## PROBLEM SET 8

1. The aim of this problem is to show an example of a graph with irrational edge weights where the Ford-Fulkerson algorithm does not halt.


Our graph is shown in the image above. The capacities are as follows:

$$
c(v, u)=c(v, w)=1, \quad c(x, w)=\tau:=\frac{\sqrt{5}-1}{2}, \quad \text { all other edges have } c=10 .
$$

1.a Compute the maximum possible flow through $G$. Give an example of a cut whose capacity equals this flow.
1.b Consider the flow $\Phi_{0}$ which is 1 on $s \rightarrow v \rightarrow w \rightarrow t$. What is the residual graph for this flow?

Define the paths $p_{1}=(s \rightarrow x \rightarrow w \rightarrow v \rightarrow u \rightarrow t), p_{2}=(s \rightarrow v \rightarrow w \rightarrow x \rightarrow t)$ and $p_{3}=(s \rightarrow u \rightarrow v \rightarrow w \rightarrow t$.
1.c Take the flow $\Phi_{0}$ in the previous example and increment it along $p_{1}$. As a check on your correctness, the most you should be able to increment along this path is $\tau$. Let $\Phi_{1}$ be the flow obtained.
1.d Draw the residual graph for $\Phi_{1}$. How much can you increment $\Phi_{1}$ along path $p_{2}$ ? Let the resulting flow be $\Phi_{2}$.
1.e Continuing as above, increment $\Phi_{2}$ along $p_{1}$, then increment the resulting flow along $p_{3}$. Then continue with $p_{1}, p_{2}, p_{1}, p_{3}, \ldots$, with the pattern repeating with period 4 . At each step, how much do you increase the total flow?
1.f Show that, no matter how many times you go through the procedure in part 1.e, you'll never get to even half the total capacity of the network.
2. I have heard the following algorithm proposed for the stable marriage problem:

Form a directed graph, with a vertex for every person, and with an edge from $i$ to $j$ if $j$ is $i$ 's first choice. Since this graph has $2 n$ edges and $2 n$ vertices, it has a directed cycle. Marry off the people in that directed cycle to each other, with each woman getting her first choice. (If there is more than one cycle, choose one arbitrarily.) Then repeat the algorithm with the remaining people.
Give an example where the output of this algorithm is not stable.
3. Suppose that we modify the Gale-Shipley algorithm as follows: Rather than men always proposing to woman, at each step of the algorithm, choose a random single person $p$ and have $p$ propose to his or her favorite person to whom he or she has not yet proposed. Call that person $q$. If $q$ is single, then $q$ accepts $p$ 's proposal. If $q$ prefers $p$ to her or his current partner, then $q$ leaves her or his current partner and pairs with $p$; if $q$ prefers her or his current partner then she or he rejects $p$. The process ends when everyone has a partner.

Is the resulting matching necessarily stable?
4. Let $M_{1}$ and $M_{2}$ be two stable matchings. For any man $m$, let $w_{1}(m)$ be his partner in $M_{1}$ and $M_{2}$ his partner in $M_{2}$. Define $w(m)$ to be whomever of $w_{1}(m)$ and $w_{2}(m)$ the man $m$ prefers.
4.a Show that, for any two men $m$ and $m^{\prime}$, we have $w(m) \neq w\left(m^{\prime}\right)$. So pairing $m$ with $w(m)$ is a matching.
4.b Show that this matching is stable. This matching is called $M_{1} \vee M_{2}$.

Remark: This problem gives an alternate proof that there is a matching where every man simultaneously achieves his favorite among all women he could be stably matched to. Let $M_{1}, M_{2}$, $\ldots, M_{r}$ be all the stable matchings, and consider the matching $M_{1} \vee M_{2} \vee \cdots \vee M_{r}$.

