ON POSTNIKOV COMPLETENESS FOR REPLETE TOPOI

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Abstract. We show that the hypercomplete ∞-topos associated with any replete topos is Postnikov complete, positively answering a question of Bhatt and Scholze. Along the way, we construct Milnor sequences for sheaves of spaces on replete topoi. As a corollary, we generalize a result of Toën on affine stacks.

Contents

1. Introduction

Postnikov towers or Postnikov systems are a widely used construction in algebraic topology, introduced by Postnikov [Pos51]. For example, they can be used to compute certain homotopy groups of spheres. The technique of Postnikov towers is used outside the context of homotopy theory of spaces as well, such as in the study of spectra. The goal of our paper is to answer a question of Bhatt–Scholze [BS15] regarding Postnikov towers in the context of ∞-topoi (see [Lur09, Ch. 6]).

From the perspective of algebraic geometry, one may think of an ∞-topos as a higher categorical generalization of the notion of a topos introduced by Grothendieck. For instance, the notion of ∞-topoi captures "sheaves of ∞-groupoids on a Grothendieck site" as an example. As demonstrated in [Lur09, Ch. 6], ∞-topoi can also be viewed as alternate universes for doing homotopy theory: one can talk about homotopy groups, Postnikov towers, Eilenberg–MacLane spaces etc. in this generality.

Roughly speaking, for an object $X \in \mathcal{X}$ of an ∞-topos $\mathcal{X}$, the technique of Postnikov towers allows one to construct a sequence of truncations $\tau_{\leq n}X$ for every $n \geq 0$, which can be studied inductively via the natural maps $\tau_{\leq n}X \to \tau_{\leq n-1}X$ between them. In order to effectively extract information about X from this sequence of objects, one needs to know that they converge in a precise sense (see Definition 3.5). While Postnikov towers always converge in the ∞-category of spaces, they need not converge in a general ∞-topos. The question of when an ∞-topos is Postnikov complete has been investigated by many authors; we mention the work of Jardine [Jar87, Lem. 3.4] (see also [Toë06, Prop. 1.2.2]) and Lurie [Lur09, Prop. 7.2.1.10] which gives a criterion for Postnikov completeness under certain finiteness hypothesis (see Remark 3.18 for more details). In [MV99], Morel–Voevodsky constructed an example showing that Postnikov towers may not converge even if the ∞-topos is hypercomplete (see Remark 3.16 and Example 3.17).

In their seminal work [BS15], Bhatt–Scholze introduce the pro-étale site of a scheme, which simplifies many foundational constructions in the theory of ℓ-adic cohomology. One of the key properties of the associated topos is captured by the notion of replete topoi, which they study in [BS15, § 3]. We recall this definition here.

Definition 1.1 ([BS15, Def. 3.1.1])**.** A Grothendieck topos $\mathcal{X}$ is *replete* if for every diagram $F\colon \mathbf{Z}_{\geq 0}^{\mathrm{op}} \to \mathcal{X}$ with the property that $F_{n+1} \to F_n$ is surjective for all n, the natural map $\varprojlim F \to F_n$ is surjective for every n.

Considering sheaves of spaces on a Grothendieck topos $\mathcal{X}$ ([Lur09, § 6.5]), one obtains an ∞-topos, which we denote by $\mathrm{Shv}_\infty(\mathcal{X})$. Let us denote the associated hypercomplete ∞-topos by $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$. One may think of objects of $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$ as sheaves of spaces that satisfy hyperdescent. In [BS15, Qn. 3.1.12], the authors ask the following question.

Question 1.2. Let $\mathcal{X}$ be a replete topos. Are Postnikov towers convergent in the hypercomplete ∞-topos $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$?

One important feature of the above question is that it does not impose any finiteness assumptions (such as homotopy dimension $\leq n$ [Lur09, Def. 7.2.1.1] or cohomological dimension $\leq n$ [Lur09, Cor. 7.2.2.30]) on the ∞-topoi that appear in the previously known criteria for convergence of Postnikov towers (see Remark 3.18 and Remark 3.20).

Certain stable analogs of Question 1.2 can already be found in the literature. In [BS15, Prop. 3.3.3], the authors show that if $\mathcal{X}$ is a replete topos, the derived ∞-category $D(\mathcal{X}, \mathbf{Z})$ (or in other words, hypercomplete sheaves of $H\mathbf{Z}$-module spectra) is left complete. In [Mat21, Prop. A.10], the case of sheaves of spectra on certain large sites (which satisfy an additional "quasi-compactness" condition and admit countable filtered limits) is addressed and used in the context of the arc-topology. Regarding Question 1.2 itself, in [BS15, Prop. 3.2.3], the authors show that Postnikov towers converge in $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$, when $\mathcal{X}$ is additionally assumed to be locally weakly contractible ([BS15, Def. 3.2.1]). The pro-étale topos of a scheme is an example of a locally weakly contractible topos ([BS15, Prop. 4.2.8]). In this paper, we answer Question 1.2 in general.

Theorem A. *Let $\mathcal{X}$ be a replete topos. Then the hypercomplete ∞-topos $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$ is Postnikov complete.*

Remark 1.3. In Theorem A, we prove that $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$ is Postnikov complete in the sense of Definition 3.5; this is stronger than just requiring that the natural map $X \to \varprojlim_n \tau_{\leq n} X$ is an equivalence.

Remark 1.4. Let $\mathcal{X}$ be a replete topos. One might wonder whether $\mathrm{Shv}_\infty(\mathcal{X})$ is automatically hypercomplete. In Example 3.29, we show that this is false even if one assumes $\mathcal{X}$ to be locally weakly contractible. This also implies that $\mathrm{Shv}_\infty(\mathcal{X})$ is not Postnikov complete in general (see Remark 3.16).

Example 1.5. Many topoi naturally appearing in algebraic geometry are replete. Theorem A is applicable in all such contexts, without having to make any assumptions on the finiteness of homotopy or cohomological dimension. For example (after fixing set theoretic issues, see e.g. the footnote in [BS15, Ex. 3.1.7]), topoi coming from the fpqc topology, v-topology ([SP22, Tag 0EVM]), and the quasisyntomic topology ([BMS19, Def. 4.10]) are all replete.

Example 1.6 ([BS15, § 4.3])**.** Let G be a profinite group. Let $(BG)_{\text{proét}}$ denote the site of profinite sets with a continuous G-action, with covers given by continuous surjections. Then $\mathrm{Shv}((BG)_{\text{proét}})$ is replete. As a special case of Theorem A, $\mathrm{Shv}_\infty((BG)_{\text{proét}})^\wedge$ is Postnikov complete (*cf.* Example 3.17).

Let us briefly explain the main ingredients in the proof of Theorem A. Let $\mathcal{X}$ be a replete Grothendieck topos. A crucial technical role in our proof is played by the Milnor sequences in $\mathrm{Shv}_\infty(\mathcal{X})$, which we develop in Proposition 3.22. The main stumbling point in their construction is the classical fact that sheafification does not commute with infinite limits in general (see [Toë06, p. 21] and Remark 3.23). In our paper, we overcome this difficulty by working with a class of presheaves that we call *multiplicative presheaves* (see Definition 2.5): in Proposition 2.13 (also see Proposition 2.12), we show that if $\mathcal{T}$ is a Grothendieck site such that $\mathrm{Shv}(\mathcal{T})$ is replete, then sheafification preserves $\mathbf{Z}_{\geq 0}^{\mathrm{op}}$-indexed limits when restricted to the category of multiplicative presheaves. It turns out that the latter is enough to construct Milnor sequences because the presheaves arising from homotopy theory in this context are all multiplicative presheaves (see Proposition 3.21).

After explaining the technical ingredients, let us now give a brief sketch of how they enter into the proof of Theorem A. Let $X \in \mathrm{Shv}_\infty(\mathcal{X})^\wedge$. One of the main steps in the proof is to show that the natural map $p\colon X \to \varprojlim \tau_{\leq n} X$ is ∞-connective (Definition 3.12). In Lemma 3.24, for $0 \leq n \leq \infty$, we give a useful criterion for a map to be n-connective in terms of certain "slicing families of points" (see the second condition in Lemma 3.24) when the ∞-topos arises as sheaves of spaces on a Grothendieck site. Since repleteness is preserved under passage to slice topoi, one can now use Lemma 3.24 along with the Milnor sequences for replete topoi constructed in Proposition 3.22 to compute the homotopy groups of $\varprojlim \tau_{\leq n} X$ based at each of these "slicing families of points" and deduce that p is ∞-connective; note that checking the surjectivity of $\pi_0(X) \to \pi_0(\varprojlim \tau_{\leq n} X)$ here requires a little more care. Since X is hypercomplete, the ∞-connective map $p\colon X \to \varprojlim \tau_{\leq n} X$ must then be an equivalence. As pointed out in Remark 1.3, one needs additional arguments to prove Theorem A, for which we again appeal to similar tools.

Remark 1.7. Let X be a scheme. As a consequence of our proof of Theorem A, we see that the ∞-category of hypercomplete sheaves of spaces on $X_{\text{proét}}$ is Postnikov complete without having to find enough weakly contractible objects. Our proof therefore circumvents the reliance on certain results from algebra [BS15, § 2, Prop. 4.2.8] in order to obtain the latter statement.

To exhibit another consequence of Theorem A, let us mention the following application to affine stacks in the sense of [Toë06, § 2.2].

Corollary 1.8 (Corollary 3.28)**.** *Let F be an affine stack over* $\operatorname{Spec} B$ *for any ring B. Then the natural map $F \to \varprojlim_n \tau_{\leq n} F$ is an equivalence.*

Previously, the above corollary was only known under certain assumptions (Remark 3.27) and the proof relied on vanishing of cohomology groups of affine schemes with coefficients in unipotent group schemes in degrees > 1. As we see now, a more general result follows from Theorem A simply as a consequence of repleteness. The Milnor sequences for replete topoi that appears in this paper will also be used in [MR22].

