PROBLEM SET 2

I will use the notation $[n]$ for $\{1, 2, \ldots, n\}$.

1. Partitions

1. (*A course in combinatorics*, Problem 15C) Let a_1, a_2, $\ldots$, a_t be positive integers with GCD 1. Let $f(n)$ denote the number of ways to write n as $x_1a_1 + x_2a_2 + \cdots x_ta_t$ with the x_i nonnegative integers. Show that $\lim_{n\to\infty} f(n)/n^{t-1} = c$ for some constant c, and compute c. Where did you use the assumption that the a_i were relatively prime?

2. Show that

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{k=1}^{\infty} \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

$$\prod_{n=1}^{\infty} \left(1+q^k\right) = \sum_{k=1}^{\infty} \frac{q^{\binom{k}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{k=1}^{\infty} \frac{q^{k^2}}{\left((1-q)(1-q^2)\cdots(1-q^k)\right)^2}.$$

3.: Let $x(n)$ be the number of ordered triples (α, β, γ), where each of α, β and γ is a partition with distinct parts, and $|\alpha|+|\beta|+|\gamma| = n$. For example, $x(2) = 6$: there are 3 permutations of $((2), \emptyset, \emptyset)$ and 3 permutations of $((1), (1), \emptyset)$. Notice that $((1, 1), \emptyset, \emptyset)$ does not occur, because $(1, 1)$ does not have distinct parts.

Let $x_o(n)$ and $x_e(n)$ be the number of such triples $\#\alpha + \#\beta + \#\gamma$ is odd or even respectively.[1]. So $x_o(2) = x_e(2) = 3$.

Write a product formula for $\sum x(n)q^n$ and for $\sum(x_e(n) - x_o(n))q^n$.

Compute enough terms of $\sum(x_e(n) - x_o(n))q^n$ to conjecture a simpler formula for it. (A computer may help here.) Prove that formula.

[1]Here $\#\lambda$ is the number of parts of λ

2. Catalan numbers

***4. (A course in combinatorics)*, Problem 14I**: We have $2n$ points on the boundary of a circle. We want to join them by n chords which do not cross or share an endpoint. In how many ways can this be done?

5. (*Enumerative Combinatorics II*, Problem 6.19.ii): You have a deck of n cards, numbered 1 through n, which you would like to sort into order. You have only enough room on your desk for one stack of cards, in addition to the output deck. You can take the top card of your deck and place it on the top of the stack, or put the card on the top of the stack into the output deck. How many permutations of the initial deck can be sorted in this manner?

6. Take your favorite bijection between triangulations of an n-gon and binary trees with $n-2$ leaves. (In a binary tree, each internal node has at most two children, and the children are labeled `LEFT` and `RIGHT`; even when there is only one child, it is labeled.)

Take a triangulation T, with corresponding tree τ. Take two triangles of T which border along a common diagonal and flip that diagonal to give a new triangulation T'. Let τ' be the corresponding tree. Describe τ' in terms of τ.

Remark: This operation is called "rotation", and is frequently used as a primitive operation when reorganizing tree-like data structures. For a nice challenge, prove that, for $m \geq 11$, it is possible to get from any m leaf binary tree to any other by using $\leq 2m-10$ rotations.[2]

[2]This bound is optimal, see Sleator, Tarjan and Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, Journal of the AMS, **1** (1988) no. 3, p. 647-681.