A New Method for Computing Performance of Choked Reacting Flows and Ram-to-Scram Transition

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An improved method has been developed to compute the thrust of a dual-mode scramjet, which is an engine with a combustor that operates both subsonically and supersonically. This strategy applies to any internal flow which is predominantly one-dimensional in character. To handle the mathematical singularity at the location of thermal choking, the simple method of Shapiro is replaced with a new method that includes the solution of ordinary differential equations that model finite-rate chemistry and high-temperature gas properties. A forward shooting method is employed to find appropriate initial conditions for integration of the governing equations which result in a unique sonic condition. Solutions of the governing equations are computed using the propulsion code MASIV, which has been integrated into a hypersonic vehicle flight dynamics code and computations for both ram-mode and scram-mode operation are compared to experimental results. Benefits of a pseudo-one-dimensional approach are that the solution time is small and that physical reasons for observed engine performance can be examined. Limitations are that behavior of oblique shocks and other two-dimensional and three-dimensional phenomena in the combustor cannot be computed directly.

I. Introduction

When the engine of a hypersonic vehicle changes from a supersonic combustion mode to a subsonic combustion mode, new physics are introduced in the flowpath which can cause changes in the thrust and moment produced by the engine. It has been experimentally observed [1, 2] that wall pressures in a scram profile are dramatically different from those of a ram profile and that components can have fundamentally different performance depending on the thermodynamic and chemical characteristics of the incoming flow [2]. Although it is important to compute thrust accurately in either mode, it is perhaps more important to understand the qualitative changes associated with mode transition. It is useful to predict conditions leading to transition and flow properties on either side of a transition event in order to design stabilizing controllers that are viable over the whole range of operation of the vehicle. Vehicle design studies also depend on understanding the various operating regimes the engine may encounter. Finally, transition between modes may cause more fundamental component problems such as flame blowout or insufficient thrust.

To investigate the ram-scram transition, a new model was developed which is based on the one-dimensional analysis of Shapiro [3]. This new approach requires an internal solver for one-dimension flow equations, which the authors have previously developed [4] and which is part of a larger vehicle dynamics model. The flow solver (called Michigan-Air Force Scramjet In Vehicle code, or MASIV) is a low-order model that is designed specifically to be suitable for control design and multidisciplinary optimization (MDO). Lookup tables are used to tabulate results of high-fidelity reacting-flow calculations and create a reduced-order model of the turbulent mixing and finite-rate chemistry (a flamelet approach). Several simplifying assumptions are made to reduce the problem to the ordinary differential equations (ODEs) listed in Section IIIC. This type of low-order model is limited to quasi-one-dimensional applications and steady conditions. The fuel must be injected as a jet in cross flow and reaction rates must be computed in advance. Since the model only predicts steady-state operation, the transient process connecting ram-mode and scram-mode operation is not predicted.

A similar effort was previously described by Merker [5], using the unsteady form of the same conservation equations. Although an unsteady simulation has the advantage of allowing transients and one-dimensional waves to be considered, its principal downside is increased simulation time. The method presented in this paper requires 5 to 30 steps to converge, while the scheme presented in [5] would require at least as many time steps as spatial steps in order to converge. Thus, it would take about 500 steps to reach steady state for the full-vehicle-length simulations shown here. Since the present effort seeks short runtimes, it was deemed appropriate to use only the steady conservation equations. Frequently in continuous-flow devices, the unsteady equations are used to consider the effect of perturbations to steady-state operation, extending the model. The method presented here is compatible with such an approach.

II. Methodology

General 1-D flow solution for hypersonic flowpaths has been covered in the literature. O’Brien et al. [6] use the basic conservation equations in one dimension, adding finite-rate chemistry, and Torrez et al. [4] implement a jet-in-cross-flow turbulence and diffusion flame model. However, since these results are obtained by solving a set of Ordinary Differential Equations (ODEs) they do not admit wave-type solutions. This difficulty becomes important if we wish to consider subsonic combustion cases, since the stability of the flow relies on the propagation of information upstream from the choking location in the combustor up to end of the supersonic inflow portion of the duct. This physical process leads to the creation of a Pre-Combustion Shock Train (PCST), which in turn allows the dual-mode flowpath to operate both sub- and supersonically. The following is a brief discussion of how steady, 1-D flows can be solved in a way that allows for choking conditions to be considered.

A. Simple cases using Mach number forcing

Previous authors [3, 7] have proposed a method to solve the thermal choking problem using several limiting assumptions. Governing equations and boundary conditions were initially proposed by Shapiro, who considered the case of heat addition and friction in a single-component gas in a variable-area duct. We build upon Shapiro’s simplified analysis by adding high-temperature gas properties, gases comprising multiple species, finite-rate chemistry, and turbulent mixing of a jet in cross flow. However, let us begin with Shapiro’s analysis in order to explain the procedure.

