Alternating curves

**Proposition**

Let $f : [0, 1] \to \mathbb{R}^k$ be a continuous curve. Then no hyperplane through 0 contains $k$ points on the curve iff the determinants

$$\det[ f(t_1) | \cdots | f(t_k) ] \quad (0 \leq t_1 < \cdots < t_k \leq 1)$$

are either all positive or all negative.

**Proof**

Since $\{ (t_1, \cdots, t_k) \in \mathbb{R}^k : 0 \leq t_1 < \cdots < t_k \leq 1 \} \subseteq \mathbb{R}^k$ is connected, its image $\{ \det[ f(t_1) | \cdots | f(t_k) ] : 0 \leq t_1 < \cdots < t_k \leq 1 \} \subseteq \mathbb{R}$ is connected.

How can we discretize this result?

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Sign variation, the Grassmannian, and total positivity  
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Theorem (Gantmakher, Krein (1950); Schoenberg, Whitney (1951))

Let $x_1, \ldots, x_n \in \mathbb{R}^k$ span $\mathbb{R}^k$. Then the following are equivalent:

(i) the piecewise-linear path $x_1, \ldots, x_n$ crosses any hyperplane through 0 at most $k - 1$ times;

(ii) the sequence $(a^T x_1, \ldots, a^T x_n)$ changes sign at most $k - 1$ times for all $a \in \mathbb{R}^n$; and

(iii) the $k \times k$ minors of the $k \times n$ matrix $[x_1 | \cdots | x_n]$ are either all nonnegative or all nonpositive.

The set of such point configurations $(x_1, \ldots, x_n)$, modulo linear automorphisms of $\mathbb{R}^k$, is the totally nonnegative Grassmannian.

Can we characterize the maximum number of hyperplane crossings of the path $x_1, \ldots, x_n$ in terms of the $k \times k$ minors of $[x_1 | \cdots | x_n]$?
The Grassmannian $\text{Gr}_{k,n}$

- The Grassmannian $\text{Gr}_{k,n}$ is the set of $k$-dimensional subspaces $V$ of $\mathbb{R}^n$.

- Given $V \in \text{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for $I \in \binom{[n]}{k}$ let $\Delta_I(V)$ be the $k \times k$ minor of $V$ with columns $I$. The Plücker coordinates $\Delta_I(V)$ are well defined up to multiplication by a global nonzero constant.

- We say that $V \in \text{Gr}_{k,n}$ is totally nonnegative if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$, and totally positive if $\Delta_I(V) > 0$ for all $I \in \binom{[n]}{k}$. Denote the set totally nonnegative $V$ by $\text{Gr}_{k,n}^{\geq 0}$, and the set of totally positive $V$ by $\text{Gr}_{k,n}^{> 0}$.
For \( v \in \mathbb{R}^n \), let \( \text{var}(v) \) be the number of sign changes in the sequence \((v_1, v_2, \cdots, v_n)\), ignoring any zeros.

\[
\text{var}(1, -4, 0, -3, 6, 0, -1) = \text{var}(1, -4, -3, 6, -1) = 3
\]

Similarly, let \( \overline{\text{var}}(v) \) be the maximum of \( \text{var}(w) \) over all \( w \in \mathbb{R}^n \) obtained from \( v \) by changing zero components of \( w \).

\[
\overline{\text{var}}(1, -4, 0, -3, 6, 0, -1) = 5
\]

**Theorem (Gantmakher, Krein (1950))**

Let \( V \in \text{Gr}_{k,n} \).

(i) \( V \) is totally nonnegative iff \( \text{var}(v) \leq k - 1 \) for all \( v \in V \).

(ii) \( V \) is totally positive iff \( \overline{\text{var}}(v) \leq k - 1 \) for all nonzero \( v \in V \).

- e.g. \[
\begin{bmatrix}
1 & 0 & -4 & -3 \\
0 & 1 & 3 & 2
\end{bmatrix}
\in \text{Gr}^{>0}_{2,4}.
\]

- Note that every \( V \in \text{Gr}_{k,n} \) contains a vector \( v \) with \( \text{var}(v) \geq k - 1 \).
A history of sign variation and total positivity

- Descartes’s rule of signs (1637): The number of positive real zeros of a real polynomial \( \sum_{i=0}^{n} a_i t^i \) is at most \( \text{var}(a_0, a_1, \ldots, a_n) \).