Acknowledgments. We thank Bhargav Bhatt, Peter Haine, Akhil Mathew, and Peter Scholze for helpful comments and conversations. We gratefully acknowledge funding through the Max Planck Institute for Mathematics in Bonn, Germany, during the preparation of this work. The first named author additionally acknowledges support from the University of Michigan, the NSF Grant DMS #1801689 and FRG #1952399.

2. Multiplicative presheaves on Grothendieck sites

In this section, we introduce a key class of objects for our paper, which we call "multiplicative presheaves." Roughly speaking, they capture the notion of a presheaf that takes arbitrary coproducts to products. However, a site may fail to have arbitrary coproducts. To make the notion precise, we will therefore need to phrase this condition in the appropriate category. Before formally introducing multiplicative presheaves, let us fix some notation.

Notation 2.1. Let $\mathcal{T}$ be a Grothendieck site. The Yoneda embedding gives a natural fully faithful functor

$$h\colon \mathcal{T} \to \mathrm{PShv}(\mathcal{T}).$$

Composition with the sheafification functor yields a functor $h^\sharp\colon \mathcal{T} \to \mathrm{Shv}(\mathcal{T})$. If there is no risk of confusion, we write $h_X := h(X)$ and $h^\sharp_X := h^\sharp(X)$ for $X \in \mathcal{T}$. We may view a presheaf P on $\mathcal{T}$ as a functor $P\colon \mathcal{T} \to \mathrm{Set}^{\mathrm{op}}$. By left Kan extension along $h^\sharp$, we obtain an extended functor $\mathrm{Lan}_{h^\sharp}(P)\colon \mathrm{Shv}(\mathcal{T}) \to \mathrm{Set}^{\mathrm{op}}$.

Lemma 2.2. *Let $\mathcal{T}$ be a Grothendieck site and $X \in \mathcal{T}$. Then for all $F \in \mathrm{Shv}(\mathcal{T})$, we have $(\mathrm{Lan}_{h^\sharp}(h_X))(F) = \mathrm{Hom}_{\mathrm{Shv}(\mathcal{T})}(F, h^\sharp_X)$; that is, $\mathrm{Lan}_{h^\sharp}(h_X)$ is represented by $h^\sharp_X$.*

Proof. Follows from the universal property of left Kan extension. For example, see [Mac71, Exercise X.3.2], which states that the left Kan extension of a (co)representable functor must be (co)represented by the image of the former (co)representing object. □

The following two lemmas are well known and follow from the universal property of left Kan extensions and the adjoint functor theorem.

Lemma 2.3. *Let $\mathcal{T}$ be a Grothendieck site. If a functor $\overline{P}\colon \mathrm{Shv}(\mathcal{T}) \to \mathrm{Set}^{\mathrm{op}}$ preserves all small colimits, then $\overline{P} \circ h^\sharp\colon \mathcal{T} \to \mathrm{Set}^{\mathrm{op}}$ is a sheaf. Further, there is a natural isomorphism $\mathrm{Lan}_{h^\sharp}(\overline{P} \circ h^\sharp) \simeq \overline{P}$.*

Lemma 2.4. *Let $\mathcal{T}$ be a Grothendieck site. If $P\colon \mathcal{T} \to \mathrm{Set}^{\mathrm{op}}$ is a sheaf, then the functor $\mathrm{Lan}_{h^\sharp}(P)\colon \mathrm{Shv}(\mathcal{T}) \to \mathrm{Set}^{\mathrm{op}}$ preserves all small colimits. Further, there is a natural isomorphism $\mathrm{Lan}_{h^\sharp}(P) \circ h^\sharp \simeq P$.*

Now we are ready to introduce the notion of multiplicative presheaves.

Definition 2.5 (Multiplicative presheaves)**.** Let $\mathcal{T}$ be a Grothendieck site. Let P be a presheaf on $\mathcal{T}$. We say that P is a *multiplicative presheaf* if for every set of objects $\{X_j\}_{j\in J} \in \mathcal{T}$ for an indexing set J, the natural map of presheaves (induced by sheafification)

$$\coprod_{j\in J} h_{X_j} \to \Big(\coprod_{j\in J} h_{X_j}\Big)^\sharp$$

induces an isomorphism

$$\mathrm{Hom}_{\mathrm{PShv}(\mathcal{T})}\left(\left(\coprod_{j\in J} h_{X_j}\right)^{\sharp}, P\right) \to \mathrm{Hom}_{\mathrm{PShv}(\mathcal{T})}\left(\coprod_{j\in J} h_{X_j}, P\right) \simeq \prod_{j\in J} P(X_j).$$

Remark 2.6. It follows directly from the definition that limits of multiplicative presheaves are again multiplicative.

Example 2.7. By the universal property of sheafification, every sheaf is an example of a multiplicative presheaf.

The following proposition provides a useful source of multiplicative presheaves.

Proposition 2.8. *Let $\mathcal{T}$ be a Grothendieck site. Let $P\colon \mathcal{T} \to \mathrm{Set}^{\mathrm{op}}$ be a presheaf such that P is naturally isomorphic to $\overline{P} \circ h^{\sharp}$ for some coproduct preserving functor $\overline{P}\colon \mathrm{Shv}(\mathcal{T}) \to \mathrm{Set}^{\mathrm{op}}$. Then P is a multiplicative presheaf.*

Proof. Let $\varphi\colon P \simeq \overline{P} \circ h^{\sharp}$. Note that we have a natural bijection

$$\mathrm{Hom}\left(\left(\coprod_{j\in J} h_{X_j}\right)^{\sharp}, P\right) \xrightarrow[\simeq]{\varphi} \mathrm{Hom}\left(\left(\coprod_{j\in J} h_{X_j}\right)^{\sharp}, \overline{P} \circ h^{\sharp}\right).$$

Further, by the universal property of left Kan extensions, we have a natural bijection

$$\mathrm{Hom}\left(\left(\coprod_{j\in J} h_{X_j}\right)^{\sharp}, \overline{P} \circ h^{\sharp}\right) \xrightarrow{\simeq} \mathrm{Hom}\left(\mathrm{Lan}_{h^{\sharp}}\left(\left(\coprod_{j\in J} h_{X_j}\right)^{\sharp}\right), \overline{P}\right).$$

However, Lemma 2.3 implies that $\mathrm{Lan}_{h^{\sharp}}((\coprod_{j\in J} h_{X_j})^{\sharp})$ is represented by $(\coprod_{j\in J} h_{X_j})^{\sharp}$. This shows that

$$\mathrm{Hom}\left(\mathrm{Lan}_{h^{\sharp}}\left(\left(\coprod_{j\in J} h_{X_j}\right)^{\sharp}\right), \overline{P}\right) \simeq \overline{P}\left(\left(\coprod_{j\in J} h_{X_j}\right)^{\sharp}\right) \simeq \prod_{j\in J} (\overline{P} \circ h^{\sharp})(X_j) \xrightarrow[\simeq]{\varphi^{-1}} \prod_{j\in J} P(X_j),$$

where the middle isomorphism follows from the facts that $\overline{P}$ preserves coproducts and sheafification preserves all colimits. Therefore, the natural map $\mathrm{Hom}\Big((\coprod_{j\in J} h_{X_j})^{\sharp}, P\Big) \to \prod_{j\in J} P(X_j)$ is an isomorphism, which shows that P is multiplicative, as desired. □

Example 2.9. Let $\mathcal{T}$ be a Grothendieck site and let F be a sheaf of abelian groups on $\mathcal{T}$. For any integer $n \geq 0$, the functor that sends an object $c \in \mathrm{Ob}(\mathcal{T})$ to the group $H^n(c, F)$ is a multiplicative presheaf on $\mathcal{T}$. If $n \geq 1$, the sheafification of the multiplicative presheaf $c \mapsto H^n(c, F)$ is zero.

Example 2.10. Let $\mathcal{T}$ be the Grothendieck site of schemes with the v-topology. Let $\mathcal{O}$ denote the structure presheaf on $\mathcal{T}$. We claim that $\mathcal{O}$ is not multiplicative.

Indeed, for any scheme X, the closed immersion of the reduced subscheme $X_{\mathrm{red}} \hookrightarrow X$ is a cover in the v-topology. Since the associated morphism of representable presheaves becomes an effective epimorphism after sheafification,

$$h^{\sharp}_{X_{\mathrm{red}} \times_X X_{\mathrm{red}}} \rightrightarrows h^{\sharp}_{X_{\mathrm{red}}} \to h^{\sharp}_X$$

is a coequalizer diagram of v-sheaves. However, as $X_{\mathrm{red}} \hookrightarrow X$ is a closed immersion, $X_{\mathrm{red}} \times_X X_{\mathrm{red}} \simeq X_{\mathrm{red}}$ and the two projections $X_{\mathrm{red}} \times_X X_{\mathrm{red}} \to X_{\mathrm{red}}$ are identified with the identity. Thus, $h^{\sharp}_{X_{\mathrm{red}}} \xrightarrow{\sim} h^{\sharp}_X$ is an isomorphism for all X. In particular, for a multiplicative presheaf P the natural map $P(X) \to P(X_{\mathrm{red}})$ must be an isomorphism. On the other hand, when $X = \operatorname{Spec} A$ for a nonreduced ring A, we have $\mathcal{O}(X) = A \neq A_{\mathrm{red}} = \mathcal{O}(X_{\mathrm{red}})$, so that $\mathcal{O}$ cannot be a multiplicative presheaf.

