PROBLEM SET 6
DUE OCTOBER 26!

1. Let G be a graph with n vertices, and every vertex of degree d. Let A be the adjacency matrix of G, and let L be the Laplacian $L = d \cdot \mathrm{Id}_n - A$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A, with $\lambda_n = d$.

1.a Let a_k be the number of cycles of length k in G. Let $F(x) = \sum_{k=1}^{\infty} a_k x^k$. Express $F(x)$ in terms of the λ_i. (Note that I start the sum at $k = 1$. If you include the $k = 0$ term, you'll have difficulty in all the later parts of this problem.)

1.b Set

$$G(x) = e^{-\int_0^x F(t)/t dt}.$$

Show that G is a polynomial of degree n.

1.c Recall from class that the number of spanning trees of G is $1/n \prod_{i=1}^{n-1}(d - \lambda_i)$. Express this quantity in terms of d, n and G. Thus, if we know the number of cycles in G, we also know the number of spanning trees.

Remark: It appears that this method requires us to know all the infinitely many a_k. In fact, you only need to know $a_1, a_2, \ldots, a_n$. Do you see why?

2. This problem continues Problem 2 from Problem Set 4. Let G be a graph where every vertex has degree d, and where there are no loops, multliple edges, 3-cycles or 4-cycles. In the previous problem, you showed that this graph had at most $d^2 + 1$ vertices. In this problem, we will assume that it has exactly $d^2 + 1$ vertices, and deduce the consequences thereof.

Let A be the adjacency matrix of this graph; so $A_{ij} = 1$ if there is an edge between i and j and 0 otherwise. Let I be the $(d^2 + 1) \times (d^2 + 1)$ identity matrix, and let J be the $(d^2 + 1) \times (d^2 + 1)$ matrix which is all ones.

2.a Show that

$$A^2 + A = (d - 1)I + J.$$

2.b Compute the eigenvalues of $(d - 1)I + J$.

2.c Let λ_+ and λ_- be $\frac{-1 \pm \sqrt{4d-3}}{2}$. Show that the eigenvalues of A are d, with multiplicity 1, and λ_+ and λ_-, with multiplicities we don't know. Let m_+ and m_- be the multiplicities of λ_+ and λ_-.

2.d The aim of this part of the problem is to find two linear equations for m_+ and m_-. First, explain why

$$m_+ + m_- = d^2.$$

Next, notice that the diagonal entries of A are all 0, so $\mathrm{Tr} A = 0$. Express the trace of A in terms of 1, $m_\pm$ and $\lambda_\pm$ to give another linear equations for the m's.

2.e Solve these linear equations for m_+ and m_-. Plug in $d = 2$ through 10. For which of these values of d can you immediately conclude no graph like G exists?

Remark: If you know some number theory, you might enjoy showing that the only possible values of d are 2, 3, 7 and 57. I'm not sure whether or not the latter two are possible.