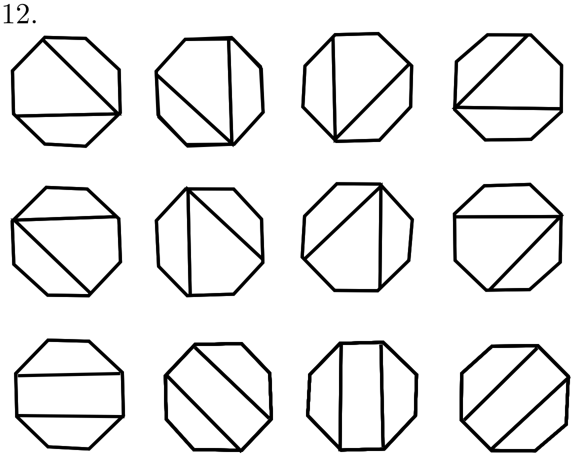
PROBLEM SET 3

1. In *Topological Methods in Algebraic Geometry*, Hirzebuch winds up studying a power series $F(z) = F_{-1}z^{-1} + F_0 + F_1 z + F_2 z^2 + \cdots$ with the property that, for every n, the coefficient of z^{-1} in $F(z)^n$ is 1. Use Lagrange inversion to find out what F is.

2. Let a_n be the number of ways to divide a $2n$-gon into quadrilaterals. For example, $a_4 = 12$.



Let $A(x) = \sum a_n x^n$. Find an equation obeyed by $A(x)$. Find a formula for a_n. (You can do this with Lagrange inversion, but there are many other ways.)

3. Let t_n be the number of planar trees where, for every vertex, each child is labeled as LEFT or RIGHT, but we are allowed to have an arbitrary number of left and an arbitrary number of right vertices. Let $T(x) = \sum t_n x^n$. Find a function W such that $T(x) = xW(T(x))$. Find a formula for t_n.

4. Consider rooted trees where every vertex has at most 2 children, but the trees are *not* planar, so we don't keep track of the order of the children. Let $F(x)$ be the corresponding generating function. Show that

$$F(x) = x + \frac{x}{2}\left(F(x)^2 + F(x^2)\right).$$

(Harder) Work out the corresponding relation when a vertex may have ≤ 3 children, and we again don't care about order. This example is important in chemistry, where such trees occur as the configurations of carbon atoms in alkane radicals.

5. On a one way street with m parking spaces, m drivers drive through looking for parking. The k-th driver starts looking when he reaches position p_k and then takes the next available spot. The sequence $(p_1, p_2, \dots, p_m)$ is called a *parking function* if all the drivers successfully park. So, if $m = 2$, then $(1, 1)$, $(1, 2)$ and $(2, 1)$ are all parking sequences, but $(2, 2)$ is not.

(a) For any sequence $(p_1, p_2, \dots, p_m)$ of numbers between 1 and m, let c_k by the number of times that k appears in the sequence (p_i). Set $(d_1, \dots, d_{m+1}) = (0, c_m, c_{m-1}, \dots, c_1)$. Show that $(p_1, \dots, p_m)$ is a parking sequence if and only if $(d_1, d_2, \dots, d_{m+1})$ is a Lukasiewicz word. (Recall that this means $1 - d_1$, $2 - d_1 - d_2$, $\dots$, $(m+1) - d_1 - d_2 - \dots - d_m)$ are all nonnegative and $(m+1) - \sum d_i = 1$.)

(b) Consider a Lukasiewicz word $(d_1, d_2, \dots, d_{m+1})$. Show that there are $\frac{m!}{(d_1)!(d_2)!\cdots(d_{m+1})!}$ sequences $(p_1, \dots, p_m)$ which correspond to it.

6. (continues ideas from Problem 5) Let P_m be the number of parking sequences. Problem 5 shoes that$P_m = \sum \frac{m!}{(d_1)!(d_2)!\cdots(d_{m+1})!}$, where the sum is over Lukasiewicz words. As we learned in class, $\sum \frac{(m+1)!}{(d_1)!(d_2)!\cdots(d_{m+1})!} = (m+1)^m$, so $P_m = (m+1)^m/(m+1) = (m+1)^{m-1}$. We will now provide a direct proof that $P_m = (m+1)^{m-1}$.

(a) Imagine that we now have m cars parking on a circular road with $m + 1$ spots. Again, the k-th driver starts looking for parking in position p_k, and then drives around the ring until she finds parking. After all the cars are parked, there will be one empty space. Show that $(p_1, p_2, \dots, p_m)$ is a parking sequence if and only if that empty space is in position $m + 1$.

(b) Show that, of the $(m + 1)^m$ possible ways for the m cars to try parking on a circular road, precisely $(m + 1)^{m-1}$ wind up leaving the empty space in position $m + 1$.

Remark for computer scientists: We can think of the parking procedure is attempting to insert m items into a hash table of size m. Of the m^m possible hashes, we have just shown that $(m + 1)^{m-1}$ are inserted with overflowing the table; that is to say, we avoid an overflow with probability $(1+1/m)^m(m+1)^{-1} \approx em^{-1}$. Of course, the question of practical interest is to insert a number of entries that is much less than the size of the table; that question can be addressed by more sophisticated versions of these methods.