Remark 2.11. Let us denote the category of multiplicative presheaves by $\mathrm{PShv}_{\mathrm{mult}}(\mathcal{T})$. There is a natural inclusion functor $\mathrm{PShv}_{\mathrm{mult}}(\mathcal{T}) \to \mathrm{PShv}(\mathcal{T})$. As noted in Remark 2.6, the category $\mathrm{PShv}_{\mathrm{mult}}(\mathcal{T})$ has all small limits, and they are preserved by the inclusion functor $i_1\colon \mathrm{PShv}_{\mathrm{mult}}(\mathcal{T}) \to \mathrm{PShv}(\mathcal{T})$. Similarly, the inclusion functor $i_2\colon \mathrm{Shv}(\mathcal{T}) \to \mathrm{PShv}_{\mathrm{mult}}(\mathcal{T})$ also preserves all small limits. Therefore, by the adjoint functor theorem, the sheafification functor $\mathrm{PShv}(\mathcal{T}) \to \mathrm{Shv}(\mathcal{T})$ can be expressed as a composition of the left adjoints $i_1^{\mathcal{L}}\colon \mathrm{PShv}(\mathcal{T}) \to \mathrm{PShv}_{\mathrm{mult}}(\mathcal{T})$ and $i_2^{\mathcal{L}}\colon \mathrm{PShv}_{\mathrm{mult}}(\mathcal{T}) \to \mathrm{Shv}(\mathcal{T})$. Our main observation, which is recorded in Proposition 2.12, is that when $\mathrm{Shv}(\mathcal{T})$ is replete, the functor $i_2^{\mathcal{L}}$ enjoys good formal properties.

Proposition 2.12. *Let $\mathcal{T}$ be a Grothendieck site such that the associated topos* $\mathrm{Shv}(\mathcal{T})$ *is replete. Then sheafification commutes with countable direct products of multiplicative presheaves on $\mathcal{T}$.*

Proof. Let $P_i \in \mathrm{PShv}(\mathcal{T})$, $i \in \mathbf{Z}_{>0}$, be a family of multiplicative presheaves. Let $\eta\colon \prod P_i \to (\prod P_i)^\sharp$ denote the sheafification map. We show that the natural morphism of sheaves $\theta\colon (\prod P_i)^\sharp \to \prod P_i^\sharp$, which is induced by the maps $\eta_i\colon P_i \to P_i^\sharp$ and the universal property of sheafification is injective and surjective [SP22, Tag 00WN].

We begin with injectivity. Let $X \in \mathcal{T}$ and $s, t \in (\prod P_i)^\sharp(X)$ with $\theta_X(s) = \theta_X(t)$. By the definition of sheafification, we can find a covering $\{X_{j_0} \to X\}_{j_0 \in \mathcal{L}^0}$ and $s_{j_0}, t_{j_0} \in (\prod P_i)(X_{j_0})$ whose images under the map $\eta_{X_{j_0}}\colon (\prod P_i)(X_{j_0}) \to (\prod P_i)^\sharp(X_{j_0})$ are given by $s|_{X_{j_0}}$ and $t|_{X_{j_0}}$, respectively, i.e., $\eta_{X_{j_0}}(s_{j_0}) = s|_{X_{j_0}}, \eta_{X_{j_0}}(t_{j_0}) = t|_{X_{j_0}}$. By the product decomposition $\theta \circ \eta = \prod \eta_i$, we have $\eta_i(s_{j_0}^{(i)}) = \eta_i(t_{j_0}^{(i)})$ for the i-th components $s_{j_0}^{(i)}$ and $t_{j_0}^{(i)}$ and all $i \in \mathbf{Z}_{>0}$. Thus, for each $j_0 \in \mathcal{L}$ the definition of sheafification produces again a covering

$$\{X_{j_0 j_1} \to X_{j_0}\}_{j_1 \in \mathcal{L}_{j_0}}$$

such that $s_{j_0 j_1}^{(1)} = t_{j_0 j_1}^{(1)}$, where $s_{j_0 j_1} := s_{j_0}|_{X_{j_0 j_1}}$ and $t_{j_0 j_1} := t_{j_0}|_{X_{j_0 j_1}}$. Here $\mathcal{L}_{j_0}$ is an index set, which depends on $j_0 \in \mathcal{L}^0$.

Inductively, for each $n \in \mathbf{Z}_{>0}$, we can obtain families of coverings

$$\{X_{j_0 \cdots j_{n-1} j_n} \to X_{j_0 \cdots j_{n-1}}\}_{j_n \in \mathcal{L}_{j_0 \cdots j_{n-1}}}$$

such that $s_{j_0 \cdots j_n} := s_{j_0 \cdots j_{n-1}}|_{X_{j_0 \cdots j_n}}$ and $t_{j_0 \cdots j_n} := t_{j_0 \cdots j_{n-1}}|_{X_{j_0 \cdots j_n}}$ satisfy $s_{j_0 \cdots j_n}^{(i)} = t_{j_0 \cdots j_n}^{(i)}$ for all $1 \leq i \leq n$. The index set $\mathcal{L}_{j_0 \cdots j_{n-1}}$ depends on the choice of $j_0 \in \mathcal{L}, j_1 \in \mathcal{L}_{j_0}, \cdots, j_{n-1} \in \mathcal{L}_{j_0, \cdots, j_{n-2}}$. Let us denote the set of all such n-tuples $(j_0, \cdots, j_{n-1})$ by the set $\mathcal{L}^{n-1}$. Then there are natural maps $\gamma_n\colon \mathcal{L}^n \to \mathcal{L}^{n-1}$ for all $n \geq 1$ such that fiber over $(j_0, \cdots, j_{n-1})$ identifies with $\mathcal{L}_{j_0 \cdots j_{n-1}}$.

Let us translate the above construction in terms of morphisms. The sections $s, t \in (\prod P_i)^\sharp(X)$ correspond to morphisms of presheaves still denoted as $s, t\colon h_X \to (\prod P_i)^\sharp$ with $\theta \circ s = \theta \circ t$. Similarly, for each $n \in \mathbf{Z}_{\geq 0}$, and every $(j_0, \cdots j_n) \in \mathcal{L}^n$, we obtain maps

$$s_{j_0 \cdots j_n}, t_{j_0 \cdots j_n}\colon h_{X_{j_0 \cdots j_n}} \to \prod P_i$$

which agree after composing with the projection map $\prod P_i \to \prod_{1\leq j\leq n} P_j$. This implies that we have maps

$$\overline{s}(n), \overline{t}(n) \colon \coprod_{(j_0,\cdots,j_n)\in\mathcal{L}^n} h_{X_{j_0,\cdots,j_n}} \to \prod P_i$$

which agree after composing with the map $\prod P_i \to \prod_{1\leq j\leq n} P_j$. Since each P_i is a *multiplicative presheaf*, the above maps factor uniquely to give maps

$$s(n), t(n) \colon \left(\coprod_{(j_0,\cdots,j_n)\in\mathcal{L}^n} h_{X_{j_0,\cdots,j_n}}\right)^\sharp \to \prod P_i$$

which agree after composing with the projection map $\prod P_i \to \prod_{1\leq j\leq n} P_j$.

The maps $\gamma_n \colon \mathcal{L}^n \to \mathcal{L}^{n-1}$ now induce maps $\overline{\gamma}_n \colon \left(\coprod_{j\in\mathcal{L}^n} h_{X_j}\right)^\sharp \to \left(\coprod_{j\in\mathcal{L}^{n-1}} h_{X_j}\right)^\sharp$ (compatible with $s(n), s(n-1)$ and $t(n), t(n-1)$) which are epimorphisms by [SP22, Tag 00WT] and stability of epimorphisms under coproducts. These maps define a prosystem $\Gamma \colon \mathbf{Z}_{\geq 0} \to \mathrm{Shv}(\mathcal{T})$ where all the transition maps are surjective. Taking the inverse limit, we obtain two maps

$$s(\infty), t(\infty) \colon \varprojlim \Gamma \to \prod P_i,$$

which are now equal. The maps $s, t \colon h_X \to (\prod P_i)^\sharp$ factor uniquely to give maps $\overline{s}, \overline{t} \colon h_X^\sharp \to (\prod P_i)^\sharp$. We also obtain a commutative diagram

$$\begin{array}{ccc} \varprojlim \Gamma & \xrightarrow{\pi} & h_X^\sharp \\ \downarrow{\scriptstyle s(\infty)} & & \downarrow{\scriptstyle \overline{s}} \\ \prod P_i & \xrightarrow{\eta} & (\prod P_i)^\sharp \end{array}$$

which implies $\eta \circ s(\infty) = \overline{s} \circ \pi$. Similarly, we obtain $\eta \circ t(\infty) = \overline{t} \circ \pi$. Since $\mathrm{Shv}(\mathcal{T})$ is replete, it follows that π is an epimorphism. On the other hand, since $s(\infty) = t(\infty)$, it follows that $\overline{s} \circ \pi = \overline{t} \circ \pi$. This implies $\overline{s} = \overline{t}$, which implies $s = t$. This shows injectivity, as desired.

Now we proceed in a similar fashion to prove the surjectivity of θ. Let $X \in \mathcal{T}$ and $s \in \prod P_i^\sharp(X)$. The description of the sheafification $P_1^\sharp$ yields a covering $\{X_{j_1} \to X\}_{j_1\in\mathcal{L}}$ and $s_{j_1} \in (P_1 \times \prod_{i>1} P_i^\sharp)(X_{j_1})$ whose images under $\eta_1 \times \mathrm{id}$ equal $s|_{X_{j_1}}$. Continuing recursively, for each $n \in \mathbf{Z}_{>0}$, we may find families of coverings $\{X_{j_1\cdots j_n} \to X_{j_0\cdots j_{n-1}}\}_{j_n\in\mathcal{L}_{j_0\cdots j_{n-1}}}$ and compatible sections $s_{j_1\cdots j_n} \in (\prod_{1\leq j\leq n} P_j \times \prod_{i>n} P_i^\sharp)(X_{j_0\cdots j_n})$ whose images under $\prod_{1\leq j\leq n} \eta_j \times \mathrm{id}$ equal $s_{j_1\cdots j_{n-1}}|_{X_{j_1\cdots j_n}}$. As before, we can define the sets $\mathcal{L}^n$ equipped with natural maps $\gamma_n \colon \mathcal{L}^n \to \mathcal{L}^{n-1}$ such that $\mathcal{L}^1 := \mathcal{L}$ and $\mathcal{L}^n$ is the set of all n-tuples $(j_1, \cdots, j_n)$ such that $j_1 \in \mathcal{L}, j_2 \in \mathcal{L}_{j_1}, \cdots, j_n \in \mathcal{L}_{j_1\cdots j_{n-1}}$.

