Tropical Geometry Minicourse

Day 4

July 22, 2021

Outline

- Day 1: introduction
- Day 2: hypersurfaces
- Day 3: trop($V(I)$), tropical linear spaces
- Day 4: more tropical linear spaces
- Day 5: tropical polytopes

Tropical Linear Spaces

Goal: Describe tropical lines.

- Review
- What is a linear space?
- Matroids and linear spaces
- Tropicalizing linear spaces
- Grassmannians and Dressians
- Tropical Lines in $\mathbb{R}^3$

Review

There are three ways we can think about a tropical hypersurface:

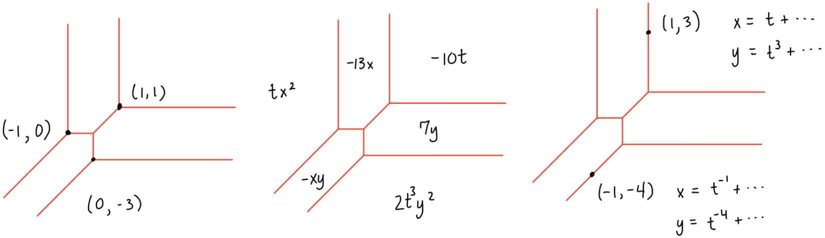

Figure: $f(x,y) = tx^2 + 2t^3y^2 - xy - 13x + 7y - 10t$

The **Gröbner complex** of an ideal is a polyhedral complex with cells whose relative interiors represent the regions on which $\text{in}_w(I)$ is constant.

Last time, we saw how to construct a polynomial g to get the Gröbner complex of any ideal.

We defined matroids in terms of independent sets, bases, and circuits.

An Important Fact

Q: Projective space uses homogeneous coordinates, so multiplying each entry of a coordinate by λ doesn't change the point in projective space. How does this translate to tropical geometry?

Non-homogeneous ideal in $x_1, \ldots, x_n \to \mathbb{R}^n$

Homogeneous ideal in $x_0, \ldots, x_n \to \mathbb{R}^{n+1}/\mathbb{1}\mathbb{R}$

Linear Spaces

A **linear space** is a k-dim'l plane in $\mathbb{P}^n \cap (K^*)^{n+1}$, which can be described three ways:

- As the **row span** of a $(k+1) \times (n+1)$ matrix $A = [b_0 \cdots b_n]$,
- A vector of Plücker coordinates,
- As the vanishing of a homogeneous **linear ideal**

$$I = \langle \ell_i : i \in J \rangle \subseteq K[x_0^{\pm}, \ldots, x_n^{\pm}]$$

An Example of Linear Spaces

Let L be the span of the row vectors of A below:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -3 & -3 \end{bmatrix}$$

L is a 2-plane in 5-space. Its Plücker coordinates are:

$$\begin{aligned} p(L) &= (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) \\ &= (\ 1\ : -4 : -4\ : -5 : -5\ :\ 0\ ) \end{aligned}$$

A is a full rank 2×4 matrix, so it's kernel has dimension 2. That means we can describe it with three independent linear equations. Here are the first two that come to mind:

$$5x_1 - 4x_2 - x_3$$
$$5x_1 - 4x_2 - x_4$$

Matroids and Linear Spaces

The non-zero Plücker coordinates of a linear space give the bases of the corresponding matroid.

The **support** of a linear relation is the set of indices with non-zero coefficients. Relations with minimal support are called **circuit relations**.

The support of the circuit relations of a linear ideal give the circuits of the corresponding matroid.

Fact: The circuit relations are a tropical basis for their linear ideal.

A **tropical basis** is a finite subset $\mathcal{T}$ of I so that $\mathrm{trop}(V(I)) = \bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f))$.

Tropicalizing Linear Spaces

The **matroid polytope** P_M is $\mathrm{conv}(e_B : B \text{ is a basis of } M)$.

Example.

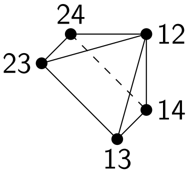

Figure: The matroid polytope for the matroid with bases $12, 13, 14, 23, 24$.

Prop. The tropicalization of a linear space is the dual through loopless faces of the corresponding matroid polytope.

Why?

Grassmannians

The classical Grassmannian $\mathrm{Gr}(r, m)$ paramatarizes $(r-1)$-planes in $\mathbb{P}^{m-1}$. It can be described via the Plücker embedding:

$$\mathrm{Gr}(r,m) \longrightarrow \mathbb{P}^{\binom{m}{r}-1}$$

$$\left[v_1|\cdots|v_m\right] \longrightarrow (\det(v_{i_1}\cdots v_{i_r})_{i_j\in\mathcal{I}})_{\mathcal{I}\in\binom{[m]}{r}}$$

$$I_{r,m} = \langle \sum_{j\in J\setminus I} \mathrm{sgn}(j;J,I)p_{J\setminus j}p_{I\cup j} : |J| = r+1, |I| = r-1\rangle$$

We like tropicalizing subvarieties of the torus, so we define:

$$\mathrm{Gr}^0(r,m) = \mathrm{Gr}(r,m) \cap (K^*)^{\binom{m}{r}-1}$$

Note: $\mathrm{Gr}^0(r, m)$ paramaterizes **uniform linear spaces**, i.e. linear spaces with matroid $U_{r,m}$ with all subsets of size r of $[m]$ bases.

Other Grassmannians and Dressians

We can also define the **Grassmannian for other matroids** by modifying the Plücker relations:

$$I_M = (I_{r,m} + \langle p_\sigma : \sigma \text{ is not a basis of } M \rangle) \subset K[p_B^{\pm} : B \text{ is a basis of } M]$$

The **Dressian of a matroid** M is the intersection of the tropical hypersurfaces given by the quadratic Plücker relations only.

The Dressian is a **tropical prevariety**, but not always a tropical variety.

The tropical Grassmannian paramatarizes **tropicalized linear spaces**, and the Dressian paramatarizes **tropical linear spaces.**

Fact: $\text{Gr}(2, n) = \text{Dr}(2, n)$ for all n.

An Example

The tropical Grassmannian $\mathrm{trop}(\mathrm{Gr}(2,5))$ is the cone over Petersen graph:

Classification of Tropical Lines in $\mathbb{R}^3$

Q: How can we turn this into a question about matroids?

Q: Which matroids do we need to use?