# Lecture I: Introduction to Tropical Geometry

David Speyer

## The field of Puiseux series

$\mathbb{C}[[t]]$ is the ring of formal power series $a_0 + a_1 t + \cdots$ and $\mathbb{C}((t))$ is the field of formal Laurent series: $a_{-N}t^{-N} + a_{-N+1}t^{-N+1} + \cdots$.

$$\mathcal{K} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n})), \quad \mathcal{R} = \bigcup_{n \geq 1} \mathbb{C}[[t]].$$

$$v : \mathcal{K}^{\times} \to \mathbb{Q},\ v\left(\sum a_i t^{i/N}\right) = \min\left(i/n : a_i \neq 0\right).$$

We should think of $\mathcal{R}$ as functions of $t$ in some small neighborhood $[0, \epsilon)$ and $\mathcal{K}$ as functions on $[0, \epsilon)$ with some pole.

If the relevant sums converge, then

$$v(f) = \lim_{t \to 0^+} \frac{\log f(t)}{\log t}.$$

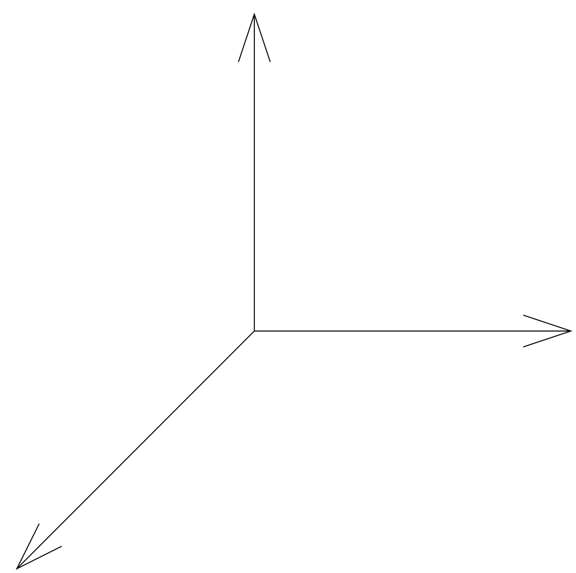
$\mathcal{K}$ is conveniently algebraically closed.

For $X$ in $(\mathcal{K}^\times)^n$, set $\operatorname{Trop} X$ to be $v(X) \subseteq \mathbb{Q}^n$. You'll also see $\overline{v(X)} \subseteq \mathbb{R}^n$. Each of these creates some notational awkwardness at the beginning, but we will soon have theorems telling us not to worry about it.

Everyone's first example: $x + y + 1 = 0$

If $x(t) + y(t) + 1 = 0$, then either

- $v(x) \geq 0$ and $v(y) = 0$.
- $v(y) \geq 0$ and $v(x) = 0$.
- $v(x) = v(y) \leq 0$.

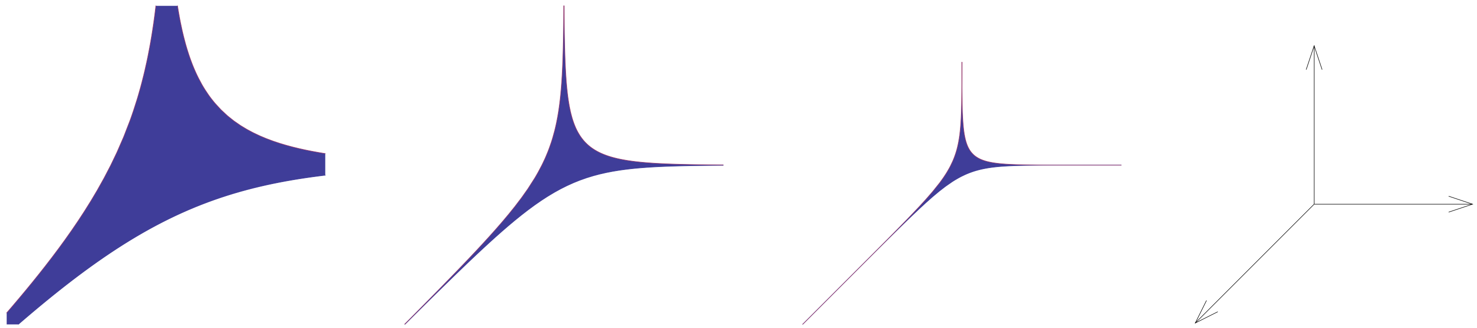


This example has "constant coefficients", meaning that there are no $t$'s in the equation $x + y + 1 = 0$. We'll stick to constant coefficient examples for a while.

