# DAY 1: TROPICAL LANGUAGE AND APPLICATIONS

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## 1. Tropical Language

**References:** !!!

### 1.1. The Tropical Semiring.

Tropical algebra takes place in the ***tropical semiring***, which is denoted

$$(\mathbb{R} \cup \{+\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{+\infty\}, \min, +)$$

The tropical plus operation is min, and the tropical multiplication operation is classical addition. For example,

$$7 \oplus 1 \odot 3 = \min(7, 1+3) = 4$$

The additive identity is $\infty$ and the multiplicative identity is 0. Most elements don't have additive inverses. I will usually write out tropical operations using min and + to avoid confusion.

**Remark 1.** *The tropical semiring defined above is sometimes called the min-plus tropical semiring. Another convention is to use the semiring* $(\mathbb{R} \cup \{-\infty\}, \max, +)$, *which is called the max-plus tropical semiring.*

### 1.2. Tropical Polynomials in One Variable.

Taking powers tropically is the same as scalar multiplication. For example, consider the following ***tropical polynomial***

$$f(x) = x^2 \oplus 1 \odot x \oplus 4 = \min(2x, 1+x, 4)$$

which is graphed in Figure 1. The changes in slope, or "break points", are called ***tropical roots***. In classical algebraic geometry, we associate a function with its vanishing set (roots in one dimension), and for tropical polynomials we will usually only graph the tropical roots.

Here are some more facts about tropical polynomials in one variable:

(1) The quadratic formula for tropical polynomials is:

$$x^2 \oplus a \odot x \oplus b = \begin{cases} (x \oplus a) \odot (x \oplus (b-a)) & \text{if } 2a \le b \\ \left(x \oplus \frac{b}{2}\right)^2 & \text{otherwise} \end{cases}$$

(2) Multiplicity of tropical roots is counted by the change in slope between the two line segments on either side.

(3) The tropical semiring is algebraically closed; every tropical polynomial in one variable of degree $n$ has $n$ tropical roots when counted with multiplicity.

(4) Factoring general tropical polynomials in one variable is a lot easier than factoring general classical polynomials in one variable.

(5) The (tropical) product of the roots of a tropical polynomial is the constant term; the (tropical) sum of the roots is the $\deg f - 1$ coefficient.

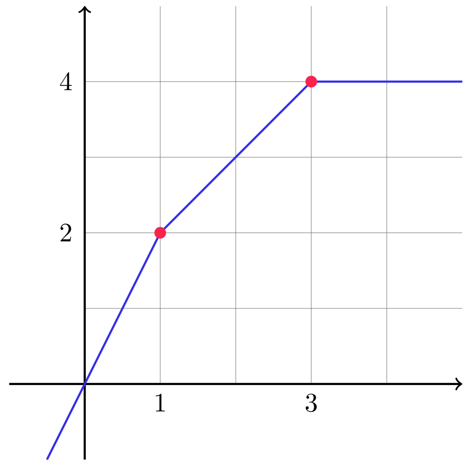



FIGURE 1. The graph of $x^2 \oplus 1 \odot x \oplus 4$. The "break points" highlighted in red are called ***tropical roots***.

**Example.** Factor

$$g(x) = x^3 \oplus 2 \odot x^2 \oplus 6 \odot x \oplus 11 = \min(3x, 2 + 2x, 6 + x, 11)$$

The rightmost root will be the point when one of the line segments with positive slope ($3x$, $2 + 2x$, or $6 + x$) intersects the line $y = 11$:

$$3x = 11 \implies x = \frac{11}{3}$$
$$2 + 2x = 11 \implies x = \frac{9}{2}$$
$$6 + x = 11 \implies x = 5$$

The rightmost solution is $x = 5$, where $y = 6 + x$ and $y = 11$ intersect, so there is a tropical root at $x = 5$ (of multiplicity 1). Continuing in this way, we find that:

$$g(x) = (x \oplus 5) \odot (x \oplus 4) \odot (x \oplus 2)$$

**Remark 2.** *The set of tropical roots is the set of points where the minimum in the tropical polynomial is achieved at least twice.*

### 1.3. Tropical Polynomials in Two Variables.

**Definition 1.** The ***vanishing set*** of a tropical polynomial is the set of points where the minimum is achieved twice.

