Topology of positive spaces

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The vertices of the permutohedron are \((\pi(1), \cdots, \pi(n)) \in \mathbb{R}^n\) for \(\pi \in \mathfrak{S}_n\).

The edges of the permutohedron are

\[
(\cdots, i, \cdots, i + 1, \cdots) \leftrightarrow (\cdots, i + 1, \cdots, i, \cdots).
\]

These correspond to cover relations in the weak Bruhat order on \(\mathfrak{S}_n\).
Permutohedron for the strong Bruhat order?

\[ S_3 \] (strong order)

Using \textit{total positivity}, we can define a space whose \( d \)-dimensional faces correspond to intervals of length \( d \) in the strong Bruhat order on \( S_n \).

This space is not a polytope! However, topologically it is just as good:

1. it is partitioned into faces \( F \), each homeomorphic to an open ball;
2. the boundary \( \partial F \) of each face \( F \) is a union of lower-dimensional faces;
3. the closure \( \overline{F} \) of each face \( F \) is homeomorphic to a closed ball.

Such a space is called a \textit{regular CW complex}. 
A matrix is *totally positive* if every submatrix has positive determinant.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 3 & 9 & 27 \\
1 & 4 & 16 & 64
\end{bmatrix}
\begin{align*}
\lambda_1 &= 71.5987 \cdots \\
\lambda_2 &= 3.6199 \cdots \\
\lambda_3 &= 0.7168 \cdots \\
\lambda_4 &= 0.0646 \cdots
\end{align*}
$$

Gantmakher, Krein (1937): the eigenvalues of a square totally positive matrix are all real, positive, and distinct.

Totally positive matrices are a discrete analogue of *totally positive kernels* (e.g. $K(x, y) = e^{xy}$), introduced by Kellogg (1918).

Lusztig (1994): total positivity for algebraic groups $G$ (e.g. $G = \text{SL}_n$) and partial flag varieties $G/P$ (e.g. $G/P = \text{Gr}_{k,n}, \text{Fl}_n$).


Postnikov (2006): *totally nonnegative Grassmannian* $\text{Gr}_{k,n}^{\geq 0}$. It has been related to the ASEP, the KP equation, Poisson geometry, quantum matrices, scattering amplitudes, mirror symmetry, singularities of curves, . . .
The Grassmannian $\text{Gr}_{k,n}$

- The Grassmannian $\text{Gr}_{k,n}$ is the set of $k$-dimensional subspaces of $\mathbb{R}^n$.

Given $V \in \text{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for $k$-subsets $I$ of $\{1, \ldots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of $V$ in columns $I$. The Plücker coordinates $\Delta_I(V)$ are well defined up to a common nonzero scalar.

- We call $V \in \text{Gr}_{k,n}$ totally nonnegative if $\Delta_I(V) \geq 0$ for all $k$-subsets $I$. The set of all such $V$ forms the totally nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq 0}$.

- We can think of $\text{Gr}_{k,n}^{\geq 0}$ as a compactification of the space of $k \times (n - k)$ totally positive matrices, or as the Grassmannian notion of a simplex.
Compactifying the space of (totally positive) matrices

- The closure of the space of $k \times \ell$ totally positive matrices is not compact, e.g. consider
  \[
  \begin{bmatrix}
  1 & 1 \\
  1 & t
  \end{bmatrix}
  \text{ as } t \to \infty.
  \]

- We can embed the space of $k \times \ell$ matrices inside $\text{Gr}_{k,k+\ell}$ as the subset where the Plücker coordinate $\Delta\{1,2,\ldots,k\}$ is nonzero:
  \[
  \begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
  \end{bmatrix}
  \mapsto
  \begin{bmatrix}
  1 & 0 & 0 & g & h & i \\
  0 & 1 & 0 & -d & -e & -f \\
  0 & 0 & 1 & a & b & c
  \end{bmatrix}
  \in \text{Gr}_{3,3+3}.
  \]

- This identifies the space of $k \times \ell$ totally positive matrices with the totally positive part of $\text{Gr}_{k,k+\ell}$, whose closure is the compact space $\text{Gr}_{k,k+\ell}^{\geq 0}$.

- Passing to the Grassmannian also reveals certain hidden symmetries, notably the cyclic symmetry given by the action
  \[
  \begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n
  \end{bmatrix}
  \mapsto
  \begin{bmatrix}
  v_2 \\
  \vdots \\
  v_n \\
  (-1)^{k-1}v_1
  \end{bmatrix}
  \text{ on } \text{Gr}_{k,n}^{\geq 0}.
The cell decomposition of $\text{Gr}_{k,n}^{\geq 0}$

$\text{Gr}_{k,n}^{\geq 0}$ has a cell decomposition due to Rietsch (1998) and Postnikov (2006). Each *positroid cell* is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

Postnikov showed that the face poset of $\text{Gr}_{k,n}^{\geq 0}$ is given by *circular Bruhat order* on *decorated permutations* with $k$ anti-excedances.

$$\text{Gr}_{1,3}^{\geq 0} \cong \frac{\Delta_2, \Delta_3 = 0}{\Delta_3 = 0 \Delta_2 = 0} \cong \frac{\Delta_1, \Delta_2, \Delta_3 > 0}{\Delta_1, \Delta_3 = 0 \Delta_1 = 0 \Delta_1, \Delta_2 = 0}$$
The topology of $\mathbf{Gr}_{k,n}^{\geq 0}$

**Conjecture (Postnikov (2006))**

The cell decomposition of $\mathbf{Gr}_{k,n}^{\geq 0}$ is a regular CW complex homeomorphic to a ball. That is, the closure of every cell is homeomorphic to a closed ball.

