponding real cases. For example, consider the intersection of the surfaces defined by $z = 1$ and $x^2 + y^2 + z^2 = 1$ (see the diagram below). Points in the intersection must satisfy $z = 1$ and $x^2 + y^2 = (x + iy)(x - iy) = 0$. In the real case, there is only one such point. In the complex case, we get a pair of lines as the intersection. In essence, things can intersect more than expected, but not less.



We now prove that our notion of dimension using fraction fields coincides with Krull dimension from commutative algebra.

Theorem. *Let $X_e \subseteq X_d$ with X_e irreducible of dimension e and X_d irreducible of dimension d. Then $\exists X_e \subsetneq X_{e+1} \subsetneq \cdots \subsetneq X_d$ with X_j irreducible of dimension j.*

Proof. We first reduce to the affine case. We will induct on $d - e$. The base case of $e = d$ is obvious (we need not find any additional X_j's). Now suppose $e < d$. Take a regular function f on X_d so that $f|_{X_e} = 0$. Note that $X_e \subseteq Z(f)$. Since X_e is irreducible, $X_e \subseteq X_{d-1}$ for some irreducible component X_{d-1} of $Z(f)$ with $\dim X_{d-1} = d - 1$. We have now reduced to $d - e - 1$, so we induct. $\square$

In general, Krull dimension is more robust than transcendence degree. For example, we may consider the chain of ideals in $\mathbb{Z}$: $(0) \subseteq (x) \subseteq (x, 3)$. In $\mathbb{Q}$, this chain collapses to

$(0) \subseteq (x)$. Thus, passing to the fraction field can lose information about the structure of a ring. Krull dimension also allows us to reason about more complicated rings (e.g. the Krull dimension of entire functions on $\mathbb{C}$ is 1). However, treating Krull dimension in general brings up additional concerns that we do not want to worry about for now.

We now wish to prove some results about dimensions of fibers of regular maps.

Theorem. *Let X and Y be irreducible of dimensions m and n, respectively. Let $\pi : Y \to X$ be a regular map. Then*

(1) $\forall x \in X$, $\pi^{-1}(x) = \emptyset$ *or* $\dim \pi^{-1}(x) \geq n - m$.
(2) *Suppose* $\overline{\pi(Y)} = X$. *Then there exists a nonempty $U \subseteq X$ s.t.* $\pi^{-1}(x) \neq \emptyset$ *and* $\dim \pi^{-1}(x) = n - m$ $\forall x \in U$.

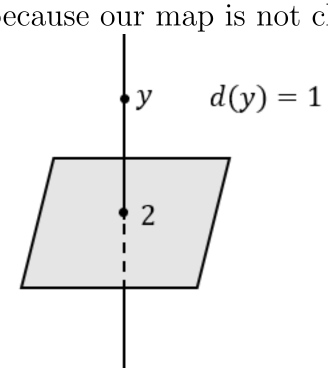
Remark. Note that (1) is not true in $\mathbb{R}$. Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. Take $\pi(x, y) = x^2 + y^2$. Then $\pi^{-1}(0) = \{(0, 0)\}$, which has dimension $0 < 2 - 1$.

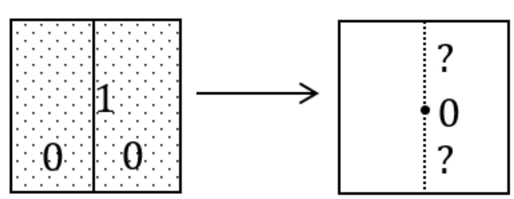
Theorem. *Let X and Y be quasi-projective varieties and $\pi : Y \to X$ a regular map.*

(3) *For $y \in Y$, define $d(y)$ to be the maximal dimension of an irreducible component of $\pi^{-1}(\pi(y))$ containing y. Then $\forall k$, $\{y \in Y : d(y) \geq k\}$ is closed.*
(4) *If $\pi : Y \to X$ is closed, then $\forall k$, $\{x : \pi^{-1}(x) \neq \emptyset$ and $\dim \pi^{-1}(x) \geq k\}$ is closed.*

As an example of $d(y)$ (or dimension at the point y), see the diagram below on the left. We have a union of a line and a plane. The dimension at any point on the plane (including the intersection point) is 2. The dimension at a point y off of the plane is 1.

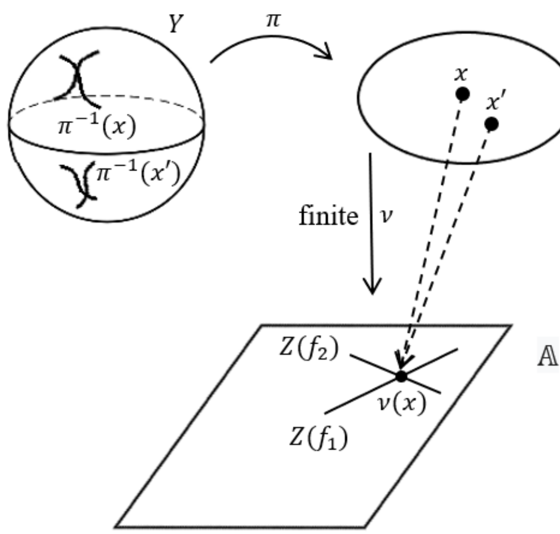
(3) and (4) essentially state that fiber dimension is upper semi-continuous. That is, dimension can only go up as we approach a point. In (4), we have some additional considerations due to the fact that some points may not be hit by our map at all. As an example, consider $\pi : \mathbb{A}^2 \to \mathbb{A}^2$ with $\pi(x, y) = (x, xy)$ (shown in the diagram below on the right). We have that the fibers of most points are points (dimension 0), but the fiber of $(0, 0)$ is a line (dimension 1). Thus, the dimension 1 points form a closed set. In the image, we see that we run into trouble because our map is not closed.





We now prove these claims.

Proof of (1). The statement is local on X, so we may assume X is affine. Take a Noether normalization $\nu : X \to \mathbb{A}^m$. The following argument can be visualized with the diagram below:



Note that

$$\pi^{-1}(\nu^{-1}(\nu(x))) = \bigsqcup_{x' \in \nu^{-1}(x)} \pi^{-1}(x').$$

Thus, irreducible components of $\pi^{-1}(x)$ must also be irreducible components of $(\nu \circ \pi)^{-1}(\nu(x))$. We can write $\nu(x) = Z(f_1, \ldots, f_m)$ where $f_1, \ldots, f_m$ are linear functions on $\mathbb{A}^m$. Then $(\nu \circ \pi)^{-1}(\nu(x)) = Z(\pi^* f_1, \ldots, \pi^* f_m)$, so every irreducible component has dimension $\geq n - m$. □

Proof of (2). We first reduce to the affine case. Recall from Wednesday, October 10, that if Y and X are irreducible and $\pi : Y \to X$ is dominant, then there exists a nonempty open $U \subseteq X$ s.t. $\pi^{-1}(U) \to U$ factors through a finite, surjective map as $\pi^{-1}(U) \to U \times \mathbb{A}^d \to U$. Notice $\dim \pi^{-1}(U) = \dim Y = n$ and $\dim U \times \mathbb{A}^d = \dim U + d = m + d$, so $n = m + d$. So for $x \in U$, $\pi^{-1}(x)$ is finite and surjective over $\mathbb{A}^{n-m}$. Thus, $\dim \pi^{-1}(x) = n - m$. □

Proof of (3). We induct on $\dim X$. If $\dim X = 0$, the statement says $d(y)$, defined to be the maximal dimension of an irreducible component of Y through y, is uppper semicontinuous. Note that

$$\{y : d(y) \geq k\} = \bigcup_{\substack{Z \text{ irreducible component of } Y, \\ \dim Z \geq k}} Z.$$

Each irreducible component is closed, and the union of closed sets is closed, so $\{y : d(y) \geq k\}$ is closed. Hence, $d(y)$ is upper semicontinuous, finishing our base case.

Note that if Y is not irreducible, we can write $Y = \bigcup Y_j$ for irreducible Y_j. The theorem then follows from the theorem from each $Y_j \to X$. Thus, we may assume that Y is irreducible. We may also replace X by $\overline{\pi(Y)}$, so we may assume X is irreducible.

Let $\dim X = m$ and $\dim Y = n$. If $k \leq n - m$, we are done, since $d(y) \geq k$ $\forall y \in Y$. Thus, we may assume $k > n - m$. Choose $U \subseteq X$ as in (2). Then $d(y) = n - m$ for $y \in \pi^{-1}(U)$. Replace X by $X' = X - U$ and Y by $\pi^{-1}(X')$. Since $\dim X' < \dim X$, we may induct. □

Proof of (4). Note that

$$\{x \in X : \pi^{-1}(x) \neq \emptyset \textit{ and } \dim \pi^{-1}(x) \geq k\} = \pi(\{y \in Y : d(y) \geq k\}).$$

Thus, the statement follows from (3) and the condition that π is closed. □

October 29: Hilbert functions and Hilbert polynomials. Today we study Hilbert functions, a subject which connects dimension to combinatorial commutative algebra. We will have the problem that we want to talk both about *dimension* and *degree* a lot today; we'll try to stick to the convention that dimensions are called "d" and degrees are called "δ".

Let $A = k[x_0, \dots, x_n]$ with its usual grading, and let M be a finitely generated graded A-module. We define the ***Hilbert function*** of M to be:

$$h_M^{\text{func}}(t) := \dim_k M_t.$$

We have the following basic commutative algebra lemma:

Lemma. There is a polynomial $h_M^{\text{poly}}(t)$, of degree $\leq n$, such that $h_M^{\text{func}}(t) = h_M^{\text{poly}}(t)$ for t sufficiently large.

Proof (omitted in class). Our proof is by induction on n. In our base case, $n = -1$, we have $A = k$, so M is a simply a finite dimensional graded vector space. Then $M_t = 0$ for $t \gg 0$, so we can take $h_M^{\text{poly}}(t) = 0$.

Now for the inductive case. Let $A' = k[x_0, \dots, x_{n-1}] = A/x_n$. Let K and Q be the kernel and cokernel of multiplication by x_n on M, so we have an exact sequence

$$0 \to K \to M \xrightarrow{x_n} M \to Q \to 0.$$

Degree by degree, we have an exact sequence

$$0 \to K_t \to M_t \xrightarrow{x_n} M_{t+1} \to Q_{t+1} \to 0$$

so we deduce

$$h_M^{\text{func}}(t+1) - h_M^{\text{func}}(t) = h_Q^{\text{func}}(t+1) - h_K^{\text{func}}(t).$$

(This is several applications of the rank-nullity theorem, but by now you should learn the more general fact that, in any exact sequence of vector spaces, the alternating sum of dimensions is 0.)

