Tropical Geometry Minicourse

Day 2

July 20, 2021

Outline

- Day 1: introduction
- Day 2: hypersurfaces
- Day 3: tropical linear spaces
- Day 4: tropical polytopes
- Day 5: tropical toric varieties (maybe?)

Hypersurfaces

Goal: Learn how to compute the tropicalization of a hypersurface.

- Review
- Subdivisions
- More general varieties

Review of Tropical Language

$(\mathbb{R}, \min, +) = (\mathbb{R}, \oplus, \odot)$

- tropical addition is $\oplus = \min$, and
- tropical multiplication is $\odot = +$.

A **valuation** is a map v from a field K to $\mathbb{R} \cup \{+\infty\}$ satisfying the following rules:

1. $v(a) = +\infty \iff a = 0$,
2. $v(ab) = v(a) + v(b)$, and
3. $v(a + b) \geq \min(v(a), v(b))$.

A **tropical polynomial** is min of several linear functions.

Given a classical polynomial f,
$\mathrm{trop}(f)(w) = \bigoplus v(c_i) \odot x^{a_i} = \min\{v(c_i) + w \cdot a_i\}$

The **vanishing set** of a tropical polynomial is the set of points where the min is achieved *at least twice*.

Subdivisions

Reference: *Triangulations : Structures for Algorithms and Applications* by De Loera, Rambau and Santos.

A **polyhedral complex** is a collection of polyhedra $\mathcal{C}$ with the properties that:

1. If $P \in \mathcal{C}$, then every face of P is also in $\mathcal{C}$,
2. If $P, Q \in \mathcal{C}$, then $P \cap Q$ is a face of both P and Q.

A **subdivision** of a polytope P is a polyhedral complex $\mathcal{C}$, the union of whose polytopes is P (as sets of points).

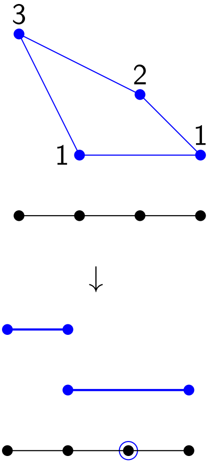
Regular Subdivisions

Regular subdivisions are obtained by lifting the points of the polytope, and then projecting the lifted polytope back down.

- A lattice polytope: $P \subset \mathbb{R}^n$, and
- a weight function: $w : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$.

Define the **lift** of P with respect to w to be:

$$\tilde{P} = \operatorname{conv}\{(v, w(v)) : v \in P \cap \mathbb{Z}^n\} \subseteq \mathbb{R}^{n+1}$$



Faces of $\tilde{P}$ look like

$$\operatorname{face}_v(\tilde{P}) = \{x \in \tilde{P} \mid v \cdot x \le v \cdot y, \text{ for all } y \in \tilde{P}\}$$

A face of $\tilde{P}$ is a **lower face** if for some v, $v_{n+1} > 0$.

The **regular subdivision** of P with respect to w has maximal cells the projected lower faces of $\tilde{P}$.

Examples of Regular Subdivisions

More Examples of Regular Subdivisions

Use Polymake!

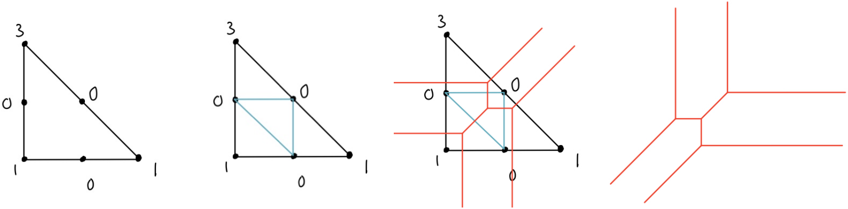
```
##run this first if you are on windows
script("$ENV{HOME}/wslviewer.pl");

##declare the points and
##HOMOGENIZE with a 1 in the first entry for each point
$M = new Matrix<Rational>([[1,p_11,p_12,...,p_1n],...,[1,p_

##declare the weight vector
$w = new Vector<Rational>([w_1,...,w_k]);

##make the subdivision
$S = new fan::SubdivisionOfPoints(POINTS=>$M, WEIGHTS->$w);

##visualize
$S->VISUAL;
```

Connection to Hypersurfaces

Prop. Given a classical polynomial f, the tropicalization of $V(f)$ is the $(n-1)$-skeleton of the (inner normal) dual to the regular subdivision of the Newton polytope of f induced by weighting the lattice points by the valuation of the corresponding coefficient.

