PROBLEM SET 9

1. Let G be a graph whose edges are assigned lengths. Let t be the length of the shortest spanning tree of G. Let s be the length of the shortest path which visits every vertex.

1.a Show that $t \leq s$.

1.b Show that $s \leq 2t$.

Computing s is known as the traveling salesman problem, and is famously difficult. Nonetheless, this argument shows that it is easy to get a crude approximation to s.

2. Give an example of a graph G, with lengths assigned to edges with the following property: For any spanning tree T of G, there is some pair of vertices so that the distance from u to v in T is ≥ 100 times larger than the distance between u and v in G.

3. We showed in class that, if a graph has all edge lengths distinct, then there is a unique minimal spanning tree. Give an example of a graph where all edges are of distinct lengths, but the second and third best spanning tree have the same length.

4. The point of this problem is to prove a lemma we used in the course of verifying the Fürer-Raghavachari algorithm. Let G be a graph, T as spanning tree of G, and k the maximum degree of any vertex in T. Let S be the set of degree k vertices of T and B a subset of the degree $k-1$ vertices of T; write s for $|S|$ and b for $|B|$. Suppose that there are no edges of G joining distinct components of $T \setminus (S \cup B)$.

The result we will be establishing is that *any* spanning tree T' of G has a vertex of degree $\geq k-1$.

4.a Show that there are at most $s+b-1$ edges of T connecting one vertex in $S \cup B$ to another vertex in $S \cup B$.

4.b Show that there are at least $(k-2)s + (k-3)b + 2$ edges connecting a vertex in $S \cup B$ to a vertex not in $S \cup B$.

Let H be the graph formed from G by deleting any edge of G which has one or more endpoints in $S \cup B$. So every vertex in $S \cup B$ forms a connected component in H.

4.c Show that H has at least $(k-1)s+(k-2)b+2$ connected components, including the isolated vertices from $S \cup B$.

4.d Now, let T' be any spanning tree of G. Show that there must be at least $(k-1)s+(k-2)b+1$ edges of T' with one more endpoints in $S \cup B$.

4.e Show that there is a vertex in $S \cup B$ with T'-degree $\geq k-1$.