

IND-ÉTALE VS FORMALLY ÉTALE

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ABSTRACT. We show that when A is a reduced algebra over a characteristic zero field k and the module of Kähler differentials $\Omega_{A/k} = 0$, then A is ind-étale, partially answering a question of Bhatt. As further applications of this result, we deduce a rigidity property of Hochschild homology and special instances of Weibel's conjecture [Wei80] and Vorst's conjecture [Vor79] without any noetherian assumptions.

CONTENTS

1. Introduction	1
2. Main results	3
3. Application to Hochschild homology	7
4. Application to K -regularity	9
Funding	11
Acknowledgements	11
References	11

1. INTRODUCTION

In this article, all rings are commutative and unital, unless otherwise mentioned.

Fix a field k . Recall that a finite type k -algebra R is *étale over k* if the module of Kähler differentials $\Omega_{R/k}$ is zero.

Definition 1.1. A k -algebra is said to be *ind-étale* if as a k -algebra, it is isomorphic to a direct limit of some direct system of étale algebras over k .

If R is an étale algebra, then the cotangent complex $\mathbb{L}_{R/k}$ is exact, i.e., $H^i(\mathbb{L}_{R/k}) = 0$ for all $i \in \mathbb{Z}$. We refer to [Sta, Tag 08P5] for the definition and basic properties of the cotangent complex. When A is a smooth algebra over k , then the cotangent complex agrees with the module of Kähler differentials. But in general, the cotangent complex is a complex of A -modules or more naturally, an object in the derived category $D(A)$ of chain complexes over A . Since the formation of cotangent complex commutes with taking direct limits, it follows that for an ind-étale algebra A , the cotangent complex $\mathbb{L}_{A/k}$ is exact.

In [Bha], Bhargav Bhatt raised the following question asking whether conversely exactness of $\mathbb{L}_{A/k}$ – i.e., A being *formally étale* implies ind-étaleness of A .

Question 1.2 (Bhatt). Let k be a field of characteristic zero. Does there exist a k -algebra A , such that the cotangent complex $\mathbb{L}_{A/k}$ is exact, yet A is not ind-étale over k ? – see [Bha, Question 0.3] and [Mor19, Question C.3].

Note that ind-étale algebras are necessarily reduced. In this note, we answer the question above when A is additionally assumed to be reduced.

Theorem 1.3. (see Theorem 2.10) Let k be a field of characteristic zero and A be a reduced k -algebra – not assumed to be noetherian. If $\Omega_{A/k} = 0$, then A is ind-étale.

Since the module of Kähler differentials is the zeroth cohomology of the cotangent complex, the exactness of $\mathbb{L}_{A/k}$ implies $\Omega_{A/k} = 0$. So our theorem partially answers Question 1.2 by showing that when the algebra A is known to be reduced, then the much *weaker* assumption of $\Omega_{A/k} = 0$, implies that A is ind-étale. Thus, Theorem 1.3 now reduces Question 1.2 to the following question:

Question 1.4. Let k be a field of characteristic zero. Does there exist a k -algebra A such that the cotangent complex $\mathbb{L}_{A/k}$ is exact, yet A is not reduced?

Remark 1.5. Note that, merely assuming $\Omega_{A/k} = 0$ and k is a field of characteristic zero does *not* imply reducedness of A , as is shown by an example originally due to Ofer Gabber. See Remark 2.12 and [MS21, Theorem 2.2].

The main difficulty in proving Theorem 1.3 is in dealing with the lack of any finiteness or noetherian assumptions. The key new ingredient in the proof is the observation of the following general result, whose proof, in turn makes judicious use of localization constructions and minimality tricks to circumvent issues caused by the lack of noetherian assumption.

Proposition 1.6 (see Proposition 2.6). Let k be a field of characteristic zero and A be a k -algebra – not assumed to be noetherian. Suppose that there is a k -algebra injection from the polynomial ring, $\iota : k[t_1, \dots, t_s] \hookrightarrow A$ for some $s \geq 1$. Then $d\iota(t_1) \wedge d\iota(t_2) \wedge \dots \wedge d\iota(t_s)$ is a nonzero element of $\wedge_A^s \Omega_{A/k}$.

When phrased in terms of *transcendence cardinality* introduced in Definition 2.1, Proposition 1.6 implies that vanishing of $\wedge_A^s \Omega_{A/k}$ forces the k -transcendence cardinality of A to be at most $s - 1$.

Remark 1.7. The analogue of Question 1.2 in positive characteristics has a negative answer. There is an example due to Bhatt of a positive characteristic field k and a k -algebra A such that the cotangent complex $\mathbb{L}_{A/k}$ is exact, but A is not reduced, thus not ind-étale – see [Bha, Proposition 0.2], where the example is attributed to Ofer Gabber.