Example. $K = \mathbb{Q}$ with the 2-adic valuation,

$$f(x,y) = 2x^2 - xy + 24y^2 - 13x + 7y - 10$$



sketch of proof: $(w,1)\cdot(u, v(c_u)) = w \cdot u + v(c)$

Examples of Hypersurfaces

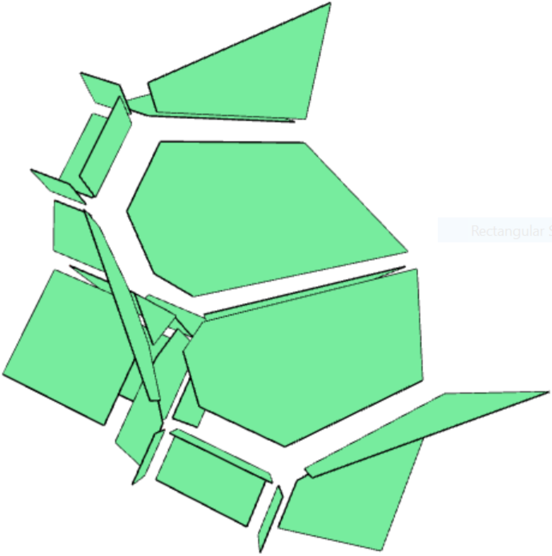


Figure: A tropical quadric.

More General Varieties

To talk about more general varieties, we need to bring in some more commutative algebra.

$$f \in I \subset K[x_1^{\pm}, \dots, x_n^{\pm}], \quad K = (K, v) \text{ is a valued field}$$

A **splitting** of a valuation is a map $\phi : (\Gamma_v, +) \to (K^*, \times)$ so that

$$\Gamma_v \xrightarrow{\phi} K^* \xrightarrow{v} \Gamma_v \equiv \Gamma_v \xrightarrow{id} \Gamma_v$$

Q: What is a splitting of the p-adic valuation? Puiseux valuation?

Initial Forms and Ideals

$$\mathrm{in}_w(f) = \overline{t^{-\mathrm{trop}(f)(w)} f(t^{w_1}x_1, \ldots, t^{w_n}x_n)} \in \Bbbk[x_1^{\pm}, \ldots, x_n^{\pm}]$$

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$$

Example. $K = \mathbb{Q}$ with 2-adic valuation, $w = (2, -1)$,

$$\begin{aligned} f &= -\frac{31}{16}x^2 + 24xy + 12y^2 + 2 \\ \mathrm{trop}(f)(2,-1) &= \min\{-4+4, 3+2-1, 2-2, 1\} \\ &= \min\{0, 4, 0, 1\} = 0 \end{aligned}$$

$$\begin{aligned} \to \mathrm{in}_w(f) &= \overline{t^{-0}\left(-\frac{31}{16}2^4x^2 + 24\cdot 2^2\cdot 2^{-1}xy + 12\cdot 2^{-2}y^2 + 2\right)} \\ &= x^2 + y^2 \in \mathbb{F}_2[x^{\pm}, y^{\pm}] \end{aligned}$$

Tropicalizing $V(I)$

Note: If an ideal in a Laurent polynomial ring contains a monomial, it's the unit ideal.

Let $X = V(I) \subset K^*$ be any subvariety of a torus.

$$\mathrm{trop}(V(I)) = \{w \in \mathbb{R}^n : \mathrm{in}_w(I) \text{ does not contain a monomial}\}$$

Also equivalent:

- The intersection of $\mathrm{trop}(V(f))$ for all $f \in I$, or
- $\overline{(v(y_1, \ldots, v(y_n)) : y \in V(I)(K)}$

Using Hypersurfaces

Fix a homogeneous ideal I in $S = K[x_0, \ldots, x_n]$.

1. For $d \in \mathbb{N}$, find a K-basis for I_d (the dth homogeneous part of I), $\{f_1, \ldots, f_{r_d}\}$. Let A_d be the matrix of coefficients of the f_i.
2. $$g_d = \sum_{\substack{J \subseteq M_d \\ |J| = r}} \det(A_d^J) \prod_{u \in J} x^u$$
3. There exists $D \in \mathbb{N}$ such that any initial ideal $\mathrm{in}_w(I)$ has generators in at most degree D.
$$g = \prod_{d \le D} g_d$$
4. $\Sigma(I) = \Sigma(g)$