Using elementary conservation equations, it is possible to solve for the evolution of the flow properties in any pseudo 1-D duct [4]. In order to compute the change in Mach number, Shapiro proposes a fictitious process $G$. The definition of the $G$-function is simply the forcing term on the right-hand side of the Mach number equation for 1-D flow. For example, for 1-D flow with heat addition and area change
that be indeterminate at the singular point; the numerator must go there is a singularity in the right-hand side of the equation.

In general, \( G \) represents all the terms in the right-hand side of (1), so Shapiro writes that (1) can be written more generally as

\[
\frac{1}{M^2} \frac{dM^2}{dx} = \frac{G(x)}{1 - M^2}
\]  

(2)

and so in (1), \( G \) is

\[
G = \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \left[ -2 \frac{dA}{A dx} + \frac{1 + \gamma M^2}{T_0} \frac{dT_0}{dx} \right]
\]  

(3)

Numerous authors show how (3) can be derived from the conservation equations. Here, \( M \) is Mach number, \( G \) is Shapiro’s forcing function, \( A(x) \) is the cross-sectional area distribution in the duct, \( T_0 \) is the stagnation temperature, and \( \gamma \) is the ratio of specific heats. The function \( G \) is more complex when other effects are added, such as mixtures of gases and finite-rate combustion.

To apply (2), Shapiro assumes that the mass flow rate is constant in the duct, and that the heat added per unit length and duct area profile \( A(x) \) are known. Assuming constant specific heats, the stagnation temperature is only a function of heat added, \( dQ/dx = \dot{m}_Q(T_0/dx) \), which allows (2) to be integrated directly. In this case the solution, \( M(x) \), of (2) is simply an initial-value problem. This approach is sufficient except when \( M = 1 \) somewhere in the domain, in which case there is a singularity in the right-hand side of the equation.

In order to solve the equations through the sonic point it is required that (2) be indeterminate at the singular point; the numerator must go to zero when the denominator goes to zero. This implies that \( G = 0 \) when \( M = 1 \). Using L'Hôpital’s rule

\[
\left( \frac{1}{M^2} \frac{dM^2}{dx} \right) * = \frac{[G'(x)]^*}{-(\frac{dA}{A dx})^*}
\]  

(4)

where * indicates the location where \( M = 1 \).

Now, consider a slightly rearranged version of (2),

\[
\frac{M^3}{(M - 1)^2} \frac{d(M - 1)^2}{dx} = -\frac{G(x)}{(M - 1)^2(M + 1)}
\]  

(5)

where \((M - 1)^2\) can be interpreted as the Euclidean distance between the current Mach number and the sonic point. Therefore, (5) indicates that \( G > 0 \) always forces the Mach number toward 1 and \( G < 0 \) always forces the Mach number away from 1. It is clear that in order for the flow to proceed through the sonic point, \( G \) must be initially positive, 0 when \( M = 1 \) and negative after that.

Usually, \( A(x) \) is fixed, and in general \( \frac{dQ}{dx} \) cannot be arbitrarily specified, since it represents heat added by reaction (the rates of which are functions of the flow states) or by heat transfer from the walls (the rate of which is a function of the wall temperature and the flow temperature, among other parameters). When sufficient heat is added to cause thermal choking, the downstream boundary condition is that \( M = 1 \) at some axial location where \( G = 0 \).

In the simple case of no friction and no mass addition thermal choking occurs when \( G = 0 \) in (3) due to the correct combination of heat release and wall divergence [7]:

\[
\frac{1}{A} \frac{dA}{dx} = 1 + \gamma \left( \frac{1}{T_0} \frac{dT_0}{dx} \right)
\]  

(6)

Thus, Heiser and Pratt determine the location of thermal choking for simple cases by finding the minimum of the effective area distribution, [7].

As Shapiro notes, however, since \( G(x) \) depends on \( M^2 \), \( G'(x) \) will depend on \( dM^2/dx \), which means that it is usually impossible simply to solve the equations through the sonic point. There are at least two ways around this problem. The first is to solve the governing equations backward from \( M^- \), which is a Mach number slightly below unity and from \( x^- \), which is slightly upstream of the predicted sonic location, \( x_s \). Downstream of the sonic point the equations are solved in the forward direction from \( x^+ \) and \( M^+ \), which are similarly defined. The second approach is to use a shooting method to find solutions that approach \( M = 1 \) at \( x_s \) arbitrarily close to the sonic point and to estimate \( G'(x_s) \), which is then used to step through the singularity and proceed with the forward solution. Details of a shooting method approach are explained in Section IIIB.

