$\mathbb{G}_a^{\mathbf{perf}}$–MODULES AND DE RHAM COHOMOLOGY

SHUBHODIP MONDAL

Abstract

We prove that algebraic de Rham cohomology as a functor defined on smooth $\mathbb{F}_p$–algebras is formally étale in a precise sense. To prove this, we define and study the notion of a pointed $\mathbb{G}_a^{\text{perf}}$–module and its refinement which we call a quasi-ideal in $\mathbb{G}_a^{\text{perf}}$ following Drinfeld. Our results show that given de Rham cohomology, one obtains the theory of crystalline cohomology as its *unique* functorial deformation.

Contents

1 Introduction

1.1 Overview of the results

Let X be a scheme over a field k. Grothendieck defined the algebraic de Rham cohomology of X to be the hypercohomology of the algebraic de Rham complex Ω^*_X [Gro66]. When k is a field of characteristic zero, de Rham cohomology forms a Weil cohomology theory for smooth proper varieties over k. But when k has positive characteristic, for example $k = \mathbb{F}_p$, then the theory of de Rham cohomology does not form a Weil cohomology theory. In particular, the de Rham cohomology groups are killed by p. To rectify this situation, Grothendieck [Gro68] and Berthelot [Ber74] devised the theory of crystalline cohomology. For a smooth algebraic variety X over $\mathbb{F}_p$, its (n-truncated) crystalline cohomology $R\Gamma_{\mathrm{crys}}(X/W_n)$ is a deformation of de Rham cohomology; in the sense that $R\Gamma_{\mathrm{crys}}(X/W_n) \otimes^L_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{F}_p \simeq R\Gamma(X, \Omega^*_X)$. However, potentially there could exist some other cohomology theory which is also a deformation of de Rham cohomology. Our goal is to show that this does not happen. In particular, we show that de Rham cohomology theory for varieties over $\mathbb{F}_p$ is *formally étale*. Thus, given the theory of de Rham cohomology, one can realize crystalline cohomology as its *unique* deformation. To make this precise, we let $\mathrm{Alg}^{sm}_{\mathbb{F}_p}$ denote the category of smooth $\mathbb{F}_p$–algebras and $\mathrm{CAlg}(D(A))$ denote the ∞–category of commutative algebra objects in the derived ∞–category $D(A)$ of an Artinian local ring A with residue field $\mathbb{F}_p$. We show the following

Theorem 1.1.1 (Theorem 5.0.1)**.** *Let*

$$dR : Alg^{sm}_{\mathbb{F}_p} \to CAlg(D(\mathbb{F}_p))$$

be the algebraic de Rham cohomology functor. Given an Artinian local ring $(A, \mathfrak{m})$ with residue field $\mathbb{F}_p$, the functor dR admits a unique deformation

$$dR' : Alg^{sm}_{\mathbb{F}_p} \to CAlg(D(A)).$$

Further, the deformation dR' is unique up to unique isomorphism.

The de Rham cohomology functor takes values in coconnective commutative algebra objects in the derived category $D(\mathbb{F}_p)$. In order to avoid talking about deformation theory in such a context, it would be convenient for us if we could work with discrete rings instead. In order to do that, instead of working with de Rham cohomology theory, we work with *derived de Rham* cohomology theory as defined and studied in [Ill72] and [Bha12]. We will write dR to denote derived de Rham cohomology as well; it agrees with the usual algebraic de Rham cohomology for smooth schemes, so the notation is consistent. For our purposes, it has the technical advantage that derived de Rham cohomology theory can be completely understood by its values on a certain class of rings introduced by Bhatt, Morrow and Scholze [BMS19] called *quasiregular semiperfect* (QRSP) algebras. We point out that somewhat similar class of rings appeared in the work of Fontaine and Jannsen as well [FJ13]. If S is a QRSP algebra, its derived de Rham cohomology $\mathrm{dR}(S)$ is then a discrete ring. Therefore, we are equivalently led to the study of dR as a functor from QRSP algebras to discrete $\mathbb{F}_p$–algebras. In fact, after some reductions that are carried out in Section 5.1, Theorem 1.1.1 would follow from the following statement formulated in purely 1–categorical language. Below, QRSP denotes the category of QRSP algebras and Alg_A denotes the category of discrete A–algebras. We show the following

