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The Impact of Operational Decisions on the Design of Salesforce Incentives

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When facing high levels of overstock inventories, firms often push their salesforce to work harder to attract more demand, and one way to do that is to offer attractive incentives. Most research on the optimal design of salesforce incentives, however, ignore this dependency and assume that operational decisions of production/inventory management are separable from the design of salesforce incentives. We study such a dependency and consider the problem of joint salesforce incentive design and inventory/production control. We develop a dynamic Principal-Agent model with both Moral Hazard and Adverse Selection, and demonstrate the impact of operational decisions on the design of a compensation package. The optimal strategy is characterized by a menu of inventory-dependent salesforce compensation contracts. We find that the optimal compensation package highly depends on the operational decisions; when inventory levels are high, (a) the firm needs to offer a more attractive contract, and (b) the contract is effective in inducing the salesforce to work harder than usual. In contrast, when inventory levels are low, the firm can offer a less attractive compensation package, but still expect the salesforce to work hard enough. Moreover, although the inventory/production management and the design of salesforce compensation package are highly correlated, we show that the firm does not need to follow a complex inventory/production control policy; a market-based state-dependent policy (with a constant base-stock level when the inventory is low) that benefits from the extracted market condition from the agent is optimal. Moreover, contrary to the well-known notion of safety-stock in classical inventory theory in which salesforce incentives are neglected, we show that a higher demand uncertainty may result in a lower optimal stock level.

Key words: Dynamic Incentive Design; Principal-Agent; Moral Hazard; Adverse Selection; Inventory Control
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1. Introduction

Marketing and Operations Management (OM) represent two important sections of most firms. Marketing personnel try to increase demand and operations personnel try to effectively match supply with the attracted demand. Both of these two sections' activities also represent major investments for firms. For instance, while operational activities (e.g., logistics, warehousing, production/inventory, etc.) are widely known to be a major cost for firms, a salesforce typically costs 5 to 40 percent of sales for a firm (Zoltners et al. (2001)).

Because of their importance, a vast stream of research has been devoted to generate insights into improving the marketing or operational activities of firms. However, the literature on effective mechanisms to integrate these activities is still immature and growing (see, for instance, O'Leary-Kelly and Flores (2002) and Jerath et al. (2011)). The importance of filling this gap and the need for research on integrating these activities have been highlighted in several studies including the *Harvard Business Review* titled "Can marketing and manufacturing coexist?" (Shapiro (1977)). Such studies (see also Jerath et al. (2011)) specifically illuminate the importance of considering marketing decisions when controlling production/inventory activities, and vice versa.

In this paper, we consider the joint problem of (a) designing appropriate *inventory-dependent* contracts to induce salesforce to attract *sufficient demand* (marketing), and (b) controlling production/inventory to generate *enough supply* (operations). In the next section, we will review the literature. As will be seen, the problem of salesforce incentive design has been widely studied, especially in the Marketing literature. Most of the research in this area, however, does not consider the interrelation between salesforce incentive design and production/inventory control. Hence, the incentive design problems studied in the literature are typically separated from such operational decisions, and consequently, lacking any coordination, the optimal contract design and the optimal production/inventory management become independent decisions. In applications, however, salesforce compensation plans may highly depend on such operational decisions. For instance, when firms face overstock inventories (e.g., as a result of a higher-than-required productions or lower-than-expected sales in previous period), they push their workforces to work hard to enhance customer demand, and offering the salesforce a more attractive compensation plan is a viable and often utilized option. By contrast, when firms are understocked, they are less interested in boosting demand. Hence, they offer a less attractive compensation package, since there is no need for a significant effort from the salesforce. Indeed, altering the compensation plans has been empirically found to be an effective method in practice to increase profit through increasing sales, and thereby decreasing the consequences of keeping high levels of unsold inventories (see, e.g., Levenson et al. (2010)). We further note that contract design is a realistic approach used in practice for effective salseforce compensation (see, for example, Gonik (1978) for a menu of incentive contract designed by IBM with regard to the salesperson's sales forecast).

The problem we consider in this paper (i.e., the joint inventory control and the design of salesforce incentives) is faced by many firms. An example is LKQ Corporation that provides a variety of products to collision repair shops as well as mechanical repair shops. The firm needs to effectively control the inventory of its items, which includes new products produced by original equipment manufacturing (OEM), products produced by other companies rather than OEMs (aftermarket products), and recycled and refurbished items. Same as what we consider in this paper, LKQ faces nonstationary and seasonal demand: *"During the winter months, we tend to have higher demand for our products because there are more weather related accidents, which generate repair"* (Cybernet Data Systems (2011)). Moreover, as is our focus in this study, it is reported by the company that (a) most of their sales personnel are paid on a commission basis, and (b) the firm has to design effective incentive mechanism for their salesforce and regularly evaluate its effectiveness: *"Our objective is to continually evaluate our sales force, . . ., and utilize appropriate measurements to assess our selling effectiveness"* (Cybernet Data Systems (2011)).

To gain insights into how firms can effectively integrate their production/inventory decisions with the design of effective compensation plans, we develop a dynamic Principal-Agent model in which the salesforce (representing the agent) sells the product on behalf of the firm (representing the principal). In each period, the firm (1) designs a menu of compensation contracts for the agent based on the current level of inventory to (a) induce him to work hard to attract enough demand, and (b) extract information about the market condition, and (2) makes production/inventory control decisions based on the extracted information to provide enough supply. The agent, possessing more information about the market than the firm, decides whether to accept a contract, and if so, how much selling effort to exert for that period.

As is the case in most real-world situations, we assume the firm cannot directly observe the sales effort exerted by the sales agent, and only observes the demand/sales in each period. Hence, the firm has to compensate the agent based on the realized sales. Therefore, the problem facing the firm in our framework is a mixture of *moral hazard* (post-contractual opportunism associated with the effort decision) and *adverse selection* (pre-contractual asymmetric information regarding the market condition). A well-known solution to this type of combined and often complex problems is to offer a menu of contracts[1] to the agent who privately knowing the market condition chooses one to sign. The signed contract, if effective, can induce the agent to work hard enough (generating enough demand). Moreover, by screening the choice made by the agent, the firm can learn about the market condition, and this knowledge can be then used by the firm in making better production/inventory decisions (generating enough supply).

We first analyze our model in the last period, and completely characterize the optimal menu of inventory-dependent contracts that should be offered by the firm. We find that the optimal contracts are (1) more attractive when the inventory level is high or the salesman is not highly risk averse, (2) effective in inducing the salesforce to work harder when inventory level is higher, and (3) suitable in providing sufficient information for the firm to effectively make production/inventory decisions. Point (1) supports the widely observed behavior of firms in offering higher commission rates when facing high levels of over-stock inventories. It also supports the empirical studies that find (a) changing compensation plans is effective in increasing sales (see, e.g., Levenson et al. (2010)), and (b) more risk-averse individuals prefer a fixed salary and are less productive under incentive pays (see, e.g., Cadsby et al. (2007)). Points (2) and (3) shed light on two typically ignored aspects of a good compensation plan.

We then analyze the problem in an arbitrary period and show that the optimal menu of contracts bear a similar structure but are more complex compared to the last period. Hence, using the result for the optimal contracts in the last period, we propose a heuristic menu of inventory-dependent contracts, and show its effectiveness through numerical examples. We also consider the optimal inventory-independent menu of contracts as another heuristic, compare the performance of both of the heuristics with the optimal one, and highlight the importance of considering inventory levels in the design of salesforce incentives. Furthermore, by analyzing the problem in an arbitrary period, we show that although the inventory/production control and the design of the compensation package are highly interconnected, the firm does not need to implement a complex inventory/production

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[1] For a real-world example of use of menu of contracts, see Agrell and Bogetoft (2003) who provide a report on the menu of incentive contracts commissioned by the Norwegian Water Resources and Energy Directorate.

management policy; the optimal inventory control policy for the firm is a *market-based state-dependent* policy that benefits from the extracted market condition from the agent. In addition, in periods with low inventory levels, this policy translates to the traditional base-stock policy, except that the firm needs to make use of the the extracted information from the agent. Moreover, in contrast to the well-known notion of safety-stock in classic inventory theory where salesforce incentives do not play a role, we show that a higher uncertainty in demand may result in a lower optimal stock level. Indeed, a higher demand uncertainty reduces the optimal incentive that is provided for the risk-averse agent to work hard. This lower incentive level in turn decreases the attracted demand. As a result, the firm will need to set a lower base-stock level for replenishing its inventory.

The rest of the paper is organized as follows. We review the literature in the next section. In Section 3, we present the model, and in Section 4, we study the optimal production and incentive design problem for the last period. In Section 5, we then analyze the model for an arbitrary period, and in Section 6, we conclude the paper with a discussion of our results and highlighting some possible directions for future research. All of the proofs are provided in an appendix at the end of the paper.

2. Literature Review

Basu et al. (1985) is usually credited as the first to implement the agency paradigm from Economics to the problem of salesforce compensation in the Marketing literature. Basu et al. (1985) discuss the impact of uncertainty, risk aversion, and some other factors on the design of compensation packages. There are, however, several other studies on the design of salesforce incentives in the Marketing literature. Comprehensive reviews of this stream can be found in Coughlan and Subrata (1989) and Coughlan (1993). A fundamental feature in this stream of research is that the salesforce compensation package should be designed based on the observed realization of the demand. In contrast, the operational decisions such as inventory or production control widely considered in the OM literature should be made a priori based on some knowledge about the demand process.

An overview of research on the integration of manufacturing and marketing decisions can be found in O'Leary-Kelly and Flores (2002). Within this broad stream of research, some studies have tried to fill the gap between the Marketing literature and that of OM by incorporating the operational decisions such as production/inventory control in the design of incentives. The first work in this vein appears to be the study of Atkinson (1979), which considers the incentive design between a single owner and a single manager under a newsvendor setting. Porteus and Whang (1991) consider the incentive design between a manufacturer manager (who make capacity and inventory decisions) and several product managers (who make sales effort decisions). Chen (2000) discusses the impact of sales-force incentives (over multiple time periods) on a manufacturing firm's production and inventory decisions. Chen (2000) proposes a moving-window plan to induce salespeople to exert selling effort to smoothen the demand process. Heese and Swaminathan (2010) consider a setting in which a retailer makes ordering and salesforce effort exertion decisions. They find that, similar to the result of Lariviere and Porteus (1999), the retailer should provide a higher service level to increase the effect of learning from uncensored demand information (the so-called "information stalking"). Jerath et al. (2011) implement agency theory to determine compensation plans for sales and operations managers to coordinate their activities.

Chen (2005) studies the salesforce compensation problem together with the firm's production and inventory decision in a single-period (i.e., static) setting, in which the compensation plan offered by the firm is independent of its production/inventory decision. Our work differs from Chen (2005) mainly in two regards: (1) we consider a multi-period (i.e., dynamic) setting, where the market condition and inventory levels change over time, and (2) we allow the compensation plan (i.e., the contract between the firm and the salesforce agent) to depend on the inventory level which itself is affected by the agent's effort in the previous periods as well as the firm's production/inventory replenishment decision. Using this dynamic framework, we show the importance of considering inventory levels when designing a compensation package for the salesforce, and generate novel insights into how this dependency can dynamically be considered by firms to increase profit. To the best of our knowledge, our work is the first attempt in this vein.

Similar to Chen (2005) and some other studies, the relationship between the firm and the salesforce in our model is of a Principal-Agent type. Principal-Agent models have been widely studied in field of Microeconomics. Mas-Colell et al. (1995) provide an introductory textbook treatment of Principal-Agent models. The book by Laffont and Martimort (2001) provides a comprehensive study of Principal-Agent models, and the books by Salanie (1997) and Bolton and Dewatripont (2005) cover broader issues in the theory of contracts, including strategic commitment and renegotiation in dynamic settings.

In general, Principal-Agent models in dynamic settings possess several technical challenges, and the underlying theory to handle those is still immature. This is despite the fact that dynamic Principal-Agent models can be effective in providing insights into many areas including OM. One of the first applications of dynmic Principal-Agent models in OM is the work of Plambeck and Zenios (2000) who consider a setting without adverse selection (i.e., moral hazard only), where the transition probabilities depend on the actions chosen by the agent, and the principal seeks to design a payment that maximizes her expected discounted profit. Plambeck and Zenios (2000) derive a set of assumptions under which two difficulties of dynamic Principal-Agent models can be handled: (1) history dependence of compensation schemes which requires accumulation of information from all the previous periods, and (2) strategic commitment of the principal and the agent where the two parties initially agree to be bound to terms for future payments. When (a) the manager's (agent's) utility is exponential and additively separable over time, (b) the manager can transfer consumptions between periods (through saving and withdrawing income), and (c) the owner (principal) is a risk-neutral profit maximizer, Plambeck and Zenios (2000) show that the problem is amenable to a two-step dynamic programming formulation and the optimal payment scheme satisfies the Bellman principle of optimality.

In another study, Plambeck and Zenios (2003) consider a principal-agent version of the classical make-to-stock single-server queueing system and show that an optimal compensation scheme consists of piece rates and inventory penalties that depend on inventory levels. The work of Plambeck and Zenios (2000) and Plambeck and Zenios (2003) both can be classified in the context of dynamic moral hazard settings.

Addressing a dynamic Principal-Agent model in an adverse selection context, Zhang and Zenios (2008) consider a setting in which the underlying system is a Markov Decision Process (MDP), the sate of the system can only be observed by the agent, but the agent's action is publicly observable.

Zhang and Zenios (2008) establish that, under their dynamic setting, it is sufficient to only consider the revelation contracts. Zhang et al. (2010) consider a dynamic adverse selection model between a supplier and a retailer in a supply chain where the supplier does not know the inventory level at the downstream retailer.

There are also a few studies on dynamic Principal-Agents models with both moral hazard and adverse selection. For this stream of research, we refer interested readers to Ding et al. (2003), Doepke and Townsend (2006), and the references therein. We contribute to this stream of research by considering the salesforce incentive design problem in a setting where the agent does not know the inventory of the firm, and the firm cannot observe the effort level exerted by the agent, but she can alter the agent's incentives depending on the inventory level.

