## PROBLEM SET 1

### 1. Who are you?

Are you an undergraduate or graduate student?

What is your major?

What math courses have you taken previously?

Which of the following notations and concepts are you familiar with: Generating functions,  $\binom{n}{k}$ , category theory,  $O(n \log n)$ , representation theory,  $\bigwedge^k V$ , hashing, regular expressions, trees? I plan to hold office hours Monday 10-11 and Thursday 2:30-3:30. Does at least one of these times work for you?

### 2. Basic enumeration and binomial coefficients

Answer at least 3.

1. (A Course in Combinatorics, Problem 13A) We want to place the integers 1, 2, ..., r into a circular array with n positions so that they occur in order, clockwise, and such that consecutive integers (including the pair (r, 1)) are not adjacent. Arrangements which are rotations of each other are considered the same. In how many ways can this be done?

2. (A Course in Combinatorics, Problem 13D) Consider the set S of all ordered k-tuples of subsets of [n]. What is

$$\sum_{(A_1,A_2,\ldots,A_k)\in S} |A_1\cup A_2\cup\cdots\cup A_k|?$$

3. (A Course in Combinatorics, Problem 13H) Let  $A_n$  be the  $n \times n$  matrix whose (i, j) entry is  $\binom{i}{i}$ , with rows and columns numbered starting from 0. So, for example,

$$A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

Compute  $A_2^{-1}$ ,  $A_3^{-1}$  and  $A_4^{-1}$ . Find and prove a formula for  $A_n^{-1}$ .

4. (Putnam, 1985, A-1) How many ordered triples of sets (A, B, C) are there such that  $A \cup B \cup C = [n]$  and  $A \cap B \cap C = \emptyset$ ?

5. Mathematics for the analysis of Algorithms, C.1 In a bubble sort, one is given a list of n distinct numbers to sort into order. One first compares the first two numbers, interchanging them if they are out of order. One then compares the second and third, then the third and fourth and so on. After one has done n - 1 comparisons, one returns to the start of the list and does it again.

Of the n! possible starting arrangement, how many will be correctly sorted at the end of the first pass? At the end of the first two passes? (I recommend you gather some data before you try to answer this.)

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(Harder) In a cocktail-shaker sort, one first compares the first two numbers, then the second and third and so forth as before but, after comparing the (n-1)st and nth numbers, one backtracks to compare (n-2)nd and (n-1)st, then (n-3)rd and (n-2)nd and so forth. Let  $x_n$  be the number of arrangements that are sorted by one pass each way. Deduce the recurrence:

 $x_n = x_{n-1} + x_{n-1} + 2x_{n-2} + 4x_{n-3} + 8x_{n-4} + 16x_{n-5} + \cdots$ 

# 3. Linear Recurrences

Answer at least 3.

6. You may want to use a computer for this one: Amber likes to choose pronounceable passwords. This means that her passwords are made of uppercase letters of the alphabet and they never contain two (or more) consecutive vowels nor three (or more) consecutive consonants<sup>1</sup> For example, one of her passwords might be RADSOTANKEDA. Let  $a_n$  be the number of passwords of length n that Amber can choose. Find  $\sum a_n x^n$ . The quantity  $a_n$  grows like  $\alpha^n$  for some  $\alpha$ : find it. Beth composes her passwords out of uppercase letters as well, but obeys no other restrictions. How long should Amber choose a password, in order to have as many options as Beth gets from an ten character password?

**7.** Let  $F_n$  be the Fibonacci numbers, defined by  $F_0 = 0$ ,  $F_1 = 1$  and  $F_n = F_{n-1} + F_{n-2}$ . Show that there are constants a, b and c such that  $F_0 + F_1 + \cdots + F_n = aF_n + bF_{n-1} + c$ . For elegance points, prove this without going to the trouble of computing (a, b, c).

# 8. (Concrete Mathematics, Exercise 7.7) Solve the recurrence:

$$g_0 = 1$$
  
 $g_n = g_{n-1} + 2g_{n-2} + \dots + ng_0$ 

9. (Concrete Mathematics, Exercise 7.23) In how many ways can a  $2 \times 2 \times n$  tower be built out of  $1 \times 1 \times 2$  bricks?

10. In problem 10, you derived the recursion

$$x_n = x_{n-1} + x_{n-1} + 2x_{n-2} + 4x_{n-3} + 8x_{n-4} + 16x_{n-5} + \cdots$$

with  $x_n = 1$ . Find a closed form for  $x_n$ .

<sup>&</sup>lt;sup>1</sup>For simplicity, treat Y as a consonant.