Theorem 1.1.2. *Let $dR : QRSP \to Alg_{\mathbb{F}_p}$ be the derived de Rham cohomology functor. Given an Artinian local ring $(A, \mathfrak{m})$ with residue field $\mathbb{F}_p$, the functor dR admits a deformation $dR' : QRSP \to Alg_A$ which is unique up to unique isomorphism (cf. Section 5.1).*

In Section 3, more generally, we study the category $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})$, where QRSP denotes the category of QRSP algebras. We show that a certain class of functors, which includes the de Rham cohomology functor, can be realized as some kind of "unwinding" (*cf.* Construction 3.4.3) of a much smaller and more tractable structure which we call a *pointed $\mathbb{G}_a^{perf}$–module.* In order to make sure that the process of "unwinding" is well-behaved, we will need to study a special class of pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules, which we call *quasi-ideals* following Drinfeld [Dri20, Def. 3.1.3].

Definition 1.1.3. The functor $\mathrm{Alg}_{\mathbb{F}_p} \to \mathrm{Alg}_{\mathbb{F}_p}$ that sends $S \mapsto S^\flat := \varprojlim_{x \mapsto x^p} S$ can be represented by an affine ring scheme which we denote as $\mathbb{G}_a^{\mathrm{perf}}$. The underlying affine scheme is given by $\mathrm{Spec}\, \mathbb{F}_p[x^{1/p^\infty}]$.

Definition 1.1.4. A pointed $\mathbb{G}_a^{\mathrm{perf}}$–module is the data of a $\mathbb{G}_a^{\mathrm{perf}}$–module scheme X equipped with a map of $\mathbb{G}_a^{\mathrm{perf}}$–module schemes $X \to \mathbb{G}_a^{\mathrm{perf}}$. The data of the map $X \to \mathbb{G}_a^{\mathrm{perf}}$ will be referred to as a *point* (*cf.* Section 2.2). In Remark 2.1.14, we give some justifications for the terminology "point" in this context.

Definition 1.1.5. A pointed $\mathbb{G}_a^{\mathrm{perf}}$–module is called a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ if the data of the point denoted as $d : X \to \mathbb{G}_a^{\mathrm{perf}}$ sits in a commutative diagram as below (*cf.* Definition 3.3.8).

$$\begin{array}{ccc} X \times X & \xrightarrow{\mathrm{id}\times d} & X \times \mathbb{G}_a^{\mathrm{perf}} \\ \downarrow{\scriptstyle d\times \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{action}} \\ \mathbb{G}_a^{\mathrm{perf}} \times X & \xrightarrow{\mathrm{action}} & X \end{array}$$

The commutativity of the above diagram ensures that for an algebra R, the map $X(R) \to \mathbb{G}_a^{\mathrm{perf}}(R)$ viewed as a complex, where $\mathbb{G}_a^{\mathrm{perf}}(R)$ sits in degree zero, has the structure of a differentially graded algebra [Dri20, Remark 3.1.2].

Remark 1.1.6. Later on, we will need to work with $\mathbb{G}_a^{\mathrm{perf}}$–module schemes defined over an Artinian local base ring A with residue field $\mathbb{F}_p$. Most of our constructions are also defined in this generality. However, for the overview, we assume that A is always $\mathbb{F}_p$.

Example 1.1.7. Let W denote the ring scheme of p-typical Witt vectors. Then W has an endomorphism F which is called the Frobenius on W. In this situation the kernel of F on W, written as $W[F]$ can be equipped with the structure of a $\mathbb{G}_a$–module scheme. We note that there is a natural map $W[F] \to \mathbb{G}_a$ of $\mathbb{G}_a$–module schemes. Pulling $W[F]$ back along the map $u : \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$ of ring schemes (Proposition 2.2.16) produces a $\mathbb{G}_a^{\mathrm{perf}}$–module scheme which we call $u^*W[F]$. Then $u^*W[F]$ can be equipped with the structure of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module scheme which is further also a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$. We point out that the group scheme $W[F]$ is isomorphic to the divided power completion of the additive group scheme $\mathbb{G}_a$, which is denoted as $\mathbb{G}_a^\sharp$ in [Dri20]. This isomorphism is also proven in [Dri20, Lemma 3.2.6].