In closing this section, we notice that while we consider the design of incentive plans for salesperson, recent work on the design of incentives for wholesale-salesperson are also relevant to our study. For this stream of research, we refer interested readers to Hopp et al. (2011), and the references therein.

3. The Model

A firm sells a single product through a sales agent over $N \in \mathbb{N}$ periods (e.g., if $N = 4$ it may represent the 4 quarters of the year, and if $N = 7$ it may represent the 7 days of the week). The demand/sales in each period depends on the market condition as well as the effort exerted by the salesforce. The attracted demand in period $n \in \{1, 2, \cdots, N\}$ is

$$D_n = \Theta_n + a_n + \mu_n + \epsilon_n,$$

where Θ_n is the market condition in period n, $a_n \in \mathbb{R}^+$ is the agent's advertising/sales effort, μ_n is a period-dependent number which may, for example, reflect seasonal effects, and ϵ_n is a random noise which is assumed to be normally distributed with mean 0 and variance σ_n^2. The variance in the noise, σ_n^2, may depend on period n, since in applications the demand may be seasonal and affected by some uncontrolled factors differently in each season. The demand variance in our setting is mainly determined by σ_n^2. We assume, for tractability, that the effort level exerted by the agent has only a first order affect and does not significantly affect σ_n^2.

Consistent with the literature and for tractability reasons, we assume that the market can be in either of two possible conditions, High (H) or Low (L), which is equivalent to Θ_n taking two possible values, θ_H and θ_L with $\theta_H \geq \theta_L$. At the beginning of each period, the salesman, being closer to the market, learns the market condition, while the firm only has an estimate of the market condition (i.e., some prior belief that the market condition is of type H or L). We assume the market condition process $\{\Theta_n, n=1,2,\cdots,N\}$ evolves as a two-state Markov chain with transition probabilities ρ_{ij} for $i,j=L,H$. That is,

$$P\{\Theta_{n+1}=\theta_H \mid \Theta_n=\theta_H\}=\rho_{HH}, \qquad P\{\Theta_{n+1}=\theta_L \mid \Theta_n=\theta_H\}=\rho_{HL},$$

$$P\{\Theta_{n+1}=\theta_H \mid \Theta_n=\theta_L\}=\rho_{LH}, \qquad P\{\Theta_{n+1}=\theta_L \mid \Theta_n=\theta_L\}=\rho_{LL}.$$

The salesman's utility in each period depends on the compensation he receives as well as the sales effort he exerts. If $s_i(D)$ is the compensation for a type $i \in \{H,L\}$ agent when the realized sales is D, and a_i is his exerted effort level, then the salesman's utility during the period is represented by the negative exponential function $u(s_i(D),a_i)=-e^{-\gamma(s_i(D)-v(a_i))}$, where $v(a_i)$ is an increasing[2] convex function representing the agent's cost of exerting effort/adervtising level a_i, and γ is a measure of risk attitude. Similar to Chen (2005), we assume $v(a)=a^2/2$, which is an example of an increasing convex function; however, our analysis can be easily extended to other forms of effort cost functions. Negative exponential utility (which is increasing concave) is prevalent in literature and describes that the agent is risk averse. Moreover, exponential utility is the natural choice of utility when a person's utility is invariant under any wealth translation (i.e., $u(w+k)=f(k)\,u(w)$ for any wealth w and k, and a function $f(\cdot)$). In our framework, the agent maximizes his expected utility in each period and is willing to accept a compensation contract if, and only if, his utility in each period can reach a minimum level $-U_0$.

The firm's problem is as follows. For each period, it incurs a unit cost c to purchase one item, and the selling $1+c$, where the marginal profit is normalized to one (these parameters can be extended to the case that they are period-dependent, see the last section for a discussion). If the

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[2] Throughout the paper, we use increasing and decreasing in non-strict sense, i.e., they represent non-decreasing and non-increasing respectively.

demand in a period is less than the inventory level, there is a holding cost of h per unit of stock. However, if the demand exceeds the on-hand inventory, then the excess demand must be satisfied via an emergency order at a cost of p per unit. (Similar assumptions have been made in Chen (2005), and it has also been used in several other scenarios. For instance, in Lutze and Ozer (2008), when stockout occurs it is assumed that the firm borrows inventory from alternative source. It is noted that this is equivalent to a lost-sales scenario, with the emergency ordering cost being the shortage cost.) These cost parameters satisfy $h < c < p < 1 + c$, but other cases can also be easily handled, and we allow for some relaxation in our numerical experiments.

The firm is risk neutral and her objective is to offer a menu of contracts to the salesperson dynamically to maximize her expected total profit (sales revenue minus compensation and production/inventory costs) over the planning horizon of N periods.

The relationship between the firm and the salesperson is that of a dynamic principal-agent in the sense that the latter in each period sells the product on behalf of the former. In each period, the principal (i.e., the firm) designs a menu of compensation contract for the agent (i.e., the salesman) and makes production/ordering decision, while the agent, possessing market information ahead of the principal, decides whether to accept a contract, and if so, how much selling effort to exert for the period. In this setting, through a suitable menu of contracts, the firm would like to (1) invoke the salesman to work hard enough to attract enough demand, and (2) extract information about the market condition that can be used for a better production decision to replenish the inventory.

The sequence of events in each period is as follows. (1) The firm reviews the inventory level (at the beginning of the period), and based on that and her belief about the market condition, the firm offers a menu of compensation contracts. (2) The market condition θ is realized and is privately observed by the agent. (3) The agent decides whether to participate, and if so, which contract to sign. (4) Under a signed contract, the firm decides the production quantity and the agent chooses the sales/advertising effort level. (5) Both parties observe the total sales (i.e., the realized value of demand), and the firm receives the selling revenue and pays the compensation to the agent. (6) The inventory shortage or overage cost is accrued. The resulting problem is a dynamic contract design problem. It should be noted that, in general, dynamic mechanism design is a very challenging problem, and the usual revelation principle may fail to hold (see, e.g., the discussions in Oh and Ozer (2011)). In this paper we focus on the set of contracts that do not provide any employment guarantee for the sales agent in the future periods. Under such contracts, the agent becomes a myopic optimizer, and the usual revelation principle applies. However, it should be noted that our approach also provides an approximation for analysis with contracts that provide short to middle-term employment guarantees.

4. Analysis of the Last Period

To gain insights, we start by analyzing the problem in the last period, N. For simplicity here we suppress the time index denoting the last stage. We also assume that, after period N there will be no cost or salvage value. That is, at the end of period N, the usual holding and shortage cost incurs and there is no additional cost afterwards.

Suppose that at the beginning of period N, and before any decision is made, the firm's inventory level is x, and the salesman privately observes the market condition, θ_H or θ_L. For convenience, we call the salesman who observes θ_H (θ_L) the H (L) type agent. By the revelation principle (Myerson (1982)), the firm needs to only consider the class of incentive compatible and direct revaluation contracts. Also, under a signed contract $s_i(D)$, to decide how much effort to exert, the type $i \in H, L$ agent needs to solve the optimization problem

$$\max_{a_i} \mathbb{E}[u(s_i(D), a_i)] = \max_{a_i} \mathbb{E}[-e^{-\gamma(s_i(D)-v(a_i))}]. \tag{1}$$

The compensation scheme we consider is the most commonly used form of compensation, i.e., an affine form, which represents a commission rate of α_i plus a fixed/salary payment of β_i: $s_i(D) = \alpha_i D + \beta_i$. By the certainty equivalence principle, since $s_i(D)$ is normally distributed, the optimization problem (1) has the solution

$$a_i^*(\theta) \triangleq \operatorname*{argmax}_{a_i} \mathbb{E}\,[-e^{-\gamma[s_i(D)-a_i^2/2]}] = \operatorname*{argmax}_{a_i} \left\{\mathbb{E}\,[s_i(D)] - \frac{\gamma}{2}\, Var\,[s_i(D)] - a_i^2/2\right\}. \tag{2}$$

Furthermore, the type i agent has the choice of selecting contract s_H or s_L. If he selects contract j $(= L$ or $H)$, then his optimal effort level is

$$a^*(s_j, i) = \operatorname*{argmax}_{a_i} \mathbb{E}[s_j(D)] - \frac{1}{2}\gamma\, \mathrm{Var}\,[s_j(D)] - \frac{a_i^2}{2}$$

$$
\begin{aligned}
&= \operatorname*{argmax}_{\alpha_i} \alpha_j(\theta_i + \mu + a_i) + \beta_j - a_i^2/2 - \frac{1}{2}\gamma\alpha_j^2\sigma^2 \\
&= \alpha_j. \qquad (3)
\end{aligned}
$$

Therefore, if the salesman observes the market condition i and selects contract s_j, then his optimal effort level to be exerted is α_j. The certainty equivalence of the salesperson's expected utility, denoted by $\hat{u}(s_j,i)$, is therefore

$$
\begin{aligned}
\hat{u}(s_j,i) &= \alpha_j(\theta_i + \mu + \alpha_j) + \beta_j - \frac{1}{2}\alpha_j^2 - \frac{1}{2}\gamma\alpha_j^2\sigma^2 \\
&= (\theta_i + \mu)\alpha_j + \beta_j + \frac{1-\gamma\sigma^2}{2}\alpha_j^2. \qquad (4)
\end{aligned}
$$

To be able to extract information about the market condition, the firm, when designing a compensation package, needs to consider an *incentive compatibility* condition to ensure that the salesman who observes market condition i will indeed select contract s_i. This entails $\hat{u}(s_i,i) \geq \hat{u}(s_j,i)$, or

$$
(\theta_i + \mu)\alpha_i - \frac{1-\gamma\sigma^2}{2}\alpha_i^2 \geq (\theta_i + \mu)\alpha_j - \frac{1-\gamma\sigma^2}{2}\alpha_j^2, \quad i \neq j.
$$

Furthermore, for the salesperson to sign one of the contracts, the firm has to impose an *individual rationality* condition to ensure that the selected contract provides a minimum utility value of $-U_0$ (or equivalently, a minimum certainty quittance utility of $(-\ln U_0)/\gamma$) for the agent.

To analyze the firm's problem, we let π_s be the firm's maximum expected profit under signed contract $s(.)$, x be her initial inventory level, and $y \geq x$ be her inventory level right after production (and before demand realization). If D is the demand for the period, then

$$
\begin{aligned}
\pi_s &= \max_{y\geq x} \mathbb{E}\left[(1+c)D - s(D) - c(y-x) - h(y-D)^+ p(D-y)^+\right] \\
&= cx + \mathbb{E}[D] - \mathbb{E}[s(D)] - \min_{y\geq x}\big\{(h+c)\mathbb{E}[(y-D)^+] + (p-c))\mathbb{E}[(D-y)^+]\big\}. \qquad (5)
\end{aligned}
$$

If the demand distribution is known to the firm, and $x=0$, the inventory optimization is a typical newsvendor problem, and its optimal solution, y^*, is the solution to $P\{D \leq y^*\} = (p-c)/(p+h)$. In particular, if D can be written as $D = \nu + \epsilon$, where ν is a known constant and ϵ is the random noise with cumulative distribution Φ, then the optimal order-up-to level can be written as $y^* = \nu + q^*$, where

$$
q^* = \Phi^{-1}\left(\frac{p-c}{p+h}\right). \qquad (6)
$$

Clearly, q^* is the optimal newsvendor solution when the random demand is ϵ, and we shall assume in what follows $q^* \geq 0$. If we let $G(y) = (h+c)\,\mathbb{E}[(y-D)^+] + (p-c)\,\mathbb{E}[(D-y)^+]$, where the expectations are taken with respect to ϵ, and $(x)^+ = \max\{x, 0\}$, then the firm's corresponding optimal cost is $\pi_s = cx + \mathbb{E}[D] - \mathbb{E}[s(D)] - G((x-\nu) \vee q^*)$, where $x \vee y = \max\{x, y\}$.

Suppose the firm offers two contracts $s_H(\cdot)$ and $s_L(\cdot)$ such that the high type agent signs s_H, and the low type agent signs s_L. If s_H is chosen, then the firm knows that $\Theta = \theta_H$, and therefore, from (3), the agent will exert $a_H^* = \alpha_H$. Hence, it follows from the analysis above that the firm's expected profit, if s_H is signed, is

$$\begin{aligned}\pi_{s_H} &= cx + \mathbb{E}[D|\theta = H, a = a_H] - \mathbb{E}[s(D)|\theta = H, a = a_H] - G(x - \theta_H - \mu - \alpha_H) \vee q^*) \\ &= cx + (1-\alpha_H)(\theta_H + \mu + a_H) - \beta_H - G((x - \theta_H - \mu - \alpha_H) \vee q^*).\end{aligned}$$

Similarly, if contract s_L is signed by the agent, the firm's expected profit is

$$\pi_{s_L}(x) = cx + (1-\alpha_L)(\theta_L + \mu + \alpha_L) - \beta_L - G((x - \theta_L - \mu - \alpha_L) \vee q^*).$$

The firm offers contracts s_H and s_L. Under the signed contract s_i, the salesperson's optimal certainty equivalence is

$$\alpha_i\,(\theta_i + \mu) + \alpha_i^2\,(1 - \gamma\sigma^2)/2 + \beta_i. \tag{7}$$

Therefore, if the firm believes that the market condition is of type H with probability ρ, then the firm's problem of designing the optimal contract can be written as

$$\begin{aligned} V_N(x) = \max_{\alpha_H, \beta_H, \alpha_L, \beta_L} \quad & cx + \rho\,[(\theta_H + \mu + \alpha_H)(1-\alpha_H) - \beta_H - G(q^* \vee [x - \theta_H - \mu - \alpha_H])] && (8)\\ & + (1-\rho)\,[(\theta_L + \mu + \alpha_L)(1-\alpha_L) - \beta_L - G(q^* \vee [x - \theta_L - \mu - \alpha_L])] \\ s.t. \quad & \\ & \alpha_H\,(\theta_H + \mu) + \alpha_H^2\,(1-\gamma\sigma^2)/2 + \beta_H \geq (-\ln U_0)/\gamma \qquad (IR-H) && (9)\\ & \alpha_L\,(\theta_L + \mu) + \alpha_L^2\,(1-\gamma\sigma^2)/2 + \beta_L \geq (-\ln U_0)/\gamma \qquad (IR-L) && (10)\\ & \alpha_H\,(\theta_H + \mu) + \alpha_H^2\,(1-\gamma\sigma^2)/2 + \beta_H \geq \alpha_L\,(\theta_H + \mu) + \alpha_L^2\,(1-\gamma\sigma^2)/2 + \beta_L \qquad (IC-HL) && (11)\\ & \alpha_L\,(\theta_L + \mu) + \alpha_L^2\,(1-\gamma\sigma^2)/2 + \beta_L \geq \alpha_H\,(\theta_L + \mu) + \alpha_H^2\,(1-\gamma\sigma^2)/2 + \beta_H \qquad (IC-LH) && (12)\\ & \alpha_H, \alpha_L \geq 0. && (13)\end{aligned}$$

In the optimization problem above, constraints (9) and (10) are the individual rationality (IR) constraints for the H type and L type agents, respectively, which guarantee that the agent will participate regardless of his type. Constraints (11) and (12) are the incentive compatibility (IC) constraints, and they guarantee that that an H type agent will not choose a L type contract and a L type agent will not choose an H type contract, respectively. Constraint (13) assures that the commission rates are non-negative.