### B. Shooting method for reacting flows

In the present study we build upon Shapiro’s simplified analysis by replacing the Mach number influence coefficient method, (3), with a set of differential equations in terms of velocity, \( u \). The details of this method can be found in [4]. We have found that the frozen-flow Mach number, \( M = u/\sqrt{\gamma p/\rho} \), is appropriate for predicting the location of the singularity. This allows \( x_s \) to be estimated as the ODEs are solved, or (as here) a shooting method to be used for finding transonic solutions.

The conservation equations in Section IIC contain temperature-dependent gas properties, multi-component gas mixtures, finite-rate chemistry, turbulent mixing of a jet in cross flow, and wall heat transfer and viscous drag. To solve transonic equations (2) the shooting method is preferred because the heat release rate profile, \( q(x) \), is not specified a priori. Instead, it depends on the finite rate chemistry, which is a function of \( p(x) \) and \( T(x) \). An example of the shooting method is shown in Fig. 4.

In order to perform the shooting method, it was necessary to define a residual function, \( y \), which is used as a metric for the deviation between the present solution iteration and the correct solution. Values of \( y \) indicate the direction and distance from the current guess, \( M_{\text{id}} \), to the correct solution, where \( M_{\text{id}} \) is the Mach number at the entrance to the combustor. The need for such a function is evident if we consider what happens when the present guess, \( M_{\text{id}} \), is too large. In this case, the Mach number will approach unity, but \( G \) will not cross zero, so this is not a valid solution. Figure 1 shows an example of this behavior. Note further that when the solution reaches a sonic condition without having the appropriate conditions on \( G \) it is impossible to continue solution of the ODEs in the duct. This becomes clear when considering (3), since for \( G > 0 \) at \( M = 1 \), the flow will never be able to depart from the sonic condition, and solution of the ODEs will fail.

![Figure 1: An iteration (same as Fig. 4a) in which \( M = 1 \) but solution of the ODEs fails due to inappropriate conditions on \( G \).](image-url)

When the present guess, \( M_{\text{id}} \), is too small, the maximum Mach number on the interval will be less than unity, but we still require a way to estimate the distance away from the proper value of \( M_{\text{id}} \). A function is required which is defined both when the maximum Mach number in the domain is less than unity and when the solution fails due to a sonic condition occurring too early in the duct. Having testing
different options extensively, we propose the function
\[
y = \begin{cases} 
(x_{\text{choking}} - x^*_i)^2 & \text{if choking occurs} \\
(M_{\text{max}}^2 - 1 + \varepsilon)^2 & \text{otherwise}
\end{cases} \tag{7}
\]
where \(x_{\text{choking}}\) is the \(x\)-location of choking when a solution fails, \(x^*_i\) is the best estimate at iteration \(i\) of the sonic location, \(M_{\text{max}}\) is the maximum Mach number achieved in a fully subsonic solution, and \(\varepsilon > 0\) is some small number. The convergence tolerance of the iterative scheme (below, section G) and the error tolerance of the ODE integrator must both be smaller that \(\varepsilon\) so that the error, \(y\), can be driven to 0. The use of \(\varepsilon\) prevents the ODEs from having to be evaluated at the singular point, \(M = 1\).

The sonic location can be found using \(G\). As \(M \rightarrow 1\), \(x(G = 0) \rightarrow x_s\). Thus, if the location of choking is driven toward \(x_s\) (the point at which transonic solutions are allowed), the point at which \(M = 1\) will be driven to the point at which \(G = 0\) by selection of the appropriate upstream boundary condition, \(M_{\text{sa}}\).

C. Thermal choking of a reacting flow

If heat is added to a flow such that \(dT_0/dx\) is known \textit{a priori} then the function \(G\) is known from (3). However, in a reacting flow the gas composition changes and \(G\) cannot be computed ahead of time. The principle is the same and \(G\) can be defined in terms of primitive variables:
\[
G = \left(1 - \frac{\rho W}{\rho u}\right) \left(\frac{2}{\gamma} \frac{dU}{d\rho} - \frac{1}{\rho} \frac{dP}{d\rho} \right) \tag{8}
\]