We will use Theorem 1.3 to prove the following result about vanishing of Hochschild homology. We point out that in [AV92], certain vanishing of Hochschild homology $\mathrm{HH}_i(A)$ for an algebra A had been used to give a criteria of smoothness.

Proposition 1.8 (see Proposition 3.5). Let k be a field of characteristic 0. Let A be a reduced commutative k algebra – not assumed to be noetherian. If $\mathrm{HH}_1(A/k) = 0$, then $\mathrm{HH}_i(A/k) = 0$ for all $i \geq 1$.

In Section 3, we briefly review the definition of Hochschild homology and prove the above proposition. The proof is manifestly based on techniques from commutative algebra and we point out that the commutativity assumption in Proposition 1.8 is very sharp. Example 3.6 shows that a similar assertion is false without the commutativity assumption; we learnt this example from Antieau.

Proposition 1.6 also has some unexpected consequences. It can be used to deduce new special instances of a question of Weibel [Wei80, Question 2.9] and Vorst’s conjecture [Vor79] without any finiteness or noetherian assumptions. The applications in this direction were pointed out by Morrow after we shared a draft version of our paper with him; we thank him heartily for generously sharing his observations with us. These questions belong to the

area of K -regularity, which roughly speaking, uses algebraic K -theory to study regularity of commutative rings. We briefly recall some necessary definitions and results in K -theory in Section 4 and include the new applications. In what follows, Proposition 1.9 is related to the question of Weibel and Proposition 1.10 is a non-noetherian case of Vorst's conjecture.

Proposition 1.9 (see Proposition 4.5). Let A be a commutative k -algebra over a field k of characteristic zero such that the module of $d + 1$ -forms $\wedge^{d+1}\Omega_{A/k} = 0$. Then

- (1) For $n < -d$, the K -groups $K_n(A) = 0$.
- (2) A is K_n -regular (see Definition 4.2) for all $n \leq -d$.

Proposition 1.10 (see Proposition 4.6). Let A be a commutative k -algebra over a field k of characteristic zero such that the module of Kähler differentials $\Omega_{A/k} = 0$. If A is K_1 -regular, then A is ind-étale. Moreover, A is K_n -regular for all integers n .

2. MAIN RESULTS

Fix a field k . In this section, we first define and explore the notion of k -transcendence cardinality of a k -algebra. Transcendence cardinality shows up later in Proposition 4.5. The main result of this section, Proposition 2.6, provides a criteria for finite transcendence cardinality in terms of the vanishing of module of differential forms.

For the definition of *cardinal numbers* appearing in the definition below refer to [Bou04, Chapter III].

Definition 2.1. Let A be a k -algebra. Given a cardinal number J , we say that A has k -transcendence cardinality at least J , if there is a set S of cardinality J and a k -algebra injection $k[\{t_j \mid j \in S\}] \hookrightarrow A$. The k -transcendence cardinality of A is $\text{Sup}\{J \mid A \text{ has } k\text{-transcendence cardinality at least } J\}$. Here the supremum is taken in the set of all cardinal numbers originating from subsets of A and the supremum is also a cardinal number- see Remark 2.2.

When the k -transcendence cardinality of A is a natural number, we call that natural number the k -transcendence degree of A .

Remark 2.2. The cardinal numbers of subsets of a given set form a well ordered set- see Theorem 1, [Hön54]. Hence the supremum in Definition 2.1 is again a cardinal number.

We establish some basic properties of the notion of transcendence cardinality listed below.

Proposition 2.3. Let A be a k -algebra, not necessarily noetherian.

- (1) If A is finite type over k , then the k -transcendence cardinality of A is finite and is the same as the Krull dimension of A .
- (2) If A has k -transcendence cardinality at most $n \in \mathbb{N}$, then every finite type k -subalgebra of A has Krull dimension at most n . Moreover the Krull dimension of A is at most n .
- (3) Let B be a k -algebra such that B contains A and is module finite over A . Then the transcendence cardinality of B is finite if and only if the transcendence cardinality of A is finite. When both the transcendence cardinalities are finite, those are the same.
- (4) Let $\phi : A \rightarrow C$ be a finite k -algebra homomorphism. If the k -transcendence cardinality of A is $n \in \mathbb{N}$, then the k -transcendence cardinality of C is at most n .
- (5) The k -transcendence cardinalities of A and A_{red} are the same.
- (6) Let A be a domain with finite k -transcendence cardinality. The transcendence cardinality of the fraction field of A is the same as that of A .

- (7) If k has characteristic zero and $\wedge^s \Omega_{A/k} = 0$, then A has k -transcendence cardinality at most $s - 1$.

Remark 2.4. The Krull dimension can be much lower than the k -transcendence cardinality. For example, a field extension L of k can have arbitrarily large k -transcendence cardinality, while the Krull dimension of L is zero.