- e.g. non-regular CW complex
- regular CW complex
- regular CW complex

- Williams (2007): The face poset of $\mathbf{Gr}_{k,n}^{\geq 0}$ is *thin* and *shellable*. Thus it is the face poset of *some* regular CW complex homeomorphic to a ball.
- Postnikov, Speyer, Williams (2009): $\mathbf{Gr}_{k,n}^{\geq 0}$ is a CW complex (via *matching polytopes* of plabic graphs).
- Rietsch, Williams (2010): $\mathbf{Gr}_{k,n}^{\geq 0}$ is a regular CW complex up to homotopy (via discrete Morse theory).

**Theorem (Galashin, Karp, Lam)**

*Postnikov’s conjecture is true.*
Motivation 1: combinatorics of regular CW complexes

- Every convex polytope (decomposed into its open faces) is a regular CW complex. We can think of a regular CW complex as the ‘next best thing’ to a convex polytope.

\[
\text{link}_{l_3}(U_3^{>0}) = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : \begin{align*} x + z &= 1, \\
\text{all minors} &\geq 0 \end{align*} \right\}
\]

\( \mathcal{S}_3 \) (strong order)

- Note that \( \mathcal{S}_n \) is not the face poset of a polytope. However, \( \mathcal{S}_n \) is shellable due to Edelman (1981), so it is the face poset of a regular CW complex homeomorphic to a ball, by work of Björner (1984).

- Bernstein, Björner: Is there such a regular CW complex ‘in nature’?

- Fomin and Shapiro (2000) conjectured that \( \text{link}_{l_n}(U_n^{>0}) \) is such a regular CW complex. This was proved by Hersh (2014), in general Lie type.
Motivation 2: amplituhedra and Grassmann polytopes

By definition, a polytope is the image of a simplex under an affine map:

A Grassmann polytope is the image of a map $\text{Gr}_{k,n}^{\geq 0} \to \text{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

When the matrix $Z$ has positive maximal minors, the Grassmann polytope is called an amplituhedron. Amplituhedra generalize cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by the physicists Arkani-Hamed and Trnka (2014), and inspired Lam (2015) to define Grassmann polytopes.
Motivation 2: amplituhedra and Grassmann polytopes

- Arkani-Hamed, Bai, Lam (2017): a positive geometry is a space equipped with a canonical differential form, which has logarithmic singularities at the boundaries of the space. Examples include convex polytopes:

\[ \pm \frac{dx dy}{xy(1-x-y)} \]

- The amplituhedron is conjecturally a positive geometry, whose canonical form for \( m = 4 \) is the tree-level scattering amplitude in planar \( \mathcal{N} = 4 \) SYM.
- Intuition from physics: the geometry determines the canonical form, and vice-versa. In order to understand amplituhedra (and more generally, Grassmann polytopes), we first need to understand \( \text{Gr}_{k,n}^{>0} \).
- Other physically relevant positive geometries include associahedra, cosmological polytopes, Cayley polytopes, halohedra, Stokes polytopes, ...
Technique 1: contractive flows

Theorem

Every compact, convex subset of $\mathbb{R}^d$ is homeomorphic to a closed ball.

Proof

This proof does not directly work for $\text{Gr}_{k,n}^{>0}$, since it is not totally geodesic.
Cyclic symmetry of $\text{Gr}^{\geq 0}_{k,n}$

- Define the (left) cyclic shift map $S$ on $\mathbb{R}^n$ by

  $$S(x) := (x_2, x_3, \cdots, x_n, (-1)^{k-1}x_1) \quad \text{for } x = (x_1, \cdots, x_n) \in \mathbb{R}^n.$$ 

Then $S$ gives the cyclic action on $\text{Gr}_{k,n}$:

$$S \cdot \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_2 & \cdots & v_n & (-1)^{k-1}v_1 \end{bmatrix}.$$

- We regard $S$ as a vector field on $\text{Gr}_{k,n}$, which sends $V \in \text{Gr}_{k,n}$ along the trajectory $\exp(tS) \cdot V$ for $t \geq 0$. This contracts all of $\text{Gr}^{\geq 0}_{k,n}$ onto the attractor $V_0 \in \text{Gr}^{\geq 0}_{k,n}$, giving a homeomorphism onto a closed ball as before.

