Tropical Geometry Minicourse

Day 5

July 23, 2021

Outline

- Day 1: introduction
- Day 2: hypersurfaces
- Day 3: trop($V(I)$), tropical linear spaces
- Day 4: more tropical linear spaces
- Day 5: tropical polytopes

Tropical Linear Spaces

Goal: Describe tropical lines.

- Review
- Tropical convexity
- What is a tropical polytope?
- Classifying polytopes
- A surprising theorem

Review

Tropical lines in the plane look like this:

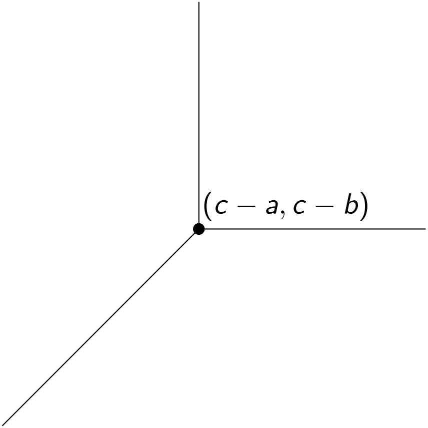


Figure: The vanishing set of $h(x) = a \odot x \oplus b \odot y \oplus c \odot z$, normalized wrt $z = 0$.

Tropical Convexity

A subset S of $\mathbb{R}^n$ is **tropically convex** if for any $x, y \in S$ and any $a, b \in \mathbb{R}$,

$$a \odot x \oplus b \odot y \in S$$

Note: Taking $a = b$, $x = y$, we have

$$a \odot x \oplus a \odot x = a \odot x \in S$$

so $S + \mathbb{1} \cdot \mathbb{R} \subseteq S$. So we can consider S in $\mathbb{R}^n / \mathbb{R} \cdot \mathbb{1}$

The **tropical convex hull** of a subset of points V is the smallest tropically convex set containing V.

If V is finite, then its convex hull is a **tropical polytope**.

Some Important Facts

Tropical versions of the following thoerems hold:

- Caratheodory's Theorem,
- Farkas Theorem

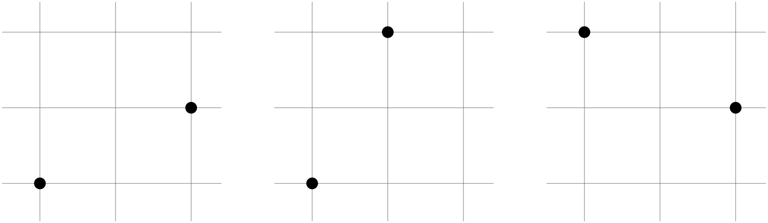
A tropical polytope is a union of classical polytopes.

A set which is both a classical and tropical polytope is called a **polytrope**.

Tropical Line Segments

Prop. Each tropical line in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbb{1}$ is the convex hull of at most n classical line segments.

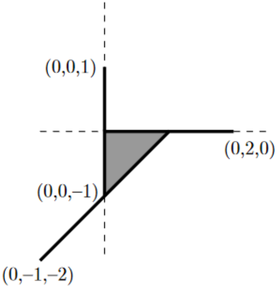
Draw: The tropical line segments between the points in each of the three pairs of points.

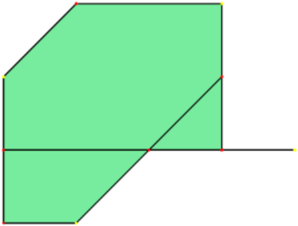


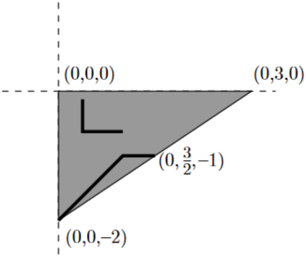
These are the three classes of examples in the plane.

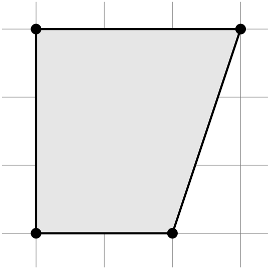
Pop-Quiz!

Q: Which of the following are tropically convex?





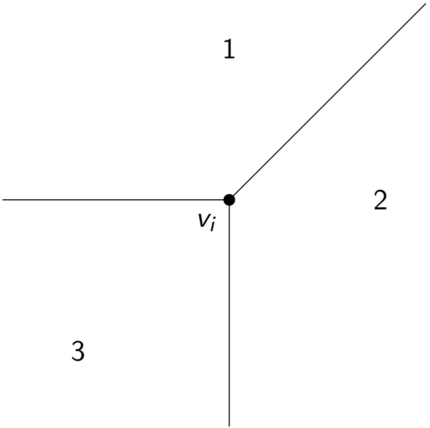




Polymake Code

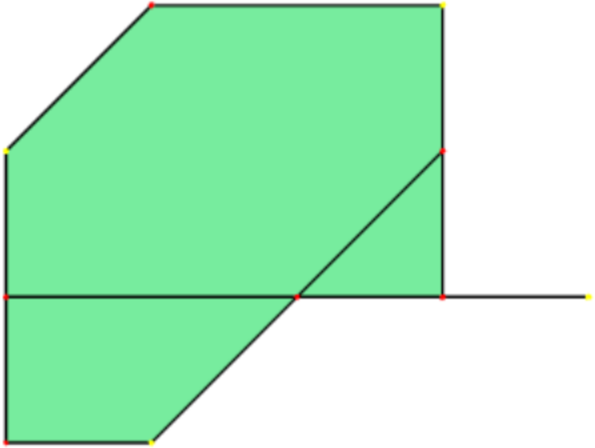
```
$A = new Matrix<Rational>([[p_11,...,p_1n],...,[p_k1,...,p_
$P = new tropical::Polytope<Min>(POINTS=>$A);
$P->VISUAL;
```

Classifying Tropical Polytopes



$$\text{type}(x) = (\{i : \text{type}_i(x) \ni 1\}, \ldots, \{i : \text{type}_i(x) \ni n+1\})$$

An Example



A Surprising Theorem

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

This is not a coincidence!

Theorem. Given any matrix $A \in \mathbb{R}^{r \times n}$ the tropical complex generated by its column vectors is isomorphic to the tropical complex generated by its row vectors.

Tropical Polytopes as Eigenspaces

Bonus Slide!

Q: What is an eigenvector for a tropical matrix?

Q: How do we find eigenvectors?

Consider the matrix as the adjacency matrix of a directed graph. The smallest normalized cycle length (sum of edges/length of path) is the only eigenvalue.

Example.

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 2 & 6 & 0 \\ 8 & 3 & 1 & 3 \\ 4 & 4 & 4 & 5 \end{bmatrix}$$

Thank You!

spcox@umich.edu
EH 4828