Let us now translate the above construction in terms of morphisms. For every $n \geq 1$ and $(j_1, \cdots, j_n) \in \mathcal{L}^n$, the sections $s_{j_1\cdots j_n}$ defines morphisms of presheaves

$$\overline{s}(n) \colon \coprod_{(j_1,\cdots,j_n)\in\mathcal{L}^n} h_{X_{j_1\cdots j_n}} \to \prod_{1\leq j\leq n} P_j \times \prod_{i>n} P_i^\sharp.$$

Since each P_i is a multiplicative presheaf, the above maps factor uniquely as

$$s(n)\colon \Big(\coprod_{(j_1,\cdots,j_n)\in\mathcal{L}^n} h_{X_{j_1\cdots j_n}}\Big)^\sharp \to \prod_{1\le j\le n} P_j \times \prod_{i>n} P_i^\sharp.$$

The map of sets $\gamma_n\colon \mathcal{L}^n \to \mathcal{L}^{n-1}$ induce natural maps

$$\overline{\gamma}_n\colon \Big(\coprod_{j\in\mathcal{L}^n} h_{X_j}\Big)^\sharp \to \Big(\coprod_{j\in\mathcal{L}^{n-1}} h_{X_j}\Big)^\sharp,$$

which are epimorphisms by construction. Further, by construction, we have a commutative diagram

$$\begin{array}{ccc}
\left(\coprod_{j\in\mathcal{L}^n} h_{X_j}\right)^\sharp & \xrightarrow[s(n)]{} & \prod_{1\le j\le n} P_j \times \prod_{i>n} P_i^\sharp \\
\Big\downarrow{\scriptstyle \overline{\gamma}_n} & & {\scriptstyle \mathrm{id}\times\eta_n\times\mathrm{id}}\Big\downarrow \\
\left(\coprod_{j\in\mathcal{L}^{n-1}} h_{X_j}\right)^\sharp & \xrightarrow{s(n-1)} & \prod_{1\le j\le n-1} P_j \times \prod_{i>n-1} P_i^\sharp
\end{array}$$

The maps $\overline{\gamma}_n$ define a pro-system $\Gamma\colon \mathbf{Z}_{>0} \to \mathrm{Shv}(\mathcal{T})$ where all transition maps are surjective. Passing to the inverse limit we obtain a map $s(\infty)\colon \varprojlim \Gamma \to \prod P_j$ which fits in a commutative diagram

$$\begin{array}{ccc}
\varprojlim \Gamma & \xrightarrow{\pi} & h_X^\sharp \\
\Big\downarrow{\scriptstyle s(\infty)} & & {\scriptstyle s}\Big\downarrow \\
\prod P_j & \xrightarrow{\eta} & \prod P_j^\sharp
\end{array}$$

Since $\mathrm{Shv}(\mathcal{T})$ is replete, it follows that π is an effective epimorphism. Applying sheafification, we obtain a diagram

$$\begin{array}{ccc}
\varprojlim \Gamma & \xrightarrow{\pi} & h_X^\sharp \\
\Big\downarrow{\scriptstyle \underline{s}(\infty)} & & {\scriptstyle s}\Big\downarrow \\
(\prod P_j)^\sharp & \xrightarrow{\theta} & \prod P_j^\sharp
\end{array}$$

Since π is surjective, by possibly refining X, one can lift the tautological section of $h_X^\sharp(X)$ to $\varprojlim \Gamma$. This implies that by possibly refining X, one can lift s along θ. This shows the surjectivity of θ and finishes the proof. □

As a consequence, we obtain similar results for sheafifications of sequential limits.

Proposition 2.13. *Let $\mathcal{T}$ be a Grothendieck site such that the associated topos* $\mathrm{Shv}(\mathcal{T})$ *is replete. Let $(P_n)_{n\in\mathbf{N}}$ be an inverse system of multiplicative presheaves of sets on $\mathcal{T}$. Then the natural map $(\varprojlim_n P_n)^\sharp \to \varprojlim_n P_n^\sharp$ is an isomorphism.*

Proof. The assertion follows formally since sheafification preserves finite limits and by Proposition 2.12, sheafification of multiplicative presheaves preserves countable products. □

Construction 2.14. We will construct a version of the $\varprojlim^1_n$ functor in the generality of group objects in a topos $\mathcal{X}$. More precisely, we construct a functor

$$\varprojlim_n{}^1\colon \operatorname{Fun}(\mathbf{N}^{\mathrm{op}}, \operatorname{Grp}(\mathcal{X})) \to \mathcal{X}_*,$$

where the latter denote the category of pointed objects on $\mathcal{X}$. To this end, let $(P_n)_{n\in\mathbf{N}}$ be an inverse system of group objects in a topos $\mathcal{X}$. Let $f_{n+1}\colon P_{n+1} \to P_n$ denote the transition map. Let G denote the group $\prod_n P_n$ and Z denote the object of $\mathcal{X}$ underlying G. Let $\alpha_n\colon G \to P_n$ and $\beta_n\colon Z \to P_n$ denote the projection maps. Let us denote the composite map

$$G \times Z \xrightarrow{\mathrm{pr}_2} Z \xrightarrow{\alpha_{n+1}} P_{n+1} \xrightarrow{f_{n+1}} P_n \to P_n$$

by ψ_n. Here, the last map $P_n \to P_n$ is the inverse operation on the group object P_n. Let us consider the maps

$$G \times Z \xrightarrow{\alpha_n \times \beta_n \times \psi_n} P_n \times P_n \times P_n \to P_n,$$

where the last map is induced by the (associative) multplication operation on P_n. Let us denote the composition of the above maps by $\theta_n\colon G \times Z \to P_n$. Using the maps θ_n for all n, we obtain a morphism

$$\theta\colon G \times Z \to Z,$$

which defines an action of the group object G on Z. This gives a diagram in $\mathcal{X}$ of the form

$$G \times Z \underset{\mathrm{pr}_2}{\overset{\theta}{\rightrightarrows}} Z. \tag{2.1}$$

We define $\varprojlim^1_n P_n$ to be the coequalizer of the above diagram. Note that Z is naturally a pointed object of $\mathcal{X}$ (via the unit morphisms $* \to P_n$ for all n); therefore $\varprojlim^1_n P_n$ is also naturally pointed and is an object of $\mathcal{X}_*$.

Remark 2.15. Construction 2.14 above extends the classical construction for inverse systems of groups (see [BK72, § IX.2]) to the generality of group objects in topoi. When $\mathcal{X}$ is replete and the P_n are abelian, it recovers the usual construction of $\varprojlim^1_n$ as the first derived functor of $\varprojlim_n$. Indeed, in this case, $\varprojlim^1_n P_n$ is the cokernel of the map

$$(\alpha_n - f_{n+1} \circ \alpha_{n+1})_n\colon \prod_n P_n \to \prod_n P_n$$

(*cf.* the proof of [BS15, Prop. 3.1.11]), which is the coequalizer of (2.1) in the category of abelian groups. But θ and pr_2 admit the common section $Z \simeq \{0\} \times Z \hookrightarrow G \times Z$, so the coequalizer is reflexive. Since sheaves in abelian groups are algebras over the "free abelian sheaf" monad on sheaves of sets and this monads preserves reflexive coequalizers, this is also the coequalizer in the category of sets; *cf.* [Lin69, Prop. 3]. Note that this argument does not apply to arbitrary sheaves of groups P_n because in general θ is not a map of groups, so it does not make sense to talk about the coequalizer of (2.1) in the category of groups.

Remark 2.16. We point out that even though the inclusion functor from the category of sheaves to the category of presheaves on a site preserves limits, it does not preserve $\varprojlim^1_n$.

Proposition 2.17. *Let $\mathcal{T}$ be a Grothendieck site such that the associated tops* $\mathrm{Shv}(\mathcal{T})$ *is replete. Let $(P_n)_{n\in\mathbf{N}}$ be an inverse system of multiplicative presheaves of groups on $\mathcal{T}$. Then the natural map $(\varprojlim^1_n P_n)^\sharp \to \varprojlim^1_n P_n^\sharp$ is an isomorphism.*

Proof. The assertion follows formally from Construction 2.14, the fact that sheafification preserves small colimits (in particular, all coequalizers) and the fact that sheafification of multiplicative presheaves preserves countable products (Proposition 2.12). □

3. Applications

In this section, our goal is to prove our main result Theorem A. We begin by summarizing some of the necessary background on the homotopy theory of ∞-topoi (following [Lur09, § 6.5]) that we will need and fix some notation along the way.

3.1. **Preliminaries on ∞-topoi.** Let $\mathcal{X}$ be an ∞-topos in the sense of [Lur09, Def. 6.1.0.4]. For the theory of infinity topoi we refer to [Lur09]; here we will discuss some examples and constructions that will be useful to us in Section 3.2.

Example 3.1. The ∞-category $\mathcal{S}$ of spaces (also known as "anima") is an ∞-topos.