**Example** (Tropical Line)**.** See Figure 2.

You might notice that a tropical line is dual to a triangle. This is not a coincidence:

**Proposition 1.** *The vanishing set of a tropical plane curve given by the tropical polynomial $f(x, y)$ is dual to a subdivision of the Newton polytope of $f(x, y)$.*

**Definition 2.** The ***Newton polytope*** of a polynomial $f(x, y)$ is the convex hull of the exponent vectors of non-zero terms of $f(x, y)$.

**Example** (Newton Polytopes)**.**

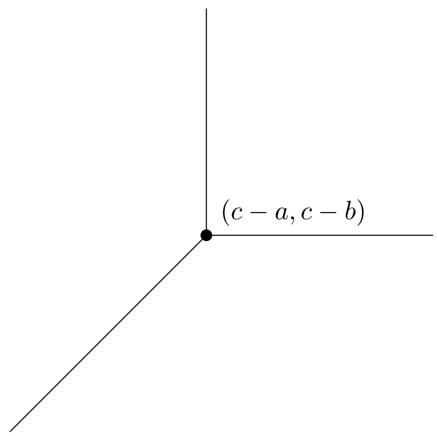
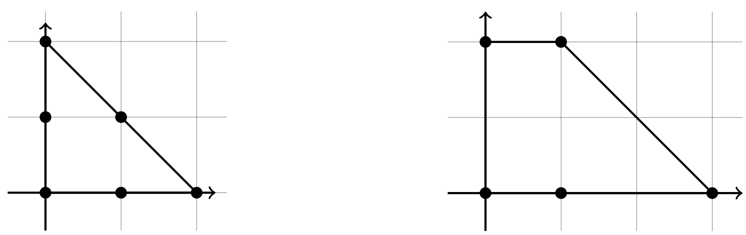



FIGURE 2. The vanishing set of $h(x) = a \odot x \oplus b \odot y \oplus c$



$3x^2 - xy + 16y^2 + x + y + 1$ $\quad$ $x^3 + xy^2 + y^2 + 2x - 3$

FIGURE 3. Some examples of Newton polytopes.

Here are some facts about tropical plane geometry:

(1) Any two general points lie on a unique tropical line. Five general points lie on a unique tropical quadratic curve.
(2) Two general tropical lines meet in a unique point. In general, two general tropical curves of degrees $c$, $d$ meet in $cd$ points (Bezout's theorem holds tropically).
(3) Pappus's theorem holds tropically when phrased in terms of constructing points, but not when phrased in terms of incidences.
(4) Any two (not necessarily general) curves have a unique ***stable intersection***. Essentially, monodromy does not exist in tropical geometry. I don't know enough about stable intersection to include it in the minicourse, but if you are interested I would be happy to talk about it!

1.4. **Tropical Structure Theorem.** Tropical varieties are:

(1) Pure dimensional **polyhedral complexes**,
(2) (weighted) ***balanced***, and
(3) ***connected through codimension 1***.

1.5. **Valuations.** So far we have discussed tropical polynomials and tropical varieties coming directly from the tropical semi-ring. We can also "tropicalize" classical

Figure 4. Left: a pure dimensional polyhedral complex; Middle: a polyhedral complex that is not pure dimensional; Right: not a polyhedral complex.

Figure 5. L: A balanced polyhedral complex; R: An unbalanced polyhedral complex.

Figure 6. L: A polyhedral complex connected through codimension 1; R: A polyhedral complex not connected through codimension 1.

varieties, for which we will need valuations. "Tropicalized" varieties are a subset of "tropical" varieties.

A ***valuation*** is a map $v$ from a field $K$ to $\mathbb{R} \cup \{+\infty\}$ satisfying the following rules:

(1) $v(a) = +\infty \iff a = 0$,
(2) $v(ab) = v(a+b)$, and
(3) $v(a+b) \geq \min(v(a), v(b))$.