- e.g. $\text{Gr}^{\geq 0}_{1,3}$
Other spaces admitting a contractive flow

- A similar argument shows the following spaces are closed balls: *cyclically symmetric* amplituhedra, Lam’s compactification of the space of electrical networks, and Lusztig’s totally nonnegative partial flag varieties \((G/P)_{\geq 0}\).

- e.g. \(\text{Fl}^{\geq 0}_{3}\) consists of complete flags \(\{0\} \subset W_1 \subset W_2 \subset \mathbb{R}^3\) such that \(W_1\) and \(W_2\) are totally nonnegative subspaces. This means that \(W_1\) is spanned by a vector \((x_1, x_2, x_3)\) and \(W_2\) is orthogonal to a vector \((y_1, -y_2, y_3)\) with

\[
x_1y_1 - x_2y_2 + x_3y_3 = 0, \quad x_1, x_2, x_3, y_1, y_2, y_3 \geq 0.
\]

This space has 4 facets, given by setting one of \(x_1, y_1, x_3, y_3\) to 0.

- Lusztig (1994), Rietsch (1999): \(\text{Fl}^{\geq 0}_n\) has a cell decomposition whose \(d\)-dimensional cells are indexed by intervals of length \(d\) in \((\mathcal{G}_n, \leq_{\text{strong}})\).
Technique 2: links

- Unfortunately, there exist cells of $\text{Gr}_{k,n}^{\geq 0}$ with no smooth contractive flow.

**Theorem (consequence of generalized Poincaré conjecture)**

*Suppose that $X$ is a compact topological manifold with boundary, whose interior $X^\circ$ is contractible and whose boundary $\partial X$ is homeomorphic to a sphere. Then $X$ is homeomorphic to a closed ball.*

- We want to show that the closure $X$ of a positroid cell in the cell decomposition of $\text{Gr}_{k,n}^{\geq 0}$ is homeomorphic to a closed ball.
- Postnikov (2006): $X^\circ$ is homeomorphic to an open ball.
- By induction, we can assume that every cell closure in the boundary of $X$ is homeomorphic to a closed ball, i.e. $\partial X$ is a regular CW complex.
- Williams (2007): The face poset of $\text{Gr}_{k,n}^{\geq 0}$ is thin and shellable, so it is the face poset of a sphere. By Björner (1984), the homeomorphism type of a regular CW complex is determined by its face poset. Therefore by induction, $\partial X$ is homeomorphic to a sphere.
It remains to show that $X$ is a topological manifold with boundary, i.e. $X$ looks like a closed half-space in $\mathbb{R}^d$ near any point on its boundary.

We adopt the framework of links from Fomin and Shapiro (2000), which they introduced to study the topology of $U^0_n$.

We prove that:

1. $\text{link}(x)$ is homeomorphic to a closed ball;
2. locally near $x$, the space $X$ looks like the cone over $\text{link}(x)$.

This implies that $X$ is a topological manifold with boundary near $x$.

We prove (1) by a similar induction. This does not reduce to a third induction, since ‘links in links are links’.

We introduce two key ideas: using Snider’s embedding, and constructing a dilation action on the small spheres centered at $x$. 
Snider’s embedding

The link framework of Fomin and Shapiro for $U_n^{\geq 0}$ involves factorization maps that do not have a direct analogue in $\text{Gr}_{k,n}$. To get around this, we employ a construction of Snider (2011).

We fix an index $I$, and embed the subset of $\text{Gr}_{k,n}$ where $\Delta_I \neq 0$ into the affine flag variety $\tilde{\text{Fl}}_n$. We can think of $\tilde{\text{Fl}}_n$ as $n$-periodic bi-infinite matrices modulo left multiplication by invertible lower-triangular matrices.

E.g. Let $I = \{1, 3\}$ with $k = 2$, $n = 4$. Then Snider’s embedding is

$$
\begin{bmatrix}
1 & a & 0 & b \\
0 & c & 1 & d
\end{bmatrix}
\mapsto

\begin{bmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 0 & a & 0 & b & 1 & 0 & \ldots \\
\ldots & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & d & 0 & c & 1 & 0 & \ldots \\
\ldots & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & a & 0 & b & 1 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
$$

We can then apply the Fomin–Shapiro framework in $\tilde{\text{Fl}}_n$.

We obtain the conic structure near $x$ by translating $x$ to a ‘hidden’ point in $\tilde{\text{Fl}}_n$ in the same cell as $x$, which does not come from a point in $\text{Gr}_{k,n}$.
Open problems

Conjecture (Williams (2007))

The cell decomposition of \((G/P)_{\geq 0}\) is a regular CW complex homeomorphic to a ball.

- Show that Lam’s compactification of the space of electrical networks forms a regular CW complex. Its face poset is the uncrossing order on matchings, which is Eulerian due to Lam (2015) and shellable due to Hersh and Kenyon (2018).
- Study the topology of amplituhedra and, more generally, Grassmann polytopes.

Thank you!