We note that K and Q are A' modules and are finitely generated (the former by Hilbert's basis theorem in M, the latter because Q is a quotient of M). So there are Hilbert polynomials h_K^{poly} and h_Q^{poly}. We deduce that, for t sufficiently large, $h_M^{\text{func}}(t+1) - h_M^{\text{func}}(t)$ is a polynomial of degree $\leq n-1$ in t. So h_M^{func} is a polynomial of degree $\leq n$ in t. $\square$

If X is Zariski closed in $\mathbb{P}^n$, with radical ideal I, we write h_X^{func} and h_X^{poly} for $h_{k[x_0,\dots,x_n]/I}^{\text{func}}$ and $h_{k[x_0,\dots,x_n]/I}^{\text{poly}}$.

We begin with several examples:

Example. If X is all of $\mathbb{P}^n$, we want to compute the Hilbert function of $k[x_0, \dots, x_n]$ itself. So we want to compute the dimension of the vector space of degree t homogenous polynomials in x_0, x_1, $\dots$, x_n. Using the obvious basis of monomials, this is $\#\{(a_0, a_1, \dots, a_n) : 0 \leq a_j,\ \sum a_j = t\}$. This is $\binom{n+t}{n}$ for $t \geq 0$. For $t < 0$, we have $h^{\text{func}}(t) = 0$, but $h^{\text{poly}}(t)$ is by definition given by $\binom{n+t}{n} = \frac{(t+n)(t+n-1)\cdots(t+1)}{n!}$. For future reference, we observe that this is a polynomial with leading term $\frac{t^n}{n!}$.

Example. Let $f(x,y,z)$ be a squarefree degree δ polynomial in $A := k[x,y,z]$; we compute the Hilbert polynomial of $Z(f)$. In other words, we must compute the dimension of the degree t part of the ring A/fA. We have a short exact sequence $0 \to A \xrightarrow{f} A \to A/fA \to 0$ which, degree by degree, gives $0 \to A_{t-\delta} \to A_t \to (A/fA)_t \to 0$. So $\dim(A/fA)_t = \dim A_t - \dim A_{t-\delta} = \binom{t+2}{2} - \binom{t-\delta+2}{2} = \delta t + \frac{3\delta - \delta^2}{2}$.

Remark. Above, we wrote $A \xrightarrow{f} A$. If you read more sophisticated sources, you will see that they write $A[-\delta] \xrightarrow{f} A$. The reason for this is that a map of graded modules, by definition, is required to preserve degree, so $A \xrightarrow{f} A$ is not a map of graded modules. The notation $A[-\delta]$ means A with shifted grading: $A[-\delta]_j := A_{j-\delta}$. I find this convention confusing and will avoid it when possible.

Example. Let's specialize the previous example to $\delta = 2$, with $Z(f)$ a smooth conic. So $\delta t + \frac{3\delta-\delta^2}{2} = 2t+1$. We note that a smooth conic is isomorphic to $\mathbb{P}^1$ (if $f = xz - y^2$, the isomorphism is $(x:y:z) = (t^2 : tu : u^2)$ for $(t:u)$ in $\mathbb{P}^1$). So the Hilbert series of $\mathbb{P}^1$ as a subvariety of itself (or as a line in $\mathbb{P}^2$) is $t+1$, but the Hilbert series $\mathbb{P}^1$ embedded as a smooth conic in $\mathbb{P}^2$ is $2t+1$.

Example. We can generalize the previous example as follows: For any positive integers m and r, we have the r-uple Veronese embedding $v : \mathbb{P}^m \to \mathbb{P}^{\binom{m+r-1}{m}-1}$, such that degree r equations in $\mathbb{P}^m$ are restrictions of linear equations from the big projective space. We looked at the 2-uple Veronese $\mathbb{P}^2 \to \mathbb{P}^5$ in a previous problem set. If X is Zariski closed in $\mathbb{P}^m$, we have $h^*_{v(X)}(t) = h^*_X(rt)$, where the $*$ could be either func or poly.

Remark. We have seen that h_X^{poly} depends on the embedding of X in $\mathbb{P}^n$, not just on the abstract isomorphism type of X. Here is something which was incredibly mysterious in the nineteenth century: $h_X^{\text{poly}}(0)$ only depends on X! We'll prove this later this term for curves; the proof for general X involves inventing sheaf cohomology.

We now want to connect Hilbert polynomials to degree.

Theorem. Let X be Zariski closed in $\mathbb{P}^n$ of dimension d. Then the leading term of h_X^{poly} is of the form $\frac{\delta}{d!}t^d$ for a positive integer d called the ***degree*** of X.

This proof is slightly restructured from the presentation in class.

Lemma. Let $A = k[x_0, \ldots, x_d]$. Let M be a finitely generated graded A-module and suppose that the action of A on M factors through A/fA for some nonzero f. Then h_M^{poly} has degree $< d$.

Proof of Theorem. Choose a Noether normalization $A/fA \to k[x_0, \ldots, x_{d-1}]$. Then M is a finitely generated graded $k[x_0, \ldots, x_{d-1}]$ module. □

Proof. Let B be the homogenous coordinate ring of X. Choose a Noether normalization $X \to \mathbb{P}^d$, and let A be the homogenous coordinate ring of $\mathbb{P}^d$. So B is a finite A-algebra. Let δ be the dimension of $B \otimes_A \mathrm{Frac}(A)$ as a $\mathrm{Frac}(A)$ vector space. So we can choose $\beta_1, \beta_2, \ldots, \beta_\delta$ in B giving a $\mathrm{Frac}(A)$ basis for $B \otimes_A \mathrm{Frac}(A)$ over $\mathrm{Frac}(A)$; let β_j have degree δ_j. This gives an injection $\bigoplus A[-\delta_j] \to B$ with some cokernel Q. We deduce that $h_B^{\text{poly}}(t) = \sum_{j=1}^{\delta} h_A^{\text{poly}}(t-\delta_j) + h_Q^{\text{poly}}(t) = \sum_{j=1}^{\delta} \binom{t-\delta_j+d}{d} + h_Q^{\text{poly}}(t)$. The sum has leading term $\frac{\delta}{d!}t^d$, so we must show that h_Q^{poly} has degree $< d$.

We have $Q \otimes_A \mathrm{Frac}(A) = 0$, and Q is finitely generate, so Q is an A/fA-module for some f. So the lemma tells us that $\deg h_Q^{\text{poly}} < d$, as desired. □

Remark. On the homework, you will establish the following: Let X be Zariski closed in $\mathbb{A}^n$ with ideal I. Let $k[x_1, \ldots, x_n]_{\leq t}$ be the set of polynomials of degree $\leq t$. Then $\dim k[x_1, \ldots, x_n]_{\leq t}/(I \cap k[x_1, \ldots, x_n]_{\leq t})$ is a polynomial in t for $t \gg 0$, of degree $\dim X$.

Remark. There is another nice result along these lines. Let X be an affine variety with coordinate ring A, and let $x \in X$ correspond to the maximal ideal $\mathfrak{m}_x \subseteq A$. Then $\dim_k A/\mathfrak{m}_x^{t+1}$ is polynomial in t for $t \gg 0$. The degree of this polynomial is the largest dimension of any irreducible component of X containing x. The leading term is $\frac{\delta}{d!}t^d$ where δ is the so-called ***muliptlicity*** of x. The function $\dim_k A/\mathfrak{m}_x^{t+1}$ is called the ***Hilbert-Samuel function***.

October 31: Bezout's Theorem. Today we discuss Bezout's theorem:

Theorem. (Imprecise version) Let $f, g \in k[x, y, z]$ be relatively prime homogeneous polynomials of degrees d and e. Then $f = g = 0$ has de solutions.

There are several caveats in the above version:

- Need k to be algebraically closed.
- Need to work in $\mathbb{P}^2$ instead of $\mathbb{A}^2$. Two curves in $\mathbb{A}^2$ may intersect at infinity, and we need to take that into account.
- Need to count multiplicity, e.g., a line tangent to a circle intersects the circle at a point of multiplicity 2.
- Need to rule out the possibility that the curves have a component in question, such as a line intersecting itself.

On commutative algebra side, the precise statement is the following:

Theorem. Let f and $g \in k[x, y, z]$be relatively prime homogeneous polynomials of degrees d and e. The Hilbert polynomial of $k[x, y, z]/(f, g)$ is the constant polynomial de.

Proof. Let $A = k[x, y, z]$. f, g being relatively prime implies that g is not a zero-divisor in A/fA and we have the following short exact sequence:

$$0 \to A/fA \xrightarrow{\cdot g} A/fA \to A/(f, g) \to 0.$$

So by results from last time, we have

$$\begin{aligned} h_{A/(f,g)}(t) &= h_{A/fA}(t) - h_{A/fA}(t-e) \\ &= \left[\binom{t+2}{2} - \binom{t-d+2}{2}\right] - \left[\binom{t-e+2}{2} - \binom{t-d-e+2}{2}\right] \\ &= de. \end{aligned}$$

∎

Note that for any ideal I, $I \subseteq \sqrt{I}$, and hence $A/I \to A/\sqrt{I}$ is surjective, which further implies that $\dim(A/I)_t \geq \dim(A/\sqrt{I})_t$. Thus $h^{poly}_{A/\sqrt{(f,g)}} = m \leq de$ for some integer m.

Claim. *This m is actually the number of geometric points of intersection.*

Proof of Claim. It suffices to show that if $p_1, \cdots, p_c \in \mathbb{P}^n$ are c distinct points, then $h^{poly}_{p_1,\cdots,p_c}(t) = c$. In fact, if all of the points are in $\mathbb{A}^n$, then $\mathcal{O}_{\{p_1,\cdots,p_c\}} \cong k^{\oplus c}$. Now choose a hyperplane $\{\lambda = 0\}$ not passing through any p_i. Let $U = \{\lambda \neq 0\} \cong \mathbb{A}^n$. Then functions regular on U are of the form $\frac{f}{\lambda^D}$ for some $f \in k[x_0, \cdots, x_n]$. Let R denote the reduced homogeneous coordinate ring of $\{p_1, \cdots, p_n\}$. For each t, we get a map $R_t \to \mathcal{O}_{\{p_1,\cdots,p_c\}}$, $f \mapsto \frac{f}{\lambda^t}$. Thus we have a sequence of maps

$$R_0 \overset{\lambda}{\hookrightarrow} R_1 \overset{\lambda}{\hookrightarrow} R_2 \overset{\lambda}{\hookrightarrow} \cdots \mathcal{O}_{\{p_1,\cdots,p_c\}}$$

It terminates since $\mathcal{O}_{\{p_1,\cdots,p_c\}}$ is finite dimensional, i.e., for large t, we have $R_t \cong \mathcal{O}_{\{p_1,\cdots,p_c\}}$. This implies that the Hilbert polynomial is c. ∎

What about $\mathbb{P}^n$? If $f_1, \cdots, f_n \in k[x_0, \cdots, x_n]$ are homogeneous polynomials of degrees $d_1, \cdots, d_n$, is $Z(f_1, \cdots, f_n)$ given by $d_1 \cdots d_n$ points with multiplicity? The answer is yes, but the commutative algebra is harder.