The above optimization problem can be significantly simplified using the following routine procedure. First, the constraints (IC-HL) and (IR-L) together imply (IR-H), because $\theta_H > \theta_L$ and $\alpha_L \geq 0$. Hence, constraint (IR-H) is redundant. Second, after eliminating (IR-H), inequality (IC-HL) must hold as an equality, because otherwise one can always decrease β_H without violating the constraints, and thereby increase the firm's expected profit. Therefore, we can write (IC-HL) as an equality. Doing that, one can observe that (IC-LH) can be substituted with $\alpha_H \geq \alpha_L$, since $\theta_H > \theta_L$. Finally, (IR-L) must hold as an equality because otherwise one can decrease β_H and β_L by the same amount without violating any of the constraints and increase the objective function. Changing (IR-L) to an equality and applying all of the above changes, we get the following equivalent problem

$$\begin{aligned}
V_N(x) = \max_{\alpha_H,\beta_H,\alpha_L,\beta_L} \quad & cx + \rho\,[(\theta_H+\mu+\alpha_H)(1-\alpha_H) - \beta_H - G(q^* \vee [x-\theta_H-\mu-\alpha_H])] \\
& + (1-\rho)\,[(\theta_L+\mu+\alpha_L)(1-\alpha_L) - \beta_L - G(q^* \vee [x-\theta_L-\mu-\alpha_L])] \\
s.t \quad & \alpha_L\,(\theta_L+\mu) + \alpha_L^2\,(1-\gamma\sigma^2)/2 + \beta_L = (-\ln U_0)/\gamma \\
& \alpha_H\,(\theta_H+\mu) + \alpha_H^2\,(1-\gamma\sigma^2)/2 + \beta_H = \alpha_L\,(\theta_H+\mu) + \alpha_L^2\,(1-\gamma\sigma^2)/2 + \beta_L \\
& \alpha_H \geq \alpha_L \geq 0.
\end{aligned}$$

The analysis above shows that, as is expected, an optimal menu of contracts makes (1) the high type agent indifferent between the two contracts, and (2) the low type agent indifferent between signing a contract and not participating. We can now use the remaining constraints (i.e., (IC-HL) and (IR-L)) to find the fixed/salary payments β_H and β_L:

$$\beta_L = -\alpha_L\,(\theta_L+\mu) - \alpha_L^2\,(1-\gamma\sigma^2)/2 - (\ln U_0)/\gamma, \tag{14}$$

$$\beta_H = +\alpha_L\,(\theta_H-\theta_L) - \alpha_H\,(\theta_H+\mu) - \alpha_H^2\,(1-\gamma\sigma^2)/2 - (\ln U_0)/\gamma. \tag{15}$$

Using these equations and simplifying the objective function, the firm's problem can simply be written as

$$\begin{aligned} V_N(x) = \max_{\alpha_H,\alpha_L} \quad & c\,x + \frac{\ln U_0}{\gamma} + \bar{\theta} + \mu + \rho\left[\alpha_H - \frac{1+\gamma\sigma^2}{2}\,\alpha_H^2 - G(q^* \vee [x-\theta_H-\mu-\alpha_H])\right] \\ & + (1-\rho)\left[\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H-\theta_L) - \frac{1+\gamma\sigma^2}{2}\,\alpha_L^2 - G(q^* \vee [x-\theta_L-\mu-\alpha_L])\right] \\ \text{s.t.} \quad & \alpha_H \geq \alpha_L \geq 0, \end{aligned} \tag{16}$$

where $\bar{\theta} = \rho\,\theta_H + (1-\rho)\,\theta_L$.

We note that the objective function of the above optimization program is separable in decision variables α_H and α_L. Moreover, it is easy to check that the function $G(\cdot)$ is convex using the Leibniz rule, and since q^* is its minimizer, it follows that $G(q^* \vee [x-\theta_H-\mu-\alpha_H])$ and $G(q^* \vee [x-\theta_L-\mu-\alpha_L])$ are both convex in x. Hence, the above objective function is concave and separable in decision variables.

To solve the mathematical program (16), we first solve two separate optimization problems (i.e., after relaxing the constraint $\alpha_H \geq \alpha_L \geq 0$). We first optimize

$$\alpha_H - \frac{1+\gamma\sigma^2}{2}\,\alpha_H^2 - G(q^* \vee [x-\theta_H-\mu-\alpha_H]). \tag{17}$$

It can be seen that when $x \leq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$, the optimal α_H is equal to $1/(1+\gamma\sigma^2)$, and when $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$, the optimal α_H is the maximizer of

$$\alpha_H - \frac{1+\gamma\sigma^2}{2}\,\alpha_H^2 - G(x-\theta_H-\mu-\alpha_H), \tag{18}$$

which is always greater than $1/(1+\gamma\sigma^2)$. We let $\alpha_H(x)$ be the maximizer of (18).

We then optimize

$$\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H-\theta_L) - \frac{1+\gamma\sigma^2}{2}\,\alpha_L^2 - G(q^* \vee [x-\theta_L-\mu-\alpha_L]).$$

It can be seen that when $x \leq q^* + \theta_L + \mu + \frac{1-\frac{\rho}{1-\rho}(\theta_H-\theta_L)}{1+\gamma\sigma^2}$, the optimal α_L is equal to $\frac{1-\frac{\rho}{1-\rho}\alpha_L(\theta_H-\theta_L)}{1+\gamma\sigma^2}$, and when $x \geq q^* + \theta_L + \mu + \frac{1-\frac{\rho}{1-\rho}(\theta_H-\theta_L)}{1+\gamma\sigma^2}$, the optimal α_L is the maximizer of

$$\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H-\theta_L) - \frac{1+\gamma\sigma^2}{2}\,\alpha_L^2 - G(x-\theta_L-\mu-\alpha_L), \tag{19}$$

which is always greater than $\frac{1-\frac{\rho}{1-\rho}\alpha_L(\theta_H-\theta_L)}{1+\gamma\sigma^2}$. We let $\alpha_L(x)$ be the maximizer of (19).

The following properties of $\alpha_H(x)$ and $\alpha_L(x)$ are useful in characterizing the optimal menu of contracts and understanding the form of optimal inventory-based commissions that should be offered by the firm.

Lemma 1 **(High Over-Stock Inventories)**. *On* $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$*, we have:*

(a) $\alpha_H(x)$, $x - \alpha_H(x)$, $\alpha_L(x)$, *and* $x - \alpha_L(x)$ *are all strictly increasing in* x*;*

(b) $\alpha_H(x)$ *and* $\alpha_L(x)$ *both converge to finite values as* $x \to \infty$*. Furthermore,*

$$\lim_{x\to\infty} \alpha_L(x) < \lim_{x\to\infty} \alpha_H(x) = \lim_{x\to\infty} \alpha_L(x) + \frac{\frac{\rho}{1-\rho}(\theta_H - \theta_L)}{1+\gamma\sigma^2};$$

(c) $\alpha_H(x)$ *and* $\alpha_L(x)$ *are concave in* x*;*

(d) $\alpha_H(x) - \alpha_L(x)$ *is strictly increasing in* x*.*

The above results show that when the firm is facing high over-stock inventories, commissions offered are increasing in the inventory level, the H type agent gets a higher commission rate than the L type agent, and the difference between commission rates becomes larger as the inventory level increases.

To provide a complete characterization of the optimal contracts, we need to provide some preliminary results. First, we need to define the following *belief threshold*:

$$\rho^* = \frac{\phi(q^*)}{1+\phi(q^*)}, \tag{20}$$

where ϕ is the density function of the Normal random variable ϵ (and hence, $0 \leq \rho^* \leq 1$). We then define another function by

$$\xi(\rho,\theta) = \Phi^{-1}\left(\frac{p-c+\frac{\rho}{1-\rho}\theta}{p+h}\right) - \Phi^{-1}\left(\frac{p-c}{p+h}\right),$$

and provide the following lemma.

Lemma 2. *If* $\rho < \rho^*$*, then there exists a unique number* $\hat{\theta}(\rho)$*,* $0 < \hat{\theta}(\rho) < \infty$ *such that* $\theta > \xi(\rho,\theta)$ *if, and only if,* $\theta \leq \hat{\theta}(\rho)$*. If* $\rho \geq \rho^*$*, then* $\theta \leq \xi(\rho,\theta)$ *for all* $\theta \geq 0$*.*

We now present two definitions that will facilitate the complete characterization of the optimal compensation plans.

Definition 1 (Perceived Market Conditions). We say that the market condition is perceived to be of high (low) type by the firm if $\rho \geq (<)\, \rho^*$ (where ρ^* is the belief threshold defined in (20)).

Definition 2 (Distinguishable Market Conditions). The market condition types are said to be distinguishable by the firm if, and only if, $\theta_H - \theta_L > \tilde{\theta}(\rho)$, where $\tilde{\theta}(\rho)$ is the unique solution to $(p+h)\Phi(\tilde{\theta}+q^*) = p - c - \tilde{\theta}\,\rho/(1-\rho)$.

Using the above definitions, the following result sheds light on the conditions under which relaxing the constraint of program (16) and separably optimizing the objective function for finding the optimal commission rates for L and H type agents is or is not problematic.

Lemma 3 **(Single Crossing)**. *If either (a) the market condition is perceived to be of high type, or (b) the market condition is perceived to be of low type but market conditions are distinguishable, then there exists a finite number* $z^* > q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$ *such that* $\alpha_H(x) < \alpha_L(x)$ *on* $z^* > x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$*,* $\alpha_H(z^*) = \alpha_L(z^*)$*, and* $\alpha_H(x) > \alpha_L(x)$ *on* $x > z^*$*. Otherwise,* $\alpha_H(x) \geq \alpha_L(x)$ *on all* $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$*.*

Before completely characterizing the optimal inventory-dependent menu of contracts, we need the following simple result. Its proof is elementary and is omitted.

Lemma 4. *Let* $g_1(x)$ *and* $g_2(y)$ *be two concave functions with maximizers* x^* *and* y^**, respectively. Let* z^* *be the maximizer of concave function* $g_1(z)+g_2(z)$*. For the program* $\max_{x\geq y}\{g_1(x)+g_2(y)\}$ *the following hold:*

(a) If $x^* \geq y^*$*, the optimal solution is* (x^*, y^*) *;*

(b) If $x^* < y^*$*, the optimal solution is* (z^*, z^*)*, where* $x^* < z^* < y^*$*.*

We are now ready to characterize the optimal inventory-dependent contracts to be offered by the firm. For the ease of notation, we let $\delta = \left[1 - \frac{\rho}{1-\rho}(\theta_H - \theta_L)\right]^+$, and define the inventory level intervals

$$I_1 = (-\infty\,,\, q^* + \theta_L + \mu + \delta\,(1+\gamma\sigma^2)^{-1}),$$

$$I_2 = [q^* + \theta_L + \mu + \delta\,(1+\gamma\sigma^2)^{-1}\,,\, z_1^*),$$

$$I_3 = [z_1^*, z_2^*),\ I_4 = [z_2^*, z_3^*),\ I_5 = [z_3^*, \infty),$$

$$I_1' = (-\infty, q^* + \theta_L + \mu + \delta(1+\gamma\sigma^2)^{-1}),$$

$$I_2' = [q^* + \theta_L + \mu + \delta(1+\gamma\sigma^2)^{-1}, q^* + \theta_H + \mu + (1+\gamma\sigma^2)^{-1}),$$

$$I_3' = [q^* + \theta_H + \mu + (1+\gamma\sigma^2)^{-1}, \infty),$$

where

$$z_1^* = \theta_L + \mu + (1+\gamma\sigma^2)^{-1} + \Phi^{-1}\left(\frac{p - c + \frac{\rho}{1-\rho}(\theta_H - \theta_L)}{p+h}\right),$$

z_2^* is the solution to $z_2 - \theta_H - \mu - \tilde{\alpha}_H(z_2) = q^*$, and z_3^* is the solution to $\tilde{\alpha}_H(z_3) = \tilde{\alpha}_L(z_3)$, where $\tilde{\alpha}_H(x)$ is the solution to

$$(p+h)\Phi(x - \theta_H - \mu - \tilde{\alpha}_H) - (1+\gamma\sigma^2)\tilde{\alpha}_H - p + c + 1 = 0,$$

and $\tilde{\alpha}_L(x)$ is the solution to

$$(p+h)\Phi(x - \theta_L - \mu - \tilde{\alpha}_L) - (1+\gamma\sigma^2)\tilde{\alpha}_L - p + c + 1 - \frac{\rho}{1-\rho}(\theta_H - \theta_L) = 0.$$

With these, we can now characterize the optimal menu of inventory-dependent contracts as follows.