Proof of Proposition 2.3. (1) Let d be the Krull dimension of A . Noether normalization (see [Sta, 00OY]) guarantees a module finite inclusion $k[x_1, \dots, x_d] \hookrightarrow A$. So the k -transcendence cardinality of A is at least d . We show that the k -transcendence cardinality of A is at most d . For that, we use the next lemma to reduce the problem to the case where A is a domain.

Lemma 2.5. Let S be a multiplicative set in a commutative ring R , such that $0 \notin S$. There is a minimal prime \mathfrak{p} of R such that $S \subseteq R - \mathfrak{p}$.

Proof. Since $0 \notin S$, the localization $S^{-1}R$ is nonzero. So $S^{-1}R$ has a minimal prime \mathfrak{q} . We can take \mathfrak{p} to be the contraction of \mathfrak{q} via the natural map $R \rightarrow S^{-1}R$. \square

Now given a k -algebra inclusion $\phi : k[x_1, \dots, x_n] \hookrightarrow A$, take $S = \phi(k[x_1, \dots, x_n] \setminus 0)$. Using Lemma 2.5, choose a minimal prime \mathfrak{p} of A such that $S \cap \mathfrak{p} = \emptyset$. This means the composition of ϕ with the quotient $A \rightarrow \frac{A}{\mathfrak{p}}$ is also injective. The last injection gives an injection of the fraction fields $k(x_1, \dots, x_n) \hookrightarrow \text{Frac}(\frac{A}{\mathfrak{p}})$. So n is at most the k -transcendence degree of $\text{Frac}(\frac{A}{\mathfrak{p}})$. Since the k -transcendence degree of $\text{Frac}(\frac{A}{\mathfrak{p}})$ is the Krull dimension of $\frac{A}{\mathfrak{p}}$ and the Krull dimension of $\frac{A}{\mathfrak{p}}$ is at most d ([Ser00, Prop. 14]), $n \leq d$.

- (2) The Krull dimension of any finite type k -subalgebra B of A is at most the k -transcendence degree of B . Since the k -transcendence degree of B is at most n , we are done.

We now prove that the Krull dimension of A is at most n . Let $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \dots \subseteq \mathfrak{p}_m$ be chain of prime ideals of A where each containment is strict. For each $j \geq 1$, choose $x_j \in \mathfrak{p}_j \setminus \mathfrak{p}_{j-1}$. Let B be the k -subalgebra of A generated by x_1, \dots, x_m . So we have a chain of prime ideals in B with strict containments,

$$\mathfrak{p}_0 \cap B \subsetneq \mathfrak{p}_1 \cap B \subsetneq \dots \subsetneq \mathfrak{p}_m \cap B.$$

So m is at most the Krull dimension of B . Since the Krull dimension of B is at most n by part 1, the Krull dimension of A is at most n .

(3) We show that if the k -transcendence cardinality of A is $n \in \mathbb{N}$, then the k -transcendence cardinality of B is also n . Given any k -algebra inclusion $\phi : k[x_1, \dots, x_m] \hookrightarrow B$, we can choose finite type k -subalgebras $B' \subseteq B$ and $A' \subseteq A$, such that $\text{Im}(\phi) \subseteq B'$, $A' \subseteq B'$ and $A' \hookrightarrow B'$ is module finite. The choice can be made as follows: for each j , $1 \leq j \leq m$, there is a nonzero monic polynomial $F_j \in A[t]$ such that $F_j(\phi(x_j)) = 0$. Take A' to be the k -subalgebra of A generated by the coefficients of F_j 's where j varies. Take B' to be A' -subalgebra of B generated by all the $\phi(x_j)$'s.

Now, by (1), the k -transcendence degree of B' is the Krull dimension of B' . Since $A' \subseteq B'$ is module finite, the Krull dimension of A' and B' are the same. By (2), the Krull dimension of A' is at most n . So the k -transcendence degree of B' is at most n . Thus $m \leq n$, proving that the k -transcendence cardinality of B is also at most n . Again since $A \subseteq B$, the k -transcendence cardinality of B is at least n .

If the k -transcendence cardinality of B is finite, the k -transcendence cardinality of A is also finite as $A \subseteq B$. Moreover the k -transcendence cardinalities of A and B coincide by the argument above.

(4) Since there is a k -algebra surjection $A \rightarrow \phi(A)$, the k -transcendence cardinality of $\phi(A)$ is at most n . Since $\phi(A) \subseteq C$ is module finite, by (3), the k -transcendence cardinality of C is at most that of $\phi(A)$ and the later is at most n .

(5) Given a set S of cardinality J and a k -algebra injection $\phi : k[\{t_j \mid j \in S\}] \hookrightarrow A$, the intersection of the image of ϕ and the nilradical of A is zero. Thus composing ϕ with the surjection $A \rightarrow A_{\text{red}}$ also gives an injection. Thus the transcendence cardinality of A_{red} is at least that of A . Given a set S and a k -algebra injection $\psi : k[\{x_j \mid j \in S\}] \rightarrow A_{\text{red}}$, lift ψ to a k -algebra map $k[\{x_j \mid j \in S\}] \rightarrow A$; the lift is necessarily injective. So the transcendence cardinality of A_{red} is at most that of A .