The equations used in MASIV are [4]:
\[
\begin{align*}
\frac{1}{\rho} \frac{d\rho}{dx} &= \frac{d\rho}{dW} \frac{dW}{dx} + \frac{dP}{dW} \frac{dW}{dx} - \frac{1}{\rho dW} \frac{dW}{dx} \tag{9} \\
\frac{1}{\rho} \frac{d\rho}{dx} &= \frac{d\rho}{dm} \frac{dm}{dx} + \frac{dP}{dm} \frac{dm}{dx} - \frac{1}{\rho dm} \frac{dm}{dx} \tag{10} \\
\frac{dY_i}{dx} &= \frac{1}{\rho u} \frac{dP}{d\rho} \frac{d\rho}{dx} + \frac{1}{\rho u} \frac{dC_f}{dS_w} \frac{dS_w}{dx} \tag{11} \\
\frac{1}{\rho u} \frac{d\rho}{dx} &= \frac{1}{\rho u} \frac{dC_f}{dS_w} \frac{dS_w}{dx} + \frac{1}{\rho u} \frac{dC_f}{dS_w} \frac{dS_w}{dx} - \frac{1}{\rho u} \frac{dC_f}{dS_w} \frac{dS_w}{dx} \tag{12} \\
&= \frac{1}{\rho m} \frac{dC_f}{dS_w} \frac{dS_w}{dx} + \frac{1}{\rho m} \frac{dC_f}{dS_w} \frac{dS_w}{dx} + \frac{1}{\rho m} \frac{dC_f}{dS_w} \frac{dS_w}{dx} \tag{13}
\end{align*}
\]

These \(4 + n\) equations \((i = 1, \ldots, n\) where \(n\) is the number of species\) represent conservation of mass, momentum, and energy, as well as conservation of mass of each species. Chemical reactions are included, which act as source terms. Conservation of atoms must be maintained by the selected reaction mechanism. In MASIV the reaction rates are computed by a flamelet model and jet mixing model [4].

To compute \(G\), \(dC_f/dS_w\) and \(dY_i/dx\) are either found by (12), (10), and (9), respectively, or can be estimated using finite differences after the ODEs have been solved. We calculate \(G\) using finite differences. A second-order scheme was used to compute \(G(p, u, p, \gamma)\) and \(dG/dx\). The location of the zero crossing was then estimated by interpolation,
\[
x_s^* = x^* - \frac{G^{-}}{dG/dx} \tag{15}
\]
where again the point just to the upstream side of the sonic point is denoted by \(-\). Figure 2 shows the estimation of the location where a sonic condition is allowed based on the location of the zero-crossing of the \(G\)-function. The apparent discontinuity in \(G\) is caused by a discrete change in the rate of change of cross-sectional area \(dA/dx\) due to a constant-area section connecting to a diverging section.

Note that although there are two locations where \(G\) crosses zero, at only one of them does \(G\) change from positive to negative. This location, \(x_s\), is then identified as the current best estimate of the location where choking is allowed, and the other location, where \(G\) goes from negative to positive, is discarded.

D. Pre-combustion shock train

Section II B provides a rapid way to compute the thermal choking location and the correct value of \(M_{sa}\) that allow transonic solutions in 1-D ducts. A process must still be identified that begins at a supersonic upstream condition, \(M_2\), and delivers condition \(M_{sa}\) to the subsonic portion of the duct. The other state variables, \(\rho_{sa}, p_{sa},\) and \(u_{sa}\), must also be computed as upstream boundary conditions for the ODE solver.

A canonical scramjet is presented in Fig. 3, featuring a constant area isolator, fuel injection location, and diverging combustor. Flame stabilization components such as cavities or steps are omitted. Station 2 is at the supersonic outflow of the inlet; station 3 is at the beginning of the combustor; station 4 is at the fuel injectors; station 5 is at the beginning of the divergence in the combustor, and station 6 is at the beginning of the external nozzle.

Heiser and Pratt [7] write that the momentum of the flow will be carried by a core region with an area smaller than the physical area of the duct, while the rest of the duct area is filled with a boundary layer and separated flow having very little momentum. Conservation of momentum states that the total impulse is conserved,
\[
I_{3a} = I_{\text{sep}} + I_{\text{core}} = I_2 = [p_{sa}(A_2 - A_{sa}) + 0] + (p_{sa}A_{3a} + \tilde{m}u_{sa}) \tag{16}
\]

which, when combined with the assumption that the isolator is adia-
batic, yields [7]

\[ \frac{p_{3a}}{p_2} = 1 + \gamma M_2^2 = \gamma M_2 M_{3a} \left[ \frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_{3a}^2} \right] \]  

(17)

Here \( A_{3a} \) is the area of the core flow at station \( \bar{c} \). The conditions at station \( \bar{c} \) (where the flow is supersonic) are computed separately by an inlet code [8] and \( M_{sa} \) is computed by the analysis of section II B, so (17) predicts the pressure at station \( \bar{c} \). The temperature and velocity at station \( \bar{c} \) are found by assuming that the stagnation temperature and gas properties do not vary in the isolator. Thus, the full state can be found at station \( \bar{c} \).