Definition. *$f_1, \ldots, f_n$ is called a* **regular sequence** *in a ring R if f_j is not a zero-divisor in $R/(f_1, \cdots, f_{j-1})$ for all j.*

This definition is exactly what we need to produce an exact sequence. Thus we have

Theorem. If $f_1, \ldots, f_n$ is a regular sequence, then $h^{\mathrm{poly}}_{k[x_0,\cdots,x_n]/(f_1,\cdots,f_n)} = d_1 \cdots d_n$.

But a more geometrically natural condition is that $\dim Z(f_1, \cdots, f_n) = 0$.

Definition. *Let R be a commutative ring of Krull dimension k. R is called* **Cohen-Macaulay** *if whenever $R/(f_1, \cdots, f_j)$ has dimension $k - j$, $f_1, \cdots, f_j$ is a regular sequence.*

This definition has roots in the following two theorems:

Theorem. (Macaulay, 1916) $k[x_0, \cdots, x_n]$ is Cohen-Macaulay.

Theorem. (Cohen, 1946) Regular rings are Cohen-Macaulay.

There is a good discussion about the Cohen-Macaulay issues at Mathoverflow .

November 2: Tangent spaces and Cotangent spaces. We define Tangent spaces at points of our variety so that we can talk about smoothness. We show that what we see in our calculus classes agrees with the commutative algebra definition.

Let $f \in k[x_1, \ldots, x_n]$ and let $\vec{v} = (v_1, \ldots, v_n) \in k^n$. Recall that the ***directional derivative*** is defined by

$$\nabla_{\vec{v}}(f) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}.$$

Lemma. Let $f_1, f_2, \ldots, f_m \in k[x_1, \ldots, x_n]$. Let I be the ideal $\langle f_1, f_2, \ldots, f_m \rangle$ and let $A = k[x_1, \ldots, x_n]/I$. Let $X = Z(I)$ and let $a \in X$, with corresponding maximal ideal $\mathfrak{m}_a \in A$.

For a vector $\vec{v}$ in k^n, the following are equivalent:

(1) The map $f \mapsto \nabla_{\vec{v}}(f)(a)$ from $k[x_1, \ldots, x_n] \to k$, descends to a map $A \to k$.
(2) For every $f \in I$, we have
$$\nabla_{\vec{v}}(f)(a) = 0.$$
(3) For each $1 \le i \le m$, we have
$$\nabla_{\vec{v}}(f_i)(a) = 0.$$

The set of such vectors $\vec{v}$ is called the ***tangent space*** $T_a X$.

Proof. The first 2 are equivalent since the map descends to $k[x_1, \ldots, x_n]/I$ if and only if I is in the kernel.

2 implies 3 is immediate. Now, we show that 3 implies 2. let $\nabla_{\vec{v}}(f_i)(a) = 0$ for a *generating* set $\{f_1, f_2, \ldots, f_m\}$ of I. Using Leibniz rule, we simplify $\nabla_{\vec{v}}(gh)(a) = g(a)\nabla_{\vec{v}}(h)(a) + h(a)\nabla_{\vec{v}}(g)(a)$. If $h = f_i$, then $f_i(a) = 0$ as $a \in Z(I)$ and $\nabla_{\vec{v}}(f_i)(a) = 0$ by assumption, implying that $\nabla_{\vec{v}}(gf_i)(a) = 0$ for any $g \in k[x_1, \ldots, x_n]$. If $f \in I$, we can write $f = g_1 f_1 + g_2 f_2 \ldots + g_m f_m$; $\nabla_{\vec{v}}(g_i f_i)(a) = 0$ along with linearity implies that $\nabla_{\vec{v}}(f)(a)$. $\square$

Let R be a commutative k-algebra and let M be an R-module. A ***derivation*** $R \to M$ ***over*** k is a map $D : R \to M$ obeying

- $D(c) = 0$, for $c \in k$.
- $D(f+g) = D(f) + D(g)$.
- $D(fg) = fD(g) + gD(f)$, where we have used the R-module structure of M.

Lemma. Let $R = k[x_1, \ldots, x_n]$ and let M be any R-module. Then, for any m_1, m_2, $\ldots$, $m_n \in M$, there is a unique derivation $D : R \to M$ with $D(x_i) = m_i$.

Proof. Using the last rule and linearity, we see that by specifying where x_i maps to, $D(f)$ is uniquely determined. In particular, $D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} D(x_i)$, which satisfies the properties of a derivation. $\square$

Lemma. Let $\mathfrak{m}$ be a maximal ideal of R. Show that every derivation $R \to R/\mathfrak{m}$ vanishes on $\mathfrak{m}^2$.

Proof. If $r, s \in \mathfrak{m}$, then $D(rs) = rD(s) + sD(r) = 0$ in $R/\mathfrak{m}$. Therefore, D vanishes on $\mathfrak{m}^2$ as any element of $\mathfrak{m}^2$ is of the form $\sum_{i=1}^{t} r_i s_i$, for $r_i, s_i \in \mathfrak{m}$. $\square$

Lemma. Let $\mathfrak{m}$ be a maximal ideal of R. Suppose that the composition $k \to R \to R/\mathfrak{m}$ is an isomorprhism $k \cong R/\mathfrak{m}$. Show that the space of derivations $R \to R/\mathfrak{m}$ is isomorphic to $\mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, k)$.

The $R/\mathfrak{m}$ vector space $\mathfrak{m}/\mathfrak{m}^2$ is called the ***Zariski cotangent space*** of $(R, \mathfrak{m})$.

Proof. We are in the following setup:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m} \arrow[r] & R \arrow[r] & R/\mathfrak{m} \arrow[r] \arrow[dl] & 0 \\ & & k \arrow[u] & & \end{tikzcd}$$

Note that the above short exact sequence splits (as $R/\mathfrak{m} \cong k$). Therefore, we can write any element $r \in R$ as a sum of an element in $\mathfrak{m}$ and k.

Let $Der(R, R/\mathfrak{m})$ denote the vector space of derivations $R \mapsto R/\mathfrak{m}$. We can restrict a derivation D in $Der(R, R/\mathfrak{m})$ to $\mathfrak{m} \subset R$, and obtain a linear map, $D|_{\mathfrak{m}}$ from $\mathfrak{m}$ to $R/\mathfrak{m} \cong k$. The previous lemma tells us that $D|_{\mathfrak{m}}$ vanishes on $\mathfrak{m}^2$, hence $D|_{\mathfrak{m}}$ induces a linear map $\widetilde{D}$ from $\mathfrak{m}/\mathfrak{m}^2$ to k. We shall show that the map $D \mapsto \widetilde{D}$ is an isomorphism from $Der(R, R/\mathfrak{m})$ to $Hom(\mathfrak{m}/\mathfrak{m}^2, k)$.

Suppose D is a nonzero derivation, i.e, $D(r) \neq 0$, for some $r \in R$. Then $D(r) = D(m) + D(\lambda) \neq 0$ for elements $m \in \mathfrak{m}$ and $\lambda \in k$. However $D(\lambda) = 0$ since $\lambda \in k$. Therefore, $D(m) \neq 0$ due to which $\widetilde{D}(m) \neq 0$, or $\widetilde{D} \not\equiv 0$. To show surjectiviy, take an element $f \in Hom(\mathfrak{m}/\mathfrak{m}^2, k)$, this gives us a map from $\mathfrak{m}$ to k (by composing with the map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$). Thus, we also obtain a map D from R to k (by composing with the map $R \to \mathfrak{m}$, which exists since the short exact sequence splits). This is easily checked to be a derivation such that $f = \widetilde{D}$. Therefore, $Der(R, R/m) \cong Hom(\mathfrak{m}/\mathfrak{m}^2, k)$. $\square$

The ***Zariski tangent space*** of $(R, \mathfrak{m})$ is defined to be $Der(R, R/\mathfrak{m})$ which we have just shown is isomorphic to the dual of the Zariski cotangent space (when $R/\mathfrak{m} \cong k$).

We see that the tangent/cotangent spaces of a variety X at a point x are both intrinsic quantities, which can be described solely in terms of the coordinate ring of R and the maximal ideal $\mathfrak{m}_x$. But they are also both very concrete quantities: If we embed X into $\mathbb{A}^n$, with ideal $f_1, \ldots, f_m$, then the tangent space is the solution to the linear equations

$$\sum v_j \frac{\partial f_i}{\partial x_j} = 0 \qquad 1 \leq i \leq m.$$

November 5: Tangent bundle, vector fields, and 1-forms. Today we define the tangent bundle, but before we do so, we list out some properties of tangent and cotangent spaces which should have been mentioned last time.

$\mathbf{T_x}$ is functorial: If we have a regular map $f : X \to Y$, where X and Y are affine varieties in $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. Then we have a map $f_* : T_x X \to T_{f(x)} Y$, given by the following Jacobian.

$$f_* = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Strictly speaking, this is a map from $T_x\mathbb{A}^n$ to $T_{f(x)}\mathbb{A}^m$. To get the map from $T_x X$, we restrict the map above to the subspace $T_x X$.

We can also define this induced map more abstractly. Recall that elements of $T_x X$ are k-linear derivations from $\mathcal{O}(X)$ to $\mathcal{O}(X)/\mathfrak{m}_x$. We can compose this with the induced map from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ to get a k-linear derivation from $\mathcal{O}(Y)$ to $\mathcal{O}(X)/\mathfrak{m}_x$, which can be canonically identified with $\mathcal{O}(Y)/\mathfrak{m}_{f(x)}$, giving an element of $T_{f(x)}Y$.

Tangent space of fibers: Suppose now we have a regular map from Y to X. Pick a point $x \in X$, and let $Z = f^{-1}(x)$ be a subvariety of Y. Pick a point $y \in Z$. The question is, what's the relation between $T_y Y$, $T_y Z$, and $T_x X$. Because X sits inside Y, we have the map from $T_y Z$ to $T_y Y$ induced by the inclusion map. We also have a map from $T_y Y$ to $T_x X$ induced by f. And if we compose the two maps, we get the map induced by the constant map from Z to X, which must necessarily be 0.

$$T_y Z \xrightarrow{i_*} T_y Y \xrightarrow{f_*} T_x X$$

The composition $f_* \circ i_* = 0$, but $T_y Z$ is not necessarily equal to kernel of the map. Consider a map f from $\mathbb{A}^1$ to $\mathbb{A}^1$ given by $y \mapsto y^2$, and look at the pre-image of 0. It's just the singleton point $\{0\}$. The tangent space of this point is a 0-dimensional space. On the other hand, the induced map f_* at $T_0\mathbb{A}^1$ sends everything to 0, that means its kernel is 1-dimensional. Next term, when we can talk about schemes, we will say that the scheme-theoretic fiber is Spec of the non-reduced ring $k[x, y]/\langle x, y^2\rangle \cong k[y]/(y^2)$. The Zeriski tangent space to this nonreduced ring is 1-dimensional, and is the kernel of the map on tangent spaces.