(6) Suppose that the transcendence cardinality of A is $n \in \mathbb{N}$. It is enough to show that the transcendence cardinality of $\text{Frac}(A)$ is at most n . By contradiction, assume that $\text{Frac}(A)$ contains elements $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{n+1}}{b_{n+1}}$, which are algebraically independent over k , where all a_i, b_i 's are in A . Then the subalgebra $k[a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}] \subseteq A$ has transcendence degree at least $n+1$ as its fraction field contains $k[\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{n+1}}{b_{n+1}}]$. Since the transcendence degree of A is n , we get a contradiction.

(7) This assertion follows from Proposition 2.6 proven below. □

Proposition 2.6. Let k be a field of characteristic zero and A be a k -algebra – not assumed to be noetherian. Suppose that there is a k -algebra injection from the polynomial ring, $\iota : k[t_1, \dots, t_s] \hookrightarrow A$ for some $s \geq 1$. Then $d\iota(t_1) \wedge d\iota(t_2) \wedge \dots \wedge d\iota(t_s)$ is a nonzero element of $\wedge_A^s \Omega_{A/k}$.

Proof. We first prove Proposition 2.6 assuming that A is a field and then deduce the general case from the field case in a few steps.

Assume that A is a field. Pick a subset $\{x_i\}_{i \in I}$ of A such that $\{\iota(t_1), \dots, \iota(t_s)\} \cup \{x_i\}_{i \in I}$ is a k -transcendence basis of A ; for example, $\{x_i\}_{i \in I}$ can be chosen to be a $k(\iota(t_1), \dots, \iota(t_s))$ -transcendence basis of A ; see [Sta, Tag 030F]. Set L to be smallest subfield of A containing k and $\{\iota(t_1), \dots, \iota(t_s)\} \cup \{x_i\}_{i \in I}$. For any finite field extension $L' \supseteq L$ where $L' \subseteq A$, since $L \subseteq L'$ is separable, we have an isomorphism,

$$(1) \quad \Omega_{L/k} \otimes_L L' \cong \Omega_{L'/k} ;$$

see [Liu02, Chapter 6, Lemma 1.13]. Varying L' over finite extensions of L such that $L' \subseteq A$, we get a direct system of isomorphisms from Equation (1); taking the direct limit of this direct system of isomorphisms we get an isomorphism

$$(2) \quad \Omega_{L/k} \otimes_L A \cong \Omega_{A/k}.$$

To get Equation (2), we have used that formation of modules of Kähler differentials commute with taking direct limit (see [Sta, Tag 00RM]) and A is the direct limit of the fields L' . Since $\Omega_{L/k}$ is isomorphic to the free L -module with basis $\{d\iota(t_1), \dots, d\iota(t_s)\} \cup \{dx_i\}_{i \in I}$, Equation (2) implies that $\Omega_{A/k}$ is a free A -module with basis $\{d\iota(t_1), \dots, d\iota(t_s)\} \cup \{dx_i\}_{i \in I}$.

Hence $du(t_1) \wedge du(t_2) \wedge \dots \wedge du(t_s)$ is a nonzero element of $\wedge_A^s \Omega_{A/k}$.

Given a k -algebra A as in Proposition 2.6, which is not necessarily a field, set A' to be A modulo the nilradical of A . Then the composition $k[t_1, \dots, t_s] \xrightarrow{\iota} A \rightarrow A'$ is also injective; denote the composition by ϕ . Set $S = \phi(k[t_1, \dots, t_s] \setminus 0)$. We note that S is a multiplicative set. Using Lemma 2.5 we can choose a minimal prime \mathfrak{p} of A' such that $S \subseteq A' - \mathfrak{p}$. So the image of any nonzero element of $k[t_1, \dots, t_s]$ under the composition $k[t_1, \dots, t_s] \xrightarrow{\phi} A' \rightarrow A'_\mathfrak{p}$ is a unit; hence the composition is also injective. Denote the last composition by ψ . We have a commutative diagram,

$$(3) \quad \begin{array}{ccc} \wedge_{k[t_1, \dots, t_s]}^s \Omega_{k[t_1, \dots, t_s]/k} & \xrightarrow{\wedge^s du} & \wedge_A^s \Omega_{A/k} \\ & \searrow \wedge^s d\psi & \swarrow \\ & \wedge_{A'_\mathfrak{p}}^s \Omega_{A'_\mathfrak{p}/k} & \end{array}$$