Heiser and Pratt also show how the core area may be computed based on conservation of mass and momentum, assuming an area deficit due to the separated region:

\[ \frac{A_{3a}}{A_2} = \frac{1}{\gamma M_{3a}^2} \left[ \frac{1}{p_{3a}/p_2} (1 + \gamma M_2^2) - 1 \right] \]  

(18)

So the thickness of the separated flow region can be calculated as a function of the Mach number, \( M_{sa} \), at the end of the isolator. Note that (16), (17), (18) are valid at all points in the PCST.

Since we know that \( A_{3a} < A_2 \) at all points in the PCST, (18) gives constraints on \( M_{sa} \). As the area ratio goes to unity, the PCST solution tends to a supersonic case (nothing happens) or to a normal-shock solution (maximum entropy generation). The actual solution is expected to be both subsonic and to generate less entropy than a normal shock, so

\[ \sqrt{\frac{1 + \frac{\gamma - 1}{2} M_2^2}{\gamma M_{3a}^2}} < M_{3a} < 1 \]  

(19)

In order to account for the thickness of the separated region downstream of the PCST, a simple interpolation was used. The core area was assumed to be \( A_{3a} \) at \( x = x_{sa} \) and \( A_{sa} = A_2 \) at \( x = x_{sb} \), which is simply an approximation that boundary layers become very thin near the fuel injector.

E. PCST length scaling

In order to use (17) to find the distribution of \( p(x)/p_2 \) in the isolator, a distribution of Mach number, \( M(x) \), is required in the PCST. Ikui [9] proposes a method for estimating the pressure rise as a function of \( x \) in the isolator. We used this method, but replaced Ikui’s “fully mixed” boundary condition with the requirement that \( \frac{d^2 u}{dx^2} = 0 \) at the end of the PCST, implying that the PCST must reach equilibrium. Ikui uses the Crocco number,

\[ w^* = \frac{\gamma - 1}{1 + \frac{\gamma - 1}{2} M_2^2} \]  

(20)

along with the following differential equation, derived from conservation of mass, assuming that only the core flow carries any mass [9].

\[ \frac{d^2 w}{dx^2} - c^2 w = 0 \]  

(21)

\[ w(0) = w_2, \, w(\ell) = w_3, \, \frac{dw(\ell)}{dx} = 0 \]  

(22)

The solution of this differential equation is

\[ w(x) = w_3 \cosh \{ b(\ell - x) \} \]  

(23)

\[ \frac{p(x) - p_2}{p_3 - p_2} = \frac{w_2 - w(x)}{w_2 - w_3} \]  

(24)

where \( w(x) \) is the Crocco number between station \( \bar{c} \) and station \( \bar{c} \) and \( p(x) \) is the static pressure in the isolator between \( p_2 \) and \( p_3 \). The ratio of specific heats \( \gamma \) is assumed not to vary in the isolator. The estimated length of the PCST, \( \ell \), is determined by using Ikui’s experimental correlation, \( \ell = \frac{1}{c} \ln(w_2/w_3) \), where \( c \) is experimentally determined. Ikui reports \( c = 0.114 \), which is the value used here.

Note that (24) differs slightly from Ikui’s result because of the assumption of different boundary conditions above. This also means that the constant changes to

\[ b = \frac{c}{\ln(w_2/w_3)} \]  

(25)

F. Expansion shock at \( M = 1 \)

Once the location of the sonic point has been found and the associated strength of PCST has been computed, the last task is to compute conditions on either side of the sonic point. Using the shooting method, the conditions just upstream of the sonic point are known to an acceptable degree of accuracy. Let us call the state at this point \([\rho^+, p^+, T^+, u^+]^T\).

In order to compute a step that proceeds through the singularity using the method described by Shapiro, it is required that a step be taken precisely at \( x_s \), and that \( G^+ \) be used at this location to compute \( dM^2/dx \). The authors attempted Shapiro’s approach using (9) to (14), but found it to be needlessly complicated for several reasons. It requires placing points very close to the singularity to obtain accurate values for \( G^+ \), requires a custom Runge-Kutta solver that allows placing a point at \( x_s \), and uses a different governing equation at this point, (4) instead of (3). Another undesirable property is that the value obtained for \( G^+ \) depends on the exact placement of \( x_s \), since in any discretized implementation \( x_s \) is a finite distance from \( x_s \) and not a true limit. The most damaging requirement, however, is that of expressing (9) through (14) in a way that is consistent with (4). The equations involved are significantly more complex than those of the non-reacting case, but solving them exactly does not provide any real advantage since they are only to be used to obtain a supersonic condition.

A much simpler approach is to jump across the singularity using the shock-jump relations. Although this process, often known as an “expansion shock” destroys entropy, the error introduced can be rendered arbitrarily small by making \( x^+ \) sufficiently close to \( x_s \). In our experience, good accuracy of the scheme is maintained using relatively coarse tolerances, \( \epsilon \) from (7).