We'll talk about

- Hypersurfaces with constant coefficients
- The "initial variety" construction
- General varieties with constant coefficients
- Nonconstant coefficients

$$\operatorname{Trop} X = \lim_{t \to 0^+} \frac{\log |X|}{\log t}.$$

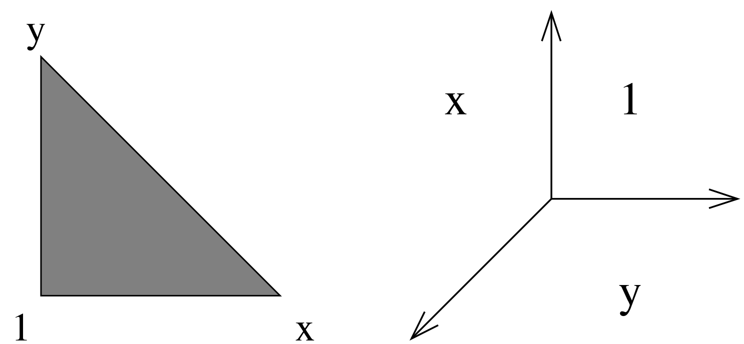


$\log t = -2$ $\quad$ $\log t = -5$ $\quad$ $\log t = -10$ $\quad$ $t = 0$

Let $F \in \mathbb{C}[x_1^{\pm}, x_2^{\pm}, \ldots, x_n^{\pm}]$ and let $X$ be the hypersurface $F = 0$. Let $\Delta(F)$ be the Newton polytope

$$\Delta(F) = \mathrm{Hull}(a \in \mathbb{Z}^n : \ x^a \text{ has nonzero coefficient in } F) \subset \mathbb{R}^n$$

For any $w \in \mathbb{R}^n$, the function $\langle w, \ \rangle$ is minimized on a face of $\Delta(F)$. Divide $\mathbb{R}^n$ up into cones according to which face of $\Delta(F)$ this minimum occurs on. This is the normal fan to $\Delta(F)$.

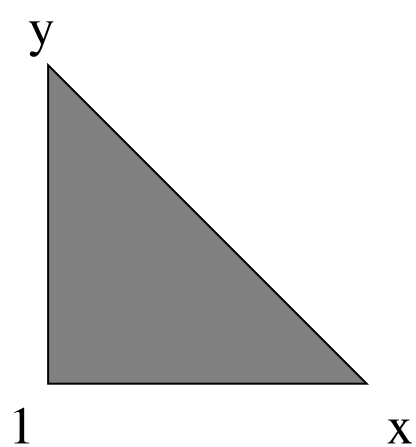


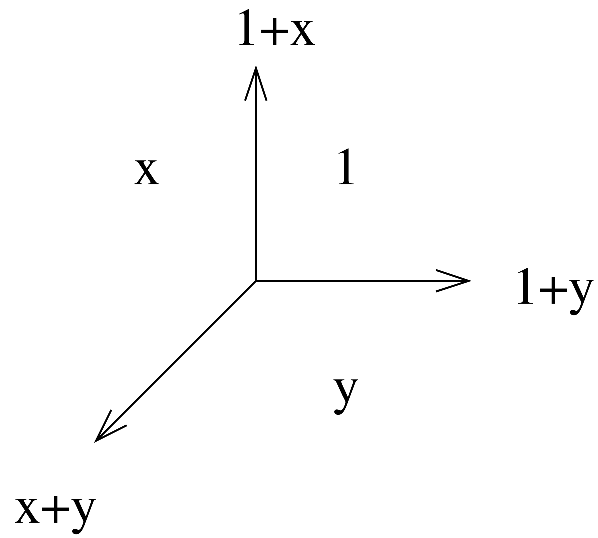

**Theorem:** $\mathrm{Trop}\, X$ is the union of the codimension one faces of the normal fan.

$\mathrm{Trop}\, X$ is a bunch of rational polyhedral cones. This is why we don't care much about the difference between $\mathbb{Q}$ and $\mathbb{R}$.

Is there a geometric meaning to this normal fan construction? Yes!

$F = \sum F_a x^a$ continues to have constant coefficients. Consider $w \in \mathbb{Q}^n$ and let $\Gamma$ be the face of $\Delta(F)$ where $\langle w, \ \rangle$ is minimized. Let $\mathrm{in}_w F$ be $\sum_{a \in \Gamma} F_a x^a$.



Let $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n$. Let $\mathrm{in}_w X$ be the hypersurface cut out by $\mathrm{in}_w X$. There is a point of $X$ of the form $(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)$ if and only if $(a_1, \ldots, a_n) \in \mathrm{in}_w(X)$.

In particular, this explains why $\operatorname{Trop} X$ is the $w$ for which $\mathrm{in}_w F$ is not a monomial.