where the unlabelled downward arrow is induced by the canonical map $A \rightarrow A'_\mathfrak{p}$. We want to show that $\wedge^s du(dt_1 \wedge \dots \wedge dt_s)$ is nonzero. To that end, first note that $A'_\mathfrak{p}$ is a field: since \mathfrak{p} is minimal, the only prime ideal of $A'_\mathfrak{p}$ namely $\mathfrak{p}A'_\mathfrak{p}$ coincides with the nilradical of $A'_\mathfrak{p}$, which is zero as A' and hence $A'_\mathfrak{p}$ is reduced. Now by the field case of Proposition 2.6, $\wedge^s d\psi(dt_1 \wedge \dots \wedge dt_s)$ is nonzero. The commutativity of diagram 3 implies that $\wedge^s d\psi(dt_1 \wedge \dots \wedge dt_s)$ is the image of $\wedge^s du(dt_1 \wedge \dots \wedge dt_s)$. So $\wedge^s du(dt_1 \wedge \dots \wedge dt_s)$ must be a nonzero element of $\wedge_A^s \Omega_{A/k}$. \square

As an immediate corollary we get,

Corollary 2.7. Let k be a characteristic zero field and A be a k -algebra – not necessarily noetherian. If $\wedge_A^s \Omega_{A/k} = 0$, then there cannot be a k -algebra injection $k[t_1, \dots, t_s] \rightarrow A$.

Remark 2.8.

- (1) The converse to Corollary 2.7 is false as the following example shows. For any characteristic zero field k , take $A = k[x]/(x^2)$. Then $\Omega_{A/k} \cong \frac{A}{xA} dx$, yet there cannot be any injection from $k[t]$ to A as A has Krull dimension zero.
- (2) Proposition 2.6 need not hold when k has positive characteristic. For example, take ι to be the inclusion $k[x] \rightarrow k[x^{1/p}]$. Then $d(\iota(x)) = 0$.

Corollary 2.9. Let k be a field of characteristic zero, A be a k -algebra – not necessarily noetherian. If $\Omega_{A/k} = 0$, then A is integral over k .

Proof. Contrary to the assertion of Corollary 2.9, assume that for $a \in A$ the k -algebra map from the polynomial ring $k[t]$ to A sending t to a is injective. Now Proposition 2.6 implies that $da \in \Omega_{A/k}$ is nonzero, contradicting our hypothesis $\Omega_{A/k} = 0$. \square

The next results partially answers Bhatt's question (Question 1.2).

Theorem 2.10. Let k be a field of characteristic zero and A is a reduced k -algebra. If $\Omega_{A/k} = 0$, then A is ind-étale.

Proof. We shall show that any finitely generated k -subalgebra of A is étale over k ; this will prove Theorem 2.10 since A is the directed union of all finitely generated k -subalgebras. Fix a finitely generated k -subalgebra B of A . The ring B is integral over k as A is integral

over k by Corollary 2.9. Therefore B has Krull dimension zero. Hence every minimal prime of B is maximal. Since B is noetherian, B has only finitely many minimal primes and hence B has only finitely many maximal ideals – say $\mathfrak{m}_1, \dots, \mathfrak{m}_r$. By the Chinese remainder theorem, we have

$$(4) \quad \frac{B}{\bigcap_{i=1}^r \mathfrak{m}_i} \cong \frac{B}{\mathfrak{m}_1} \times \dots \times \frac{B}{\mathfrak{m}_r}.$$

Since A is reduced, so is B . Hence $\bigcap_{i=1}^r \mathfrak{m}_i = 0$. Thus from Equation (4), we get that $B \cong \frac{B}{\mathfrak{m}_1} \times \dots \times \frac{B}{\mathfrak{m}_r}$. Since k has characteristic zero and B is finite type over k , for each i , $1 \leq i \leq r$, B/\mathfrak{m}_i is a finite, separable field extension of k , so $\Omega_{\frac{B}{\mathfrak{m}_i}/k} = 0$; see [Sta, Tag090W]).

Finally we conclude $\Omega_{B/k} = 0$, since as abelian groups

$$\Omega_{B/k} \cong \bigoplus_{i=1}^r \Omega_{\frac{B}{\mathfrak{m}_i}/k}.$$

Thus B is étale over k , as desired. \square

Remark 2.11. With additional restrictions on A , the reducedness hypothesis on A in Theorem 2.10 becomes redundant. For example, when k is a perfect field of any characteristic, if $\Omega_{A/k} = 0$ and additionally A is noetherian; or a local ring with maximal ideal \mathfrak{m} such that $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$; or A is an \mathbb{N} -graded k -algebra with A_0 noetherian, then A is automatically reduced; see [MS21, Theorem 3.1, Corollary 3.3, Theorem 3.6] for details.