THEOREM 1 **(Optimal Commissions - Last Period)**. *(1) If either (a) the market condition is perceived to be of high type, or (b) the market condition is perceived to be of low type but the market condition types are distinguishable by the firm, then the optimal inventory-dependent commissions are as follows.*

(i) If $x \in I_1$, *then* $\alpha_L^*(x) = \delta/(1+\gamma\sigma^2)$, *and* $\alpha_H^*(x) = 1/(1+\gamma\sigma^2)$.

(ii) If $x \in I_2$, *then* $\alpha_L^*(x) = \tilde{\alpha}_L(x)$, *and* $\alpha_H^*(x) = (1+\gamma\sigma^2)^{-1}$.

(iii) If $x \in I_3$, *then* $\alpha_L^*(x) = \alpha_H^*(x) = \alpha_{I_3}^*(x)$ *where* $\alpha_{I_3}^*(x)$ *is the solution to* $(p+h)(1-\rho)\Phi(x - \theta_L - \mu - \alpha_{I_3}) - (1+\gamma\sigma^2)\alpha_{I_3} - (1-\rho)(p-c) - \rho(\theta_H - \theta_L) + 1 = 0$.

(iv) If $x \in I_4$, *then* $\alpha_L^*(x) = \alpha_H^*(x) = \alpha_{I_4}^*(x)$ *where* $\alpha_{I_4}^*(x)$ *is the solution to* $(p+h)\big[\rho\Phi(x - \theta_H - \mu - \alpha_{I_4}) + (1-\rho)\Phi(x - \theta_L - \mu - \alpha_{I_4})\big] - (1+\gamma\sigma^2)\alpha - p + c - \rho(\theta_H - \theta_L) + 1 = 0$.

(v) If $x \in I_5$, *then* $\alpha_L^*(x) = \tilde{\alpha}_L(x)$ *and* $\alpha_H^*(x) = \tilde{\alpha}_H(x)$.

(2) Otherwise, the optimal commissions are as follows.

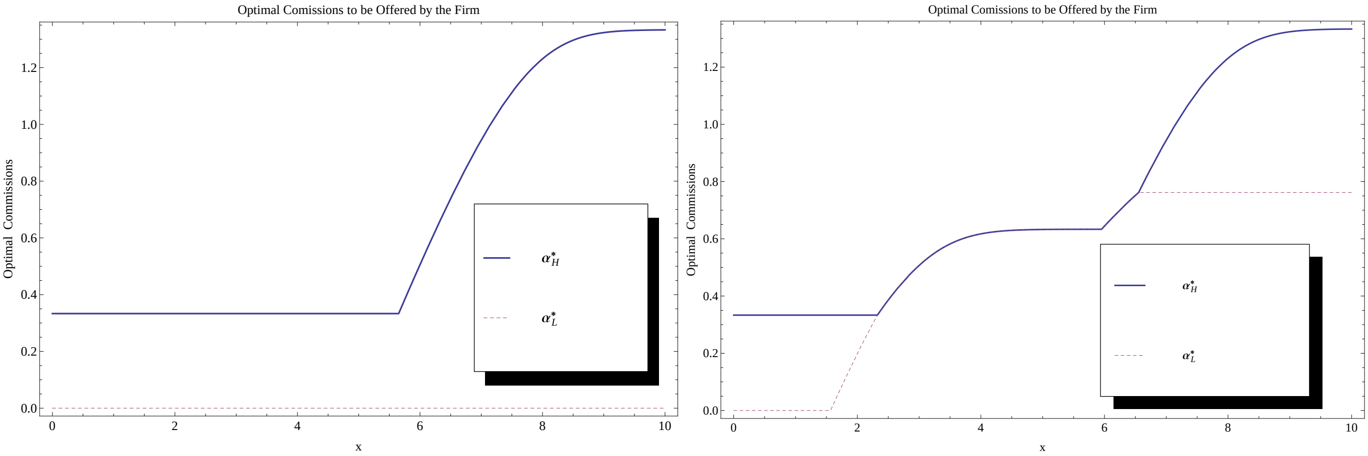

Figure 1 Optimal commissions to be offered by the firm with $\rho = 0.9$ (left) and $\rho = 0.3$ (right) [$p = 7$, $h = 1$, $c = 2$, $\sigma = 1$, $U_0 = 10$, $\gamma = 2$, $\theta_H = 5$, and $\theta_L = 1$].

(i) If $x \in I'_1$, *then* $\alpha^*_L(x) = \delta\,(1+\gamma\sigma^2)^{-1}$, *and* $\alpha^*_H(x) = (1+\gamma\sigma^2)^{-1}$.

(ii) If $x \in I'_2$, *then* $\alpha^*_L(x) = \tilde{\alpha}_L(x)$, *and* $\alpha^*_H(x) = (1+\gamma\sigma^2)^{-1}$.

(iii) If $x \in I'_3$, *then* $\alpha^*_L(x) = \tilde{\alpha}_L(x)$ *and* $\alpha^*_H(x) = \tilde{\alpha}_H(x)$.

To gain some insights, we now develop a numerical study and illustrate the optimal inventory-dependent commission rates that should be offered by the firm.

Numerical Study 1. Figure 1 illustrates the optimal inventory-based commissions offered by a firm to its salesman. The figure on left is for a case where the firm belief about the market (or similarly the agent) being of type H is $\rho = 0.9$, and the figure on right is for a case where $\rho = 0.3$. In the latter case, the structure of the optimal commissions depend on five inventory intervals, as was predicted by Theorem 1. In the former case, as can be seen in Figure 1, the firm, regardless of her inventory level, only offers a fixed/salary payment to the low type agent (i.e., $\alpha^*_L(x) = 0$ for all $x \geq 0$).

The following result presents conditions under which the low type agent is only offered a fixed/salary payment.

Corollary 1 **(Salary-Only Payment)**. *If* $\rho \geq (h+c+1)/(h+c+1+(\theta_H - \theta_L))$ *then* $\alpha^*_L(x) = 0$ *for all* x, *i.e., the low type agent is only offered a salary payment regardless of the firm's inventory level. Furthermore, when* $x \leq q^* + \theta_L + \mu + \delta\,(1+\gamma\sigma^2)^{-1}$, *the low type agent is only offered a salary payment if, and only if,* $\rho \geq 1/(1+(\theta_H - \theta_L))$.

The above result shows that when the firm is sure enough that the market is high, she does not offer a commission-based payment to the low type agent. In that case, from (14), the firm only offers the minimum salary that would make the low type agent indifferent between signing the contract and leaving.

The following result sheds more light on the characteristics of the optimal commissions.

COROLLARY 2 **(Monotone and Locally Concave Commissions)**. *The optimal inventory-based commissions to be offered by the firm are (1) increasing in the firm's inventory level, (2) decreasing in the agent's risk aversion level, and (3) concave in each inventory interval.*

For convenience, we call a contract more attractive if the commission rate is higher. Parts (1) and (2) of the above results show that (a) optimal contracts are indeed more attractive when inventory levels are high, and (b) the firm needs to offer better commissions to its less risk averse agents. These findings support the empirical studies reported in the literature that find (i) varying compensation plans is effective in increasing sales (see, e.g., Levenson et al. (2010)), and (ii) more risk-averse individuals prefer a fixed salary and are less productive under incentive pays (see, e.g., Cadsby et al. (2007)). Part 3 of Corollary 2 states that the commission rates posses a *local diminishing rate of return* property; for each inventory interval, the rate of increase in the agent's commission is decreasing in the firm's inventory level.

Another interesting insight from the optimal menu of inventory-dependent contracts offered by the firm is that it is successful in inducing the agent to exert more effort when the inventory is high. In fact, as the Corollary 2 indicates, with a higher inventory level, the firm offers a higher commission rate. As the following result states, this will indeed provide sufficient incentives for the agent to exert more effort as a means to attract more demand.

COROLLARY 3 **(Monotone Effort Level)**. *The agent's optimal effort level induced by the firm is increasing in the firm's inventory level.*

Another simple but important result that we will use in the next section is the following.

PROPOSITION 1 **(Concave Value Function - Last Period)**. *The firm's optimal expected profit as a function of the initial inventory x and belief probability ρ, denoted by $V_N(x,\rho)$, is concave in x for any fixed ρ, and $V_N(x,\rho) - cx$ is decreasing in x for any fixed ρ.*

In the next section, we analyze the problem in an arbitrary period, and show that many (but not all) of the properties discussed in this section also hold in other periods.

5. Analysis of an Arbitrary Period

We now proceed to analyze the joint contract design and inventory/production control problem for an arbitrary period n, $n < N$.

By induction, suppose that we have analyzed the problem from stages $n+1$ to the last stage. In particular, suppose we have established that the maximum expected profit starting from period $n+1$ to the end of the planning horizon under the optimal contracts, denoted by $V_{n+1}(\rho, x)$, where x is the starting inventory level at the beginning of period $n+1$ and ρ is the firm's belief probability that the market condition in period $n+1$ will be high (H), is a concave function of x for any fixed value ρ.

Recall that, at the beginning of period n, the salesman first learns the market condition, but the firm only has a belief probability that it is high (H) or low (L). Let ρ be the belief probability that the market condition will be high. If the starting inventory level at the beginning of period n is x, then the state of the system at the beginning of period n is (x, ρ). The firm reviews its inventory and offers a menu of contracts $s_i(x)$, $i = H, L$, for period n to maximize her total expected profit until the end of the planning horizon. Under this contract, the salesman's revenue for period n is $s_i(x) = \alpha_i(x)D_n + \beta_i(x)$ when he signs the contract s_i. Thus, the salesman's objective is to maximize his expected utility for the current period resulting from $s_i(D_n)$, which is[3]

$$u(s_i, a) = -\mathbb{E}[e^{-\gamma[s_i(D_n) - a_n^2/2]}].$$

The analysis of the salesman's problem is similar to that of the last period; the optimal effort level the salesman exerts is α_H when the market condition is of type H and α_L if the market condition is of type L. Hence, the certainty equivalence of the expected utility for the salesperson, if the market condition is of type $i = L, H$, is equal to

$$\alpha_i(x)(\theta_i + \mu_n) + \alpha_i^2(x)(1 - \gamma\sigma_n^2)/2 + \beta_i(x).$$

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[3] It has been reported that sometimes in practice, firms at the very first periods monitor the productivity of their sales agent. If the agent is observed to be productive, s/he is then offered a lower incentive rate in the next periods. Anticipated by the agent, s/he might have some incentives in the very first periods to deviate from optimizing his/her regular utility (the so-called "ratchet effect", see, e.g., Frexias et al. (1985)). We assume the planning horizon starts after such initial exploratory periods.

Now we consider the firm's problem of determining the optimal menu of compensation plans. By the revelation principle (Myerson (1982)), the firm only limits the search for the optimal menu of contracts to the class of incentive compatible and direct revelation contracts. Suppose the agent selects contract type i. Since the replenishment decision is made after the sales agent selects the contract, the firm would know the market condition when placing the order. If the agent selects $(\alpha_i(x), \beta_i(x))$, then the firm knows that the demand during the period is $D_n = \theta_i + \mu_n + \alpha_i(x) + \epsilon_n$, and hence, her operational decision is to replenish inventory in a way that maximizes her expected revenue from periods n to N. If the firm replenishes its inventory level to $y \geq x$, similar to (5), her current period expected profit is

$$\begin{aligned}
&cx + \theta_i + \mu_n + \alpha_i(x) - \mathbb{E}[\alpha_i(x)(\theta_i + \mu_n + \alpha_i(x) - \epsilon_n)] - \beta_i(x) \\
&-(h+c)\mathbb{E}[(y - \theta_i + \mu_n - \alpha_i(x) - \epsilon_n)^+] - (p-c)\mathbb{E}[(\theta_i + \mu_n + \alpha_i(x) + \epsilon_n - y)^+].
\end{aligned}$$

Since the market condition follows a two-state Markov chain, the market condition will be H type with probability ρ_{iH} and it will be L type with probability ρ_{iL}. Hence, the maximum expected profit from period $n+1$ onwards is

$$\rho_{iH}\mathbb{E}[V_{n+1}((y - \theta_i - \mu_n - \alpha_i(x) - \epsilon_n)^+, \rho_{iH})] + \rho_{iL}\mathbb{E}[V_{n+1}((y - \theta_i - \mu_n - \alpha_i(x) - \epsilon_n)^+, \rho_{iL})].$$

Thus, when contract $(\alpha_i(x), \beta_i(x))$ is signed at the beginning of period n, the firm's optimization problem is to find the order-up-to level $y \geq x$ to maximize

$$\begin{aligned}
&cx + \theta_i + \mu_n + \alpha_i(x) - \mathbb{E}[\alpha_i(x)(\theta_i + \mu_n + \alpha_i(x) - \epsilon_n)] - \beta_i(x) \\
&-(h+c)\mathbb{E}[(y - \theta_i + \mu_n - \alpha_i(x) - \epsilon_n)^+] - (p-c)\mathbb{E}[(\theta_i + \mu_n + \alpha_i(x) + \epsilon_n - y)^+] \qquad (21)\\
&+\rho_{iH}\mathbb{E}[V_{n+1}((y - \theta_i - \mu_n - \alpha_i(x) - \epsilon_n)^+, \rho_{iH})] + \rho_{iL}\mathbb{E}[V_{n+1}((y - \theta_i - \mu_n - \alpha_i(x) - \epsilon_n)^+, \rho_{iL})].
\end{aligned}$$

For convenience, we let $W_i(y) =$

$$-(h+c)\mathbb{E}[(y-\epsilon_n)^+] - (p-c)\mathbb{E}[(\epsilon_n - y)^+] + \rho_{iH}\mathbb{E}[V_{n+1}((y-\epsilon_n)^+, \rho_{iH})] + \rho_{iL}\mathbb{E}[V_{n+1}((y-\epsilon_n)^+, \rho_{iL})].$$

Note that, if $V_{n+1}(x, \rho) - cx$ is decreasing concave, then $W_i(y)$ is concave. To see this, note that

$$W_i(y) = -h\mathbb{E}[(y-\epsilon_n)^+] - (p-c)\mathbb{E}[(\epsilon_n - y)^+] + \rho_{iH}\mathbb{E}[(V_{n+1}((y-\epsilon_n)^+, \rho_{iH}) - c(y-\epsilon_n)^+)]$$

$$+\rho_{iL}\mathbb{E}[(V_{n+1}((y-\epsilon_n)^+,\rho_{iL})-c(y-\epsilon_n)^+)].$$

Since, $f(g(x))$ is concave when f is decreasing concave and g is convex, it follows from V_{n+1} being decreasing concave, and $(y-\epsilon_n)^+$ and $(\epsilon_n - y)^+$ being convex that $W_i(x)$ is concave. Let q_i^* be the maximizer of $W_i(x)$. We have the following result that describes the optimal production/inventory control policy of the firm.