We now define the tangent bundle of an affine variety X when it's embedded in $\mathbb{A}^n$. Let I be the ideal of polynomials in $k[x_1, \ldots, x_n]$ that vanish on X. This gives us a concrete way of describing the tangent space of X at x, namely the set of all vectors $v \in \mathbb{A}^n$ such that $\sum_i v_i \frac{\partial f}{\partial x_i}$ for all $f \in I$. This also lets us build up the tangent space as a variety, which is a collection (x, v), where $x \in X$, and $v \in T_x X$.

Definition (Tangent bundle)**.** *The tangent bundle TX of an affine variety $X \subseteq \mathbb{A}^n$ is a closed subset of $\mathbb{A}^{2n}$ (where the first n coordinates are $\{x_1, \ldots, x_n\}$ and the last n coordinates* *are $\{v_1, \ldots, v_n\}$) defined by the common zeroes of the following polynomials.*

$$f(x_1, x_2, \ldots, x_n) \ \forall f \in I(X)$$
$$\sum_i v_i \frac{\partial f}{\partial x_i} \ \forall f \in I(X)$$

The tangent bundle comes with a map π to X, which is just projection onto the first n coordinates, and the fibre of π over any $x \in X$ turns out to be (x, v), where v ranges over all elements of $T_x X$.

A (regular) vector field is a regular section of the tangent bundle, i.e. a regular map s from X to TX such that $\pi \circ s = \mathrm{id}$. Concretely, it's given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \phi_1(x), \ldots, \phi_n(x))$, where ϕ_i are regular functions such that for all x, $(\phi_1(x), \ldots, \phi_n(x)) \in T_x X$. Recall the condition for a vector v to lie in $T_x X$: $\sum_i v_i \frac{\partial f}{\partial x_i} = 0$ for all $f \in I(X)$. That means a collection of regular functions $\{\phi_1, \ldots, \phi_n\}$ comes from a regular section iff $\sum_i \phi_i(x) \frac{\partial f}{\partial x_i}$ is 0 everywhere on X, or equivalently, lies in $I(X)$ for all $f \in I(X)$.

Just like how tangent vectors at $x \in X$ were defined as k-linear derivations from $\mathcal{O}(X)$ to $\mathcal{O}(X)/\mathfrak{m}_x$, we can define vector fields in terms of derivations, this time from $\mathcal{O}(X)$ to $\mathcal{O}(X)$.

More concretely, we have the following theorem.

Proposition. *Let $\{\phi_1, \ldots, \phi_n\}$ be regular functions on X. Then the following statements are equivalent.*

1. *For all $x \in X$, $(\phi_1(x), \ldots, \phi_n(x))$ is in $T_x X$.*
2. *There is a derivation $D : A \to A$ with $D(x_i) = \phi_i$.*

Proof. Define a derivation from $k[x_1, \ldots, x_n]$ to A by setting $D(x_i) = \phi_i$. It will only factor through $\mathcal{O}(X)$ if for all $f \in I(X)$, $D(f) = 0$. Since $D(x_i) = \phi_i$, $D(f) = \sum_i \phi_i(x) \frac{\partial f}{\partial x_i}$. If (1) is true, then $\sum_i \phi_i(x) \frac{\partial f}{\partial x_i}$ must be equal to 0, which means $D(f) = 0$, and the derivation D factors through $\mathcal{O}(X)$. Conversely, if the derivation factors through $\mathcal{O}(X)$, that means $D(f) = 0$ for all $f \in I(X)$, and $\phi(x)$ is in the tangent space at all points X. This proves the result. $\square$

Now that we have defined vector fields, it's natural to try to define 1-forms as well. One way to define 1-forms is as regular maps from TX to k, such that they are linear on each tangent space, i.e. a map ω such that $\omega(x, cv + w) = c\omega(x, v) + \omega(x, w)$. To put it in naïvely a 1-form is a way of regularly/holomorphically/smoothly assigning a number to each tangent vector at each point.

A way to construct 1-forms is to take exterior derivatives of regular functions. The exterior derivative df of a regular function $f : X \to k$, is the differential form that takes the tangent vector v to $v(f)$ (recall that a tangent vector is a derivation).

There's a related notion of Kähler differentials, which treats differential forms as purely abstract objects in an $\mathcal{O}(X)$-module generated by dx_i, where x_i's are coordinate functions on the ambient $\mathbb{A}^n$, and the relations on the module are the relations generated by the rules of the exterior derivative, namely linearity and the Leibnitz property, and that $d(f) = 0$ for all $f \in I(X)$. These clearly surject onto regular differential forms, but they usually don't inject into the space of regular differential forms. On the problem set, you will see that xdy is a nonzero Kähler 1-form on $X := \{xy = 0\} \subset \mathbb{A}^2$, but vanishes at every point of TX. We will eventually be able to show that, for a smooth variety, the Kähler 1-forms and the regular 1-forms coincide. In the world of smooth functions, other things can go wrong; see the discussion at `https://mathoverflow.net/a/6138/56183` .

The following question was asked after class: Is there a cotangent bundle, which 1-forms (of either kind) are sections of? For singular X, no. Let $X = \{xy = 0\} \subset \mathbb{A}^2$. The 1-form dx is 0 in $T^*_(a,b)X$ for $a = 0$, $b \neq 0$, but not at $(0, 0)$. If there were some hypothetical $T^*X \to X$ which dx was a section of, then it would have to vanish on a closed set. For X smooth, such a cotangent bundle exists. As a sketch of the construction: Let X be smooth of dimension d. We will soon be able to show that X has an open cover U_i such that $TU_i \cong U_i \times \mathbb{A}^d$. Then TU_i and TU_j will glue by $(x, \vec{v}) \mapsto (x, g_{ij}(x)\vec{v})$ for some regular function $g_{ij} : U_i \cap U_j \to GL_n$. Then T^*X is formed by taking the varieties $U_i \times \mathbb{A}^d$ and gluing the two copies of $(U_i \cap U_j) \times \mathbb{A}^d$ to each other by $(x, \vec{v}) \mapsto (x, g_{ij}^{-T}(x)\vec{v})$. If we were allowed to talk about gluing abstract varieties, this would be a construction and, after working hard enough, we could deduce that, if X is affine then so is T^*X, using (for example) Proposition 7.3.4 in Vakil. But I don't see how to do this if I am not allowed to talk about the abstract object before I deduce that it is affine. See the discussion at `https://mathoverflow.net/questions/186396`.

November 7: Gluing Vector Fields and 1-Forms. We start with an example from the problem set. Take:

$$X = \{y^2 = x^3 + x\}, \qquad A = k[x, y]/\langle y^2 - x^3 - x\rangle, \qquad \Omega^1_A = \frac{\langle dx, dy\rangle}{2ydy - (3x^2 + 1)dx}$$

where that last is secretly the Kähler 1-forms. Since $2y$ and $3x^2 + 1$ have no common roots, we can write $X = U \cup V$, where $U = \{2y \neq 0\}$, and $V = \{3x^2 + 1 \neq 0\}$. On the intersection $U \cap V$ we have $\frac{dy}{3x^2+1} = \frac{dx}{2y}$. We checked that Ω^1_A is a free A-module with some generator ω. The idea is that we should do something like $\omega = \frac{dy}{3x^2+1} = \frac{dx}{2y}$, so that $dy = (3x^2 + 1)\omega$, $dx = 2y\omega$. By the Nullstellensatz, we know there exist f and g such that $2yf + (3x^2+1)g \equiv 1 \mod \langle y^2 - x^3 - x\rangle$. We define ω by $fdx + gdy$; conceptually, this formula is motivated by $\omega = (2yf + (3x^2 + 1)g)\omega = fdx + gdy$ since $2y\omega = dx$ and $(3x^2 + 1)\omega = dy$. So we can take one formula for ω which is valid when $2y \neq 0$ and another which is valid when $3x^2 + 1$ is nonzero, and glue them together to a 1-form defined on the union of these two open sets. We will want to repeat this construction generally.

Now lets look in more generality. If X is affine, $U \subset X$ affine open, then we have

$$\begin{array}{ccc} TX & \xrightarrow{\pi_X} & X \\ \uparrow & & \uparrow \\ TU & \xrightarrow{\pi_U} & U \end{array}$$

We observe that $TU \cong \pi_X^{-1}(U)$.

We showed a while ago that the condition of a function being regular can be checked locally. We deduce:

Theorem. If a function $\omega : TX \to k$ is linear on each $\pi^{-1}(x)$, then it is regular if and only if X has a cover $\{U_i\}$ such that $\omega|_{U_i}$ is regular, which holds if and only if it holds for all covers.

In other words, regular 1-forms glue. There is a similar result for Kähler 1-forms.

Also, the condition of a map being regular can be checked locally. We deduce:

Theorem. If we have a set theoretic section $\sigma : X \to TX$ it is regular if and only if there exists an open cover U_i such that $\sigma|_{U_i}$ is regular for all i, which holds if and only if it holds for all covers.

So regular vector fields glue.

We now use this to define regular vector fields and 1-forms on non-affine varieties.

Definition. Let X be a quasiprojective variety. A *vector field* on X is a choice of vector $\varphi(x) \in T_xX$ for each $x \in X$ such that φ restricts to a regular vector field on some (equivalently: any) affine cover. A *regular 1-form* on X is a choice for each $x \in X$ of a linear map $\omega_x : T_xX \to k$ which restricts to a regular 1-form on some (equivalently: any) affine cover.

Why equivalently any? For any pair of open covers $\{U_i\}$, $\{V_j\}$ the intersections $\{U_i \cap V_j\}$ form an affine cover. We can use this to transfer the condition from one cover to the other.