**Remark 3.** *It is possible to replace $\mathbb{R}$ with any ordered group $\Gamma$. The restriction $v : (K^*, \times) \to (\mathbb{R}, +)$ is a group morphism (because of rule 2) from the* ***multiplicative*** *group of units of the field to the* ***additive*** *group $\mathbb{R}$. Also, some people might also use a max convention in the third rule, and replace $+\infty$ with $-\infty$.*

Here are some other words related to valuations:

(1) A ***valued field*** if a field that comes with a valuation.
(2) The ***value group*** is the image of the group of units $K^*$ under the valuation, denoted $\Gamma_v$. If the value group is $\mathbb{Z}$, the valuation is said to be ***discrete***.
(3) The ***valuation ring*** is the subring of $K$ of non-negatively ($\geq 0$) valued elements. I will usually denote it by $R_v$. It is a local ring with maximal ideal the set of elements with strictly positive valuation. I will denote the maximal ideal by $m_v$.
(4) The ***residue field*** is $\Bbbk_v = R_v/m_v$.

There are basically three examples of valuations relevant to us for this minicourse, although there are many valuations that behave very differently!

**Example** (Trivial Valuation)**.** $K$ is any field, $v(a) = 0$ for $a \in K^*$, $v(0) = +\infty$.

**Example** ($p$-adic Valuation)**.** $K = \mathbb{Q}$, fix a prime $p$. Then the $p$-adic valuation of $a/b \in \mathbb{Q}$ is:

$$v_p(a/b) = \text{largest power of } p \text{ dividing } a - \text{largest power of } p \text{ dividing } b$$

**Example** (Laurent/Puiseux Series Valuation)**.** First, let $K = \Bbbk(\!(t)\!)$ be the field of formal Laurent series over a field $\Bbbk$. The Laurent valuation takes a formal Laurent series to the lowest exponent that occurs in the series:

$$v\left(\sum_{i=1}^{\infty} c_i t^{a_i}\right) = a_1$$

where $a_i < a_{i+1}$. ***Puiseux series*** are an extension of Laurent series, where we allow elements with fractional exponents as long as the exponents have a common denominator:

$$\text{Puiseux Series } = \Bbbk\{\{t\}\} = \bigcup_n \Bbbk\{\{t^{1/n}\}\} = \left\{ \sum_{i=1}^{\infty} c_i t^{a_i/n} \mid a_i < a_{i+1} \right\}$$

The valuation we defined on Laurent series also extends to Puiseux series, again by sending a series to it's lowest occuring exponent:

$$v\left( \sum_{i=1}^{\infty} c_i t^{a_i/n} \right) = \frac{a_1}{n}$$

A ***splitting*** of a valuation is a group morphism $\phi : (\Gamma_v, +) \to (K^*, \times)$ such that $v \circ \phi$ is the identity on the value group. The three examples we gave above have splittings (as do all discretely valued fields, and all algebraically closed valued fields). However, there are many valuations without splittings!

Some more relevant facts:

(1) The Puiseux series field $\Bbbk\{\{t\}\}$ is algebraically closed if $\Bbbk$ has characteristic 0 and is algebraically closed. When $\Bbbk$ has characteristic $> 0$, the Artin-Schrier polynomial has no roots: $x^p - x - t^{-1}$.

1.6. **Tropicalization.**

**Remark 4.** *Taking the valuation of 0 is bad (i.e. infinity), so we will try to avoid that by considering classical varieties in the algebraic torus $(K^*)^n$ instead of $\mathbb{A}^n_K$, and correspondingly considering ideals in the Laurent polynomial ring $K[x_1^{\pm}, \ldots, x_n^{\pm}]$ instead of the polynomial ring $K[x_1, \ldots, x_n]$.*

Fix a valued field $(K, v)$, and a classical algebraic variety $X = V(I)$ over $K$. From the perspective of Maclagan and Sturmfels' *Introduction to Tropical Geometry*, there are basically two ways of thinking about ***tropicalizing*** $X$:

(1) Take the closure of the coordinate-wise valuation of the points in the classical variety:

$$\overline{\{(v(y_1), \ldots, v(y_n)) \mid (y_1, \ldots, y_n) \in X(K)\}}$$

(2) Intersecting the tropical hypersurfaces defined by trop($f$) for $f \in I$. Below I will describe how to tropicalize a polynomial.

These constructions are equivalent (the Fundamental Theorem of Tropical Geometry), although the proof is rather technical.