PROPOSITION 2 **(Market-Based State-Dependent Policy)**. *The optimal inventory replenishing policy of the firm for an arbitrary period n is a market-based state-dependent policy. Furthermore, if contract s_i $(i=H,L)$ is signed (and hence the market if of type i), then the optimal order-up-to level is $q_i^*+\theta_i+\mu_n+\alpha_i(x)$. That is, it is optimal to order $q_i^*-x+\theta_i+\mu_n+\alpha_i(x)$ if $x-\theta_i-\mu_n-\alpha_i(x)\le q_i^*$ and order nothing otherwise.*

The above result shows that the optimal inventory/production control policy is in general characterized by an order-up-to level that depends on (a) market condition, and (b) the current inventory level (as $\alpha_i(x)$ is not constant in general). However, the following corollary shows that, although the incentive design problem and that of inventory control are highly interconnected, the optimal operational policy for periods with low inventory levels is interestingly the traditional simple base-stock policy, except that the base-stock level of a period depends on the extracted information from the agent about the market condition. However, it should be noted that the base-stock levels of two periods with the same market condition may be different due to effects such as seasonality.

COROLLARY 4 **(Market-Based Base-Stock Policy (Low Inventory Periods))**. *In periods with low inventory levels ($x\le q_i^*+\theta_i+\mu_n+(1+\delta\mathbb{1}\{i=L\})/(1+\gamma\sigma_n^2)$), the optimal production/inventory control policy is a market-based base-stock policy with base-stock level $x\le q_i^*+\theta_i+\mu_n+(1+\delta\mathbb{1}\{i=L\})/(1+\gamma\sigma_n^2)$.*

Another important observation from Proposition 2 and Corollary 4 is the following, which is in contrast with the usual notion of safety-stock in classical inventory theory where the salesforce incentives are neglected: a higher demand uncertainty can decrease the optimal stock level. To see this, notice that α_i is decreasing in demand uncertainty, σ_n[4]. Indeed, with a higher demand

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[4] This will be clear in Theorem 3 but can also be similarly seen from Theorem 1.

uncertainty, the risk averse-agent is offered less incentives, and hence, exerts less effort in attracting more demand. With a lower demand level, the firm may need to lower the stock level, although the value of q_i^* may increase. This observation sheds more light on the importance of considering the effect of a risk-averse agent's incentive when controlling the inventory levels. For detailed analysis of the effect of increased demand uncertainty in the classical inventory theory (but with risk aversion) we refer interested readers to Eeckhoudt et al. (1995).

Using Proposition 2, the optimal value function of (21) is

$$cx + \theta_i + \mu_n + \alpha_i(x) - \mathbb{E}[\alpha_i(x)(\theta_i + \alpha_i(x) + \mu_n - \epsilon_n)] - \beta_i(x) + W_i(q_i^* \vee (x - \theta_i - \mu_n - \alpha_i(x))).$$

Hence, the firm's problem at the beginning of period n is to solve

$$\begin{aligned} V_n(x,\rho) = \max_{\alpha_L,\alpha_H,\beta_H,\beta_H} \Big\{ & cx + \rho\Big([1-\alpha_H(x))[\theta_H + \mu_n + \alpha_H(x))] - \beta_H(x) + W_H(q_H^* \vee (x - \theta_H - \mu_n - \alpha_H(x)))\Big] \\ & + (1-\rho)\Big[(1-\theta_L)[\theta_L + \mu_n + \alpha_L(x))] - \beta_L(x) + W_L(q_L^* \vee (x - \theta_L - \mu_n - \alpha_L(x)))\Big]\Big\}, \end{aligned}$$

subject to IC, IR, and non-negativity constraints similar to (9)-(13).

Moreover, similar to the analysis of the previous section, it can be seen that the firm's optimization problem can be reduced to, writing $\alpha_i(x)$ as α_i,

$$\begin{aligned} V_n(x,\rho) = \max_{\alpha_H,\alpha_L:\alpha_H \geq \alpha_L \geq 0} \quad & \Big\{ cx + \frac{\ln U_0}{\gamma} + \bar{\theta} + \mu_n + \rho\left[\alpha_H - \frac{1+\gamma\sigma^2}{2}\alpha_H^2 + W_H(q_H^* \vee (x - \theta_H - \mu_n - \alpha_H))\right] \\ & + (1-\rho)\left[\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H - \theta_L) - \frac{1+\gamma\sigma^2}{2}\alpha_L^2 + W_L(q_L^* \vee (x - \theta_L - \mu_n - \alpha_L))\right]\Big\}. \end{aligned} \tag{22}$$

The following result shows that, similar to the last period, the optimal value function of the firm is concave in the inventory level.

Proposition 3 **(Concave Value Function - Arbitrary Period)**. *For any n, $V_n(x,\rho)$ is concave in x for any ρ, and $V_n(x,\rho) - cx$ is decreasing in x for any ρ.*

To find the optimal menu of contracts, we first solve the following two separate optimization problems:

$$\max_{\alpha_H}\Big\{\alpha_H - \frac{1+\gamma\sigma_n^2}{2}\alpha_H^2 + W_H(q_H^* \vee (x - \theta_H - \mu_n - \alpha_H)\Big\}, \tag{23}$$

and

$$\max_{\alpha_L \geq 0}\Big\{\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H - \theta_L) - \frac{1+\gamma\sigma^2}{2}\alpha_L^2 + W_L(q_L^* \vee (x - \theta_L - \mu_n - \alpha_L))\Big\}. \tag{24}$$

It can be easily shown that, when $x \le q_H^* + \mu_n + \theta_H + 1/(1+\gamma\sigma_n^2)$, then the optimal solution of (23) is $\alpha_H^*(x) = (1+\gamma\sigma_n^2)^{-1}$. If $x \ge q_H^* + \mu_n + \theta_H + 1/(1+\gamma\sigma^2)$, we let $\tilde{\alpha}_H(x)$ be the solution to

$$1 - (1+\gamma\sigma_n^2)\alpha_H - W_H'(x - \theta_H - \mu_n - \alpha_H) = 0, \tag{25}$$

which is increasing in x. It can also be seen that on $x \ge q_H^* + \mu_n + \theta_H + 1/(1+\gamma\sigma^2)$, the optimal solution of (23) is $\alpha_H^*(x) = \tilde{\alpha}_H(x) \ge 1/(1+\gamma\sigma_n^2)$. Similarly, for optimization problem (24), if $x \le q_L^* + \theta_L + \mu_n + \delta(1+\gamma\sigma^2)^{-1}$ where, as before, $\delta = [1 - \frac{\rho}{1-\rho}(\theta_H - \theta_L)]^+$, then the optimal solution of (24) is $\delta(1+\gamma\sigma_n^2)^{-1}$. If, however, $x \ge q_L^* + \theta_L + \mu_n + \delta\,(1+\gamma\sigma^2)^{-1}$, we let $\tilde{\alpha}_L(x)$ be the maximizer of

$$\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H - \theta_L) - \frac{1+\gamma\sigma^2}{2}\,\alpha_L^2 + W_L(x - \theta_L - \mu_n - \alpha_L),$$

which is also increasing in x. Then the optimal solution of (24) satisfies $\alpha_L^*(x) = \tilde{\alpha}_L(x) \ge \frac{\delta}{1+\gamma\sigma_n^2}$. The following lemma describes the structure of $\tilde{\alpha}_H(x)$ and $\tilde{\alpha}_L(x)$ for a specific range of interest where the firm is facing high levels of over-stock inventories.

Lemma 5 **(High Over-Stock Inventories)**. *For* $x \ge \max\{q_H^*, q_L^*\} + \theta_H + \mu_n + (1+\gamma\sigma_n^2)^{-1}$, *we have*

(a) $\tilde{\alpha}_H(x)$ *and* $\tilde{\alpha}_L(x)$ *are increasing in* x.

(b) $\tilde{\alpha}_H(x)$ *and* $\tilde{\alpha}_L(x)$ *converge to finite numbers as* $x \to \infty$, *and* $\lim_{x\to\infty} \tilde{\alpha}_L(x) < \lim_{x\to\infty} \tilde{\alpha}_H(x)$.

Unlike the analysis of the last period, we cannot prove that $\tilde{\alpha}_H(x)$ and $\tilde{\alpha}_L(x)$ are always concave for an arbitrary period. The above result, however, implies that (after some inventory level) (1) regardless of the agent type, the firm offers a more attractive commission rate, when facing a higher inventory level, and (2) the commission rate offered to the high type agent is more attractive than the one offered to the low type agent.

The following results characterizes the optimal menu of inventory-based contracts for an arbitrary period $n < N$. As expected, the optimal contracts are more complex than the last period. For the ease of notation, similar to the analysis of the last period, we define the following inventory intervals:

$$I_1 = (-\infty, q_L^* + \theta_L + \mu_n + \delta/(1+\gamma\sigma_n^2)],$$

$$I_2 = (q_L^* + \theta_L + \mu_n + \delta/(1+\gamma\sigma_n^2), z_1^*],$$

$$I_3 = (z_1^*, z_2^*],\ I_4 = (z_2^*, z_3^*],\ I_5 = (z_3^*, \infty),\ I_1' = I_1,$$

$$I_2' = (q_L^* + \theta_L + \mu_n + \delta/(1+\gamma\sigma_n^2), q_H^* + \theta_H + \mu_n + 1/(1+\gamma\sigma_n^2)],$$

$$I_3' = (q_H^* + \theta_H + \mu_n + 1/(1+\gamma\sigma_n^2), z_2^{'*}],$$

$$I_4' = (z_2^{'*}, z_3^{'*}],\ I_5' = (z_3^{'*}, \infty),$$

$$I_1'' = (-\infty, q_H^* + \theta_H + \mu_n + 1/(1+\gamma\sigma_n^2)],$$

$$I_2'' = (q_H^* + \theta_H + \mu_n + 1/(1+\gamma\sigma_n^2), q_L^* + \theta_L + \mu_n + \delta/(1+\gamma\sigma_n^2)],$$

$$I_3'' = (q_L^* + \theta_L + \mu_n + \delta/(1+\gamma\sigma_n^2), z_2^{'*}],$$

$$I_4'' = (z_2^{*'}, z_3^{*'}],\ I_5'' = (z_3^{*'}, \infty),$$

where $z_1^* \in (q_L^* + \theta_L + \mu_n + \delta/(1+\gamma\sigma_n^2), q_H^* + \theta_H + \mu_n + 1/(1+\gamma\sigma_n^2)]$ is the solution to $\tilde{\alpha}_L(z_1) = 1/(1+\gamma\sigma_n^2)$, z_2^* satisfies $z_1^* \le z_2^* < q_H^* + \theta_H + \mu_n + 1/(1+\gamma\sigma_n^2)$ and is the solution to $z_2 - \theta_L - \mu_n - \alpha_{I_3}^*(z_2) = q_L^*$ where $\alpha_{I_3}^*(x)$ is the maximizer of $\alpha - \frac{1}{1+\gamma\sigma_n^2}\alpha^2 - \rho(\theta_H - \theta_L)\alpha - (1-\rho)W_L(x - \theta_L - \mu_n - \alpha)$, $z_3^* \ge q_H^* + \theta_H + \mu_n + 1/(1+\gamma\sigma_n^2)$ is the largest solution to $\tilde{\alpha}_H(z_3) = \tilde{\alpha}_L(z_3)$, and $z_2^{'*}$ and $z_3^{'*}$ are defined as the smallest and largest points z after $q_L^* + \theta_L + \mu_n + \delta/(1+\gamma\sigma_n^2)$ with $\tilde{\alpha}_L(z) = \tilde{\alpha}_H(z)$.

THEOREM 2 **(Optimal Commissions - Arbitrary Period)**. *The optimal menu of inventory-dependent contracts for period* $n < N$ *is determined by the two numbers* q_H^* *and* q_L^*. *If* $q_H^* + \theta_H + 1/(1+\gamma\sigma_n^2) > q_L^* + \theta_L + \delta/(1+\gamma\sigma_n^2)$, *then the optimal commissions* $\alpha_H^*(x)$ *and* $\alpha_L^*(x)$ *are characterized based on two cases.* *Case 1:* $\tilde{\alpha}_L\Big(q_H^* + \theta_H + \mu_n + \frac{1}{1+\gamma\sigma_n^2}) > 1/(1+\gamma\sigma_n^2)\Big)$, *and* *Case 2:* $\tilde{\alpha}_L\Big(q_H^* + \theta_H + \mu_n + \frac{1}{1+\gamma\sigma_n^2}) \le 1/(1+\gamma\sigma_n^2)\Big)$.

Case 1. In this case the optimal commissions are as follows.

(1) If $x \in I_1$, *then* $\alpha_H^*(x) = (1+\gamma\sigma_n^2)^{-1}$, *and* $\alpha_L^*(x) = \delta(1+\gamma\sigma_n^2)^{-1}$.

(2) If $x \in I_2$, *then* $\alpha_H^*(x) = (1+\gamma\sigma_n^2)^{-1}$, *and* $\alpha_L^*(x) = \tilde{\alpha}_L(x)$.

