Lecture 2: Tropical geometry and positivity

With a potential encore about weights and the balancing condition

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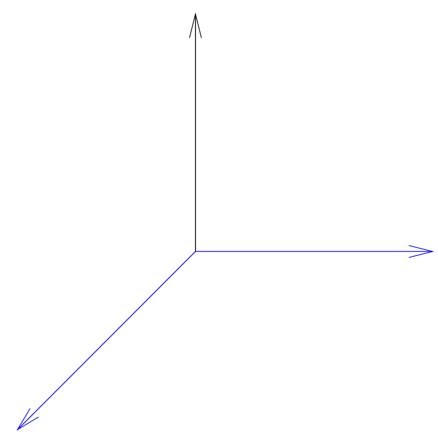
Recall $\mathcal{K} = \bigcup_{m \geq 1} \mathbb{C}((t^{1/m}))$. Let $\mathcal{R} = \bigcup_{m \geq 1} \mathbb{R}((t^{1/m})) \subset \mathcal{K}$. Let $\mathcal{R}$ be the sub-semifield of power series whose leading coefficient is positive.

Suppose our variety $X \subset (\mathcal{K}^\times)^n$ is defined by equations in $\mathcal{R}$. We'll set

$$\mathrm{Trop}_+(X) = \{w \in \mathbb{Q}^n : \mathrm{in}_w(X) \cap (\mathbb{R}_{>0})^n \neq \emptyset\}.$$

$\mathrm{Trop}_+(X)$ is a subfan of $\mathrm{Trop}\, X$



$y = x + 1$

I'm not sure this is the best definition. It would seem more elegant to me to set $\mathrm{Trop}_+(X) = v(X \cap \mathcal{R}^n_+)$. Danielle Alesandrini has studied this. It is also a fan, but possibly a smaller one.

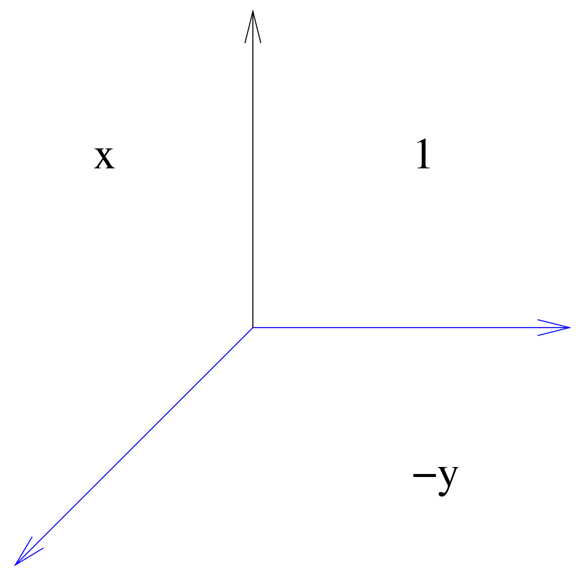
Example

$$X = \{x^2 - 2x + 1 + t^2 = 0\} = \{1 + it, 1 - it\}.$$

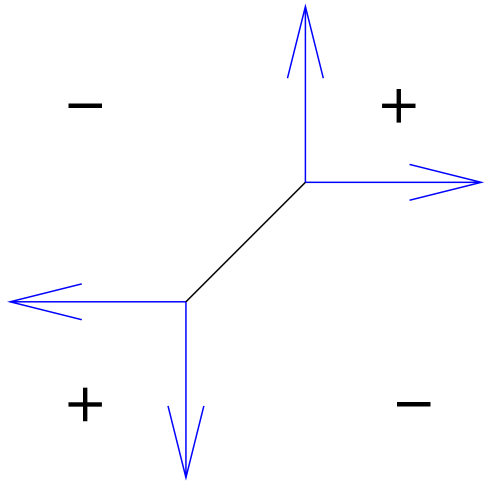
Then $\mathrm{in}_0(X) = x^2 - 2x + 1 = (x - 1)^2$, so 0 is a positive point by the definition on the previous slide, but not by the alternative here. In particular, whether $w \in \mathrm{Trop}_+(X)$ cannot be determined by $\mathrm{in}_w X$ with the proposed alternate definition, which is inconvenient in many ways.

I am going to be focusing on examples (coming from cluster algebras) where this distinction isn't important. In particular, if $\mathrm{in}_w X$ is smooth, then the two definitions are equivalent at w.

Let X be a hypersurface with constant coefficients in $\mathbb{R}$. Let F be the defining equation and let Δ be the Newton polytope. Suppose that each edge of Δ has no inner lattice points. Then $\mathrm{Trop}_+(X)$ consists of those walls of the normal fan which separate positive and negative coefficients.



If X has variable coefficients in $\mathcal{K}$, then similarly $\mathrm{Trop}_+(X)$ is dual to those edges which separate coefficients in $\mathcal{R}_+$ and $\mathcal{R}_-$



$$1 - x - y + txy$$

This is key to Viro's patchworking method.

