# Tropical Geometry Minicourse

## Day 3

July 21, 2021

# Outline

- Day 1: introduction
- Day 2: hypersurfaces
- Day 3: trop($V(I)$), tropical linear spaces
- Day 4: more tropical linear spaces
- Day 5: tropical polytopes

# Tropicalizing $V(I)$, Tropical Linear Spaces

**Goal:** Learn how to compute the tropicalization of a hypersurface.

- Review
- Wrap-Up hypersurfaces
- $\text{trop}(V(I))$
- Linear Spaces
- Matroids
- Tropical linear spaces
- Grassmannians and Dressians (if time)

# Review

Given a classical polynomial $f$,
$\mathrm{trop}(f)(w) = \bigoplus v(c_i) \odot x^{a_i} = \min\{v(c_i) + w \cdot a_i\}$

The **vanishing set** of a tropical polynomial is the set of points where the min is achieved *at least twice*.

$$\mathrm{in}_w(f) = \overline{t^{-\mathrm{trop}(f)(w)} f(t^{w_1} x_1, \ldots, t^{w_n} x_n)} \in \Bbbk[x_1^{\pm}, \ldots, x_n^{\pm}]$$

Hypersurfaces can be computed using regular subdivisions!

# Wrap-Up from last time

- A hypersurface is smooth if and only if the corresponding subdivision consists of lattice simplices with minimal volume (unimodular triangulation).
- Every lift of a smooth tropical surface is smooth.
- The Fermat cubic is defined by:

$$x^3 + y^3 + z^3 + w^3 = 0$$

  We must have $v(1) = 0$, so the corresponding subdivision of $\mathrm{conv}((3,0,0,0),(0,3,0,0),(0,0,3,0),(0,0,0,3))$ is trivial. This means the Fermat cubic is **not smooth**.
- Most hypersurfaces considered with the trivial valuation are not smooth, but you will always get a fan!
- Some tropical cubics have 27 lines, some (like the tropicalized Fermat cubic) have fewer, and some have infinitely many (see *Smooth Tropical Surfaces with Infinitely Many Tropical Lines* by Vigeland.

# Tropicalizing $V(I)$

**Note:** If an ideal in a Laurent polynomial ring contains a monomial, it's the unit ideal.

Let $X = V(I) \subset K^*$ be any subvariety of a torus.

$$\mathrm{trop}(V(I)) = \{w \in \mathbb{R}^n : \mathrm{in}_w(I) \text{ does not contain a monomial}\}$$

Also equivalent:

- The intersection of $\mathrm{trop}(V(f))$ for all $f \in I$, or
- $\overline{(v(y_1, \ldots, v(y_n)) : y \in V(I)(K)}$

# Using Hypersurfaces

Fix a homogeneous ideal $I$ in $S = K[x_0, \ldots, x_n]$.

1. For $d \in \mathbb{N}$, find a $K$-basis for $I_d$ (the $d$th homogeneous part of $I$), $\{f_1, \ldots, f_{r_d}\}$. Let $A_d$ be the matrix of coefficients of the $f_i$.
2. $$g_d = \sum_{\substack{J \subseteq M_d \\ |J| = r}} \det(A_d^J) \prod_{u \in J} x^u$$
3. There exists $D \in \mathbb{N}$ such that any initial ideal $\mathrm{in}_w(I)$ has generators in at most degree $D$.
$$g = \prod_{d \leq D} g_d$$
4. $\Sigma(I) = \Sigma(g)$

# An Example

We will work in the ring: $K = \mathbb{C}\{\{t\}\}[x_1^{\pm}, \ldots, x_4^{\pm}]$, with

$$I = \langle x_1 - x_2 + tx_3, -x_1 - x_2 + tx_4, -x_1 - x_3 + x_4, -x_2 + x_3 + x_4 \rangle$$

**Fact:** For linear ideals, $D = 1$.

$$A_1 = \begin{bmatrix} 1 & -1 & t & 0 \\ -1 & -1 & 0 & t \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

# Structure Theorem

The **Structure Theorem** tells us what tropical varieties can look like. It says $\text{trop}(V(I))$ is:

1. a pure-dimensional polyhedral complex,
2. balanced (with appropriate weights on maximal cells), and
3. connected through codimension 1.