Example 3.2. Let $\mathcal{T}$ be a small ∞-category. Then the ∞-category $\mathrm{PShv}_\infty(\mathcal{T}) := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ is an ∞-topos, called the *∞-category of $\mathcal{S}$-valued presheaves or presheaves of spaces on $\mathcal{T}$.*

Example 3.3. Let $\mathcal{T}$ be a Grothendieck site. In this context, one can define the notion of $\mathcal{S}$-valued sheaves or sheaves of spaces on $\mathcal{T}$ denoted as $\mathrm{Shv}_\infty(\mathcal{T})$. There is a natural inclusion functor $\mathrm{Shv}_\infty(\mathcal{T}) \hookrightarrow \mathrm{PShv}_\infty(\mathcal{T})$. The ∞-category $\mathrm{Shv}_\infty(\mathcal{T})$ is example of an ∞-topos. The inclusion $\mathrm{Shv}_\infty(\mathcal{T}) \hookrightarrow \mathrm{PShv}_\infty(\mathcal{T})$ admits a left adjoint, given by the (left exact) *sheafification* functor $\mathrm{PShv}_\infty(\mathcal{T}) \to \mathrm{Shv}_\infty(\mathcal{T})$.

If $\mathcal{X}$ is an ∞-topos, then so is the slice category $\mathcal{X}_{/X}$ for any $X \in \mathcal{X}$ [Lur09, Lem. 6.3.5.1]. An ∞-topos $\mathcal{X}$ is a presentable ∞-category [Lur09, Prop. 5.5.4.15] and thus admits all small limits and colimits [Lur09, Cor. 5.5.2.4, Def. 5.5.0.1]. In particular, it admits a final object $* \in \mathcal{X}$. Moreover, for any presentable ∞-category $\mathcal{C}$, there is a notion of n-truncated objects and n-truncation functors $\tau_{\leq n}$ [Lur09, Prop. 5.5.6.18]. If $\mathcal{X}$ is an ∞-topos, then the subcategory of discrete objects $\tau_{\leq 0}(\mathcal{X})$ is an ordinary Grothendieck topos ([Lur09, Rem. 6.4.1.3, Thm. 6.4.1.5]). For example, if $\mathcal{X} = \mathrm{Shv}_\infty(\mathcal{T})$ for some Grothendieck site $\mathcal{T}$, then $\tau_{\leq 0}(\mathcal{X}) \simeq \mathrm{Shv}(\mathcal{T})$.

The notion of truncated objects allows one to formulate the notion of Postnikov towers in any presentable ∞-category $\mathcal{C}$. The following definitions are taken from [Lur09, Def. 5.5.6.23]; we include them here for convenience of the reader. Let $\mathbf{Z}_{\geq 0}^{\infty}$ be the linearly ordered set $\mathbf{Z}_{\geq 0} \cup \{\infty\}$ with ∞ as the largest element. The nerve $N(\mathbf{Z}_{\geq 0}^{\infty})$ of $\mathbf{Z}_{\geq 0}^{\infty}$ is an ∞-category.

Definition 3.4. A *Postnikov tower* in $\mathcal{C}$ is a functor $X\colon N(\mathbf{Z}_{\geq 0}^{\infty})^{\mathrm{op}} \to \mathcal{C}$ such that for each $n \geq 0$, the map $X(\infty) \to X(n)$ exhibits $X(n)$ as an n-truncation of $X(\infty)$. A *Postnikov pretower* is a functor $X\colon N(\mathbf{Z}_{\geq 0})^{\mathrm{op}} \to \mathcal{C}$ such that for each $n \geq 0$, the map $X(n+1) \to X(n)$ exhibits $X(n)$ as an n-truncation of $X(n+1)$.

Let $\mathrm{Post}^+(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(N(\mathbf{Z}_{\geq 0}^{\infty})^{\mathrm{op}}, \mathcal{C})$ spanned by the Postnikov towers. Let $\mathrm{Post}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(N(\mathbf{Z}_{\geq 0})^{\mathrm{op}}, \mathcal{C})$ spanned by the Postnikov pretowers.

Definition 3.5. We say that $\mathcal{C}$ is *Postnikov complete* if the forgetful functor $\mathrm{Post}^+(\mathcal{C}) \to \mathrm{Post}(\mathcal{C})$ is an equivalence of ∞-categories.

Remark 3.6. In [Lur09, Prop. 5.5.6.23], Lurie uses the phrase "Postnikov towers in $\mathcal{C}$ are convergent" instead of "$\mathcal{C}$ is Postnikov complete." We follow the more recent convention of [Lur18, Def. A.7.2.1] (*cf.* [Lur09, Rem. 5.5.6.23]).

The following lemma gives a useful restatement of the above definition, which we will use later.

Lemma 3.7 ([Lur09, Prop. 5.5.6.26])**.** *Let $\mathcal{C}$ be a presentable ∞-category. Then $\mathcal{C}$ is Postnikov complete if and only if, for every functor $X\colon N(\mathbf{Z}^{\infty}_{\geq 0})^{\mathrm{op}} \to \mathcal{C}$, the following conditions are equivalent:*

- *(a) The diagram X is a Postnikov tower.*
- *(b) The diagram X is a limit in $\mathcal{C}$ and the restriction $X|_{N(\mathbf{Z}_{\geq 0})^{\mathrm{op}}}\colon N(\mathbf{Z}_{\geq 0})^{\mathrm{op}} \to \mathcal{C}$ is a Postnikov pretower.*

It is a classical result that the ∞-topos $\mathcal{S}$ is Postnikov complete. However, this need not be true for a general ∞-topos. A related property, which holds for the ∞-topos $\mathcal{S}$, but not for a general ∞-topos, is the notion of hypercompleteness. We refer to [Lur09, § 6.5] for the notion of hypercompleteness which will be briefly recalled below.

Construction 3.8 ([Lur09, § 6.5.1])**.** Let $\mathcal{X}$ be an ∞-topos and $X \in \mathcal{X}$ be an object. Let S^n denote the n-sphere and let $* \in S^n$ be a fixed base point. Evaluation at the base point and [Lur09, Rem. 5.5.2.6] induces a map $s\colon X^{S^n} \to X$, which we identify with an object of $\mathcal{X}_{/X}$. One can define

$$\pi_n(X \to *) := \tau_{\leq 0} s \in \mathcal{X}_{/X}.$$

A similar definition applies to the relative context: for a map $f\colon X \to Y$ in $\mathcal{X}$ (viewed as an object of $\mathcal{X}_{/Y}$), one can use the construction from the previous paragraph to define

$$\pi_n(f) \in (\mathcal{X}_{/Y})_{/f} \simeq \mathcal{X}_{/X}.$$

Remark 3.9. The projection $S^n \to *$ makes $\pi_n(X \to *)$ from Construction 3.8 naturally into a pointed object of $\tau_{\leq 0}(\mathcal{X}_{/X})$, which is a group object for $n > 0$ and a commutative group object for $n > 1$. See the discussion after [Lur09, Lem. 6.5.1.2].

Remark 3.10. In the context of Construction 3.8, for a base point $x\colon * \to X$, pullback of $\pi_n(X \to *)$ yields a (discrete) pointed object of $\mathcal{X}$, which we denote as $\pi_n(X, x)$ (or $\pi_n(X, *)$, when the data of the point is clear from the context). In the special case $\mathcal{X} = \mathcal{S}$, this recovers the usual n-th homotopy group of the pointed space (X, x) ([Lur09, Rem. 6.5.1.6]). For $n = 0$, we define $\pi_0(X) := \tau_{\leq 0} X \in \mathcal{X}$. Then $\pi_0(X \to *) \simeq (X \times \pi_0(X) \xrightarrow{\mathrm{pr}_1} X)$ in the category $\mathcal{X}_{/X}$; *cf.* [Lur09, Rem. 6.5.1.3].

Remark 3.11. Let $\mathcal{T}$ be a Grothendieck site and $\mathcal{X} := \mathrm{Shv}_\infty(\mathcal{T})$. For an object $X \in \mathcal{X}$ equipped with a base point $x\colon * \to X$, one can also describe the (group) object $\pi_n(X, x)$ for $n > 0$ in the following alternative way: we can think of X as a pointed sheaf of spaces on $\mathcal{T}$. We can then naturally obtain a presheaf of sets denoted as $\pi_n^{\mathrm{pre}}(X, x)$ which sends an object $c \in \mathcal{T}$ to $\pi_n(X(c), *)$. For $n = 0$, one can similarly define a presheaf $\pi_0^{\mathrm{pre}}(X)$ via $\mathcal{T} \ni c \mapsto \pi_0(X(c))$. By construction and [Lur09, 5.5.6.28], it follows that $\pi_n(X, x)$ (resp. $\pi_0(X)$) as defined in Remark 3.10 is simply the sheafification of $\pi_n^{\mathrm{pre}}(X, x)$ (resp. $\pi_0^{\mathrm{pre}}(X)$).

For a version for unpointed objects, see also [Lur09, Rem. 6.5.1.4] (applied to the geometric morphism $\mathrm{Shv}_\infty(\mathcal{T}) \hookrightarrow \mathrm{PShv}_\infty(\mathcal{T})$ and the canonical morphism $X \to *$).

Definition 3.12 ([Lur09, Def. 6.5.1.10])**.** A morphism $f\colon X \to Y$ in an ∞-topos $\mathcal{X}$ is ∞-connective if it is an effective epimorphism and $\pi_k(f) = *$ for $k \geq 0$. For $0 \leq n < \infty$, a morphism $f\colon X \to Y$ in $\mathcal{X}$ is said to be n-connective if it is an effective epimorphism and $\pi_k(f) = *$ for $0 \leq k < n$.