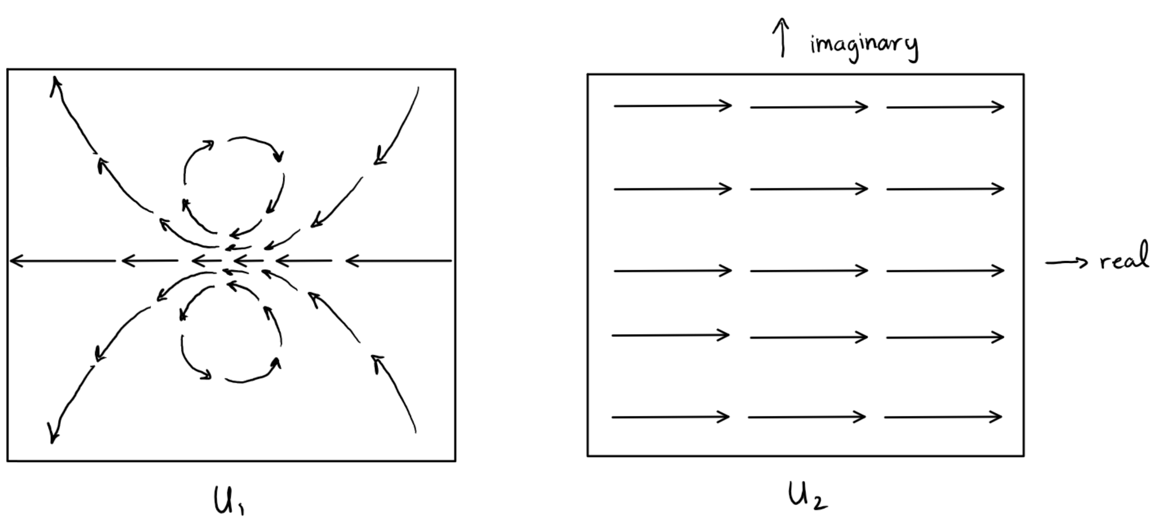
Example (Vector fields on $\mathbb{P}^1$)**.** $\mathbb{P}^1$ has homogeneous coordinate ring $k[z_1, z_2]$, so write $\mathbb{P}^1 = U_1 \cup U_2$ where $U_i = \{z_i \neq 0\}$. The regular function rings for U_1, U_2 are $k[\frac{z_2}{z_1}]$, $k[\frac{z_1}{z_2}]$ respectively. If we use Vakil's notation of $\frac{z_i}{z_j} = x_{i/j}$, then on $U_1 \cap U_2$, $x_{1/2} = (x_{2/1})^{-1}$. Vector fields on U_1 look like $p_1(x_{2/1})\frac{\partial}{\partial x_{2/1}}$, p_1 a polynomial. On U_2, vector fields look like $p_2(x_{1/2})\frac{\partial}{\partial x_{1/2}}$. On the intersection, how are these related? With intuition from differential geometry, we try writing

$$\frac{\partial}{\partial x_{1/2}} = \frac{\partial}{\partial (x_{2/1})^{-1})} = -x_{2/1}^2 \frac{\partial}{\partial x_{2/1}}.$$

If we don't have that intuition, we can instead look at

$$\frac{\partial}{\partial x_{1/2}} : x_{2/1}^n = x_{1/2}^{-n} \mapsto (-n)x_{1/2}^{-n-1} = (-n)x_{2/1}^{n+1} = -x_{2/1}^2 \frac{\partial x_{2/1}^n}{\partial x_{2/1}}$$

and so our guess was correct! Therefore we can extend $\frac{\partial}{\partial x_{1/2}}$ to a global vector field, because the $\frac{\partial}{\partial x_{1/2}}$ is regular on U_2, and $-x_{2/1}^2 \frac{\partial}{\partial x_{2/1}}$ is regular on U_1. If we draw it in $\mathbb{C}$, we get the following picture.



Other global fields on $\mathbb{P}^1$ are

$$x_{2/1}\frac{\partial}{\partial x_{2/1}} = -x_{1/2}\frac{\partial}{\partial x_{1/2}} \qquad \text{and} \qquad \frac{\partial}{\partial x_{2/1}} = -x_{1/2}^2\frac{\partial}{\partial x_{1/2}}$$

which we'll prove on the problem set is a basis for the k-vector space of regular global vector fields on $\mathbb{P}^1$. .

Can we describe this using derivations $k[z_1, z_2] \to k[z_1, z_2]$? We'll want them to be degree preserving. If $D : k[z_1, z_2] \to k[z_1, z_2]$ is a degree 0 (i.e. degree preserving) derivation and f is a non-homogeneous polynomial, then D extends to $f^{-1}k[z_1, z_2] \to f^{-1}k[z_1, z_2]$ and restricts to $(f^{-1}k[z_1, z_2])_0 \to (f^{-1}k[z_1, z_2])_0$. This extended and restricted function will be a tangent vector field to $X \setminus Z(f)$. Such degree preserving derivations have as a basis

$$z_1\frac{\partial}{\partial z_1}, \quad z_2\frac{\partial}{\partial z_1}, \quad z_1\frac{\partial}{\partial z_2}, \quad z_2\frac{\partial}{\partial z_2}$$

but this map has a kernel: $z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} \mapsto 0$. More generally, the vector space of degree preserving derivations on $k[x_0, \dots, x_n]$ has dimension $(n+1)^2$, with basis $z_j\frac{\partial}{\partial z_i}$; the vector space of vector fields on $\mathbb{P}^{n-1}$ is the surjective image of this but has dimension only $(n+1)^2-1$, as $\sum \frac{\partial}{\partial z_j}$ maps to 0.

In general, let $X \subset \mathbb{P}^n$ be a projective variety, and A the graded homogenous coordinate ring. We get a map from degree preserving derivations $A \to A$ to vector fields on X, but this map need be neither injective nor surjective. There doesn't seem to be a simple description of vector fields on X using commutative algebra of A.

November 9 : Varieties are generically smooth. This class was devoted to the dimension of tangent space at generic points of a quasi-projective variety. Here were the main results; let X be a quasiprojective variety:

Theorem. The function $x \mapsto \dim T_xX$ is upper semicontinuous, meaning that $\{x : \dim T_xX \geq k\}$ is closed.

For $x \in X$, let $d_x(X)$ be the maximum dimension of any component of X containing x.

Theorem. For all $x \in X$, we have $\dim T_xX \geq d_x(X)$.

Theorem. Let X be irreducible of dimension d. There is a nonempty (and therefore dense) subset U of X such that $\dim T_xX = d$ for $x \in U$.

To prove the first theorem, note that the result is local on X, so we may assume that X is affine, with $X \subset \mathbb{A}^n$, and we have $TX \subset \mathbb{A}^n \times \mathbb{A}^n$. Consider the variant $\mathbb{P}TX \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$, consisting of pairs $(x, [\vec{v}])$ with $\vec{v} \in T_xX$. We have the projection $\pi : \mathbb{P}TX \to X$. By the theorem on dimension of fibers, the dimension of the fibers of this map is upper semicontinuous. These fibers are precisely the projectivizations of the tangent spaces to X.

We now prepare to prove the other two results, which are largely independent.

Proof that $\dim T_xX \geq d_x(X)$. Our proof is by induction on $\dim T_xX$; the base case is actually the most interesting.

Base Case: Suppose $T_xX = \{0\}$. We must show that x is an isolated point. The claim is local, so assume X is affine with coordinate ring A and $\mathfrak{m} \subset A$, the maximal ideal corresponding to X. The hypothesis is that $\mathfrak{m}/\mathfrak{m}^2 = 0$. By Nakayama lemma, $\exists f \in A$, $f \equiv 1 \mod \mathfrak{m}$ such that $f^{-1}\mathfrak{m} = 0$. So, passing to $f^{-1}A$ we have $f^{-1}\mathfrak{m} = 0$. So, on $D(f)$, every function that vanishes at x is identically zero. So, there are no other points in $D(f)$ and x is isolated.

Inductive Case: Let $\dim T_xX = dimT_x^*X > 0$. Choose some $g \in \mathfrak{m}_X - \mathfrak{m}_X^2$. Consider $X' = X \cap Z(g)$. So, $\dim T_x^*X' \leq \dim T_xX - 1$. By the Krull's Principal Ideal Theorem, we have $d_x(X') \geq d_x(X) - 1$. Inductively, we have $\dim T_xX - 1 \geq \dim T_xX' \geq d_x(X') \geq d_x(X) - 1$ and thus $\dim T_xX \geq d_xX$. □

Finally, we show that $\dim T_xX$ is generically $\dim X$. The following key lemma is basically implicit differentiation:

Lemma. Let $Y = \mathrm{MaxSpec}(B) \to X = \mathrm{MaxSpec}(A)$ be a map of varieties. Suppose B is generated as an A-algebra by $\theta \in B$ satisfying $a(\theta) = 0$ and suppose $a'(\theta)$ is a unit in B. Then, for all $y \in Y$, $f_* : T_yY \to T_{f(y)}X$ is injective and dually, $f^* : T^*_{f(x)}X \to T^*_yY$ is surjective.

To be clear, writing $a(t) = \sum a_j t^j$, by $a'(t)$ we mean $\sum j a_j t^{j-1}$.

Proof. We'll check surjectivity in the dual spaces. Since B is generated by A and θ, T_yY is spanned by $\{da\}_{a\in A}$ and $d\theta$.

From the equation $\sum_{j=0}^{n} a_j\theta^j = 0$ we deduce $\sum(\theta^j da_j + j a_j \theta^{j-1} d\theta) = 0$. So $\sum j a_j \theta^{j-1} d\theta = -\sum \theta^j da_j$ and $d\theta = \frac{\sum a_j \theta^{j-1} d\theta}{a'(\theta)}$. Therefore, $d\theta$ is in the $B-span$ of $\{da\}_{a\in A}$. So, $[da]_{a\in A}$ span T_y^*Y and the map is surjective. □

Example. Let $\mathrm{MaxSpec}(k[x]) = X$ and $\mathrm{MaxSpec}(k[y]) = Y$ and consider the map $y \mapsto y^2$ from $Y \to X$. This corresponds to the inclusion $A = k[y^2] \subset k[y] = B$. Then, $B = A[y]$ and $a(T) = T^2 - y^2$ and $a'(T) = 2T$. Therefore, $a'(y)$ is not a unit in B. As seen in a previous class, the map of tangent spaces is zero at the point $0 \in Y$, hence, is not injective. If we modify $A' = X^{-1}A$ and $B' = Y^{-1}B$, then $a'(y)$ will be a unit and corresponds to the map of tangent spaces being injective.

We now need to appeal to

Theorem. (A strengthening of Noether normalization) If X is an irreducible d dimensional affine variety, then there is a finite surjective map $X \to \mathbb{A}^d$ such that $\mathrm{Frac}(X)/\mathrm{Frac}(\mathbb{A}^d)$ is separable.

Theorem. Let X be a d-dimensional irreducible quasi-projective variety. Then, there is a dense open subset $U \subset X$ such that $\dim T_uU \leq d$ for $u \in U$.

Proof. We may assume X is affine. Let $X = \mathrm{MaxSpec}(B)$. Choose a separable Noether Normalization $X \to \mathbb{A}^d$, let A be the coordinate ring of $\mathbb{A}^d$. Let B be generated by $\theta_1, \ldots, \theta_t$ over A. Consider the nested sequence $C_0 = A \subset C_1 = A[\theta_1] \subset \cdots \subset C_t = A[\theta_1, \ldots, \theta_t] = B$. So, C_{j+1} is generated over C_j by θ_{j+1}. Let θ_{j+1} satisfy a polynomial $a_{j+1}(T)$ over C_j. All $a'(\theta_{j+1})$ are non-zero in the domain B. By inverting them all (localizing at the product of all the a_j's), we have an open subset $U \subset X$. For $u \in U$, $f_* : T_uY \to T_{f(u)X}$ is a composition of the injective maps $T_uY \to T_uU$ and $T_uU \to T_{f(u)}X$. Therefore, is injective. Hence, $\dim T_uY \leq d$. □

November 12: Smoothness and Sard's Theorem. Recall from last time the following fact:

Theorem. If X is irreducible of dimension d, then

(1) For any $x \in X$, $\dim T_x X \geq d$.
(2) There exists a non-empty open subset $U \subseteq X$ such that $\dim T_x X = d$ for all $x \in U$.
(3) $\dim T_x X$ is an upper semi-continuous function of X, meaning
$$\{x \in X \mid \dim T_x X \geq k\}$$
is closed for all $k \geq 0$.