Let

$$f = \sum_u c_u x^u \in K[x_1^{\pm}, \ldots, x_n^{\pm}]$$

be a classical Laurent polynomial over a valued field $(K, v)$, and let $w \in \mathbb{R}^n$. I will refer to $w$ as a ***weight***. The ***tropicalization of*** $f$ ***at*** $w$ is:

$$\text{trop}(f)(w) = \bigoplus_u v(c_u) \odot w^u = \min_u \{v(c_u) + u \cdot w \mid c_u \neq 0\}$$

Here is what is happening algebraically:

(1) We replace the classical arithmetic operations with tropical ones. So the sum becomes a min, and multiplication turns into sums. Note that the exponentiation $w^u$ tropically is the standard dot product $u \cdot w$.

(2) We replace the coefficients $c_u$ with their valuation.

The points in the tropical hypersurface $tV(f)$ are the weights $w$ at which the minimum in $\text{trop}(f)(w)$ is achieved at least twice.

1.7. **Other Approaches to Tropical Geometry.** I will focus on an algebraic approach to tropical geometry, but it is worth mentioning that there are other more analytically-minded paths to tropicalization. These are described briefly in §1.4 of Maclagan and Sturmfel's book with references for further reading.

## 2. Applications of Tropical Geometry

2.1. **Curve Counting.** A famous problem in enumerative algebraic geometry is to count all rational plane curves passing through $3d-1$ general points. Grigory Mikhalkin counted tropical plane curves to solve this problem. In fact, he solved it more generally for genus $g$ plane curves passing through $g+3d-1$ points. The number of such curves is a ***Gromov-Witten*** number, denoted $N_{g,d}$. For example $N_{0,1}=1$ (a unique line passes through two points), and $N_{1,3}=1$ (a unique cubic passes through 9 points).

In the tropical setting, a curve is degree $d$ if it is dual to a subdivision of the polytope $\text{conv}((0,d),(d,0),(0,0))$ (the Newton polytope of a general polynomial of degree $d$ in two variables. The genus of a tropical curve is given by:

$$g(C)=\frac{1}{2}t(C)-\frac{1}{2}r(C)+1$$

where $t(C)$ is the number of triangles in the corresponding subdivision and $r(C)$ is the number of unbounded rays on the tropical curve.

**Theorem 1** (Mikhalkin's Correspondence)**.** *The number of simple tropical curves of degree $d$ and genus $g$ that pass through $g+3d-1$ general points in $\mathbb{R}^2$, where each curve is counted with its contribution, equals the Gromov-Witten number $N_{g,d}$ of the complex projective plane $\mathbb{P}^2$.*

We still need to define "simple" and "contribution". To do this, we use the correspondence between the tropical plane curve defined by $f(x,y)$ and a subdivision of the Newton polytope of $f(x,y)$.

**Definition 3.** A tropical plane curve defined by $f(x,y)$ is ***simple*** if all the maximal cells in the corresponding subdivision are triangles or parallelograms.

**Definition 4.** The ***contribution*** of a simple plane curve is the product of the normalized areas of the triangles in the corresponding subdivision.

2.2. **Implicitization.**

2.3. **Algebraic Statistics.**

2.4. **Wonderful Compactification.**

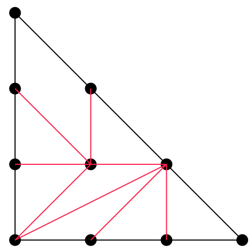


FIGURE 7. Genus 0 simple cubic with contribution 1.

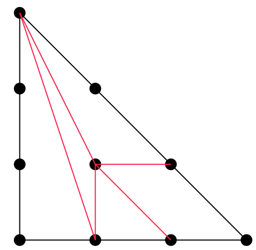


FIGURE 8. Genus 0 simple cubic with contribution 6.

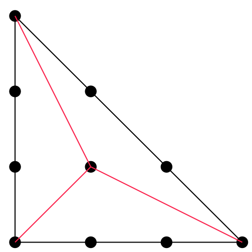


FIGURE 9. Genus 1 simple cubic with contribution 27.

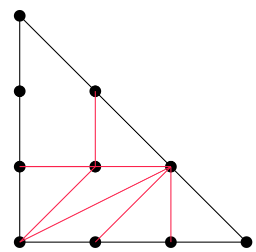


FIGURE 10. A genus 0 non-simple cubic.