Remark 2.12. For any characteristic zero field k , Gabber has constructed a k -algebra R_∞ such that $\Omega_{R_\infty/k} = 0$, but R_∞ is not reduced. The idea is to first construct a direct system $\{R_i \mid i \in \mathbb{N}\}$, of finite dimensional local k -algebras such that the maps $R_i \rightarrow R_{i+1}$ are injective and the induced maps $\Omega_{R_i/k} \rightarrow \Omega_{R_{i+1}/k}$ are all zero maps. Then R_∞ is taken to be the union of all R_i 's. See [MS21, Theorem 2.2] for the details of Gabber's construction.

3. APPLICATION TO HOCHSCHILD HOMOLOGY

We give an application of Theorem 2.10 in Hochschild homology. We begin by giving a minimal review of Hochschild homology here.

Definition 3.1. Let A be a commutative ring over a field k . Then the n -th Hochschild homology $\mathrm{HH}_n(A/k)$ is defined to be $\mathrm{Tor}_n^{A \otimes_k A}(A, A)$.

Remark 3.2. Note that Hochschild homology can be defined for any associative k -algebra which is not necessarily commutative. If we denote A° to denote the opposite algebra of A , one can in general define $\mathrm{HH}_n(A) := \mathrm{Tor}_n^{A \otimes_k A^\circ}(A, A)$. Thus, $\mathrm{HH}_n(A)$ is really a “noncommutative invariant” of A , even if A is a commutative algebra.

Remark 3.3. There is an explicit chain complex which can be used to compute Hochschild homology groups in general. It is given by

$$\dots \rightarrow A \otimes_k A \otimes_k A \rightarrow A \otimes_k A \rightarrow A \rightarrow 0,$$

where A lives in degree zero. The differentials $d : A^{\otimes_k n+1} \rightarrow A^{\otimes_k n}$ are given by

$$a_0 \otimes \dots \otimes a_n \rightarrow a_0 a_1 \otimes \dots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \dots \otimes a_n + \dots + (-1)^n a_0 \otimes \dots \otimes a_{n-1} a_n + (-1)^{n+1} a_n a_0 \otimes \dots \otimes a_{n-1}.$$

The complex described above can be viewed as an object in the derived category of A denoted as $D(A)$, where it is quasi-isomorphic to $A \otimes_{A \otimes_k A^\circ}^L A$. This object will be denoted by $\mathrm{HH}(A/k) \in D(A)$.

We recall an important result about the object $\mathrm{HH}(A/k)$. The result is phrased using the language of filtered objects in derived categories and we refer the reader to [BMS19] for the necessary definitions. The proposition below is obtained by left Kan extending the Postnikov filtration from the smooth case.

Proposition 3.4. (Hochschild–Kostant–Rosenberg (HKR) filtration) Let A be a commutative k -algebra as before. Then $\mathrm{HH}(A/k)$ – viewed as an object of (the stable ∞ -category) $D(A)$ admits a natural, complete, descending \mathbb{N} -indexed filtration, whose i -th graded piece is isomorphic to $\wedge^i \mathbb{L}_{A/k}[i]$ for $i \geq 0$.

Proof. See [Mor19, Proposition 2.28] and [BMS19, Section 2.2]. \square

Proposition 3.5. Let k be a field of characteristic 0 and A be a reduced commutative k -algebra. If $\mathrm{HH}_1(A/k) = 0$, then $\mathrm{HH}_i(A/k) = 0$ for all $i \geq 1$.

Proof. We note that $\mathrm{HH}_1(A/k) = \mathrm{Tor}_1^{A \otimes_k A}(A, A) \simeq \Omega_{A/k}$. Therefore, our hypothesis implies that $\Omega_{A/k} = 0$. Since A is reduced, it follows from Theorem 2.10 that A is in fact ind-étale and therefore $\mathbb{L}_{A/k}$ is exact. So $\mathbb{L}_{A/k}$ is isomorphic to 0 when viewed as an object of $D(A)$. Let $\mathrm{Fil}_{\mathrm{HKR}}^n(\mathrm{HH}(A/k))$ denote the HKR filtration on $\mathrm{HH}(A/k)$. Since the i -th graded piece for the HKR filtration is zero for $i \geq 1$ by Proposition 3.4., we see that

$$(5) \quad \mathrm{Fil}_{\mathrm{HKR}}^n(\mathrm{HH}(A/k)) \simeq \mathrm{Fil}_{\mathrm{HKR}}^1(\mathrm{HH}(A/k))$$

for $n \geq 1$. Thus, we have an exact triangle

$$\mathrm{Fil}_{\mathrm{HKR}}^1(\mathrm{HH}(A/k)) \rightarrow \mathrm{HH}(A/k) \rightarrow \wedge^0 \mathbb{L}_{A/k}[0] = A[0].$$