The idea of the proof of (2) is to choose a Noether normalization $\pi : X \to \mathbb{A}^d$ such that $\operatorname{Frac} X / \operatorname{Frac} \mathbb{A}^d$ is separable. This ends up giving us an open subset $U \subseteq X$ such that the induced map $f_* : T_x X \to T_{\pi(x)} \mathbb{A}^d \cong \mathbb{A}^d$ is an isomorphism for each $x \in U$.

These facts tell us that the singularities on varieties are relatively controlled, in that no "fractal" behavior can occur. We now define smoothness:

Definition. An algebraic variety X is ***smooth*** (or ***regular***, or ***non-singular***) if for all $x \in X$,
$$\dim T_x X = \max_{Y \ni x} \dim Y$$
where this max is taken over all irreducible components Y of X containing the point x.

Note that when X is irreducible of dimension d, this definition reduces to the condition that $\dim T_x X = d$ for every point $x \in X$. We also note that the notions of "smooth" and "non-singular" coincide in all contexts in which they are both defined, but the notion of "regular" is slightly more general, and slightly weaker.

The following wasn't actually said until a lot later, but belongs here:

Proposition. Let X be smooth of dimension n at x. Suppose that $f_1, f_2, \ldots, f_k$ are functions vanishing at x and that $df_1, df_2, \ldots, df_k$ are linearly independent in $T_x^* X$. Then $Z(f_1, \ldots, f_k)$ is smooth at x of dimension $n - k$.

Proof. Put $Y = Z(f_1, \ldots, f_k)$. By Krull's Principal Ideal Theorem, $\dim Y \geq n - k$. On the other hand, $T_x^* Y$ is a quotient of $T_x^* X$ under which $df_1, \ldots, df_k$ map to 0, so $\dim T_x^* Y \leq n - k$. And we know that $\dim T_x^* Y \geq \dim Y$. Concatenating these, $\dim Y = \dim T_x^* Y = n - k$. Furthermore, $f_{k+1}, \ldots, f_n$ give a basis of $T_x^* Y$. □

Example. Let $C = \{x^3 + y^3 = 1\} \subset \mathbb{A}^2$. The the projection $\pi : C \to \mathbb{A}^1$ of C onto the x-axis induces an isomorphism $T_{(x,y)} C \cong T_x \mathbb{A}^1 (\cong \mathbb{A}^1)$ for every $(x, y) \in C$ not equal to $(1, 0)$. Similarly, the projection $\pi : C \to \mathbb{A}^1$ of C onto the y-axis induces an isomorphism $T_{(x,y)} C \cong T_y \mathbb{A}^1$ for every $(x, y) \in C$ not equal to $(0, 1)$.

The following example exhibits how induced maps on tangent bundles can behave pathologically in positive characteristic:

Example. Let $\operatorname{char} k = p$. Then the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by $t \mapsto t^p$ has derivative zero at every point in $\mathbb{A}^1$.

Lets return to characteristic 0, and let X be smooth with a Noether normalization $\pi : X \to \mathbb{A}^d$ as above. Let $U \subseteq X$ be such that $U \to \mathbb{A}^d$ induces isomorphisms on each tangent space. Then $TU \cong U \times \mathbb{A}^d$, the trivial d-plane bundle on U, which illustrates the fact that TX is locally free when X is smooth. Given $f : X \to Y$, this fact allows us to explicitly compute the derivative map $f_* : TX \to TY$. To do so, we choose $U \subseteq X$ and $V \subseteq Y$ such that the restrictions $U \to \mathbb{A}^n$ and $V \to \mathbb{A}^m$ of the Noether normalizations of X and Y

induce isomorphisms on tangent spaces at each point. Then in the coordinates $\mathbb{A}^m, \mathbb{A}^n$, the derivative map is just given by the $m \times n$ matrix of partial derivatives of f.

Proposition. *Let X and Y be smooth of dimension m and n, respectively. Let $f : Y \to X$ be a regular map. Then the rank of f_* is a lower semi-continuous function on Y, in that*

$$\{y \in Y \mid \mathit{rank}(f_* : T_yY \to T_{f(y)}X) \leq k\}$$

is closed in Y for all $k \geq 0$.

To prove this, we need the following lemma:

Lemma. For any $k \geq 0$, the subset

$$\{M \in \mathrm{Mat}_{m\times n} \mid \mathrm{rank}(M) \leq k\} \subset \mathrm{Mat}_{m\times n} \cong \mathbb{A}^{mn}$$

is Zariski closed.

Proof. Consider $K := \{(M, [\vec{v}]) \mid M\vec{v} = 0\} \subseteq \mathrm{Mat}_{m\times n} \times \mathbb{P}^{n-1}$. Then we can consider the projection $\pi : K \to \mathrm{Mat}_{m\times n}$. Noting that

$$\mathrm{rank}(M) \leq k \Leftrightarrow \dim \pi^{-1}(M) \geq n - k - 1$$

it follows by upper semi-continuity of the dimension of fibers that

$$\{M \in \mathrm{Mat}_{m\times n} \mid \mathrm{rank}(M) \leq k\}$$

is closed. □

We now give a proof of the proposition:

Proof. Since the assertion is local on X and Y, we can assume $TX \cong X \times \mathbb{A}^m$ and $TY \cong Y \times \mathbb{A}^n$. The induced map $f_* : TY \to TX$ is then given by

$$\begin{aligned} TY &\xrightarrow{f_*} TX \\ (y, \vec{v}) &\mapsto (f(y), D(y)\vec{v}) \end{aligned}$$

where $D : Y \to \mathrm{Mat}_{m\times n}$ is regular. Then by the lemma, the collection of all rank $\leq k$ matrices in $D(Y)$ is a closed subset of $D(Y)$, hence its D-preimage, which is exactly $\{y \in Y \mid \mathrm{rank}(f_* : T_yY \to T_{f(y)}X) \leq k\}$, is closed in Y. □

We now state the algebraic analogue of Sard's theorem:

Theorem (Bertini)**.** Let char$k = 0$. Let X and Y be irreducible of dimensions m and n, and let $f : Y \to X$ be a regular map. Then there exists an open subset $U \subseteq X$ such that for any $x \in U$, either $f^{-1}(x) = \varnothing$ or $f^{-1}(x)$ is smooth of dimension $n - m$.

Note that this theorem trivial holds when $m > n$. We follow the statement of this theorem with two examples that illustrate why the assumption that char$k = 0$ is necessary. The first is a "moral" counterexample, which becomes a real counterexample when one takes a scheme-theoretic perspective, and the second is a true counterexample from our naive perspective.

Example. Let char$k = p$, and consider the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by $t \mapsto t^p$. Then the (scheme-theoretic) preimage of $a \in \mathbb{A}^1$ is

$$f^{-1}(a) = Z(t^p - a) = Z((t - a^{1/p})^p)$$

which is not reduced. However from our naive perspective, the preimage of a is just a single point, which is a smooth variety of dimension $1 - 1 = 0$, so the conclusion of the theorem holds in this case.

Example. Let char$k = p$ an odd prime, and consider the map $\mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto y^2 - x^p$. Then the preimage of $a \in \mathbb{A}^1$ is given by

$$\begin{aligned} f^{-1}(a) &= \{y^2 - x^p = a\} \\ &= \{y^2 = (x + a^{1/p})^p\} \end{aligned}$$

which is singular at $(x, y) = (-a^{1/p}, 0)$, so the conclusion of the theorem does not hold.

Corollary (Bertini)**.** *Let $X \subseteq \mathbb{P}^n$ be smooth of dimension d. Then for a generic (projective) hyperplane $H \subset \mathbb{P}^n$, $X \cap H$ is smooth of dimension $d - 1$.*

Proof. Let $(\mathbb{P}^n)^\vee$ be the dual projective space parametrizing hyperplanes in $\mathbb{P}^n$. Consider the closed set

$$E := \{(x, [H]) : x \in X,\ x \in H\} \subset X \times (\mathbb{P}^n)^\vee.$$

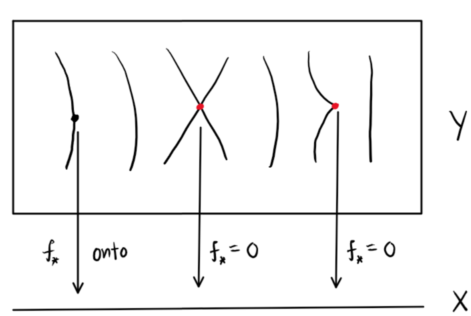
The projection $E \to X$ is a $\mathbb{P}^{n-1}$-bundle, so E is smooth of dimension $d + n - 1$. Next, we consider the projection $E \to (\mathbb{P}^n)^\vee$. By Sard's theorem, the fiber over a generic $[H] \in (\mathbb{P}^n)^\vee$ is smooth of dimension $(d + n - 1) - n = d - 1$, which proves the result. ☐

In fact, we have a stronger result:

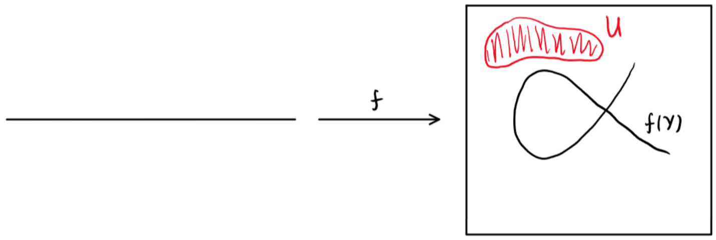
Theorem (Kleiman - Bertini)**.** Let G be an algebraic group acting transitively on a smooth variety Z of dimension n. Let X and Y be smooth subvarieties of dimensions d and e. Let char$k = 0$. Then for generic $g \in G$, $X \cap gY$ is either empty or smooth of dimension $(d + e - n)$.

November 14: Proof of Sard's theorem.

Theorem ("Sard's Theorem")**.** Let char$(k) = 0$. Let X, Y be quasi-projective varieties, and $f : Y \to X$ a regular map. Then there exists a dense, open subset $U \subseteq X$ so that $f_* : T_yY \to T_{f(y)}X$ is surjective for all $y \in U$.



Remark. If $\dim(Y) < \dim(X)$, this is obvious: we can choose an open set away from the image of f, like in the example to the right.



Proof. We proceed by induction on $\dim Y$, but first we make some preliminary reductions:

(1) First, we can reduce to the case when Y **is irreducible**. If $Y = \bigcup_j Y_j$ is the irreducible decomposition of Y, and U_j is a dense open in each Y_i satisfying the criteria of the theorem, then we can take $U = \bigcap_j U_j$.