Once the error, \( \gamma \), from section II B has been reduced to an acceptable level, the subsonic integration is discontinued at \( x_s \). Then, the shock-jump equation is applied using this subsonic condition and a new supersonic condition, \([\rho^+, p^+, T^+, u^+]^T\), is obtained. Finally, since this supersonic condition only applies downstream of \( x_s \), a new starting point is chosen, \( x_s \), slightly downstream of \( x_s \), and the integration is re-started.

G. Iterative procedure

This section describes the practical implementation of the procedure outlined in II A to F. In the shooting method outlined above, one simply searches for an initial condition, \( M_{3a} \), that causes the Mach number to equal unity precisely when \( G = 0 \). The solution that meets this requirement is the allowed ram solution for the given boundary conditions at the beginning of the isolator (which must be supersonic). A detailed procedure for implementing this method follows.

1. The inlet solution (from the SAMURI code, [8]) provides a supersonic upstream boundary condition for the isolator, \((p_2, p_3, T_2, u_2, M_3)\).

2. Assume that the flow will remain supersonic throughout the isolator and combustor. Attempt to compute a scram solution. If solution succeeds, then the engine operates in scram mode. No pre-combustion shock train is required; stop.

3. If the sonic point was approached (from above) at any \( x \)-location then this flow does not have a stable supersonic solution. Determine the ram-mode solution.

4. Assume that there is a PCST in the isolator which sets conditions at the beginning of the combustor. The PCST is of unknown strength. Bracket the condition which allows a transonic solution with \( M_{3a}^{(1)} \) set by (19) and \( M_{3a}^{(2)} = 1 - 10 \epsilon \). We have used \( \epsilon = 0.005 \) with success.
5. Perform a search for the root of the error metric, \( y \), as a function of \( M_{3a} \). Terminate when \( |(M_{3a})^{i+1} - (M_{3a})^i|/(M_{3a})^i < \varepsilon/2 \).
Let $x_s = x^*_s$, which is the best prediction of the location of choking.

6. Estimate the new state at $x^*_s$ by using the shock-jump relations for $M$, $p$, and $T$ at $x^*_s$. When the initial Mach number in the jump equation is less than unity, entropy is destroyed. However, the error incurred in this step can be made as small as required by setting tolerance $\epsilon$.

7. Re-start combustor solution at $x^*_s$, with the supersonic condition as computed above.

Figure 4 shows a typical ram-mode solution, showing only distinctive iterations. First, the solution is bracketed from below (step 4 in the iterative procedure) by guessing a value of $M_{3a}^{(1)} \approx 0.46$, which is the circled value shown in Fig. 4a. The solution is also bracketed from above in iteration 2 (not shown). Iterations 4 to 20 correspond to step 5 of the iterative procedure above. By iteration 4, the $x$-location of the sonic point can be predicted accurately, but the maximum Mach number is still less than unity, as shown in Fig. 4b. By iteration 15, as shown in Fig. 4e, $M_{3a}$ is well-bounded from above, since the maximum Mach number approaches 1. However, as iteration 16 demonstrates, there is still uncertainty in the predicted choking location (Fig. 4d). By iteration 20, the uncertainties in the maximum Mach number and the location of choking have both been reduced to an acceptable level (Fig. 4e), and choking occurs near the sonic location as predicted using the $G$-function. Figure 4f shows the final solution after completing steps 6 and 7 of the iterative procedure. The final value of $M_3$ is 0.8906, and $x_s = 17.17$ m. The actual search procedure used is a combination of bisection, Newton’s method and Mueller methods. Best results were obtained after significant tuning of the convergence parameters involved.

### III. Validation

The cases shown in Figs. 5 and 6 correspond to the geometry and run conditions of Fotaia and Driscoll [10]. This experiment is a laboratory-scale combustor with flow rates in the range of 200 g/s to 350 g/s of vitiated air. In order to compare accurately to the experiment, a chemistry set representing vitiated air at the required temperature and pressure was used. The solid lines represent computations made by the method proposed here, and the circles represent experimentally measured data. The uncertainty in the measured pressures is about 0.9 kPa [10]. The nominal run conditions are shown in Table 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>$\rho$</td>
<td>0.28 $\frac{kg}{m^3}$</td>
</tr>
<tr>
<td>$p$</td>
<td>46.7 kPa</td>
</tr>
<tr>
<td>$T$</td>
<td>521 K</td>
</tr>
<tr>
<td>$u$</td>
<td>977 $\frac{m}{s}$</td>
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<td>$M$</td>
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<td>$\gamma$</td>
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<td>$\phi$</td>
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<td>$Y_{O_2}$</td>
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</tr>
<tr>
<td>$Y_{O_3}$</td>
<td>0.22</td>
</tr>
<tr>
<td>$Y_{H_2O}$</td>
<td>0.13</td>
</tr>
<tr>
<td>$Y_{Ar}$</td>
<td>0.01</td>
</tr>
</tbody>
</table>

