The $m = 1$ amplituhedron and cyclic hyperplane arrangements

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Steven N. Karp, UC Berkeley
joint work with Lauren Williams

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The Grassmannian $\text{Gr}_{k,n}$

The Grassmannian $\text{Gr}_{k,n}$ is the set of $k$-dimensional subspaces $V$ of $\mathbb{R}^n$.

Given $V \in \text{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for $k$-subsets $I$ of $\{1, \ldots, n\}$ let $\Delta_I(V)$ be the $k \times k$ minor of $V$ in columns $I$. The Plücker coordinates $\Delta_I(V)$ are well defined up to a common nonzero scalar.

We say that $V \in \text{Gr}_{k,n}$ is totally nonnegative if $\Delta_I(V) \geq 0$ for all $k$-subsets $I$. The set of all totally nonnegative $V$ forms the totally nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq 0}$.
Sign variation

- For $v \in \mathbb{R}^n$, let $\text{var}(v)$ be the number of sign changes in the sequence $(v_1, v_2, \ldots, v_n)$, ignoring any zeros.

$$\text{var}(1, -4, 0, -3, 6, 0, -1) = \text{var}(1, -4, -3, 6, -1) = 3$$

Similarly, let $\overline{\text{var}}(v)$ be the maximum number of sign changes we can get if we choose a sign for each zero component of $v$.

$$\overline{\text{var}}(1, -4, 0, -3, 6, 0, -1) = 5$$

**Theorem (Gantmakher, Krein (1950))**

Let $V \in \text{Gr}_{k,n}$. The following are equivalent:

(i) $V$ is totally nonnegative;
(ii) $\text{var}(v) \leq k - 1$ for all $v \in V$;
(iii) $\overline{\text{var}}(w) \geq k$ for all $w \in V^\perp$.

- e.g. \[
\begin{bmatrix}
1 & 0 & -4 & -3 \\
0 & 1 & 3 & 2
\end{bmatrix}
\] $\in \text{Gr}_{2,4}^{\geq 0}$.

- The upper bound $k - 1$ and the lower bound $k$ are both ‘best possible’.
The cell decomposition of $\text{Gr}_{k,n}^{\geq 0}$

- $\text{Gr}_{k,n}^{\geq 0}$ has a cell decomposition. Each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero.

- $\text{Gr}_{1,3}^{\geq 0} = \mathbb{P}^2_{\geq 0} \cong \Delta_2, \Delta_3 = 0$

- $\text{Gr}_{1,3}^{\geq 0}$ is an $(n - 1)$-dimensional simplex in $\mathbb{P}^{n-1}$. So, one can think of the totally nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq 0}$ as a generalization of a simplex.
Cyclic hyperplane arrangements

- A cyclic polytope is a polytope (up to combinatorial equivalence) whose vertices line on the moment curve in $\mathbb{R}^k$

$$\begin{pmatrix} t, t^2, \ldots, t^k \end{pmatrix} \quad (t > 0).$$

- e.g. $k = 2$

- Cyclic polytopes achieve the upper bound in the upper bound theorem of McMullen and Stanley.
- A cyclic hyperplane arrangement consists of hyperplanes in $\mathbb{R}^k$ of the form

$$tx_1 + t^2x_2 + \cdots + t^kx_k + 1 = 0 \quad (t > 0).$$
**Faces of cyclic hyperplane arrangements**

- e.g.

- **Theorem (Karp, Williams)**

  Let $\mathcal{H}$ be a cyclic hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^k$.

  (i) The bounded faces of $\mathcal{H}$ are labeled precisely by those sign vectors $\sigma \in \{0, +, -\}^n$ (up to sign) with $\text{var}(\sigma) = k$.

  (ii) The unbounded faces of $\mathcal{H}$ are labeled precisely by $\sigma$ with $\text{var}(\sigma) < k$. 

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By definition, a polytope is the image of a simplex under an affine map:

![Diagram showing a simplex in $\mathbb{P}^{n-1}$ mapping to a Grassmann polytope in $\text{Gr}_{k,n}^{\geq 0}$ via a linear map $\mathbb{R}^n \to \mathbb{R}^{m+1}$, and another linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$ mapping to a Grassmann polytope in $\text{Gr}_{k,k+m}$.]

A Grassmann polytope is the image of a map $\text{Gr}_{k,n}^{\geq 0} \to \text{Gr}_{k,k+m}$ induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$. (Here $m \geq 0$ with $k + m \leq n$.)

- When the matrix $Z$ has positive maximal minors, the corresponding Grassmann polytope is called an amplituhedron, denoted $A_{n,k,m}(Z)$.
- Amplituhedra are a common generalization of cyclic polytopes ($k = 1$) and totally nonnegative Grassmannians ($k + m = n$). They were introduced by Arkani-Hamed and Trnka in their study of scattering amplitudes.
The amplituhedron

Conjecture (Arkani-Hamed, Trnka (2014))

The $m = 4$ amplituhedron $A_{n,k,4}(Z)$ is ‘triangulated’ by the images of certain $4k$-dimensional cells of $Gr_{k,n}^{\geq 0}$, coming from the BCFW recursion.

- This conjecture appears to be difficult, so we first considered $m = 1$.

Lemma

Let $W \in Gr_{k+m,n}$ denote the subspace spanned by the rows of $Z$. Then

$$A_{n,k,m}(Z) \cong B_{n,k,m}(W) := \{V^\perp \cap W : V \in Gr_{k,n}^{\geq 0}\} \subseteq Gr_{m}(W).$$

- Using results of Gantmakher and Krein, we obtain

$$B_{n,k,m}(W) \subseteq \{X \in Gr_{m}(W) : k \leq \text{var}(v) \leq k + m - 1 \text{ for all } v \in X \setminus \{0\}\}.$$

Problem

Does equality hold above?
The $m = 1$ amplituhedron

- We showed that equality does hold when $m = 1$:
  \[ \mathcal{B}_{n,k,1}(W) = \{ w \in \mathbb{P}(W) : \text{var}(w) = k \} \subseteq \mathbb{P}(W). \]

**Theorem (Karp, Williams)**

(i) $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to the complex of bounded faces of a cyclic hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^k$.

(ii) $\mathcal{A}_{n,k,1}(Z)$ is isomorphic to a subcomplex of cells of $\text{Gr}_{k,n}^{>0}$.

(iii) $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a closed ball of dimension $k$.

- Part (iii) follows directly from part (i) by a general result of Dong.

\[ \mathcal{A}_{6,2,1} \quad \mathcal{A}_{6,3,1} \]