(3) If $x \in I_3$, *then* $\alpha_H^*(x) = \alpha_L^*(x) = \alpha_{I_3}^*(x)$.

(4) If $x \in I_4$, *then the optimal contract depends on whether* $\tilde{\alpha}_L(x) \le \tilde{\alpha}_H(x)$ *or* $\tilde{\alpha}_L(x) > \tilde{\alpha}_H(x)$; *in the former case,* $\alpha_H^*(x) = \tilde{\alpha}_H(x)$ *and* $\alpha_L^*(x) = \tilde{\alpha}_L(x)$; *and in the latter case,* $\alpha_H^*(x) = \alpha_L^*(x) = \alpha_{I_4}^*(x)$ *where* $\alpha_{I_4}^*(x)$ *is the optimizer of*

$$\alpha - \frac{1+\gamma\sigma_n^2}{2}\alpha^2 + \rho\alpha(\theta_H - \theta_L) + \rho W_H(x - \mu_n - \theta_H - \alpha) + (1-\rho)W_L(x - \mu_n - \theta_L - \alpha). \tag{26}$$

(5) If $x \in I_5$, then $\alpha^*_H(x) = \tilde{\alpha}_H(x)$ and $\alpha^*_L(x) = \tilde{\alpha}_L(x)$.

Case 2. In this case the optimal commissions are as follows.

(1) If $x \in I'_1$ then $\alpha^*_H(x) = (1+\gamma\sigma_n^2)^{-1}$, and $\alpha^*_L(x) = \delta(1+\gamma\sigma_n^2)^{-1}$.

(2) If $x \in I'_2$ then $\alpha^*_H(x) = (1+\gamma\sigma_n^2)^{-1}$ and $\alpha^*_L(x) = \tilde{\alpha}_L(x)$.

(3) If $x \in I'_3$ then $\alpha^*_L(x) = \tilde{\alpha}_L(x)$ and $\alpha^*_H(x) = \tilde{\alpha}_H(x)$.

(4) If $x \in I'_4$, then the optimal contract depends on whether $\tilde{\alpha}_L(x) \leq \tilde{\alpha}_H(x)$ or $\tilde{\alpha}_L(x) > \tilde{\alpha}_H(x)$; in the former case, $\alpha^*_L(x) = \tilde{\alpha}_L(x)$ and $\alpha^*_H(x) = \tilde{\alpha}_H(x)$, and in the latter case, $\alpha^*_H(x) = \alpha^*_L(x) = \alpha^*_{I_4}(x)$, where $\alpha^*_{I_4}(x)$ is the optimizer of (26).

(5) $x \in I'_5$, then $\alpha^*_L(x) = \tilde{\alpha}_L(z)$ and $\alpha^*_H(x) = \tilde{\alpha}_H(x)$.

On the other hand, if $q^*_H + \theta_H + 1/(1+\gamma\sigma_n^2) \leq q^*_L + \theta_L + \delta/(1+\gamma\sigma_n^2)$, then the optimal commissions are as follows.

(1) If $x \in I''_1$, $\alpha^*_H(x) = (1+\gamma\sigma_n^2)^{-1}$, and $\alpha^*_L(x) = \delta(1+\gamma\sigma_n^2)^{-1}$.

(2) If $x \in I''_2$, then $\alpha^*_H(x) = \tilde{\alpha}_H(x)$, and $\alpha^*_L(x) = \delta(1+\gamma\sigma_n^2)^{-1}$.

(3) If $x \in I''_3$, then $\alpha^*_H(x) = \tilde{\alpha}_H(x)$, and $\alpha^*_L(x) = \tilde{\alpha}_L(x)$.

(4) If $x \in I''_4$, then the optimal contract depends on whether $\tilde{\alpha}_L(x) \leq \alpha_H(x)$ or $\tilde{\alpha}_L(x) > \alpha_H(x)$; in the former case, $\alpha^*_H(x) = \tilde{\alpha}_H(x)$ and $\alpha^*_L(x) = \tilde{\alpha}_L(x)$, and in the latter case, $\alpha^*_H(x) = \alpha^*_L(x) = \alpha^*_{I_4}(x)$ where $\alpha^*_{I_4}(x)$ is the optimizer of (26).

(5) If $x \in I''_5$, then $\alpha^*_H(x) = \tilde{\alpha}_H(x)$ and $\alpha^*_L(x) = \tilde{\alpha}_L(x)$.

Since the optimal menu of contracts for an arbitrary period n is complex and not easy to implement, we propose a heuristic menu of contracts in the next section.

5.1. A Heuristic Menu of Inventory-Based Compensation Plans

To develop a heuristic menu of contracts, we benefit from the analysis of the last period presented in Section 4. More specifically, we show how the structure of the optimal contracts in the last period can be used to generate a simple and implementable menu of inventory-dependent contracts for an arbitrary period.

Our proposed heuristic menu of contracts is as follows. In each period n, we first compute q^*_i (for $i = H, L$). Next, if it holds that $q^*_H + \theta_H + 1/(1+\gamma\sigma_n^2) \leq q^*_L + \theta_L + \delta/(1+\gamma\sigma_n^2)$, then the contracts are the same as those presented in Theorem 2, but with $z_2^{'*} = z_3^{'*} = q_L^* + \theta_L + \delta/(1+\gamma\sigma_n^2)$ (i.e., assuming $I_4'' = \emptyset$). Hence, the contract structure is extremely simple in this case and commission for type i agent is either constant or equal to $\tilde{\alpha}_i(x)$. If $q_H^* + \theta_H + 1/(1+\gamma\sigma_n^2) > q_L^* + \theta_L + \delta/(1+\gamma\sigma_n^2)$, then the contract is similar to that of the last period presented in Theorem 1. Specifically, we first check wether Case 1 or Case 2 of Theorem 2 holds. If Case 1 (Case 2) holds, the contract is similar to Part (1) (Part (2)) of Theorem 1, but after replacing μ and σ^2 of the last period with μ_n and σ_n^2 of period n, respectively.

We now investigate the performance of the proposed heuristic menu of contracts. To this end, we compare it with (1) the optimal, and (2) the inventory-independent menu of contracts of Chen (2005).[5] This will shed light on the performance of the proposed heuristic, and also on the importance of considering dynamic inventory levels when designing compensation plans: a fact that has been ignored in the literature.

Numerical Study 2. We consider a problem with planning horizon of $N = 3$, where $\mu_1 = 3$, and $\mu_n = \mu_{n-1} + \eta$ ($n = 2, 3$ and $\eta = -1, -0.5, 0, 0.5, 1$), $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $\rho_{HH} = 0.6$, $\rho_{HL} = 0.3$, $\rho_{LH} = 0.3$, $\rho_{LL} = 0.7$, and other parameters the same as those in Numerical Study 1 (i.e., $p = 7$, $h = 1$, $c = 2$, $\theta_H = 5$, $\theta_L = 1$, $\gamma = 2$, and $U_0 = 10$). Assuming that at the beginning of the horizon, the firm does not have any on hand inventory, and that the market condition is high, we first find the optimal expected profit of the firm for all of the five test cases parameterized by η (which represents an increasing or decreasing trend in the market demand). To find the optimal expected profit, we solve the constrained optimal dynamic programming equation (22) after a discretization of the continuous inventory state space with a precision of 0.2 as well as a truncation to range $[-2, 6]$ (i.e., considering a cardinality of 40 for the inventory space). Next, using the same discretization method, we compute the performance of the proposed inventory-dependent menu of contracts as well as the inventory-independent menu of contracts of Chen (2005) (i.e., with $\alpha_H^*(x) = (1+\gamma\sigma_n^2)^{-1}$, and $\alpha_L^*(x) = \delta(1+\gamma\sigma_n^2)^{-1}$) for all test cases,. The performance of the two heuristics compared to the optimal one is depicted in Figure 2 using the percentage optimality gap (defined as the percentage

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[5] Notice that in both our heuristic menu of the contract and the menu of contracts of Chen (2005), as well as in the optimal one, the firm optimizes her inventory/production policy using the extracted information about the market condition. However, the menus of contracts offered to the agent are different.

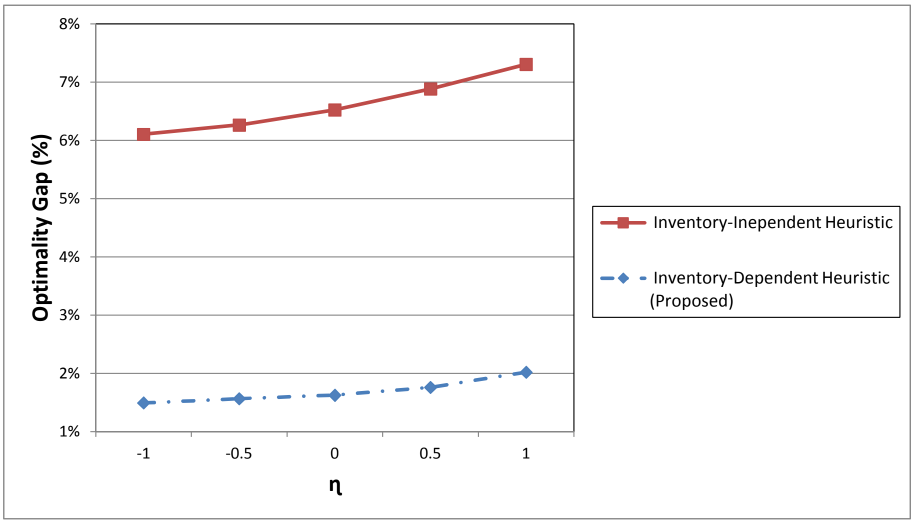

Figure 2 Performance of the proposed heuristic menu of contracts compared to (a) the optimal, and (b) the inventory-independent policy for different test cases. Parameter η represents an increase or decrease in the demand ($\mu_n = \mu_{n-1} + \eta$).

difference between total profit under the heuristic strategy and that of the optimal strategy). From this figure, the proposed inventory-dependent heuristic shows an average optimality gap the of 1.69%, while the inventory-independent heuristic shows an average optimal gap of 6.62%. This observation illuminates the disadvantage of separating production/inventory control decision with that of salesforce incentive design as is currently widely done in both research and practice. Furthermore, from Figure 2, we observe that both heuristic menus of contracts show a better performance when demand in future periods is decreasing. This is to some extent expected, as both heuristics are built upon characteristics of the optimal contracts in a single-period; if future demand is negligible compared to the current demand, the greedy heuristics are expected to work well. Furthermore, we observe that the proposed inventory-dependent policy is more *robust* to changes in the future market demand compared to the inventory-independent heuristic. In fact, the optimality gap range (the difference between maximum and minimum optimality gap in the test suite) for the proposed policy and the inventory-independent heuristic are 0.53% and 1.20%, respectively. This observation shows that the proposed heuristic is much more effective in considering the impact of the current period inventory control decisions on meeting the requirement of the future periods.

To gain further insights into the performance of the proposed heuristic menu of contracts, we conduct another numerical study to demonstrate the performance of the heuristic when the initial stock level changes.

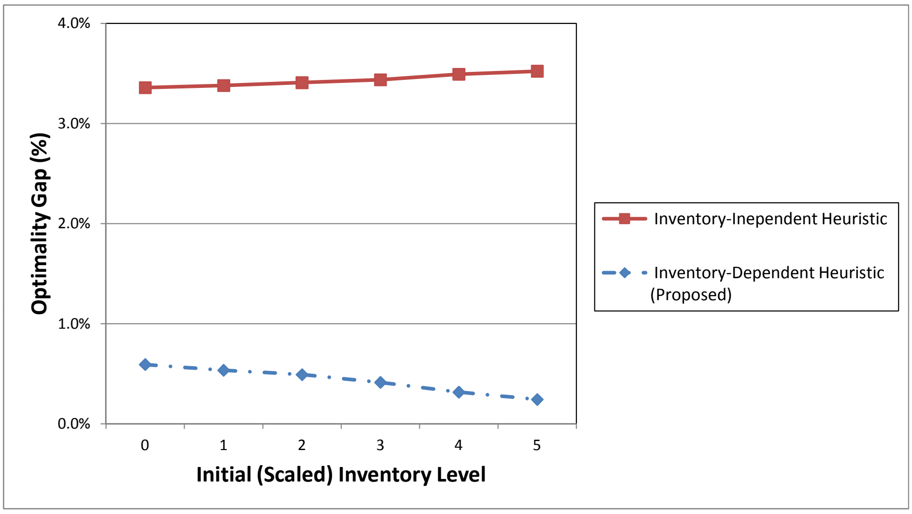

Figure 3 The effect of initial inventory level on the performance of the proposed heuristic menu of contracts compared to (a) the optimal, and (b) the inventory-independent policy.

Numerical Study 3. Consider again the problem with planning horizon of $N = 3$, and parameters $p = 5$, $h = 1$, $c = 1.5$, $\theta_H = 3$, $\theta_L = 1$, $\gamma = 2$, $U_0 = 5$, $\mu_1 = \mu_2 = \mu_3 = 3$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $\rho_{HH} = 0.5$, $\rho_{HL} = 0.5$, $\rho_{LH} = 0.3$, $\rho_{LL} = 0.7$. Assuming the market condition at the beginning of horizon is high, Figure 3 compares the performance of both the proposed inventory-dependent heuristic and the inventory-independent contract (defined in Numerical Study 2) with that of the optimal for various initial inventory levels (i.e., inventory level at the beginning of the planning horizon). From Figure 3, we observe that the optimality gap for both heuristics in this example is lower than that of Numerical Study 2. However, the proposed inventory-dependent heuristic still has a much lower average optimality gap (0.43%) compared to the inventory-independent one (3.43%). Furthermore, as is expected, the performance of the proposed heuristic becomes better (compared to the optimal policy) as the initial inventory increases, but the performance of the inventory-independent heuristic becomes worse. This is expected, when we note that (a) when the inventory level in each period is sufficiently high (the result of a very high initial inventory level), the proposed heuristic is essentially the same as the optimal one, and (b) the inventory-independent menu of contracts gets closer to the optimal for very low levels of inventory, and hence, is expected to perform its best when the inventory level in each period is low.