Using the fact that the HKR filtration is complete, we argue that $\mathrm{HH}(A/k) \simeq A[0]$; see for e.g., [BMS19, Definition 5.1] for the definition of a filtered object in the derived category being complete. Indeed, from the completeness of the HKR filtration and Equation (5), it follows that

$$0 \simeq R \varprojlim_n \mathrm{Fil}_{\mathrm{HKR}}^n(\mathrm{HH}(A/k)) \simeq \mathrm{Fil}_{\mathrm{HKR}}^1(\mathrm{HH}(A/k)).$$

However, by the exact triangle above, that implies that $\mathrm{HH}(A/k) \simeq A[0]$. This finishes the proof. \square

Example 3.6. Proposition 3.5 is false if we do not assume the ring to be commutative. A natural source of counterexamples arise from the theory of differential operators. For $n \geq 1$, let A_n denote the n -th Weyl algebra over a field k of characteristic 0; one can also think of A_n as the ring of differential operators of the polynomial ring in n variables over k . Concretely, A_n is an associative unital algebra over k generated by x_1, \dots, x_n and $\partial^1, \dots, \partial^n$ modulo the relations $x_i x_j = x_j x_i$, $\partial_i \partial_j = \partial_j \partial_i$ and $\partial_i x_j - x_j \partial_i = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol.

There is a natural increasing and multiplicative filtration on A_n called the order filtration. Since k has characteristic 0, the associated graded algebra of A_n under the order filtration is a commutative polynomial algebra in $2n$ variables. This implies that A_n is a *reduced* noncommutative k -algebra. We note that

$$\mathrm{HH}_i(A_n) = \begin{cases} k & \text{if } i = 2n, \\ 0 & \text{otherwise;} \end{cases}$$

see [Ric04, section 3.1] or [Sri61, section 5]. This gives a very natural counterexample to Proposition 3.5 if the ring is not assumed to be commutative. Note that A_n is even an “almost commutative ring” in the sense of filtered rings.

4. APPLICATION TO K -REGULARITY

We begin by very briefly recalling the definition of the higher K -groups. For any associative and unital ring A , one can define the nonconnective K -theory spectrum $K(A)$ [TT90], [Wei13]. One defines the K -groups of A , denoted by $K_n(A)$ for $n \in \mathbb{Z}$ to be the n -th homotopy group of the spectrum $K(A)$, i.e.,

$$K_n(A) := \pi_n(K(A)).$$

Remark 4.1. Let us give more elementary descriptions of some of the K -groups that are of relevance to us. We note that $K_0(A)$ is the Grothendieck group of A , which is obtained by group completing the monoid of finitely generated projective A -modules.

Now we explicitly describe $K_1(A)$; see [Wei13, Chapter III, Section 1]. For a ring A , note that we have a sequence of group inclusions

$$\mathrm{GL}_1(A) \hookrightarrow \mathrm{GL}_2(A) \hookrightarrow \dots \hookrightarrow \mathrm{GL}_n(A) \hookrightarrow \dots$$

where the inclusion $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A)$ takes a matrix M to $\begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}$. Let us denote the group obtained by taking union of the above sequence of inclusions by $\mathrm{GL}(A)$. Let $[\mathrm{GL}(A), \mathrm{GL}(A)]$ denote the derived subgroup, i.e., the subgroup generated by the commutators. Then one has

$$K_1(A) = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)].$$

The negative K -groups can also be described explicitly, in an inductive fashion, using an earlier construction of Bass. For $n < 0$, one has

$$K_n(A) = \mathrm{Coker} \left(K_{n+1}(A[t]) \times K_{n+1}(A[t^{-1}]) \rightarrow K_{n+1}(A[t, t^{-1}]) \right).$$

The above description can be obtained by covering \mathbb{P}_A^1 by the two standard affine opens $\mathrm{Spec} A[t]$ and $\mathrm{Spec} A[t^{-1}]$ and using a Mayer–Vietoris sequence argument (see [TT90, Theorem 6.1]).

Definition 4.2. A commutative k -algebra A is defined to be K_n -regular if the natural map

$$K_n(A) \rightarrow K_n(A[x_1, \dots, x_r])$$

is an isomorphism for all $r \geq 0$.

In [Wei80, Question 2.9], Weibel asked the following questions.

Question 4.3 (Weibel). Let R be a commutative noetherian ring of Krull dimension d .

- (1) Is $K_n(R) = 0$ for $n < -d$?
- (2) Does R happen to be K_n -regular for $n \leq -d$?

In [Wei80], Weibel also answered the question when $d = 0$ and 1. This question was answered in [Cor+08] by Cortiñas, Haesemeyer, Schlichting and Weibel for finite type algebras over a field of characteristic zero. The question was completely answered by Kerz, Strunk, and Tamme in [KST18]. See also [Ker18]. Note that when R is a regular noetherian ring, then all the negative K -groups of R vanish [Bas68].

It was proven by Quillen in [Qui73] that a regular noetherian ring is K_n -regular for all integers n . The following was conjectured (and proven in dimensions ≤ 1) by Vorst in [Vor79], which predicts the converse.