(2) Next, we can **reduce to the case where** $X = \overline{f(Y)}$. If we can find $V \subseteq \overline{f(Y)}$ which satisfies the criteria of the theorem, then we can take $U = V \cup (X \setminus \overline{f(Y)})$.

(3) We can assume that X **is affine** since the question is local on X.

(4) If $Y = \bigcup_j Y_j$ is a covering of Y by open affines Y_j, and we find a U_j satisfying the criteria of the theorem for $Y_j \to X$, then $U = \bigcap_j U_j$ satisfies the criteria for $Y \to X$. Therefore, we can assume Y **is affine**.

To summarize the reductions: without loss of generality, we may assume that X, Y are irreducible affines and f is dominant (i.e. has dense image). Now let $n := \dim(Y)$, $m := \dim(X)$. Since X, Y are irreducible and f is dominant, $n > m$.

By relative Nöther normalization, we can pass to a dense open subset of Y so that the map factors as:

$$\begin{tikzcd} & X \times \mathbb{A}^{n-m} \arrow[dr, two heads] & \\ Y \arrow[ur, "\text{finite}", "g"'] \arrow[rr, "f"] & & X \end{tikzcd}$$

Note that g is separable since $\mathrm{char}(k) = 0$, so there's a closed $K \subsetneq Y$ so that

$$g_* : T_y Y \to T_{g(y)}(X \times \mathbb{A}^{n-m})$$

is an isomorphism for all $y \in Y \setminus K$. Namely,

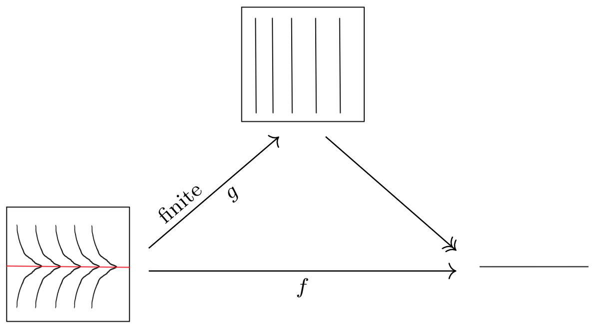
$$K = \overline{\{g \text{ is not injective}\}} \cup \{y \in Y \mid Y \text{ is singular at } y\} \cup f^{-1}(\{x \in X \mid X \text{ is singular at } x\})$$

If $y \notin K$, then $g_* : T_y Y \xrightarrow{\cong} T_{g(y)}(X \times \mathbb{A}^{n-m})$ is an isomorphism and $T_{g(y)}(X \times \mathbb{A}^{n-m}) \twoheadrightarrow T_{f(y)} X$ is a surjection, so $f_* : T_y Y \to T_{f(y)} X$ is a surjection as well.

Now $\dim(K) < n = \dim(Y)$, so by the inductive hypothesis, we can find a $U \subseteq X$ that works for K. We claim that this U also works for Y. To see this, let $y \in Y$ so that $f(y) \in U$. If $y \notin K$, then we're done. Otherwise, $y \in K$, so the composition

$$T_y K \hookrightarrow T_y Y \xrightarrow{f_*} T_{f(y)} X$$

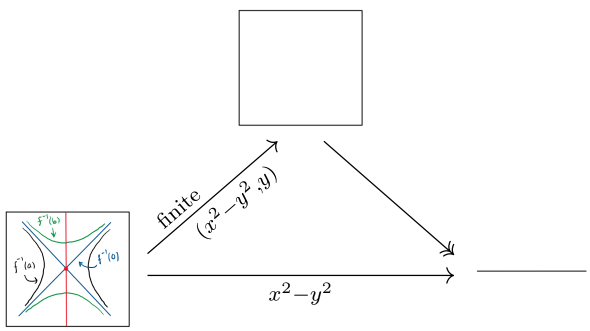
is a surjection, so f_* is a surjection as well. $\square$



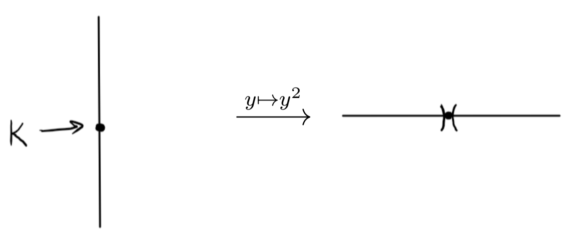
Remark. The following example shows what can go wrong if $\mathrm{char}(k) = p$.

Along the red line, g_* is not an isomorphism, but g_* is an isomorphism on the rest of Y, so we can take our dense open $V \subseteq Y$ to be the complement of the red line. But $g(V) = X$, so V doesn't translate to a dense open of X.

We also worked through an example that used the inductive step:



We see that g is singular along the red line, so we take K_1 to be the red line. In the next step, we pick K_2 to just be a point. Then we can take $U = X \setminus f(K)$.



Proposition. *Let Y be smooth of pure dimension n, X smooth of pure dimension m, $f : Y \to X$ regular. If $f_* : T_yY \to T_{f(y)}X$ is surjective, then $f^{-1}(f(y))$ is smooth at y and of dimension $n - m$.*

Proof. Let $F := f^{-1}(f(y))$ and $x = f(y)$. By the (first) theorem on the dimension of fibers, $\dim F \geq n - m$. Also, $T_yF \subseteq \ker(T_y \xrightarrow{f_*} T_xX)$, so $\dim T_yF \leq n - m$. Together, these two facts imply

$$n - m \leq \dim F \leq \dim T_yF \leq n - m \implies \dim F = \dim T_yF = n - m$$

$\square$

Corollary. *If $char(k) = 0$, Y is smooth of dimension n, X is smooth of dimension m and $f : Y \to X$ is regular, then there exists a dense open subset $U \subseteq X$ so that $f^{-1}(x)$ is smooth and of dimension $n - m$ for every $x \in U$.*

Conclusion: If $\mathrm{char}(k) = 0$, Y is smooth of dimension n, X is smooth of dimension m, and $f : Y \to X$ is regular, then there exists a dense open subset $U \subseteq X$ so that $f^{-1}(x)$ is smooth and of dimension $n - m$ for all $x \in U$.

At the end of class, Professor Speyer made a remark about choosing a Noether normalization:

Remark. Let $\dim X = d$ and $x \in X$. Someone asked: can we choose a Noether normalization $f : X \to \mathbb{A}^d$ so that $f_* : T_xX \to T_{f(x)}\mathbb{A}^d$ is an isomorphism?

Clearly, we need x to be smooth, or else the dimensions will not match. And, if x is smooth, we can indeed find such a Noether normalization!

The map π is given by a generic linear map (i.e. a $d \times n$ matrix). We showed that a generic such π will give a Noether normalization. It will also be true, for generic π, that $T_xX \hookrightarrow \mathbb{A}^n \xrightarrow{\pi} \mathbb{A}^d$ will be an isomorphism. So a generic π will have both such properties.

November 16: Completion and regularity. Today we discussed completion and regularity in commutative algebra.

Definition (I-adic completion)**.** Let A be a commutative ring. $I \subset A$ is an ideal. The I-adic completion of A is defined as

$$\hat{A} = \lim_{\leftarrow n} A/I^n,$$

where

$$\lim_{\leftarrow n} A/I^n = \{(a_1, a_2, a_3, ...)| a_j \in A/I^j, a_{j+1} \equiv a_j \mod I^j\}.$$

Example. An example element in $\lim_{\leftarrow n} \mathbb{Q}[x]/(x^n)$ would be

$$\left(1, 1+x, 1+x+\frac{1}{2}x^2, 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3, ...\right).$$

This is a ring where $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ...$ make sense.

Example. Assuming A is Noetherian, there exists an $M \in \mathbb{Z}_+$ such that

$$\sqrt{I}^M \subset I \subset \sqrt{I}.$$

Therefore I-adic and $\sqrt{I}$-adic completions are isomorphic.

Example. $k[x_1, ..., x_n]$ completed at $\langle x_1, ..x_n\rangle$ is $k[[x_1, ..., x_n]]$, the ring of power series.

Lemma. Let A be a Noetherian commutative ring and m be a maximal ideal. Then there exists a $f \equiv 1 \mod m$ such that the map $f^{-1}A \to \hat{A}$ is injective, where $\hat{A}$ is the m-adic completion.

Remark. Geometrically, if X is an variety and $x \in X$, there exists a Zariski distinguished open neighborhood U of x such that $\mathcal{O}_U \to \hat{\mathcal{O}}_X$ is injective.

Proof. The natural map $A \to \hat{A}$ which sends a to $(a, a, a, ...)$ has kernel

$$\bigcap_{j=1}^{\infty} m^j =: J.$$

Since J is an ideal of A, it is a finitely generated A-module. Note that $mJ = J$, by Nakayama's lemma, there exists a $f \equiv 1 \mod m$ such that $f^{-1}J = 0$. Note that $f^{-1}A \to \hat{A}$ has kernel $f^{-1}J$, hence this map is injective. □

Corollary. *If X is irreducible, then $\mathcal{O}_X \to \hat{\mathcal{O}}_X$ is injective.*

Remark. This is analogous to the "principle of analytic continuation": A regular function (on an irreducible variety) is determined by its power series at any point.

Staying in commutative algebra, let A be a Noetherian commutative ring. m be a maximal ideal. $F := A/m$ is a field and $V := m/m^2$ is a F-vector space. Then we have a natural map:

$$\mathrm{Sym}^d V \to m^d/m^{d+1}.$$

Lemma. The above map is surjective.

Proof. Let $\{v_1, ..., v_n\}$ be a F-basis of V. We lift them to $\{w_1, ..., w_n\} \subset m$. By Nakayama's lemma, after localizing we have $\{w_1, ..., w_n\}$ generates m as a A-module. Therefore $\{w_1^{k_1}...w_n^{k_n} | k_1 + ... + k_n = d\}$ generates m^d as an A-module. Therefore $\{v_1^{k_1}...v_n^{k_n} | k_1 + ... + k_n = d\}$ spans m^d/m^{d+1}. □

Definition (Regularity)**.** Let A be a Noetherian ring and m is a maximal ideal of A. A is regular at m if the map

$$\mathrm{Sym}^d V \to m^d/m^{d+1}$$

is an isomorphism for all d.

This definition is particularly nice if A is a k-algebra such that

$$k \to A \to A/m = F$$

is an isomorphism. In this case we can choose $\{v_1, ..., v_n\}$ as before and lift it to $\{w_1, ..., w_n\} \in A$. Then we get a map $k[x_1, ..., x_n] \to A$ such that $x_j \to w_j$.

Proposition. *In the above case, regularity is equivalent to*

$$k[x_1, ..., x_n]/\langle x_1, ..., x_n\rangle^d \to A/m^d \tag{1}$$

is isomorphism for all d.