6. Concluding Remarks

In this paper, we developed a model and analysis to address the observed behavior of firms in practice in offering higher incentives to their sales agents when facing high over-stock inventory levels. We provided several managerial insights through developing and analyzing a dynamic Principal-Agent model with both moral hazard and adverse selection. Using this model, we showed that optimal compensation offer is defined by a menu of inventory-dependent contracts. We further completely characterized the optimal menu of contracts that should be offered by the firm in each period. We found that the optimal commission rates offered by the firm are indeed increasing in the inventory levels, describing the observed behavior of firms in practice. In addition, we found that the firm needs to offer higher commission rates to its less risk averse agents. These analytical findings support the empirical studies that find (i) changing compensation plans is effective in increasing sales (e.g., Levenson et al. (2010)), and (ii) more risk-averse individuals prefer a fixed salary and are less productive under incentive pays (e.g., Cadsby et al. (2007)).

For the first time in the literature, our results shed light on the importance of considering dynamic inventory levels when designing compensation packages, a connection that is typically ignored. In addition to illuminating the optimal design of the compensation packages, we characterized the firm's optimal inventory/production control policy. We found that in general it can be characterized as a market-based state-dependent policy where the firm only needs to take advantage of the extracted information from the agent. Furthermore, making use of such information, the optimal inventory control policy translates to a simple traditional base-stock policy in periods with low inventory levels. We also showed that a higher demand uncertainty can result in a lower optimal stock level. This result is in contrast with the well-known notion of safety-stock in the classical inventory theory where salesforce incentives are completely neglected.

For the ease of exposition and analysis, we made some modeling assumptions. Although they may seem restrictive, we note that the results and insights provided are robust to many of those modeling assumptions. For instance, the system parameters, e.g., p, c, h, agent's risk parameter γ, or agent's cost of effort $v(a)$ are assumed to be period-independent. We point out that all the results can be extended to the case with period-dependent parameters. This extension is valid and useful in applications. For example, these parameters could be different for summer and winter seasons. Also, for tractability and consistent with the literature, we assumed that the market condition can only get two values. Analysis with more than two possible levels of market condition will be much more complex, but the main insights will not change.

In our model, the agent optimizes his expected utility for the current period and is assumed to be a “myopic agent.” We note that this assumption is only suitable for scenarios where the contract does not provide any guarantee to the sales gent for employment in future periods. Focusing on such contracts, we were able to apply the revelation principle in the design of the dynamic mechanism. That is, we were able to restrict our attention the class of mechanisms in which the agent is truth-telling. In scenarios where the contract provides long-term employment guarantees, the agent may become strategic and may want to disguise himself to sacrifice his utility for the current period to obtain higher utility for future periods. This gives out wrong signal to the firm and misleads the firm’s mechanism design. Dynamic mechanism design with a strategic agent is a challenging but also a very interesting setting to study (see, e.g., Oh and Ozer (2011)). Our approach should only be viewed as an approximation for such settings.

There are several possible directions for future research. In this paper, we considered the case where the selling price (and hence unit revenue) is fixed and given. One interesting direction to consider is when pricing is also a decision. In that case, the demand will depend on the selling price. We found that most of the results in the paper can be carried over to the case of additive demand, in which the demand function is defined as $D_n = \Theta_n + a_n + \mu_n - d_n(p_n) + \epsilon_n$, where $d_n(n)$ is a decreasing function of selling price p_n in period n. Under the common assumption that the revenue function $xd_n^{-1}(x)$ is concave, where $d_n^{-1}(.)$ is the inverse function of $d_n(.)$, similar inventory and pricing decisions as Federgruen and Heching (1999) can be obtained (similar to Chen and Simchi-Levi (2004), the decision of inventory and pricing need to be transformed into decision of inventory and average demand), and the optimal contract is still the same as what we presented in Section 5. Furthermore, if pricing is a decision, it will be interesting to compare the case where the pricing decision is made by the firm with the case where the pricing decision is delegated to the sales agent. There has been plenty of research in the Marketing literature on this issue (see, for example, Lal (1986), Bhardwaj (2001), or Mishra and Prasad (2005)), and it will be interesting to revisit the insights provided by such studies when operational decisions are also taken into account.

Finally, a more challenging (but perhaps less practically relevant) problem is when excess demand is not satisfied by emergency order, but is simply lost. In that case, the utility functions of the agent will depend on ordering quantity of the firm. Since the compensation of the agent is no longer normally distributed in this case, the certainty equivalence principle will no longer be applicable. One possible relaxation is to consider the case where the sales agent is risk-neutral, but even in this simplified version, there still remains the issue of dependency between decisions of the firm and the sales agent (e.g., agent's choice of contract will depend on the firm's ordering quantity, while firm's ordering decision will depend on which contract the agent selects). One plausible approach is to use the concept of "rational expectations equilibrium" (see Su (2010)), which is a strategy in which both parties' expectations are consistent.

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Appendix: Proofs

Proof of Lemma 1. (a) Note that on $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$, $\alpha_H(x)$ and $\alpha_L(x)$ are respectively the solution to

$$(p+h)\Phi(x-\theta_H-\mu-\alpha)-(1+\gamma\sigma^2)\alpha-p+c+1=0, \tag{27}$$

and

$$(p+h)\Phi(x-\theta_L-\mu-\alpha)-(1+\gamma\sigma^2)\alpha-p+c+1-\frac{\rho}{1-\rho}(\theta_H-\theta_L)=0, \tag{28}$$

Taking derivative with respect to x yields

$$\alpha'_H(x) = \frac{(p+h)\phi(x-\theta_H-\mu-\alpha_H(x))}{(p+h)\phi(x-\theta_H-\mu-\alpha_H(x))+(1+\gamma\sigma^2)}, \tag{29}$$

$$\alpha'_L(x) = \frac{(p+h)\phi(x-\theta_L-\mu-\alpha_L(x))}{(p+h)\phi(x-\theta_L-\mu-\alpha_L(x))+(1+\gamma\sigma^2)} \tag{30}$$

Thus, $0 \leq \alpha'_H(x) \leq 1$, $0 \leq \alpha'_L(x) \leq 1$. This proves that $\alpha_H(x), \alpha_L(x), x-\alpha_H(x)$, and $x-\alpha_L(x)$ are all increasing functions.

(b) We only prove that $\alpha_H(x)$ converges as $x \to \infty$, as similar analysis applies to $\alpha_L(x)$. Since $\alpha_H(x)$ and $x - \alpha_H(x)$ are both increasing, $\Phi(x - \theta_H - \mu - \alpha_H(x))$ converges to a nonnegative number less than or equal to 1, and $\alpha_H(x)$ also converges to a limit, possibly infinity. We show that this limit has to be finite. To see this, recall that by (27) and (28), $\theta_H + \mu + \alpha_H(x)$ is the solution z to

$$(p+h)\Phi(x-z) - (1+\gamma\sigma^2)z - p + c + 1 + (1+\gamma\sigma^2)(\theta_H + \mu) = 0, \tag{31}$$

while $\theta_L + \mu + \alpha_L(x)$ is the solution z to

$$(p+h)\Phi(x-z) - (1+\gamma\sigma^2)z - p + c + 1 + (1+\gamma\sigma^2)(\theta_L + \mu) - \frac{\rho}{1+\rho}(\theta_H - \theta_L) = 0. \tag{32}$$

It follows from (31) that

$$(p+h)\Phi(x - \theta_H - \mu - \alpha_H(x)) - p + c + 1 = (1+\gamma\sigma^2)\alpha_H(x).$$

Since the left hand side converges to a finite number, it follows that $\alpha_H(x)$ also converges to a finite number as $x \to \infty$.

Substituting $\theta_H - \mu - \alpha_H(x)$ and $\theta_L - \mu - \alpha_L(x)$ in (31) and (32), and subtracting the results, we obtain

$$\begin{aligned} &(p+h)[\Phi(x-\theta_H-\mu-\alpha_H(x)) - \Phi(x-\theta_L-\mu-\alpha_L(x))] - \frac{\rho}{1-\rho}(\theta_H-\theta_L) \\ &= (1+\gamma\sigma^2)(\alpha_H(x) - \alpha_L(x)). \end{aligned} \tag{33}$$

Since $\alpha_H(x)$ and $\alpha_L(x)$ converge to finite numbers, $x - \theta_H - \mu - \alpha_H(x)$ and $x - \theta_L - \mu - \alpha_L(x)$ both go to infinity as $x \to \infty$. Therefore, the first term on the left hand side of (33) goes to 0 as $x \to \infty$. Since the second term of the left hand side of (33) is a negative constant, this shows that the right hand side converges to $-\rho(\theta_H - \theta_L)/(1-\rho) < 0$ as $x \to \infty$, implying

$$\lim_{x\to\infty} \Big\{ \alpha_H(x) - \alpha_L(x) \Big\} = -\frac{\frac{\rho}{1-\rho}(\theta_H - \theta_L)}{1+\gamma\sigma^2} < 0,$$

which proves part (b).

(c) Showing $\alpha_H(x)$ is concave is equivalent to showing that $\alpha'_H(x)$ is decreasing. By (29), it suffices to prove that $\phi(x - \theta_H - \mu - \alpha_H(x))$ is decreasing in x. From part (a), $x - \alpha_H(x)$ is increasing in

x. Since $\phi(x)$ is strictly decreasing on $x \geq 0$, it suffices to prove that, in the range of consideration, we have $x - \theta_H - \mu - \alpha_H(x) \geq 0$.

From our analysis, the optimization of (17) is independent of $G(.)$ and is given by $1/(1+\gamma\sigma^2)$ on $x \leq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$, while on $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$, the optimizer of (17) satisfies $x - \theta_H - \mu - \alpha_H(x) \geq q^* \geq 0$. Therefore, $\alpha_H(x)$ is concave in x on $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$.

Similar argument shows that $\alpha_L(x)$ is concave on $x \geq \theta_L + \mu + \delta/(1+\gamma\sigma^2)$, and that on this range, $x - \theta_L - \mu - \alpha_L(x) \geq q^* \geq 0$. In particular, these results are true on $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$.
(d) We next prove $\alpha_H(x) - \alpha_L(x)$ is strictly increasing on $x \geq q^* + \theta_H + 1/(1+\gamma\sigma^2)$, or $\alpha'_H(x) > \alpha'_L(x)$. By (29) and (30) and that $\phi(x)$ is strictly decreasing on $x \geq 0$, it suffices to prove

$$\theta_H + \alpha_H(x) \geq \theta_L + \theta_L(x), \tag{34}$$

$x - \theta_H - \mu - \alpha_H(x) \geq 0$, and $x - \theta_L - \mu - \alpha_L(x) \geq 0$. The two latter inequalities are proved in part (c). Hence, in the following, we prove (34). To show (34), recall that $\theta_H + \mu + \alpha_H(x)$ is the solution to (31), while $\theta_L + \mu + \alpha_L(x)$ is the solution to (32). Because $(p+h)\Phi(x-z) - (1+\gamma\sigma^2)z$ is strictly decreasing in z, it follows from $(1+\gamma\sigma^2)\theta_H > (1+\gamma\sigma^2)\theta_L - \frac{\rho}{1+\rho}(\theta_H - \theta_L)$ that the solution to (31) is less than the solution to (32). This proves $\theta_H + \alpha_H(x) > \theta_L + \alpha_L(x)$, which completes the proof of part (c). □

Proof of Lemma 2. Recall that $q^* = \Phi^{-1}((p-c)/(p+h))$. Since Φ is strictly increasing, $\theta > \xi(\rho, \theta)$ if, and only if, $\Phi(\theta + q^*) > \frac{p-c+\frac{\rho\theta}{1-\rho}}{p+h}$. Let $g(\theta) = \Phi(\theta + q^*) - \frac{p-c+\frac{\rho\theta}{1-\rho}}{p+h}$. Then, $g(0) = 0$, and $\lim_{\theta\to\infty} g(\theta) = -\infty$. Since $q^* \geq 0$, $g''(\theta) = \phi'(\theta + q^*) < 0$ on $\theta \geq 0$. This shows that $g(\theta)$ is a concave function on $\theta \geq 0$. Hence, there exists a $\bar{\theta}$ such that $g(\bar{\theta}) = 0$ if, and only if, $g'(0) > 0$, or $\phi(q^*) > \frac{\rho}{1-\rho}$. This is the same as $\rho < \rho^*$. Thus, when $\rho < \rho^*$, we define $\bar{\theta}(\rho) > 0$ as the solution to $g(\theta) = 0$. Then on $0 \leq \theta \leq \bar{\theta}(\rho)$, we have $\theta > \xi(\rho, \theta)$, and $\theta \leq \xi(\rho, \theta)$ on $\theta \geq \bar{\theta}(\rho)$. However, if $\rho \geq \rho^*$, then $g'(0) \leq 0$. Hence, $g(\theta) \leq 0$ for all $\theta \geq 0$. This shows that, when $\rho \geq \rho^*$, $\theta \leq \xi(\rho, \theta)$ for all $\theta \geq 0$. □

Proof of Lemma 3. By Lemma 2, if one of the conditions (a) or (b) is satisfied, we have $\theta_H - \theta_L < \xi(\rho, \theta_H - \theta_L)$, or

$$q^* + \theta_H - \theta_L < \Phi^{-1}\left(\frac{p-c+\frac{\rho}{1-\rho}(\theta_H - \theta_L)}{p+h}\right). \tag{35}$$

Furthermore, by part (d) of Lemma 1, the necessary and sufficient condition for the existence of a finite number $z^* > q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$ such that $\alpha_H(x) < \alpha_L(x)$ on $z^* > x \geq q^* + \theta_H + 1/(1+\gamma\sigma^2)$, $\alpha_H(z^*) = \alpha_L(z^*)$, and $\alpha_H(x) > \alpha_L(x)$ on $x > z^*$, is

$$\alpha_L\left(q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}\right) > \frac{1}{1+\gamma\sigma^2}. \tag{36}$$

Since on $x \geq q^* + \theta_L + \mu + \delta/(1+\gamma\sigma^2)$, $\theta_L + \mu + \alpha_L(x)$ is the solution of (32) and the left hand side of (32) is decreasing in $z = \theta_L + \mu + \alpha_L$, (36) is satisfied if, and only if, when $x = q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}$, the left hand side of (32) is positive with $z = \theta_L + \mu + \frac{1}{1+\gamma\sigma^2}$. That is (36) is satisfied if, and only if,

$$\begin{aligned}(p+h)\Phi\left(q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2} - z\right) - (1+\gamma\sigma^2)z - p + c + 1 \\ +(1+\gamma\sigma^2)(\theta_L + \mu) - \frac{\rho}{1+\rho}(\theta_H - \theta_L)\,|_{z=\theta_L+\mu+\frac{1}{1+\gamma\sigma^2}} > 0.\end{aligned}$$

Canceling common terms, this is equivalent to

$$(p+h)\Phi(q^* + \theta_H - \theta_L) > p - c + \frac{\rho}{1-\rho}(\theta_H - \theta_L). \tag{37}$$

Since Φ is strictly increasing, this is equivalent to (35), and the existence of z^* is established.