# Matroids

**Slogan**: Matroids encode linear relations on the base set.

A **matroid** is a finite *base set* $E$, together with a collection of subsets of $E$ called the *independent sets* obeying the following axioms:

1. $\emptyset$ is independent,
2. Every subset of an independent set is independent,
3. If $A, B \subseteq E$ are two independent subsets and $|A| < |B|$, then there's $j \in B$ with $A \cup j$ independent.

$$\mathrm{A} = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 1 & 3 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \end{array} \quad \rightarrow \quad \begin{aligned} E &= [4] = \{1, 2, 3, 4\} \\ \mathcal{I} &= \{\emptyset, 1, 2, 3, 4, \\ &\qquad 12, 13, 14, 23, 24, 34, \\ &\qquad \cancel{123}, 124, 134, 234, \\ &\qquad \cancel{1234}\} \end{aligned}$$

# Bases

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ A = & \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 1 & 3 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Independent sets:
$\emptyset, 1, 2, 3, 4, 12, 13, 14,$
$23, 24, 34, 124, 134, 234$

- **Bases**: independent sets that are maximal wrt inclusion
  1. There is at least one basis,
  2. If $B_1 \neq B_2$ are bases, $\forall i \in B_1 \setminus B_2$, there is $j \in B_2$ with $(B_1 \setminus i) \cup j$ a basis.

**Example:** $M_A$ has three bases: 124, 134, and 234. These are the index sets of the non-zero Plücker coordinates of $A$.

# Circuits

$$\begin{array}{c} \\ A = \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 1 & 3 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Dependent sets:
123, and 1234.

- **Circuits**: dependent sets that are minimal wrt inclusion
  A circuit of size 1 is called a **loop**.
  1. $\emptyset$ is not a circuit,
  2. No circuit is properly contained in any other circuit,
  3. If $C_1 \neq C_2$ are circuits, $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus e$ contains a circuit.

**Example:** $M_A$ has just one circuit: 123.

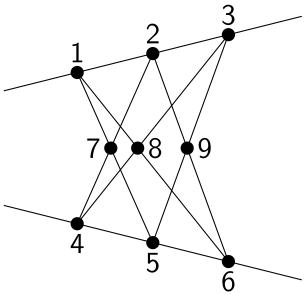
# Matroids Overview

# Warning about Matroid Realizability

- A matroid $M$ is **realizable** if there is some collection of vectors, over some field, that have matroid $M$.
- A matroid is called **regular** if it's realizable over every field.

**WARNING!** Statistically, 100% of matroids are non-realizable.

**Example:** Non-Pappus matroid




Bases:

- Triples of non-collinear points,
- 789

# Linear Spaces

A **linear space** is a $k$-dim'l plane in $\mathbb{P}^n \cap (K^*)^{n+1}$, which can be described three ways:

- As the **row span** of a $(k+1) \times (n+1)$ matrix $A = [b_0 \cdots b_n]$,
- A vector of Plücker coordinates,
- As the vanishing of a homogeneous **linear ideal**

$$I = \langle \ell_i : i \in J \rangle \subseteq K[x_0^{\pm}, \ldots, x_n^{\pm}]$$

# An Example of Linear Spaces

Let $L$ be the span of the row vectors of $A$ below:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -3 & 4 & 7 \end{bmatrix}$$

$L$ is a 2-plane in 5-space. Its Plücker coordinates are:

$$\begin{aligned} p(L) &= (p_{12} : p_{13} : p_{14} : p_{15} : p_{23} : p_{24} : p_{25} : p_{34} : p_{35} : p_{45}) \\ &= (\ 1 : 2 : -4 : 6 : -5 : 2 : 5 : 7 : 10 : 3\ ) \end{aligned}$$

$A$ is a full rank $2 \times 5$ matrix, so it's kernel has dimension 3. That means we can describe it with three independent linear equations. Here are the first three that come to mind:

$$\begin{gathered} 5x_1 - 4x_2 - x_3 \\ 2x_1 - 3x_2 + x_4 \\ 5x_1 - 6x_2 + x_5 \end{gathered}$$

# Tropicalizing a Linear Space