What happens when $X$ is not a hypersurface?

$$\operatorname{Trop} X = \bigcap_{F|_X=0} \operatorname{Trop}\{F = 0\}$$

Moreover, there is a finite set of polynomial $F_1$, $F_2$, ..., $F_r$ such that

$$\operatorname{Trop} X = \operatorname{Trop}\{F_1 = 0\} \cap \operatorname{Trop}\{F_2 = 0\} \cap \cdots \cap \operatorname{Trop}\{F_r = 0\}$$

Therefore, $\operatorname{Trop} X$ is a union of finitely many rational cones. (Rational means of the form $\{w : \langle w, a_1\rangle,\ \langle w, a_2\rangle,\ \ldots,\ \langle w, a_r\rangle \geq 0\}$ for $a_i$ vectors in $\mathbb{Z}^n$.) This is why we don't have to worry much about the difference between $\mathbb{Q}$ and $\mathbb{R}$.

The variety $\mathrm{in}_w X$ is cut out by $\mathrm{in}_w F$, for $F$ in $I$. Again, there is a point of $X$ of the form $(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)$ if and only if $(a_1, \ldots, a_n) \in \mathrm{in}_w(X)$. And we can choose the finite generating set $F_1$, $F_2$, …, $F_r$ on the previous slide so that $\mathrm{in}_w F_1$, $\mathrm{in}_w F_2$, …, $\mathrm{in}_w F_r$ cut out $\mathrm{in}_w(X)$ for all $w$.

We have $\dim X = \dim \mathrm{Trop}\, X$.

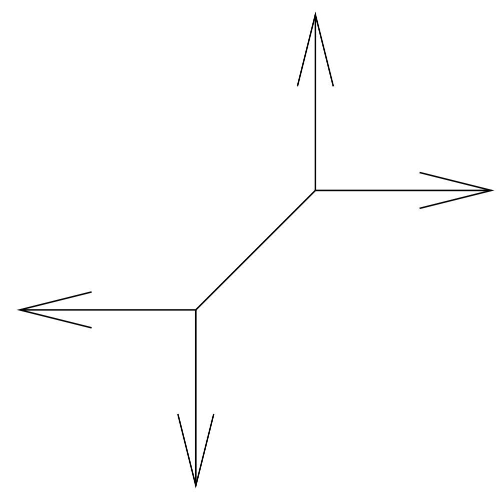
If $X$ is connected in codimension 1, so is $\mathrm{Trop}\, X$.

Near a point $w \in \mathrm{Trop}\, X$, $\mathrm{Trop}\, X$ looks like a translate of $\mathrm{Trop}\, \mathrm{in}_w X$.

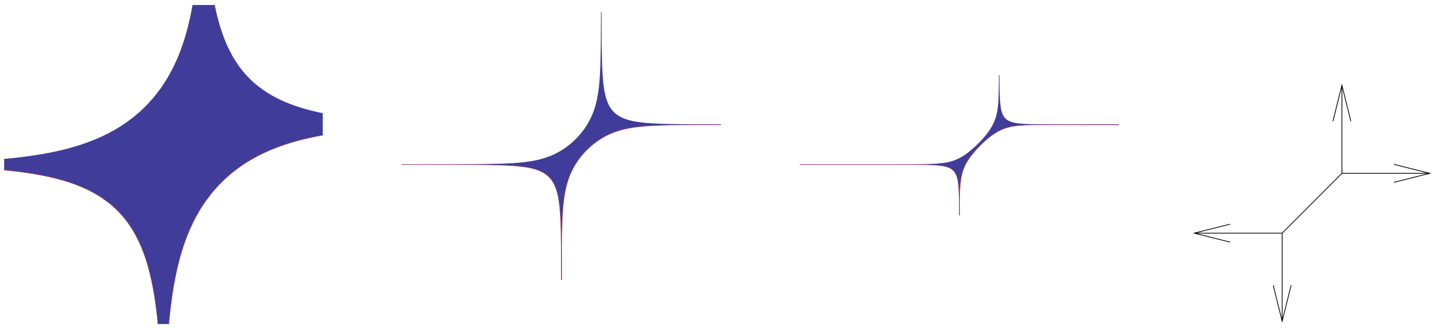
What about nonconstant coefficients? I don't expect to use these in my lectures, but I certainly expect professors Mikhalkin and Gross will.

So, what does $\mathrm{Trop}(xy + x + y + t)$ look like? It is the union of the following possibilities:

- $v(y) = 0$, $v(x) \leq 0$.
- $v(x) = 0$, $v(y) \leq 0$.
- $0 \leq v(x) = v(y) \leq 1$.
- $v(y) = 1$, $v(x) \geq 1$.
- $v(x) = 1$, $v(y) \geq 1$.