Now suppose neither of conditions (a) or (b) of Lemma 3 holds. Then, by Lemma 2, the opposite direction of (35) is satisfied. But the same argument as above shows that $\alpha_H(x) \geq \alpha_L(x)$ on all $x \geq q^* + \theta_H + \mu + 1/(1+\gamma\sigma^2)$ if, and only if, the opposite direction of (37), or equivalently, the opposite direction of (35) holds. This completes the proof. □

Proof of Theorem 1. Before proving this result, we note that the control parameters z_1^*, z_2^* and z_3^* used in defining inventory intervals $I_2, I_3, \cdots, I_5$ satisfy $z_1^* < z_2^* < q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2} < z_3^*$. To see $z_1^* < z_2^*$, by the definition of z_2^* and that $x - \tilde{\alpha}_L(x)$ is increasing, it suffices to show that when $x = z_1^*$, $x - \theta_H - \mu - \tilde{\alpha}_H(x)$ is less than q^*. Note that at $x = z_1^*$, we have

$$\begin{aligned}&x - \theta_H - \mu - \tilde{\alpha}_H(x) \\ &= \theta_L + \mu + \frac{1}{1+\gamma\sigma^2} + \Phi^{-1}\left(\frac{p - c + \frac{\rho}{1-\rho}(\theta_H - \theta_L)}{p+h}\right) - \theta_H - \mu - \frac{1}{1+\gamma\sigma^2} \\ &= \theta_L - \theta_H + \Phi^{-1}\left(\frac{p - c + \frac{\rho}{1-\rho}(\theta_H - \theta_L)}{p+h}\right).\end{aligned}$$

From Lemma 2 and conditions of part (1), this is less than q^*. Similarly, to verify that $z_2^* < q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}$, form the definition of z_2^*, it suffices to show that at $x = q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}$, $x - \theta_H - \mu - \tilde{\alpha}_H(x)$ is greater than q^*. It is known that

$$\tilde{\alpha}_H\left(q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}\right) < \frac{1}{1+\gamma\sigma^2}.$$

Hence at $x = q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}$, we have

$$\begin{aligned}
& x - \theta_H - \mu - \tilde{\alpha}_H(x) \\
&= q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2} - \theta_H - \mu - \tilde{\alpha}_H\left(q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}\right) \\
&= q^* + \frac{1}{1+\gamma\sigma^2} - \tilde{\alpha}_H\left(q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}\right) \\
&> q^*.
\end{aligned}$$

We first consider part (1), i.e., the case where condition (a) or (b) holds. In this case, $\tilde{\alpha}_H(x)$ and $\tilde{\alpha}_L(x)$ cross twice, the first time at z_1^*, and the second time at z_3^*. If $x \in I_1, I_2$, or I_5, the optimal solution of (16) is the same as that without the $\alpha_H \geq \alpha_L$ constraint (see also Lemma 4). Hence, the optimal commission rates in these ranges are as given in the theorem. Now consider the case where $x \in I_3 = [z_1^*, z_2^*)$, and examine optimization problem (16). Since the optimal solution for the unconstrained optimization satisfies $\alpha_L(x) > \alpha_H(x)$, by Lemma 4 (b) the optimal solution for (16) has to be between the two optimal solutions . That is, we need to increase $\alpha_H(x)$ but reduce $\alpha_L(x)$ until they are equal. Note that at $\alpha_H(x) = 1/(1+\gamma\sigma^2)$, $G(q^* \vee (x - \theta_H - \mu - \alpha_H(x)) = G(q^*)$ is flat as long as $x - \theta_H - \mu - 1/(1+\gamma\sigma^2) < q^*$, and that when $\alpha_H(x)$ increases, $G(q^* \vee (x - \theta_H - \mu - \alpha_H(x))$ remains constant. Hence, the optimal $\alpha_H^*(x) = \alpha_L^*(x)$ is the maximizer of

$$\rho\left[\alpha_H - \frac{1+\gamma\sigma^2}{2}\alpha_H^2\right] + (1-\rho)\left[\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H - \theta_L) - \frac{1+\gamma\sigma^2}{2}\alpha_L^2 - G(x - \theta_L - \mu - \alpha_L])\right].$$

Taking derivative we obtain the results given in part (iii). Therefore, it suffices to show that when $x \in I_3 = [z_1^*, z_2^*)$, we have $x - \theta_H - \mu - 1/(1+\gamma\sigma^2) < q^*$. This was proved earlier, where we showed $z_2^* < q^* + \theta_H + \mu + \frac{1}{1+\gamma\sigma^2}$. The result for the case where $x \in I_4$ follows a similar line of proof. Thus, it remains to to prove part (2) of the theorem. The proof for this part follows Lemma 4 (a), since $\tilde{\alpha}_H(x)$ and $\tilde{\alpha}_L(x)$ do not cross in this case. That is, the objective function is separable in $\alpha_H(x)$

and $\alpha_L(x)$ (Lemma 3). Hence, by Lemma 4 (a), we can relax the constraint $\alpha_L(x) \leq \alpha_H(x)$ of (16), which results in the solutions given in parts (i)-(iii) of part (2). This completes the proof. □

Proof of Corollary 1 Notice that when $\rho \geq (h+c+1)/(h+c+1+(\theta_H-\theta_L))$, $\lim_{x\to\infty} \tilde{\alpha}_L(x) < 0$, since $\Phi(\cdot)$ is a cdf function ($\tilde{\alpha}_L(x)$ is defined right before Theorem 1). Next, observe that from Theorem 1, for large enough inventory levels, $\alpha_L^*(x) = (\tilde{\alpha}_L(x))^+$. Hence, $\lim_{x\to\infty} \alpha_L^*(x) = 0$. Moreover, since $\alpha_L^*(x)$ is constrained to be non-negative for all x, and is also increasing in x, it follows that $\alpha_L^*(x) = 0$ for all x. Furthermore, when $x \leq q^* + \theta_L + \mu + \delta\,(1+\gamma\sigma^2)^{-1}$, from Theorem 1, $\alpha_L^*(x) = \delta\,(1+\gamma\sigma^2)^{-1}$. Hence, $\alpha_L^*(x) = 0$ if, and only if, $\delta \leq 0$, which is equivalent to $\rho \geq 1/(1+(\theta_H-\theta_L))$. □

Proof of Corollary 2. To prove part (1), notice that from Theorem 1 and Lemma 1(a), the commission offered (as a function of the inventory level) to either H or L type agent is initially constant, followed by increasing functions. Furthermore, from Theorem 1, it follows that commissions are also decreasing in γ. The local concavity (i.e., concavity in each inventory interval) of the optimal commissions, follows directly by checking the second derivative of the optimal commissions given in Theorem 1. □

Proof of Corollary 3. The proof follows directly from 3 (which shows that the effort induced by the firm is equal to the commission rate) and Corollary 2 part (1) (which shows that the commissions offered by the firm are increasing in her inventory level). □

Proof of Proposition 1. From (16), the only terms where x appears is in $G(q^* \vee (x-\theta_H-\mu-\alpha_H))$ and $G(q^* \vee (x-\theta_L-\mu-\alpha_L))$, and both are jointly concave in (x, θ_H, α_L). This shows that the objective of (16) is jointly concave in (x, α_H, α_L). Since the constraint $\alpha_H \geq \alpha_L \geq 0$ defines a convex set, it follows from Proposition B-4 of page 525 of Heyman and Sobel (1984) that resulting optimal value function $V_N(x,\rho)$ is convex in x. □

Proof of Proposition 2. Using induction, assume $V_{n+1}(\rho, x)$ is concave in x. It follows that $W_i(y)$ (defined right before this proposition) is also concave. Since q_i^* denotes the maximizer of $W_i(y)$, replacing $W_i(y)$ in the optimization problem (21) shows that it is optimal to set $y^* = q_i^* + \theta_i + \mu_n + \alpha_i(x)$ or equivalently order $(q_i^* - x + \theta_i + \mu_n + \alpha_i(x))^+$, where $(a)^+ = max\{a, 0\}$. This completes the proof. □

Proof of Corollary 4. The results follows directly from Proposition 2 and Theorem 2 for Intervals I_1 and I"1. Notice that on these intervals the optimal commission, $\alpha_i(x)$, is independent of x. □

Proof of Proposition 3. It is argued in the main body of the paper that, for $i = L, H$, $W_i(x)$ is concave. Since q_i^* is the maximizer of W_i, it follows that both $W_H(q_H^* \vee (x - \theta_H - \mu_n - \alpha_H))$ and $W_L(q_L^* \vee (x - \theta_L - \mu_n - \alpha_L))$ are decreasing concave in x. Rewrite (22) as

$$\begin{aligned} V_n(x,\rho) - cx = \max_{\alpha_H,\alpha_L:\alpha_H \geq \alpha_L \geq 0} & \left\{ \frac{\ln U_0}{\gamma} + \bar{\theta} + \mu_n + \rho \left[\alpha_H - \frac{1+\gamma\sigma^2}{2}\alpha_H^2 + W_H(q_H^* \vee (x - \theta_H - \mu_n - \alpha_H)) \right] \right. \\ & \left. + (1-\rho) \left[\alpha_L - \frac{\rho}{1-\rho}\alpha_L(\theta_H - \theta_L) - \frac{1+\gamma\sigma^2}{2}\alpha_L^2 + W_L(q_L^* \vee (x - \theta_L - \mu_n - \alpha_L)) \right] \right\}. \end{aligned} \tag{38}$$

Since, for each given feasible α_H and α_L, the objective function in (38) is decreasing in x, it follows that $V_n(x) - cx$ is also decreasing in x. Concavity of $V_n(x)$ follows from the fact that the objective function is jointly concave in (α_H, α_l, x), and the feasible region is a convex set of (α_H, α_L). Therefore, from Proposition B-4 of page 525 of Heyman and Sobel (1984) $V_n(x,\rho) - cx$, and hence $V_n(x,\rho)$, is concave in x. □

Proof of Lemma 5. (a) On the range of interest for x, $\alpha_H(x)$ is the solution to (25). Taking derivative with respect to x on both sides, we obtain

$$-(1+\gamma\sigma_n^2)\alpha_H'(x) - W''(x - \theta_H - \mu_n - \alpha_H(x))(1 - \alpha_H'(x)) = 0.$$

Solving for $\alpha_H'(x)$, and using the fact that $W(\cdot)$ is concave yields

$$1 \geq \alpha_H'(x) = \frac{-W''(x - \theta_H - \mu_n - \alpha_H(x))}{1 + \gamma\sigma_n^2 - W''(x - \theta_H - \mu_n - \alpha_H(x))} \geq 0.$$

This shows that $\alpha_H(x)$ and $x - \alpha_H(x)$ are both increasing. Similar argument shows that $\alpha_L(x)$ and $x - \alpha_L(x)$ are increasing on $x \geq \max\{q_H^*, q_L^*\} + \theta_H + \mu_n + \frac{1}{1+\gamma\sigma_n^2}$ and completes the proof for part (a). To prove part (b), we first need the following simple result. Suppose y is a vector with convex feasible set C that is independent of x, and that $g(x,y)$ is decreasing in x for any given y, then $\max_{y \in C} g(x,y)$ is also decreasing in x. Using this fact and (22) we can show by induction that

$$\frac{\partial W(x,\rho)}{\partial x} \geq -\frac{h}{1-\alpha}$$

for all x. Now write (25) as

$$(1+\gamma\sigma^2)\alpha_H(x) = 1 - W'(x - \theta_H - \alpha_H(x)).$$

Since both $\alpha_H(x)$ and $x-\alpha_H(x)$ are increasing and that $W'(.)$ is decreasing, it follows that both sides of equation above have a limit. But the right hand side is less than or equal to $1+h/(1-\alpha)$. Hence, $\alpha_H(x)$ is increasing and converges to a finite number. Similar argument proves the result for $\alpha_L(x)$. □

Proof of Theorem 2. The proof is similar to that of Theorem 1. First consider Case 1, and suppose $x \in I_1$, I_2 or I_5. The optimal solution of (22) in these intervals is the same as that without the constraint $\alpha_H \geq \alpha_L$ (see also Lemma 4). Hence, the optimal commissions in these ranges can be obtained from separately optimizing (23) and (24). Doing so, proves the result of parts (1), (2), and (5) of Case 1. When $x \in I_3$ or $x \in I_4$, the optimizers of (23) and (24) do not satisfy the constraint $\alpha_H \geq \alpha_L$. Hence, by Lemma 4, the optimizers of (22) can be obtained by setting $\alpha_H = \alpha_L$. This yields the results provided in parts (3) and (4) of of Case 1. The proof for Case 2 and also for the case where $q_H^* + \theta_H + 1/(1+\gamma\sigma_n^2) \leq q_L^* + \theta_L + \delta/(1+\gamma\sigma_n^2)$, follows the same line of argument by considering cases where separately optimizing (23) and (24) does or does not yield the optimizers of (22). □