Proof. LHS of (1) is filtered by $\langle x_1, ..., x_n\rangle^j/\langle x_1, ..., x_n\rangle^d$ and RHS of (1) is filtered by m^j/m^d. Therefore (1) is an isomorphism if and only if each

$$\langle x_1, ..., x_n\rangle^j/\langle x_1, ..., x_n\rangle^{j+1} \to m^j/m^{j+1}$$

is an isomorphism. □

Remark. In this setting, A is regular at m is equivalent to

$$\hat{A} \cong k[[x_1, ..., x_n]].$$

Theorem. Let X be an affine variety, $x \in X$. The ring of regular functions $\mathcal{O}_X$ is regular at m_x if and only if $\dim T_x X$ is equal to the dimension near x.

Proof. This proof was skipped in class. We abbreviate $\mathcal{O}_X$ to A and the maximal ideal of A corresponding to x to $\mathfrak{m}$.

Suppose that A is regular. We always have $d_1 := \dim T_x X \geq \dim X =: d_2$. Choose a Noether normalization $\pi : X \to \mathbb{A}^{d_2}$, corresponding to $R \subset A$. Let $\mathfrak{n}$ be the maximal ideal of R corresponding to $\pi(x)$. So we have a surjection $R^{\oplus r} \to A$ for some r and, since $\mathfrak{n}R \subseteq \mathfrak{m}$, we obtain a surjection $(R/\mathfrak{n}^N)^{\oplus r} \to A/\mathfrak{m}^N$. So $\dim A/\mathfrak{m}^N \leq r \dim R/\mathfrak{n}^N$. The right hand side is a polynomial in N of degree $\dim R = d_2$. If A is regular, then the left hand side is a polynomial of degree d_1 in N, so $d_1 \leq d_2$.

In the reverse direction, this theorem appears as Theorem 4 in Section II.2.2 of Shavarevich. Suppose that $\dim T_x^* X = \dim X = d$. Choose $f_1, \dots, f_d$ mapping to a basis of $T_x^* X$. We know that we have a surjection $k[[t_1, \dots, t_d]] \to \hat{A}$, we need to show that it is injective. In other words, given any nonzero degree k polynomial $g(t_1, \dots, t_k)$, we must show that $g(f_1, \dots, f_d) \notin \mathfrak{m}^{k+1}$. Suppose otherwise. After a change of coordinates, we may assume that the coefficient of t_d^k in g is nonzero. Let $C = Z(f_1, \dots, f_{d-1})$, so C is smooth of dimension 1, with f_d mapping to a basis of the one dimensional vector space $T_x^* C$. So, writing $\mathfrak{m}_C$ for the maximal ideal of x in C and passing to an open neighborhood, we have $\mathfrak{m}_C = (f_d)$ Restricting the equation $g(f_1, \dots, f_d) \notin \mathfrak{m}^{k+1}$ to C, we get that $f_d^k \in \mathfrak{m}_C^{k+1}$ on C. But that shows that $f_d^{k+1} | f_d^k$ on C, a contradiction. $\square$

Corollary. *If X is smooth at x, then X has an irreducible Zariski open neighborhood.*

Proof. Since the result is local, we can assume X is affine. There exists a neighborhood U of x such that the map $\mathcal{O}_U \to \hat{\mathcal{O}}_X \cong k[[x_1, ..., x_n]]$ is injective, hence $\mathcal{O}_U$ is a domain, which implies U is irreducible. $\square$

Finally, we extend the result which should have been stated on November 12: Let X be smooth of dimension n at x. Suppose that $f_1, f_2, \dots, f_k$ are functions vanishing at x and that $df_1, df_2, \dots, df_k$ are linearly independent in $T_x^* X$. We noted before that $Z(f_1, \dots, f_k)$ is smooth at x of dimension $n - k$. We now show that,

Proposition. After passing to an open neighborhood of x, the functions $f_1, \dots, f_k$ will generate the reduced ideal of Y.

Proof. Let A be the ring of regular functions on a neighborhood of x. We must show that, after passing to a possible smaller neighborhood of x, the ring $A/\langle f_1, \dots, f_k \rangle$ is reduced. We know that, after passing to such an open neighborhood, it injects into the completion $\hat{A}/\langle f_1, \dots, f_k \rangle$. But this is simply $k[[f_1, \dots, f_d]]/\langle f_1, \dots, f_k \rangle \cong k[[f_1, \dots, f_d]]$, which is reduced. $\square$

November 19: Divisors and valuations. Let X be an ambient (quasiprojective) variety, which we will assume to be irreducible.

Definition. *A **divisor** of X is an irreducible subvariety $D \subset X$ of codimension* 1.

Our goal for today is to define, for a rational function $f \in K(X)$, its "order of vanishing" along a divisor $D \subset X$. The intuition here is that D, being codimension 1, should be locally a hypersurface, i.e. after passing to an open subset $U \subset X$, we should have $Y \cap U = Z(g)$ for some $g \in \mathcal{O}(U)$. We can then ask, very roughly speaking: given an arbitrary rational function $f \in K(X)$, what is the "largest power of g dividing f"? That we can really make sense of this productively and in a way that does not depend on the choice of open set U is today's work. First, we show that indeed divisors are locally hypersurfaces; in fact, we show something more general.

Proposition. *Let $Y \subset X$ be a subvariety. Suppose that at a point $z \in Y$, we have X is smooth of dimension m, and Y is smooth of dimension n. Then, there exists an open neighborhood $z \in U \subset X$ such that $I(Y \cap U) \subset \mathcal{O}(U)$ is generated by $m-n$ regular functions.*

The catchy mnemonic version of the above result is "*smooth inside smooth is a locally complete intersection.*"

Proof. The statement is local, so we may as well assume X and Y are affine with $X = \mathrm{MaxSpec}(A)$ and $Y = \mathrm{MaxSpec}(B)$. Setting $I := I(Y) \subset A$, we have an exact sequence

$$0 \to I \to A \to B \to 0$$

of A-modules. Let $\mathfrak{m}_A \subset A$ and $\mathfrak{m}_B \subset B$ be the maximal ideals in A and B corresponding to the point z (i.e. regular functions vanishing at z). The above exact sequence restricts to an exact sequence

$$0 \to I \to \mathfrak{m}_A \to \mathfrak{m}_B \to 0$$

and then, tensoring with the A-module $A/\mathfrak{m}_A$, we obtain a right exact sequence

$$I/\mathfrak{m}_A I \to \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to 0$$

The inclusion $Y \hookrightarrow X$ induces a map $T_X^* \to T_Y^*$ of cotangent spaces and, by exactness of the sequence above, it follows that the inclusion $I \hookrightarrow A$ is a surjection onto $\mathrm{Ker}(T_X^* \to T_Y^*)$.

Choose $f_1, \cdots, f_{m-n}$ in I mapping onto a basis of $\mathrm{Ker}(T_X^* \to T_Y^*)$, and let

$$Y' := Z(f_1, \cdots, f_{m-n}) \subset X$$

Then $Y \subset Y'$ and Y' is smooth near z (of dimension n). Let $U \subset X$ be an open subset such that $U \cap Y'$ is irreducible (take e.g. U to be the complement of all the irreducible components of Y' not containing z). Since $U \cap Y' \supset U \cap Y$ and both are irreducible of the same dimension, it follows that $U \cap Y' = U \cap Y$. Hence $I(Y \cap U) \subset \mathcal{O}(U)$ is precisely

$$I(Y' \cap U) = \sqrt{\langle f_1, \cdots, f_{m-n} \rangle} = \langle f_1, \cdots, f_{m-n} \rangle$$

where the latter equality follows from the fact that $f_1, \cdots, f_{m-n}$ form a system of parameters for the local ring $\mathcal{O}_{Y,z}$. ☐

In particular, if $D \subset X$ is a divisor and $z \in D$ is a smooth point of *both* D and X, then D is locally a hypersurface near z. (In fact, something stronger holds: the same result is true if z is only a smooth point of X, and not necessarily a smooth point of D.)

Definition. *Let D be a principal divisor of an irreducible affine open set $U \subset X$, i.e. $I(D) = \langle f \rangle \subset \mathcal{O}(U)$ for some $f \in \mathcal{O}(U)$. If $g \in \mathcal{O}(U)$, we define its* ***order along*** *D as*

$$v_{D,U}(g) = \max\{n \,:\, g \in I(D)^n\}$$

That this maximum is well-defined follows from Proposition A.12 in Shafarevich. In the setting of the definition above, if $v_{D,U}(g) = n$, then $g = f^n u$ for some $u \notin I(D)$, i.e. $u|_D$ is nonzero. If we restrict to any smaller irreducible affine $U' \subset U$ with $U' \cap D \neq \emptyset$, then $u|_{D \cap U'}$ is still nonzero, so $v_D(g)$ stays the same. It follows that if we compute $v_D(g)$ using an open affine irreducible $U \subset X$ and a different open affine irreducible $V \subset X$, then we obtain the same result by passage $U \hookleftarrow U \cap V \hookrightarrow V$. Accordingly, we can refine the definition as follows:

Definition. *Let D be a divisor of X such that there exists some irreducible affine open $U \subset X$ in which D is a hypersurface. Then for any regular function $g \in \mathcal{O}(V)$ for some open $V \subset X$, we can define the* ***order along*** *D by $v_{D,U\cap V}(g)$. The above argument shows this does not depend on U, so we can just write $v_D(g)$.*

It is relatively straightforward to check that the valuation v_D satisfies the properties:

$$v_D(g_1g_2) = v_D(g_1) + v_D(g_2)$$
$$v_D(g_1 + g_2) \geq \min(v_D(g_1), v_D(g_2))$$

Accordingly, v_D can be extended to $K(X)^*$ via $v_D : K(X)^* \to \mathbb{Z}$ given by

$$v_D(g/h) := v_D(g) - v_D(h)$$

Notice that if $\dim(\mathrm{Sing}(X)) \leq \dim(X) - 2$, then any $D \subset X$ satisfies the condition that it is locally a hypersurface in some open neighborhood: just choose $z \notin \mathrm{Sing}(X) \cup \mathrm{Sing}(D)$ (possible by dimension), then apply the proposition above to obtain a neighborhood of z in which D is principal.

With this machinery, we can now talk about ramification indices. Suppose $\pi : Y \to X$ is a finite surjection; assume X and Y are smooth in codimension ≥ 1. Let $E \subset Y$ be a divisor. Then, since π is closed, $D := \pi(E)$ will be a divisor in X. We have valuations $v_E : K(Y) \to \mathbb{Z}$ and $v_D : K(X) \to \mathbb{Z}$.

Definition. *The* ***ramification index of*** *π* ***at*** *E is the positive integer r such that $v_E(\pi^\star f) = rv_D(f)$.*

To justify that such an r exists: pass to an open neighborhood on which D and E are principal. Let g_D be the local equation for D, and define $r := v_E(\pi^\star g_D)$. For any $f \in K(X)$, write $f = g_D^k u$ where $k = v_D(f)$ and $u|_D \neq 0$. Then $\pi^\star(f) = \pi^\star(g_D)^k\pi^\star(u)$, hence

$$v_E(\pi^\star f) = kv_E(\pi^\star g_D) = rv_D(f)$$

as desired.