Once again, we can see this as $\lim_{t\to 0^+} \frac{\log|X(t)|}{\log t}$.



$t = 0.5$ $t = 0.1$ $t = 0.01$ $t = 0$

Let $F = \sum F_a x^a$ be a polynomial in $\mathcal{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$, where $a$ ranges over some finite subset $A$ of $\mathbb{Z}^n$. Let $w \in \mathbb{Q}^n$.

Let

$$u = \min_{a \in A} v(F_a) + \langle a, w \rangle.$$

Let $F_a = g_a t^{u - \langle a, w \rangle} +$ higher order terms, so $g_a \neq 0$ if and only if the minimum defining $u$ is achieved at $a$. Set

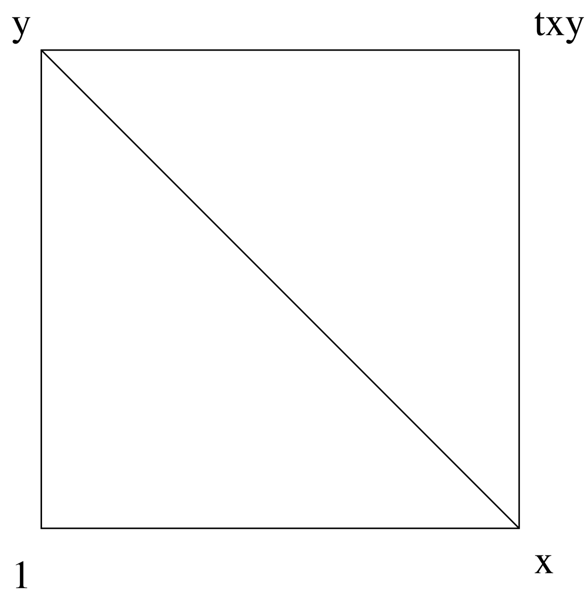
$$\mathrm{in}_w F = \sum g_a x^a$$

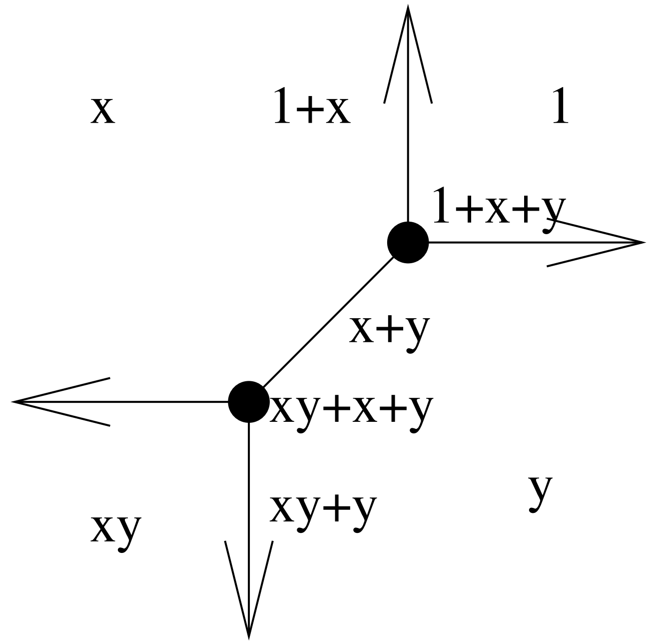
and

$$\mathrm{in}_w X = \{\mathrm{in}_w F = 0\}.$$

There is a point of $X$ of the form $(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)$ if and only if $(a_1, \ldots, a_n) \in \mathrm{in}_w(X)$. So $w \in \mathrm{Trop}\, X$ if and only if $\mathrm{in}_w F$ is not a monomial.

Combinatorially, this involves working with a convex subdivision of the Newton polytope of $F$.



For nonhypersurfaces, let $I \subset \mathcal{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ be the ideal of $X$.

Define $\mathrm{in}_w X$ to be $\{x : (\mathrm{in}_w F)(x)\} = 0$, for $F \in I$. Again, we can find a finite subset $G$ of $I$ such that $\mathrm{in}_w X$ is generated by $(\mathrm{in}_w F)_{F \in G}$ for all $w$.

There is a point of $X$ of the form $(a_1 t^{w_1} + \cdots, a_2 t^{w_2} + \cdots, \ldots, a_n t^{w_n} + \cdots)$ if and only if $(a_1, \ldots, a_n) \in \mathrm{in}_w(X)$. So $w \in \mathrm{Trop}\, X$ if and only if $\mathrm{in}_w F$ is not a monomial.

Again, $\mathrm{Trop}\, X$ is a finite rational polyhedral complex, of dimension $\dim X$. One can recover the degree of $X$, and limited information about the cohomology of $X$, from $\mathrm{Trop}\, X$.