The great thing about positivity is that interacts so well with *parameterization*.

Suppose that we have a rational function $f/g \in \mathbb{R}(x_1, \ldots, x_n)$, where both f and g have positive coefficients.

Let $y_1, \ldots, y_n \in \mathcal{R}_+$. Then

$$v\left(\frac{f(y_1, \ldots, y_n)}{g(y_1, \ldots, y_n)}\right)$$

is determined by $v(y_1)$, $v(y_2)$, $\ldots$, $v(y_n)$.

The recipe:

- Replace $+$ by min
- Replace $\times$ by $+$ and $/$ by $-$.
- Replace constants (members of $\mathbb{R}$) by 0

Example $f(x_1, x_2) = x_1 + x_2 + 1$ becomes $\min(x_1, x_2, 0)$. If $v(x_1) = v_1$ and $v(x_2) = v_2$, with x_1 and $x_2 \in \mathcal{R}_+$, then $v(x_1 + x_2 + 1) = \min(v_1, v_2, 0)$.

No need to assume that f/g is in lowest terms, or that all polynomials are fully expanded out.

If we have varying coefficients in $\mathcal{R}_+$, then replace the coefficient a by $v(a)$.

For example, let X be the variety

$$x_1x_3 = x_2 + 1,\ x_2x_4 = x_3 + 1,\ x_3x_5 = x_4 + 1$$

$$x_4x_1 = x_5 + 1,\ x_5x_2 = x_1 + 1$$

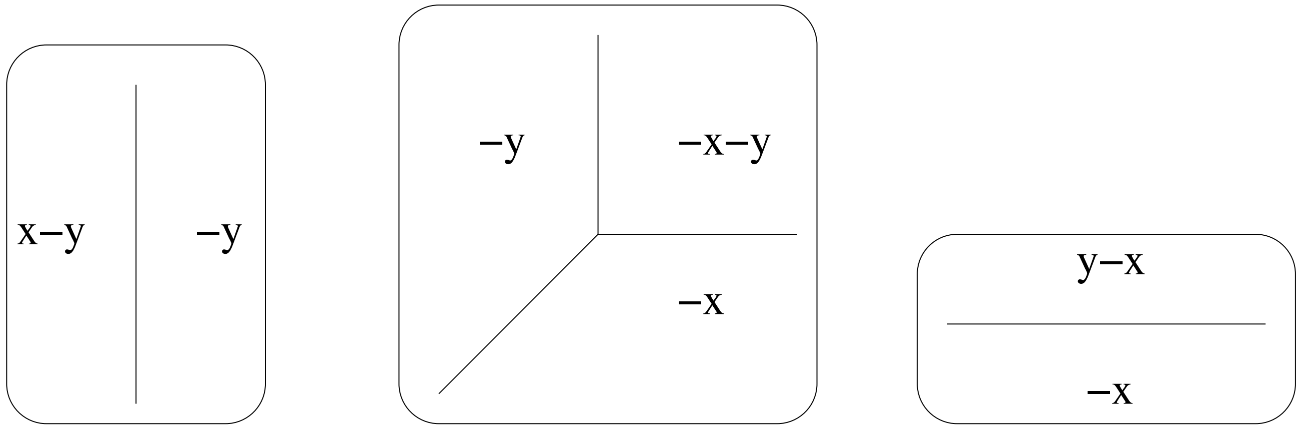
in $(\mathcal{K}^\times)^5$. This is $M_{0,5}$, and also is the cluster variety of type A_2 with no frozen variables.

We can rewrite

$$x_3 = \frac{x_2+1}{x_1},\ x_4 = \frac{x_1+x_2+1}{x_1x_2},\ x_5 = \frac{x_1+1}{x_2}.$$

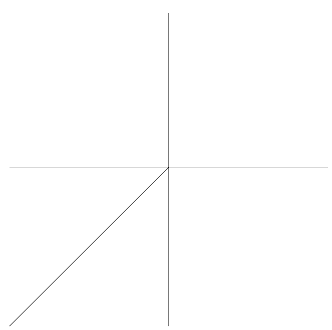
So we can parametrize

$$\mathrm{Trop}_+(X) = (x_1, x_2, \min(x_2, 0) - x_1, \min(x_1, x_2, 0) - (x_1 + x_2), \min(x_1, 0) - x_2)$$



$$(x, \quad y, \quad \min(x,0) - y, \quad \min(x,y,0) - x - y, \quad \min(y,0) - x)$$