The fuel in the experiment and simulation was hydrogen, and although the experiment used both jet and cavity fueling, the simulation only considers the main jet fuel source since the cavity fueling was only used to ignite the flow at the beginning of the run. Equivalence ratios ranged from 0.198 to 0.313, which spans the ram-scram transition, and provides a resolution of about 0.01 in equivalence ratio. This corresponds to fuel mass addition rates of 1.6 g/s to 2.1 g/s.

Boundary layers internal to the duct were accounted for by using the initial value of $\delta^*$ as reported in the experiment. Although this treatment is relatively simple, it provides reasonable agreement in terms of qualitative prediction of pressure rise in the combustor as a function of equivalence ratio.

Figures 5a–e show the computed flow profiles in which the combustor operated in ram mode. Note that as the equivalence ratio decreases, the pressure rise in the combustor also decreases. The location of choking can be estimated in the experimental results by the presence of a rapid drop in pressure just before $x/H = 5$ in the ram-mode plots.

The constants of the MASIV model [4] are shown in Table 2. The constants were optimized to match the pressure rise of the $\phi = 0.313$ (highest ram equivalence ratio) case and the $\phi = 0.215$ (highest scram equivalence ratio) case. This provided relatively good agreement in all cases where the routine converged (convergence was not achieved for $\phi = 0.234$). Note that this set of parameters provides a good prediction of the maximum pressure rise in the isolator (just before the fuel injector) for most cases and good agreement of the location of choking for all cases. The maximum pressure rise prediction is within the experimental error for cases 5b, 5c, 6a, and 6b, within 2 $\sigma$ the experimental error for cases 5a and 5d, and within 3 $\sigma$ the error for case 5e. Note as well that the length of the pressure rise, based on the equation of Ikui [9] is fairly accurate for all cases except case 5b. The constants used here differ slightly from those used previously [4].

<table>
<thead>
<tr>
<th>Constant</th>
<th>Experimental Range</th>
<th>MASIV value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>1.2 to 2.6 $[11]$</td>
<td>2.0</td>
</tr>
<tr>
<td>$c_2$</td>
<td>0.28 to 0.34 $[11]$</td>
<td>0.7</td>
</tr>
<tr>
<td>$c_3$</td>
<td>0.68-0.95 $[12]$</td>
<td>1.3</td>
</tr>
<tr>
<td>$c_4$</td>
<td>0.76 $[11]$</td>
<td>0.86</td>
</tr>
<tr>
<td>$c_5$</td>
<td>0.0084-0.0093 $[12]$</td>
<td>0.009</td>
</tr>
</tbody>
</table>

The small peak in pressure in each of the simulation results around $x/D = 0$ is due to the simplistic assumption that the boundary layer goes from its very thick value, computed using (18) at the end of the PCST to an assumed value of zero near the injector. Also, the artificial division of the combustor into components (isolator and combustor) means that the PCST is not allowed to overlap with the fuel injection, which adds a small displacement in the upstream direction to the pressure-rise predictions. In the experiment it is observed that the end of the PCST sometimes, but not always, overlaps the injector. The prediction of the location of maximum pressure rise is not very good. It is known that 2-D and 3-D effects influence the location of maximum pressure rise [10], but the results shown here indicate that a 1-D model is sufficient to predict the pressure distribution values except for a displacement in the $x$-direction.

Figures 6a–b show the series of cases in which the combustor operated in scram mode. Note that as the equivalence ratio decreases, the pressure rise in the combustor also decreases, although less dramatically than in ram mode. These plots can be used directly to assess the heat release model, since in a fully supersonic flow there is no PCST to affect the modeling of pressure rise and all pressure rise is due to heat addition or geometric divergence. The agreement between computations and experiment in both plots here is good. The main discrepancy is due to the effect of facility shocks in the experiment, present in both plots, between $(x - x_{inj})/H = -10$ and $(x - x_{inj})/H = 0$ (apparent as oscillatory behavior).