Lemma 3.13. *Let $\mathcal{X}$ be an ∞-topos. Let $f\colon X \to Y$ be a map in $\mathcal{X}$ such that f is $(n+1)$-connective and Y is n-truncated. Then the map f is an n-truncation.*

Proof. Let us consider the map $p\colon X \to \tau_{\leq n}X$. By the sequence of pointed objects in [Lur09, Rmk. 6.5.1.5] applied to the maps $X \xrightarrow{p} \tau_{\leq n}X \to *$ along with [Lur09, Rmk. 6.5.1.7] and [Lur09, Lem. 6.5.1.9], it follows that $\pi_k(p) = *$ for $k \leq n$ and $\pi_k(p) = \pi_k(X \to *)$ for $k > n$. This implies that the map p is $(n+1)$-connective. Since Y is n-truncated, the map f factors through p, i.e., there is a map $q\colon \tau_{\leq n}X \to Y$ such that there is a diagram

$$\begin{array}{ccccc} & & \tau_{\leq n}X & & \\ & \overset{p}{\nearrow} & & \overset{q}{\searrow} & \\ X & & \xrightarrow{\quad f \quad} & & Y \end{array}$$

in $\mathcal{X}$. By [Lur09, Prop. 6.5.1.16], it follows that q is $(n+1)$-connective. Since $\tau_{\leq n}X$ and Y are both n-truncated, it follows that q is n-truncated. But q is also $(n+1)$-connective. Therefore q is an equivalence ([Lur09, Prop. 6.5.1.12]). This finishes the proof. ☐

Definition 3.14. An ∞-topos $\mathcal{X}$ is called *hypercomplete* if every ∞-connective morphism in $\mathcal{X}$ is an equivalence.

Remark 3.15. Every ∞-topos $\mathcal{X}$ admits a hypercompletion denoted by $\mathcal{X}^\wedge$, which may be characterized by a universal property [Lur09, Prop. 6.5.2.13].

Remark 3.16. An object of an $\mathcal{X}$ is said to be hypercomplete if it is local with respect to the ∞-connective morphisms. It follows that $\mathcal{X}$ is hypercomplete if and only if every object of $\mathcal{X}$ is hypercomplete. Any n-truncated object of $\mathcal{X}$ is hypercomplete [Lur09, Lem. 6.5.2.9]. As a consequence, it follows that if $\mathcal{X}$ is Postnikov complete, then $\mathcal{X}$ is hypercomplete.

It is however much more subtle to give a criterion for Postnikov completeness. In general, the hypercompleteness of $\mathcal{X}$ does not imply Postnikov completeness of $\mathcal{X}$. To illustrate this, we mention the following example due to Morel–Voevodsky (which we learned from Marc Hoyois); see also [BGH18, Warn. 3.11.13].

Example 3.17 ([MV99, Ex. 1.30])**.** Let $G = \prod_\infty \mathbf{Z}/2$. Let T be the site of finite G sets and let us consider the ∞-topos $\mathcal{X} := \mathrm{Shv}_\infty(T)^\wedge$, where the latter denotes hypercompletion of $\mathrm{Shv}_\infty(T)$ (see discussion before [Lur09, Lem. 6.5.2.12]). Let $F \in \mathcal{X}$ be the hypersheafification of the presheaf $\prod_{i>0} K(\mathbf{Z}/2, i)$. Since hypersheafifications preserve truncations ([Lur09, Prop. 5.5.6.28]), $\pi_0(F) = \tau_{\leq 0}(F)$ is trivial.

On the other hand, hypersheafification preserves finite limits, so by [Lur09, Prop. 5.5.6.28, Lem. 6.5.1.2] $\varprojlim_n \tau_{\leq n}F \simeq \prod_{i>0} K(\mathbf{Z}/2, i)^\sharp$, where $K(\mathbf{Z}/2, i)^\sharp$ is the (hyper)sheafification of $K(\mathbf{Z}/2, i)$. Thus, $\pi_0(\varprojlim_n \tau_{\leq n}F)$ is the sheaf associated to the presheaf

$$U \mapsto \prod_{i>0} H^i(U, \mathbf{Z}/2)$$

for $U \in T$. When $U = *$ is the final object of T, the element in the latter group obtained taking products of pull back of $\tau^i \in H^i(\mathbf{Z}/2, \mathbf{Z}/2)$ along the i-th projection $G \to \mathbf{Z}/2$ for a generator $\tau \in H^1(\mathbf{Z}/2, \mathbf{Z}/2)$ gives an element that is not killed by any finite cover. Therefore, $\pi_0(\varprojlim_n \tau_{\leq n} F)$ is nontrivial, showing that Postnikov towers do not converge in the hypercomplete ∞-topos $\mathcal{X}$. Additionally, we point out that the cohomology ring $H^*(\mathbf{Z}/2, \mathbf{Z}/2)$ is a symmetric algebra on $H^1(\mathbf{Z}/2, \mathbf{Z}/2)$.

Remark 3.18. Let us mention some known criteria for an ∞-topos $\mathcal{X}$ to be Postnikov complete. If $\mathcal{X}$ is locally of homtopy dimension $\leq n$, then $\mathcal{X}$ is Postnikov complete [Lur09, Prop. 7.2.1.10]. Another criterion for Postnikov completeness appears in [Jar87, Lem. 3.4] (see also [Toë06, Prop. 1.2.2]), which relies on certain cohomology vanishing assumptions above a fixed degree. We point out that the notion of finite cohomological dimension [Lur09, Def. 7.2.2.18] is weaker that finite homotopy dimension, i.e., an ∞-topos with homotopy dimension $\leq n$ also has cohomological dimension $\leq n$ [Lur09, Cor. 7.2.2.30].

3.2. **A question of Bhatt–Scholze.** As discussed in Remark 3.18, all the criteria for Postnikov completeness discussed before rely on making certain finiteness assumptions. However, in many cases, these finiteness assumptions do not hold. The following question of Bhatt–Scholze asks if one can relax such finiteness assumptions in certain situations.

Question 3.19 ([BS15, Qn. 3.1.12])**.** Do Postnikov towers converge in the hypercomplete ∞-topos of sheaves of spaces on a replete topos?

Remark 3.20. As discussed in Remark 3.16, the hypercompleteness assumption in the above question is necessary and cannot be removed; in fact, as we will show in Example 3.29, the ∞-topos of sheaves of spaces on a replete topos need not be hypercomplete. Also, as Example 3.17 shows, Postnikov towers do not converge in a hypercomplete ∞-topos in general.

In Theorem A, we will answer this question affirmatively. This will require the notion of multiplicative presheaves (Definition 2.5) and also some additional preparations. An important observation is the following proposition that shows that certain naturally occuring presheaves are multiplicative.

Proposition 3.21. *Let $\mathcal{T}$ be a Grothendieck site and F be a pointed object of* $\mathrm{Shv}_\infty(\mathcal{T})$. *Then $\pi_n^{\mathrm{pre}}(F, *)$ (Remark 3.11) is a multiplicative presheaf.*

Proof. Note that F represents a colimit preserving functor

$$\underline{F}\colon \mathrm{Shv}_\infty(\mathcal{T}) \to \mathcal{S}^{\mathrm{op}}.$$

The natural inclusion functor $\mathrm{Shv}(\mathcal{T}) \to \mathrm{Shv}_\infty(\mathcal{T})$ preserves coproducts. Therefore, by restriction, we obtain a coproduct preserving functor

$$\underline{\underline{F}}\colon \mathrm{Shv}(\mathcal{T}) \to \mathcal{S}^{\mathrm{op}}.$$

Since F is a pointed object, so are $\underline{F}$ and $\underline{\underline{F}}$. By taking the associated homotopy groups, we obtain a coproduct preserving functor

$$\pi_n^{\mathrm{pre}}(\underline{\underline{F}}, *)\colon \mathrm{Shv}(\mathcal{T}) \to \mathrm{Set}^{\mathrm{op}}.$$

By construction, it follows that $\pi_n^{\mathrm{pre}}(F, *)$ is naturally isomorphic to $\pi_n^{\mathrm{pre}}(\underline{\underline{F}}) \circ h^\sharp$, where $h^\sharp\colon \mathcal{T} \to \mathrm{Shv}(\mathcal{T})$ is the map from Notation 2.1. By Proposition 2.8, we see that $\pi_n^{\mathrm{pre}}(F, *)$ is multiplicative, as desired. $\square$

Proposition 3.22 (Milnor sequences in a replete topos)**.** *Let $\mathcal{X}$ be a* 1*-topos. Assume that $\mathcal{X}$ is replete. Let us consider the ∞-topos* $\mathrm{Shv}_\infty(\mathcal{X})$*. Let $(F_n)_{n\in\mathbf{N}}$ be an inverse system of pointed objects of* $\mathrm{Shv}_\infty(\mathcal{X})$*. Then we have the following exact sequence (of group objects for $q \geq 1$, of pointed objects for $q = 0$) in $\mathcal{X}$:*

$$* \to \varprojlim{}^1 \pi_{q+1}(F_n, *) \to \pi_q(\varprojlim F_n, *) \to \varprojlim \pi_q(F_n, *) \to *. \tag{3.1}$$

Proof. Let us explain how to construct such a sequence. We may pick a Grothendieck site $\mathcal{T}$ such that $\mathrm{Shv}(\mathcal{T}) \simeq \mathcal{X}$. Then the objects F_n maybe viewed as pointed sheaves of spaces on $\mathcal{T}$. By the usual Milnor sequence for the ∞-category $\mathcal{S}$ [BK72, Thm. IX.3.1], we have the following exact sequece (of presheaves of groups for $q \geq 1$, of pointed presheaves of sets for $q = 0$):

$$* \to \varprojlim{}^1 \pi^{\mathrm{pre}}_{q+1}(F_n, *) \to \pi^{\mathrm{pre}}_q(\varprojlim F_n, *) \to \varprojlim \pi^{\mathrm{pre}}_q(F_n, *) \to *. \tag{3.2}$$

Here, for any pointed (pre)sheaf of spaces F on $\mathcal{T}$, the set valued presheaf $\pi^{\mathrm{pre}}_q(F, *)$ denotes the functor $\mathcal{T} \ni U \mapsto \pi_q(U, *)$. If F is a sheaf, then by Remark 3.11, $\pi_q(F, *)$ is simply the sheafification of $\pi^{\mathrm{pre}}_q(F, *)$. Moreover, by Proposition 3.21, we know that since F is a sheaf, the presheaf $\pi^{\mathrm{pre}}_q(F, *)$ is a multiplicative presheaf (Definition 2.5). Therefore, we obtain the desired Milnor sequence (3.1) by sheafifying (3.2) and using Proposition 2.13 and Proposition 2.17. □

Remark 3.23. In [Toë06, p. 21], it is mentioned that in general there is no Milnor sequence for sheaves on a site, because sheafification does not commute with inverse limits in general. However, since the fpqc site is used in [Toë06], which is replete, sheafification commutes with "enough" inverse limits (Proposition 2.13) and one in fact always has a Milnor sequence in this context.