- Pólya (1912) asked when a linear map \( A : \mathbb{R}^k \rightarrow \mathbb{R}^n \) diminishes variation, i.e. satisfies \( \text{var}(Ax) \leq \text{var}(x) \) for all \( x \in \mathbb{R}^k \). Schoenberg (1930) showed that an injective \( A \) diminishes variation iff for \( j = 1, \ldots, k \), all nonzero \( j \times j \) minors of \( A \) have the same sign.

- Gantmakher, Krein (1935): The eigenvalues of a totally positive square matrix (whose minors are all positive) are real, positive, and distinct.


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A natural mathematical base is proposed for the investigation of the so-called oscillation properties of small harmonic oscillations of linear elastic continua, such as, transverse oscillations of strings, rods, and multiple-span beams, and torsional oscillations of shafts. The book is
A history of sign variation and total positivity

- Whitney (1952): The totally positive matrices are dense in the totally nonnegative matrices.
- Aissen, Schoenberg, Whitney (1952): Let $r_1, \ldots, r_n \in \mathbb{C}$. Then $r_1, \ldots, r_n$ are all nonnegative reals iff $s_\lambda(r_1, \ldots, r_n) \geq 0$ for all partitions $\lambda$.
- Lusztig (1994) constructed a theory of total positivity for $G$ and $G/P$.

One of the main tools in our study of $G_{\geq 0}$ and $G_{> 0}$ is the theory of canonical bases in [L1]. Thus, our proof of the fact that $G_{\geq 0}$ is closed in $G$ (Theorem 4.3) is based on the positivity properties of the canonical bases (in the simply-laced case), proved in [L1],[L2], which is a non-elementary statement, depending ultimately on the Weil conjectures. The Rietsch (1997) and Marsh, Rietsch (2004) developed the theory for $G/P$.

- Fomin and Zelevinsky (2000s) introduced cluster algebras.
- Postnikov (2006) and others studied the combinatorics of $\text{Gr}_{k,n}^{\geq 0}$.
- Kodama, Williams (2014): A $\tau$-function $\tau = \sum_{I \in \binom{[n]}{k}} \Delta_I(V)s_\lambda(I)$ associated to $V \in \text{Gr}_{k,n}$ gives a regular soliton solution to the KP equation iff $V$ is totally nonnegative.
How close is a subspace to being totally positive?

- Can we determine \( \max_{v \in V} \text{var}(v) \) and \( \max_{v \in V \setminus \{0\}} \text{var}(v) \) from the Plücker coordinates of \( V \)?

**Theorem (Karp (2015))**

Let \( V \in \text{Gr}_{k,n} \) and \( s \geq 0 \). Then \( \text{var}(v) \leq k - 1 + s \) for all nonzero \( v \in V \) iff

\[
\text{var}((\Delta_{J \cup \{i\}}(V))_{i \in J}) \leq s
\]

for all \( J \in \binom{[n]}{k-1} \) such that the sequence above is not identically zero.

e.g. Let \( V := \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \text{Gr}_{2,4} \) and \( s := 1 \). The fact that

\( \text{var}(v) \leq 2 \) for all \( v \in V \setminus \{0\} \) is equivalent to the fact that the sequences

\[
(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (2, 1, 1), \quad (\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (1, 4, -6),
\]

\[
(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) = (2, 4, -8), \quad (\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) = (1, -8, -6)
\]
each change sign at most once.
How close is a subspace to being totally nonnegative?

**Theorem (Karp (2015))**

Let $V \in \text{Gr}_{k,n}$ and $s \geq 0$.

(i) If $\var(v) \leq k - 1 + s$ for all $v \in V$, then

$$\var((\Delta_{J \cup \{i\}}(V))_{i \notin J}) \leq s \quad \text{for all } J \in \binom{[n]}{k-1}.$$  

The converse holds if $V$ is generic (i.e. $\Delta_{I}(V) \neq 0$ for all $I$).