### IV. Results

The cases shown in Figure 7 represent two different cases having the same flight condition but different amounts of fueling. The flight condition represents a case on the boundary of ram- and scram-mode operation. The flight Mach number is $M_{inj} = 4.18$ and the altitude is $h = 25$ km. The equivalence ratio was 0.215 for the supersonic combustion case and 0.267 for the subsonic combustion case; this value is large enough to cause choking in the combustor. These values correspond to the test conditions shown in Fig. 6a and Fig. 5e, respectively, so they represent fueling conditions on either side of ram-scram transition. The flowpath geometry was the same as the geometry of the val-
Figure 5. Pressure profiles through the isolator and combustor for a range of equivalence ratios corresponding to ram-mode operation. Distance from the injector, \( x - x_{\text{inj}} \) is normalized by the duct height, \( H \).

Figure 5 shows the pressure profiles through the isolator and combustor for a range of equivalence ratios corresponding to ram-mode operation. The pressure profiles are displayed for various equivalence ratios: (a) \( \phi = 0.313 \), (b) \( \phi = 0.301 \), (c) \( \phi = 0.290 \), (d) \( \phi = 0.278 \), and (e) \( \phi = 0.267 \).

The validation laboratory experiment shown in Figs. 5 and 6 and the flowpath was integrated into an X-43-like vehicle geometry. The full flowpath geometry is shown at the top of Fig. 7. Thrust was 1.7 kN in the ram case and -0.24 kN in the scram case.
Figure 6. Pressure rise through the combustor for a range of equivalence ratios corresponding to scram-mode operation. Distance from the injector, \( x - x_{\text{inj}} \) is normalize by the duct height, \( H \).

Figure 7. Computed ram-mode (black) and scram-mode (gray) flow properties at the same flight condition, \( M_\infty = 4.2 \), altitude \( h = 25 \text{ km} \), \( \alpha = 3.6^\circ \). For ram mode, the equivalence ratio is \( \phi = 0.267 \) and for scram mode, \( \phi = 0.215 \). Note that these equivalence ratios correspond to the equivalence ratios in Fig. 5e and 6a.
This example shows that vehicle thrust can drop abruptly during a ram-scram transition. The negative scram mode thrust is not desirable and it indicates that the inlet geometry used for this example is not appropriate for the flight condition. Specifically, the angle of attack is too high (which creates too much drag) and the Mach number is too low (which does not cause enough compression in the inlet) to allow scram-mode operation with positive thrust.

Extending the analysis to a range of flight conditions, Fig. 8 shows that the thrust in ram mode is greater than the thrust in the scram mode for a range of conditions. Most significant on this plot is the difference in slope, $\Delta \phi / \Delta M$, between ram-mode and scram-mode operation. The discrete jump in thrust and the difference in slope of thrust with respect to flight Mach number on either side presents a serious control issue. Finally, Fig. 9 shows the equivalence ratio at which choking is predicted to occur for a range of Mach numbers, from the lowest Mach number at which ram operation was possible, the highest Mach number at which enough heat could be added to choke the flow. This figure indicates that the allowed margin of $\phi$ before choking occurs is mostly a function of the Mach number at the entrance to the isolator, itself a function of the flight Mach number and the inlet geometry. It is impossible to compare Figs. 8 and 9 to experiment since, to the authors' knowledge, no data are available for dual-mode scramjets in flight.

V. Conclusions

A method has been presented for solving steady-state reacting internal flow problems through a singularity in Mach number without time-stepping. The principal advantage of this method is that it converges in about 5 to 30 steps, whereas time-stepping an unsteady set of partial differential equations to steady state can require many more iterations.

Validation was performed by comparing computations to results from an experimental, laboratory-scale combustor. Agreement was good, in general. We conclude that most of the behavior of the device can be captured using a one-dimensional approximation and that engineering performance metrics, such as pressure evolution and location of thermal choking point, can be predicted by such a model. Effects which are two- or three-dimensional in nature cannot be predicted by this model, as some of the comparisons indicated. However, agreement in pressure evolution suggests that the model can be used to predict thrust, and agreement in choking location suggests that the model can be used to predict whether the engine will operate in ram mode or in scram mode. Thrust comparisons cannot be made here because the test section is not installed on a thrust stand. Other experiments have thrust information, but lack detailed pressure profiles as a function of equivalence ratio.

Finally, the model is used to predict performance of a similar flow-path for conditions which cannot at present be tested experimentally. The model predicts that ram-mode operation will generate more thrust than scram mode under certain conditions (low $M$, high angle-of-attack), although this is not necessarily a general result. It also predicts that the thrust produced will undergo a discrete change when a mode change occurs, which agrees with the discrete change in pressure profile observed experimentally. The derivative of thrust with respect to Mach number, $\Delta \phi / \Delta M$, is also different on different sides of the transition point. This combination of factors indicates a potential problem area for vehicle control, as successful control strategies must either avoid mode transition at an inconvenient flight condition or handle the discrete change in thrust and derivatives of thrust that occur across a transition event.

**References**