Finally, we will need the following lemma.

Lemma 3.24. *Let $\mathcal{T}$ be a Grothendieck site. Let $f\colon F \to F'$ be a map of sheaves of spaces on $\mathcal{T}$. Let $0 \leq n \leq \infty$. Suppose that the following two conditions hold.*

(1) The map $\pi_0(F) \to \pi_0(F')$ is a surjection of sheaves on $\mathcal{T}$.
(2) For every object $U \in \mathcal{T}$ and every morphism $* \to F|_{\mathcal{T}_{/U}}$ in $\mathcal{T}_{/U}$, the induced map on homotopy groups

$$\pi_m(f, *)\colon \pi_m(F|_{\mathcal{T}_{/U}}, *) \to \pi_m(F'|_{\mathcal{T}_{/U}}, *)$$

is a surjection for $m = n$, an isomorphism for $0 < m < n$ and a map with trivial kernel (as pointed objects) for $m = 0$.

Then the map f is n-connective.

Proof. First, we note that f is an effective epimorphism. This is equivalent to showing that the induced map $\pi_0(F) \to \pi_0(F')$ is an effective epimorphism [Lur09, Prop. 7.2.1.14], which follows from hypothesis (1).

Second, we need to prove that $\pi_k(f) = *$ as an object of $\tau_{\leq 0}(\mathrm{Shv}_\infty(\mathcal{T})_{/F})$ for all $0 \leq k < n$. We may think of $\pi_k(f)$ as a map $\varphi_k\colon \pi_k(f)^{\mathrm{tot}} \to F$ in $\mathrm{Shv}_\infty(\mathcal{T})$. Our goal is to show that φ_k is an equivalence. Let $U \in \mathcal{T}$. We obtain an induced map $\varphi_k(U)\colon \pi_k(f)^{\mathrm{tot}}(U) \to F(U)$. Let $p'\colon * \to F(U)$ be a map. It corresponds to a map $p\colon * \to F|_{\mathcal{T}_{/U}}$ in $\mathrm{Shv}_\infty(\mathcal{T})$. Let $\mathrm{Fib}_{p,U}$ be the fiber of $f|_{\mathcal{T}_{/U}}\colon F|_{\mathcal{T}_{/U}} \to F'|_{\mathcal{T}_{/U}}$ along

$f|_{\mathcal{T}_{/U}} \circ p\colon * \to F'|_{\mathcal{T}_{/U}}$. Then by construction, the fiber of $\varphi_k(U)\colon \pi_k(f)^{\mathrm{tot}}(U) \to F(U)$ along $p'\colon * \to F(U)$ is equivalent to $\pi_k(\mathrm{Fib}_{p,U}, p)(U)$. Now by our hypothesis, it follows that $\pi_k(\mathrm{Fib}_{p,U}, p)$ is trivial. This implies that $\pi_k(\mathrm{Fib}_{p,U}, p)(U)$ is also trivial. Since p' was arbitrary, this implies that the map $\varphi_k(U)$ is an equivalence. Since $U \in \mathcal{T}$ was arbitrary, we obtain that φ_k is an equivalence, as desired. □

We are now ready to prove our main theorem.

Theorem A. *Let $\mathcal{X}$ be a replete topos. Then the hypercomplete ∞-topos $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$ is Postnikov complete.*

Proof. Pick a Grothendieck site $\mathcal{T}$ with $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{T})$. To show that $\mathcal{C} := \mathrm{Shv}_\infty(\mathcal{T})^\wedge$ is Postnikov complete, we prove that conditions (a) and (b) from Lemma 3.7 are equivalent.

(a) $\implies$ (b): Pick an object $X \in \mathcal{C}$. We will show that the natural map $X \to \varprojlim_n \tau_{\leq n} X$ is an equivalence. Since $\mathcal{C}$ is a hypercomplete ∞-topos, it would be enough to show that $X \to \varprojlim_n \tau_{\leq n} X$ is ∞-connective (Definition 3.12). For this, we check the two hypotheses of Lemma 3.24.

First, we check that $\eta\colon \pi_0(X) \to \pi_0(\varprojlim_n \tau_{\leq n} X)$ is a surjection of sheaves on $\mathcal{T}$. To this end, let us consider the map $\pi_0^{\mathrm{pre}}(X) \to \pi_0^{\mathrm{pre}}(\varprojlim_n \tau_{\leq n} X)$ of presheaves on $\mathcal{T}$. Let $U \in \mathcal{T}$ and $s \in \pi_0^{\mathrm{pre}}(\varprojlim_n \tau_{\leq n} X)(U)$. We may lift s to a map $\tilde{s}\colon * \to (\varprojlim_n \tau_{\leq n} X)\Big|_{\mathcal{T}_{/U}}$. Now Proposition 3.22 yields an exact sequence of pointed objects

$$* \to \varprojlim_n{}^1 \pi_1(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \to \pi_0(\varprojlim_n \tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \to \varprojlim_n \pi_0(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \to *.$$

Note that $\varprojlim^1_n \pi_1(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \simeq *$ and $\varprojlim_n \pi_0(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \simeq \pi_0(X|_{\mathcal{T}_{/U}})$. Thus, the exact sequence of pointed objects degenerates to

$$* \to \pi_0(\varprojlim_n \tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \xrightarrow{\zeta} \varprojlim_n \pi_0(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \to *. \tag{3.3}$$

Now we consider the maps

$$\pi_0(X|_{\mathcal{T}_{/U}}) \xrightarrow{\eta} \pi_0(\varprojlim_n \tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \xrightarrow{\zeta} \varprojlim_n \pi_0(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *),$$

whose composition is an isomorphism. Taking sections, we obtain the maps

$$\pi_0(X)(U) \xrightarrow{\eta_U} \pi_0(\varprojlim_n \tau_{\leq n} X)(U) \xrightarrow{\zeta_U} \varprojlim_n \pi_0(\tau_{\leq n} X)(U)$$

whose composition, which we call θ, is again an isomorphism. Let $s^\sharp$ be the image of s under the natural map $\pi_0^{\mathrm{pre}}(\varprojlim_n \tau_{\leq n} X)(U) \to \pi_0(\varprojlim_n \tau_{\leq n} X)(U)$ induced by sheafification. By the exactness of the sequence in (3.3), it follows that $\eta_U(\theta^{-1}(\zeta_U(s^\sharp))) = s^\sharp$. Since $s \in \pi_0^{\mathrm{pre}}(\varprojlim_n \tau_{\leq n} X)(U)$ and $U \in \mathcal{T}$ was arbitrary, this shows that the map of sheaves $\pi_0(X) \to \pi_0(\varprojlim_n \tau_{\leq n} X)$ is a surjection, yielding (1).

Now we check hypothesis (2) from Lemma 3.24. Let $U \in \mathcal{T}$ and $x\colon * \to X|_{\mathcal{T}_{/U}}$. Using Proposition 3.22, we obtain for every $m \geq 0$ an exact sequence

$$* \to \varprojlim_n{}^1 \pi_{m+1}(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \to \pi_m(\varprojlim_n \tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \to \varprojlim_n \pi_m(\tau_{\leq n} X|_{\mathcal{T}_{/U}}, *) \to *.$$

Again, $\varprojlim^1_n \pi_{m+1}(\tau_{\leq n}X|_{\mathcal{T}_{/U}}, *) \simeq *$ and $\varprojlim_n \pi_m(\tau_{\leq n}X|_{\mathcal{T}_{/U}}, *) \simeq \pi_m(X|_{\mathcal{T}_{/U}}, *)$. This implies that the natural map $\pi_m(X|_{\mathcal{T}_{/U}}, *) \to \pi_m(\varprojlim_n \tau_{\leq n}X|_{\mathcal{T}_{/U}}, *)$ is an isomorphism for $m > 0$ and injective for $m = 0$. Since $\mathcal{C}$ is a hypercomplete ∞-topos, we obtain that the natural map $X \to \varprojlim_n \tau_{\leq n}X$ is an equivalence. This proves that (a) $\implies$ (b) from Lemma 3.7 holds in $\mathcal{C}$.