(ii) We can perturb $V$ into a generic $W$ with

$$\max_{v \in V} \var(v) = \max_{v \in W} \var(v).$$

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*Note:* $\var$ is *increasing* while $\overline{\var}$ is *decreasing* with respect to genericity.

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**Example:** Consider

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0.1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0.01 \\ 0 & 1 & 0.1 & 1.001 \end{bmatrix}.$$

The 4 sequences of Plücker coordinates are

$$\begin{align*}
(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) &= (1, \emptyset, 1), \\
(\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) &= (\emptyset, -1, 1), \\
(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) &= (1, -1, \emptyset), \\
(\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) &= (1, \emptyset, 1).
\end{align*}$$

*Note:* $\var$ is *increasing* while $\overline{\var}$ is *decreasing* with respect to genericity.
Oriented matroids

- An oriented matroid is a combinatorial abstraction of a real subspace, which records the Plücker coordinates up to sign, or equivalently the vectors up to sign.

- These results generalize to oriented matroids.
Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ be a linear map, and $Z_{\text{Gr}} : \text{Gr}_{k,n}^{\geq 0} \to \text{Gr}_{k,k+m}$ the map it induces on $\text{Gr}_{k,n}^{\geq 0}$. In the case that all $(k + m) \times (k + m)$ minors of $Z$ are positive, the image $Z_{\text{Gr}}(\text{Gr}_{k,n}^{\geq 0})$ is called a (tree) amplituhedron.

E.g. Let $Z := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 1 & 1 & 4 \end{bmatrix}$ and $k := 1$. Then $Z_{\text{Gr}}(\text{Gr}_{1,5}^{\geq 0})$ equals

$$\begin{cases} (1 : -2a - b + d + 2e : a, b, c, d, e \geq 0, \\ 4a + b + c + d + 4e) : a + b + c + d + e = 1 \end{cases} \subseteq \mathbb{P}^2.$$
When $k = 1$, amplituhedra are precisely cyclic polytopes. Cyclic polytopes achieve the maximum number of faces (in every dimension) in Stanley’s upper bound theorem (1975).

Lam (2015) proposed relaxing the positivity condition on $Z$, and called the more general class of images $Z_{Gr}(Gr_{k,n}^{\geq 0})$ Grassmann polytopes. When $k = 1$, Grassmann polytopes are precisely polytopes.

Arkani-Hamed and Trnka (2013) introduced amplituhedra in order to study scattering amplitudes, which they compute as an integral over the amplituhedron $Z_{Gr}(Gr_{k,n}^{\geq 0})$ when $m = 4$.

A scattering amplitude is a complex number whose modulus squared is the probability of observing a certain scattering process, e.g. a process involving $n$ gluons, $k + 2$ of negative helicity and $n - k - 2$ of positive helicity.
When is $Z_{Gr}$ well defined?

- Recall that $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ is a linear map, which induces a map $Z_{Gr} : \text{Gr}^{\geq 0}_{k,n} \to \text{Gr}_{k,k+m}$ on $\text{Gr}^{\geq 0}_{k,n}$. How do we know that $Z_{Gr}$ is well defined on $\text{Gr}^{\geq 0}_{k,n}$, i.e. $\dim(Z_{Gr}(V)) = k$ for all $V \in \text{Gr}^{\geq 0}_{k,n}$?
- Note: $\dim(Z_{Gr}(V)) = k \iff Z(v) \neq 0$ for all nonzero $v \in V$.

Lemma

$\bigcup \text{Gr}^{\geq 0}_{k,n} = \{v \in \mathbb{R}^n : \text{var}(v) \leq k - 1\}$.

- $\subseteq$ follows from Gantmakher and Krein’s theorem. $\supseteq$ is an exercise.

\[
(2, 0, 5, -1, -4, -1, 3) \in \begin{bmatrix} 2 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \in \text{Gr}^{\geq 0}_{3,7}
\]
When is $Z_{Gr}$ well defined?