Conjecture 4.4 (Vorst). If R is a commutative ring of dimension d , essentially of finite type over a field k , then K_{d+1} -regularity implies regularity.

When R is essentially of finite type over a field k of characteristic zero, Cortiñas, Haesemeyer, and Weibel proved that the above conjecture holds [CHW08]. Positive characteristic variants have been studied by Geisser and Hesselholt in [GH12] and Kerz, Strunk, and Tamme in [KST21].

We point out that all the results above makes certain finiteness assumptions. However, as Vorst mentions in [Vor79], it is not clear if the finiteness assumptions are necessary in Conjecture 4.4. As a consequence of Theorem 2.10, we will deduce some instances of Weibel's question (see Proposition 4.5) and Vorst's conjecture (see Proposition 4.6) without any finiteness or even noetherian assumptions.

Proposition 4.5. Let A be a commutative k -algebra over a field k of characteristic zero such that the module of $(d+1)$ -forms $\wedge^{d+1}\Omega_{A/k} = 0$. Then

- (1) $K_n(A) = 0$ for $n < -d$.
- (2) A is K_n -regular for all $n \leq -d$.

Proof. By Proposition 2.6, the k -transcendence cardinality of A is $\leq d$. It then follows from (2), Proposition 2.3 that any finite type k -subalgebra of A must have Krull dimension $\leq d$. Therefore, by using [Cor+08, Theorem 6.2] and taking filtered colimits over all finite type k -subalgebras of A , we obtain the desired conclusion. \square

Proposition 4.6. Let A be a commutative k -algebra over a field k of characteristic zero such that the module of Kähler differentials $\Omega_{A/k} = 0$. If A is K_1 -regular, then A is ind-étale. Moreover, A is K_n -regular for all integers n .

Proof. One observes that A being K_1 -regular implies that A is reduced. In order to see this, we note some well-known general constructions. For any commutative ring R , taking determinant induces a natural group homomorphism $\det : \mathrm{GL}(R) \rightarrow R^\times$, which factors to give a map

$$\det : K_1(R) \rightarrow R^\times.$$

Here R^\times is the abelian group of units of R . Note that there is also a natural map $R^\times = \mathrm{GL}_1(R) \rightarrow K_1(A)$ which admits a section provided by $\det : K_1(R) \rightarrow R^\times$. Coming back to our situation, the K_1 -regularity of A in particular implies that the map $K_1(A) \rightarrow K_1(A[t])$ induced by the natural inclusion $A \hookrightarrow A[t]$ is an isomorphism. We have the following commutative diagram where the vertical arrows are given by the 'det' maps and the horizontal maps are induced by the inclusion $A \hookrightarrow A[t]$.

$$\begin{array}{ccc} A^\times & \longrightarrow & (A[t])^\times \\ \uparrow & & \uparrow \\ K_1(A) & \longrightarrow & K_1(A[t]) \end{array}$$

Since the vertical maps and the bottom horizontal maps are surjective, the inclusion $A^\times \hookrightarrow (A[t])^\times$ is also surjective. If there were a nonzero nilpotent element $a \in A$, the element $1+at \in (A[t])^\times$ would not come from A^\times . Thus we conclude that A is reduced. Theorem 2.10

now implies that A is ind-étale. For the last part of the proposition, we again note that the K -groups commute with taking direct limits and étale algebras are K_n -regular for all n , which yields the claim. \square

Note that Proposition 4.6 provides a criteria for an algebra being ind-étale in terms of certain condition on the differential forms and K_1 -regularity. It seems to be an interesting question to find higher dimensional generalizations of this proposition, which would give a criteria for ind-smoothness. Motivated by proposition 4.6, we formulate the following question which we do not know how to answer.

Question 4.7. Let k be a field of characteristic zero and A be a k -algebra such that $\wedge^{d+1}\Omega_{A/k} = 0$. Suppose that A is K_n regular for all n . Is A necessarily a direct limit of smooth k -algebras?

The above question imposes K_n -regularity condition on the algebra A , which is motivated by Vorst’s conjecture. However, we point out that there is a difference between the formulation of classical Vorst conjecture and Question 4.7 or Proposition 4.6. In the classical version (see Conjecture 4.4) the K -regularity assumption and the conjectured regularity, both involve absolute notions such as K -groups and regular rings; the essentially finite type hypothesis serves as an assumption making other techniques (such as the crucial usage of the *cdh* topology) applicable in the problem. But, in Question 4.7, the K -regularity assumption involves absolute notions whereas the desired conclusion in the question, namely the ind- k -smoothness is a relative notion (as it refers to the base k); the $\wedge^{d+1}\Omega_{A/k} = 0$ assumption however is again a relative assumption. The latter ensures for example, that all finite type k -subalgebras of A have dimension at most d (by Proposition 2.3, (2) and (7)).

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