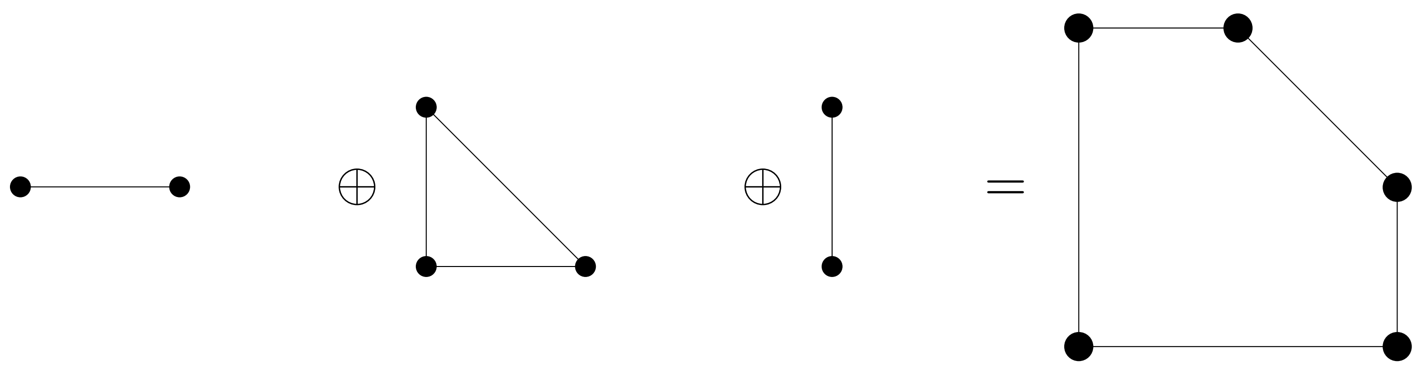
In general, if we have a parameterization $\phi : \mathbb{R}_+^m \to (X \cap \mathbb{R}_+^n)$ given by positive rational functions, then we get a piece-wise linear parametrization $\operatorname{Trop} \phi$ of $\operatorname{Trop}_+(X)$ by $\mathbb{R}^m$.



$$\longrightarrow \mathbb{R}^5$$

If the parametrization $\phi : \mathbb{R}^m_+ \to (X \cap \mathbb{R}^n_+)$ is by positive Laurent polynomials, then the components of $\operatorname{Trop} \phi$ are concave piecewise linear functions.

Let Δ_i be the Newton polytope of ϕ_i. Then $\operatorname{Trop} \phi_i$ is linear on the regions of the normal fan of Δ_i and $\operatorname{Trop} \phi$ is linear on the regions of the normal fan to the Minkowski sum of all the Δ_i.



Let X have dimension d, and let w be in the interior of a d-dimensional face σ of $\operatorname{Trop} X$. Let H be the d-dimensional linear space spanned by $\langle u - w \rangle_{u \in \sigma}$.

Then $\operatorname{in}_w X$ is invariant under the torus $\exp(H)$. Let $Y := \operatorname{in}_w / \exp(H)$. By dimension counting, Y has dimension 0. The weight $w(\sigma)$ of σ is the length of the scheme Y. These are the weights in Mikhalkin's talk.

Intuitively, if we have an amoeba which is "near $\operatorname{Trop} X$" then the fiber of the $\log | \ |$ map will be $w(\sigma)$ copies of $(S^1)^d$.

Now suppose that w lies in a face τ of dimension $d-1$; let H be the corresponding $d-1$ plane. So $\mathrm{in}_w X = \exp(H) \times Y$ where Y has dimension 1.

Remember that the neighborhood of w in $\operatorname{Trop} X$ looks like $\operatorname{Trop} \mathrm{in}_w X$. In this case, $\operatorname{Trop} \mathrm{in}_w X = H \times \operatorname{Trop} Y$. To understand what tropical varieties look like near codimension 1 faces, we just need to understand what $\operatorname{Trop} Y$ looks like for Y a constant coefficient curve.

If Y is a planar curve defined by a degree d polynomial F with the standard Newton polytope, then $\operatorname{Trop} Y$ looks like a tropical line, with some multiplicities. What are the multiplicities?

Recall that $\operatorname{in}_w F$ is the polynomial gotten by looking at one side of the triangle. It has degree d. The corresponding scheme is d points (possibly with multiplicity). So the weight of each ray is d.

More generally, if Y is any curve with a map $(\phi_1, \ldots, \phi_n) : Y \to (\mathbb{C}^*)^n$, then $\operatorname{Trop} \phi(Y)$ has a ray for each puncture of Y. Letting z be a local coordinate on Y around a puncture, the direction of that ray is $(\operatorname{ord}(\phi_1(z)), \operatorname{ord}(\phi_2(z)), \ldots, \operatorname{ord}(\phi_n(z))$ where ord is the order of zero or pole.

The balancing condition says

$$\sum_z (\operatorname{ord}(\phi_1(z)), \operatorname{ord}(\phi_2(z)), \ldots, \operatorname{ord}(\phi_n(z)) = 0$$

The balancing condition expresses that a meromorphic function on a compact curve has equally many zeroes and poles!