**Theorem (Karp (2015))**

Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ have rank $k + m$, and $W \in Gr_{k+m,n}$ be the row span of $Z$. The following are equivalent:

(i) the map $Z_{Gr}$ is well defined, i.e. $\dim(Z_{Gr}(V)) = k$ for all $V \in Gr_{k,n}^\geq$;

(ii) $\var(v) \geq k$ for all nonzero $v \in \ker(Z) = W^\perp$; and

(iii) $\overline{\var}((\Delta_{J\setminus\{i\}}(W))_{i \in J}) \leq m$ for all $J \in \binom{[n]}{k+m+1}$ with $\dim(W_J) = k + m$.

- e.g. Let $Z := \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 3 \end{bmatrix}$, so $n = 4$, $k + m = 2$. The 4 relevant sequences of Plücker coordinates (as $J$ ranges over $\binom{[4]}{3}$) are

$(\Delta_{\{2,3\}}, \Delta_{\{1,3\}}, \Delta_{\{1,2\}}) = (-1, -3, 5)$,

$(\Delta_{\{3,4\}}, \Delta_{\{1,4\}}, \Delta_{\{1,3\}}) = (4, 5, -3)$,

$(\Delta_{\{2,4\}}, \Delta_{\{1,4\}}, \Delta_{\{1,2\}}) = (-5, 5, 5)$,

$(\Delta_{\{3,4\}}, \Delta_{\{2,4\}}, \Delta_{\{2,3\}}) = (4, -5, -1)$.

The maximum number of sign changes among these 4 sequences is 1, which is at most $2 - k$ iff $k \leq 1$. Hence $Z_{Gr}$ is well defined iff $k \leq 1$. 

When is $Z_{\text{Gr}}$ well defined?

**Theorem (Karp (2015))**

Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ have rank $k + m$, and $W \in \text{Gr}_{k+m,n}$ be the row span of $Z$. The following are equivalent:

1. The map $Z_{\text{Gr}}$ is well defined, i.e. $\dim(Z_{\text{Gr}}(V)) = k$ for all $V \in \text{Gr}_{\geq 0}^{k,n}$;
2. $\text{var}(v) \geq k$ for all nonzero $v \in \ker(Z) = W_{\perp}$; and
3. $\overline{\text{var}}\left(\left(\Delta_{J \setminus \{i\}}(W)\right)_{i \in J}\right) \leq m$ for all $J \in \binom{[n]}{k+m+1}$ with $\dim(W_J) = k+m$.

If $m = 0$, then (ii) $\iff$ (iii) is a ‘dual version’ of Gantmakher and Krein’s theorem: $V \in \text{Gr}_{k,n}$ is totally positive iff $\text{var}(v) \geq k$ for all $v \in V_{\perp} \setminus \{0\}$.

Arkani-Hamed and Trnka’s condition on $Z$ (for $Z$ to define an amplituhedron) is that its $(k+m) \times (k+m)$ minors are all positive. In this case, $Z_{\text{Gr}}$ is well defined by either (ii) or (iii).

Lam’s condition on $Z$ (for $Z$ to define a Grassmann polytope) is that $W$ has a totally positive $k$-dimensional subspace. This is sufficient by (ii).

Open problem: is Lam’s condition also necessary?
Further directions

- Is there an efficient way to test whether a given $V \in \text{Gr}_{k,n}$ is totally positive using the data of sign patterns? (For Plücker coordinates, in order to test whether $V$ is totally positive, we only need to check that some particular $k(n - k)$ Plücker coordinates are positive, not all $\binom{n}{k}$.)

- Is there a simple way to index the cell decomposition of $\text{Gr}_{k,n}^{\geq 0}$ using the data of sign patterns?

- Is there a nice stratification of the subset of the Grassmannian
  $$\{ V \in \text{Gr}_{k,n} : \text{var}(x) \leq k - 1 + s \text{ for all } x \in V \}$$
  for fixed $s$? (If $s = 0$, this is $\text{Gr}_{k,n}^{\geq 0}$.)

- I determined when $Z_{\text{Gr}}$ is well defined on the totally positive Grassmannian $\text{Gr}_{k,n}^{> 0}$. When is $Z_{\text{Gr}}$ well defined on a given cell of $\text{Gr}_{k,n}^{\geq 0}$?

Thank you!