(b) $\implies$ (a): Let $X: N(\mathbf{Z}^{\infty}_{\geq 0})^{\mathrm{op}} \to \mathcal{C}$ be a limit diagram such that the restriction $X|_{N(\mathbf{Z}_{\geq 0})^{\mathrm{op}}}$ is a Postnikov pretower. In such a situation, we have $X(\infty) \simeq \varprojlim_n X(n)$. There are also natural maps $\mu_n: X(\infty) \to X(n)$. Since $X(n)$ is n-truncated, to prove that μ_n is an n-truncation, by Lemma 3.13, it suffices to show that μ_n is $(n+1)$-connective. We first argue that μ_n is an effective epimorphism. We note that the natural map $\pi_0^{\mathrm{pre}}(X(\infty)) \simeq \pi_0^{\mathrm{pre}}(\varprojlim_n X(n)) \to \varprojlim_n \pi_0^{\mathrm{pre}}(X(n))$ is surjective as presheaves on $\mathcal{T}$. However, since $\pi_0^{\mathrm{pre}}(X(n))$ is a multiplicative presheaf, by repleteness of $\mathcal{X}$ and Proposition 2.13, it follows that the map

$$\pi_0(X(\infty)) \simeq \pi_0(\varprojlim_n X(n)) \to \varprojlim_n \pi_0(X(n)) \simeq \pi_0(X(n))$$

is surjective. Therefore, μ_n is an effective epimorphism. Now, in order to check that μ_n is $(n+1)$-connective, we can apply Lemma 3.24 and compute homotopy groups by using the Milnor sequences from Proposition 3.22 in a manner similar to the previous paragraph. This finishes the proof. □

We include two applications of Theorem A; the necessary notations and background are explained before the statements of the corollaries.

Notation 3.25. For any ∞-topos $\mathcal{X}$ and any ∞-category $\mathcal{C}$, we denote by $\mathrm{Shv}_\infty(\mathcal{X}, \mathcal{C})$ the ∞-category of $\mathcal{C}$-valued sheaves [Lur18, Def. 1.3.1.4]. If R is a connective $\mathbf{E}_1$-ring and $\mathcal{C} = \mathrm{LMod}_R$ is the stable ∞-category of left R-modules ([Lur17, § 7.1.1]), then $\mathrm{Shv}_\infty(\mathcal{X}, \mathrm{LMod}_R)$ is naturally equipped with a t-structure (by extending the construction of [Lur18, Prop. 2.1.1.1]).

We thank Peter Haine for pointing out the following corollary of Theorem A, which generalizes [BS15, Prop. 3.3.3].

Corollary 3.26. *Let $\mathcal{X}$ be a replete topos and R be a connective $\mathbf{E}_1$-ring. Then the stable ∞-category $D(\mathcal{X}, R) := \mathrm{Shv}_\infty(\mathrm{Shv}_\infty(\mathcal{X})^\wedge, \mathrm{LMod}_R)$ equipped with the t-structure from Notation 3.25 is left complete (see* [Lur17, § 1.2.1]*).*

Proof. Let Sp be the ∞-category of spectra. Set $\mathrm{Shv}_\infty(\mathcal{X}, \mathrm{Sp})^\wedge := \mathrm{Shv}_\infty(\mathrm{Shv}_\infty(\mathcal{X})^\wedge, \mathrm{Sp})$. Using the notations from [Lur18, § 1.3.2], we have a conservative, limit preserving functor

$$\mathrm{Shv}_\infty(\mathcal{X}, \mathrm{Sp})^\wedge = \mathrm{Shv}_\infty(\mathrm{Shv}_\infty(\mathcal{X})^\wedge, \mathrm{Sp})_{\geq 0} \xrightarrow{\Omega^\infty} \mathrm{Shv}_\infty(\mathrm{Shv}_\infty(\mathcal{X})^\wedge, \mathcal{S}) \simeq \mathrm{Shv}_\infty(\mathcal{X})^\wedge$$

that commutes with the natural truncation functors. Since $\mathcal{X}$ is replete, by Theorem A, $\mathrm{Shv}_\infty(\mathcal{X})^\wedge$ is Postnikov complete. Now, using the properties of the functor Ω^∞ stated above, the condition (a) $\iff$ (b) from Lemma 3.7 continues to hold in $\mathrm{Shv}_\infty(\mathcal{X}, \mathrm{Sp})^\wedge_{\geq 0}$. Thus $\mathrm{Shv}_\infty(\mathcal{X}, \mathrm{Sp})^\wedge$ is left complete. For an arbitrary connective $\mathbf{E}_1$-ring R, it similarly follows that $D(\mathcal{X}, R)$ is left complete because the natural functor $D(\mathcal{X}, R) \to \mathrm{Shv}_\infty(\mathcal{X}, \mathrm{Sp})^\wedge$ is conservative, limit preserving and t-exact (*cf.* [Lur18, Prop. 2.1.1.1.(a)] and [Lur18, Prop. 2.1.0.3.(3)]). □

Next, we discuss an application of Theorem A in the theory of affine stacks [Toë06, § 2.2].

Remark 3.27. In [Toë06, p. 55], Toën shows that if F is a pointed connected affine stack over a field, then the natural map $F \to \varprojlim_n \tau_{\leq n} F$ is an equivalence. The proof uses the criterion from [Toë06, Prop. 1.2.2] and requires the representability of $\pi_n(F, *)$ by unipotent affine group schemes and certain vanishing of cohomology with coefficients in such group schemes. Since affine stacks are hypercomplete and the fpqc topos (after fixing set-theoretic issues, as in [Toë06]) is replete, one obtains as a corollary of Theorem A the following generalization, which removes the assumptions that the stack F is connected (i.e., $\pi_0(F, *) \simeq *$) and defined over a field.

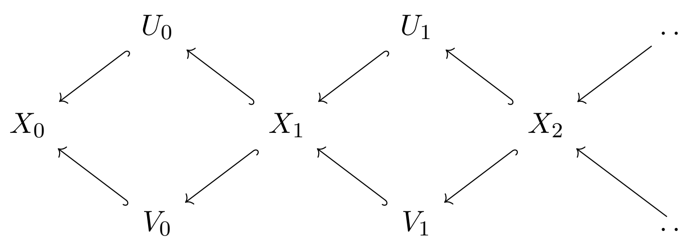
Corollary 3.28. *Let F be an affine stack over* $\operatorname{Spec} B$ *for any ring B. Then the natural map $F \to \varprojlim_n \tau_{\leq n} F$ is an equivalence.*

We end our paper with an example which shows that the hypercompleteness assumption in Theorem A is crucial.

Example 3.29. We use a topos due to Dugger–Hollander–Isaksen [DHI04, Ex. A.9] (modifying a suggestion of Carlos Simpson). Let $\mathcal{T}$ be the following Grothendieck site:

(1) Objects of $\mathcal{T}$ are the open subintervals $X_n := (\frac{3^n-1}{2\cdot 3^n}, \frac{3^n+1}{2\cdot 3^n})$, $U_n := (\frac{3^n-1}{2\cdot 3^n}, \frac{3^{n+1}+1}{2\cdot 3^{n+1}})$, and $V_n := (\frac{3^{n+1}-1}{2\cdot 3^{n+1}}, \frac{3^n+1}{2\cdot 3^n})$ of $(0,1)$.
(2) Morphisms are inclusions of underlying sets.
(3) Covers are given by jointly surjective maps.

One may think of the category $\mathcal{T}$ in terms of the following poset:



We claim that the 1-topos $\operatorname{Shv}(\mathcal{T})$ is locally weakly contractible in the sense of [BS15, Def. 3.2.1]. Indeed, the site $\mathcal{T}$ is subcanonical and every object is qcqs. Since every covering family of the objects U_n (resp. V_n) contains U_n (resp. V_n), the sheaves h_{U_n} (resp. h_{V_n}) are weakly contractible. Further, given any $Z \in \mathcal{T}$, there exists a surjection $(h_{U_n} \coprod h_{V_n}) \twoheadrightarrow h_Z$ for some $n \geq 0$. This implies that $\operatorname{Shv}(\mathcal{T})$ is locally weakly contractible. In particular, $\operatorname{Shv}(\mathcal{T})$ is replete ([BS15, Prop. 3.2.3]).

However, $\mathcal{X} := \operatorname{Shv}_\infty(\mathcal{T})$ is not hypercomplete. In fact, [DHI04, Ex. A.9] exhibits an example of a non-hypercomplete sheaf G with values in $D(\mathbf{Z})_{\geq 0}$; it is given by

$$G(X_n) := \mathbf{Z}[\operatorname{Sing}(S^{n-1})], \quad G(U_n) := \mathbf{Z}[\operatorname{Sing}(D^n_+)], \quad G(V_n) := \mathbf{Z}[\operatorname{Sing}(D^n_-)],$$

where D^n_+ (resp. D^n_-) denotes the upper (resp. lower) hemisphere of S^n. We use the convention that $S^{-1} := \varnothing$; in particular, $\mathbf{Z}[\operatorname{Sing}(S^{-1})] = 0$. The maps induced by the standard inclusions of topological spaces $D^n_+ \hookrightarrow S^n$ and $D^n_- \hookrightarrow S^n$ naturally equip G with the structure of a presheaf. The sheaf property amounts to checking Čech descent, which essentially follows from the Mayer–Vietoris theorem for the open cover $S^n = D^n_+ \cup D^n_-$. To show that G is not hypercomplete, [DHI04, Ex. A.9] constructs an explicit hypercover for which G does not satisfy descent. Alternatively, one can simply observe that for all $q > 0$, we have

$$\pi_q(G) \simeq \left(\pi_q^{\mathrm{pre}}(G)\right)^\sharp \simeq 0.$$

Here, the latter isomorphism follows from the isomorphisms $H_i(D^n_+, \mathbf{Z}) \simeq H_i(D^n_-, \mathbf{Z}) = 0$ for $i > 0$ and the fact that every object of the site $\mathcal{T}$ admits a refinement by U_n and V_n. Therefore, the natural map $G \to \tau_{\leq 0} G$ is ∞-connective. However, it cannot be an equivalence since G is not 0-truncated (e.g., $\pi_n(G(X_{n+1})) \simeq H_n(S^n, \mathbf{Z}) \simeq \mathbf{Z}$). This shows that G is not hypercomplete.

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