Tropical Geometry Minicourse

Day 1

July 19, 2021

What is Tropical Geometry?

The Tropical Semiring

$$(\mathbb{R}, \min, +) = (\mathbb{R}, \oplus, \odot)$$

$$\text{addition} = \min$$
$$\text{multiplication} = +$$

- **Additive identity**: ∞
- **Multiplicative identity**: 0
- All elements except ∞ have a multiplicative inverse, no elements except 0 have an additive inverse.

$$\begin{aligned} 3 \odot 5 &= 3 + 5 \\ &= 8 \\ 7 \oplus 9 &= \min(7, 9) \\ &= 7 \\ 1 \odot 2 \odot 3 &= 1 + 2 + 3 \\ &= 6 \end{aligned}$$

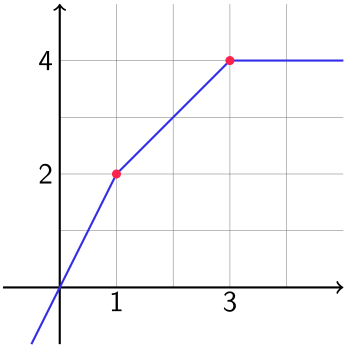
$$\begin{aligned} 13 \odot -11 \oplus 3 &= \min(13 - 11, 3) \\ &= \min(2, 3) \\ &= 2 \\ 9 \oplus 2 \odot 3 \oplus 8 &= \min(9, 2 + 3, 8) \\ &= \min(9, 5, 8) \\ &= 5 \end{aligned}$$

Tropical Polynomials

Q: What is 6^4 tropically?

$$f(x) = x^2 \oplus 1 \odot x \oplus 4$$

$$= \min(2x, 1 + x, 4)$$



The **tropical roots** are the break points in graph of the tropical polynomial, which is where the **min is achieved twice**.

Tropical Polynomials in Two Variables

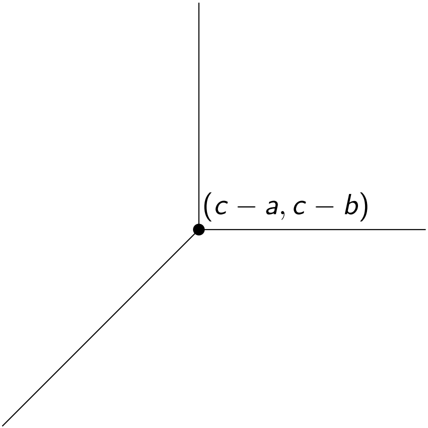
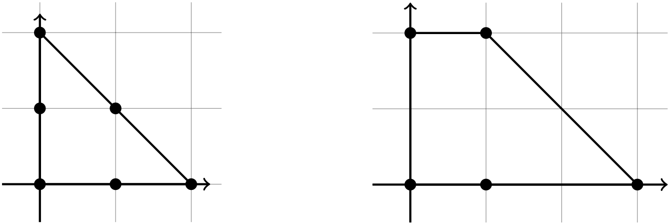


Figure: The vanishing set of $h(x) = a \odot x \oplus b \odot y \oplus c$

Newton Polytopes



$3x^2 - xy + 16y^2 + x + y + 1$ $\qquad$ $x^3 + xy^2 + y^2 + 2x - 3$

Figure: Some examples of Newton polytopes.

Tropicalization

Q: How do we go from an algebraic variety to a tropical variety?

- **A1:** Translate classical polynomials into tropical ones.
  translate: $3x^2 + 2ix + 1$

- **A2:** Take the coordinate-wise valuation of points in $V(X)$.
  $\text{val}(i/3)$, $\text{val}(-i)$ are the tropical roots.

Valuations

A **valuation** is a map v from a field K to $\mathbb{R} \cup \{+\infty\}$ satisfying the following rules:

1. $v(a) = +\infty \iff a = 0$,
2. $v(ab) = v(a + b)$, and
3. $v(a + b) \geq \min(v(a), v(b))$.

- A **valued field** if a field that comes with a valuation.
- The **value group** is the image of the group of units K^* under the valuation, denoted Γ_v. If the value group is $\mathbb{Z}$, the valuation is said to be **discrete**.
- The **valuation ring** is the subring of K of non-negatively (≥ 0) valued elements. I will usually denote it by R_v. It is a local ring with maximal ideal the set of elements with strictly positive valuation. I will denote the maximal ideal by m_v.
- The **residue field** is $\Bbbk_v = R_v/m_v$.

Valuation Examples

- **Trivial Valuation**. $v(a) = 0$ if $a \neq 0$, $v(0) = +\infty$.
- p**-adic valuation**.
- **Laurent/Puiseux Valuation**.

Other Approaches

Curve Counting

Q: How many rational degree d curves pass through $3d - 1$ pts? How many genus g curves pass through $g + 3d - 1$ points?

A: $N_{0,1} = 1$, a unique line passes through 2 general points. $N_{0,2} = 1$, a unique quadric passes through 5 general points.

The $N_{g,d}$ are called **Gromov-Witten** numbers.

Theorem (Mikhalkin's Correspondence). **Gromov-Witten** numbers can be counted tropically.

Mikhalkin's Correspondence

- A general tropical plane curve of degree d is dual to a subdivision of the triangle with vertices $(0, d)$, $(d, 0)$, and $(0, 0)$.
- A tropical plane curve is **simple** if the corresponding subdivision consists of triangles and parallelograms.
- $t(C)$ is the number of triangles in the subdivision for C.
- $r(C)$ is the number of unbounded rays in C.
- The **genus** of a tropical plane curve is:

$$g(C) = \frac{1}{2}t(C) - \frac{1}{2}r(C) + 1$$

- The **contribution** of a simple tropical plane curve is the product of the normalized areas of triangles in the subdivision.

Examples

Questions?

- Grigory Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$.
- Andreas Gathman and Hannah Markwig, *The Caporaso-Harris formula and relative Gromov-Witten invariants in tropical geometry*, and *The numbers of tropical plane curves through points in general position*.