Our goal is to use the data of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module to produce a functor such as de Rham cohomology in a lossless manner. Note that there is a natural functor $\mathfrak{G} : \mathrm{QRSP} \to \mathrm{Alg}_{\mathbb{F}_p}$ which sends $S \mapsto S^\flat$, where $S^\flat$ denotes the tilt of S defined as $S^\flat := \varprojlim_{x \mapsto x^p} S$. In Construction 3.4.3, we construct the (contravariant) unwinding functor denoted by Un which takes in the data of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module as input and produces a functor from $\mathrm{QRSP} \to \mathrm{Alg}_{\mathbb{F}_p}$. As a basic example, we note that the functor $\mathfrak{G}$ is the unwinding of the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module given by $\mathbb{G}_a^{\mathrm{perf}}$ itself. Other examples are noted in Example 1.1.9 and Theorem 1.1.10 below. Restricting our attention to quasi-ideals satisfying a particular property, which we call *nilpotent quasi-ideals* (Definition 3.4.10), we obtain the following.

Theorem 1.1.8 (Proposition 3.4.20)**.** *There is a fully faithful (contravariant) embedding of the category of nilpotent quasi-ideals in* $\mathbb{G}_a^{perf}$ *inside* $Fun(QRSP, Alg_{\mathbb{F}_p})_{\mathfrak{G}/}$ *given by the unwinding functor Un.*

Example 1.1.9. We note that $\operatorname{Spec} \mathbb{F}_p[x^{1/p^\infty}]/x$ can be equipped with the structure of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module which we denote as $\alpha^\flat$. Another way to describe $\alpha^\flat$ is to say that it is the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module underlying the kernel of the map $u : \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$. It is also the same as $u^*\operatorname{Spec} \mathbb{F}_p$ where $\operatorname{Spec} \mathbb{F}_p$ is the pointed $\mathbb{G}_a$–module underlying the zero group scheme. Applying the unwinding functor to $\alpha^\flat$ gives the functor $\mathrm{QRSP} \mapsto \mathrm{Alg}_{\mathbb{F}_p}$ that sends $S \mapsto S$.

Theorem 1.1.10. *Derived de Rham cohomology naturally viewed as an object dR of* $Fun(QRSP, Alg_{\mathbb{F}_p})_{\mathfrak{G}/}$ *is naturally isomorphic to the unwinding of the nilpotent quasi-ideal given by* $u^*W[F]$.

The above results indicate that properties of certain objects of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})_{\mathfrak{G}/}$ can be deduced by studying nilpotent quasi-ideals or more generally pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules which is the subject of Section 2. For example, we define a full subcategory of pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules which we call *fractional rank* 1 pointed $\mathbb{G}_a^{\mathrm{perf}}$–module (*cf.* Definition 2.2.17) which has an initial object given by $\alpha^\flat$. By applying the unwinding functor, using Example 1.1.9 and the universal property of $\alpha^\flat$ mentioned before, one gets the following result.

Theorem 1.1.11 (Proposition 4.0.5)**.** *The natural transformation $gr^0 : dR \to id$ coming from gr^0 of the Hodge filtration in derived de Rham cohomology is the unique natural transformation between dR and id viewed as objects of the category $Fun(QRSP, Alg_{\mathbb{F}_p})_{\mathfrak{G}/}$.*

We study a more refined class of objects which we call *pure fractional rank 1* $\mathbb{G}_a^{\mathrm{perf}}$–module (*cf.* Definition 2.5.7). The full subcategory of pure fractional rank 1 $\mathbb{G}_a$–module has an initial object given by $u^*W[F]$. By applying the unwinding functor, one gets a universal property of the de Rham cohomology functor which we loosely state below.

Theorem 1.1.12 (Universal property of dR)**.** *Derived de Rham cohomology is a final object of a certain full subcategory of $Fun(QRSP, Alg_{\mathbb{F}_p})_{\mathfrak{G}/}$ (cf. Proposition 4.0.4).*

As an application of the universal property, we can deduce the following result [BLM20, Prop. 10.3.1].

Theorem 1.1.13 (Bhatt-Lurie-Mathew)**.** *If we consider algebraic de Rham cohomology as a functor defined on smooth $\mathbb{F}_p$–algebras denoted as dR, then any endomorphism of dR that commutes with the gr^0 map of the Hodge filtration $dR \to id$ is identity.*

Outline of the proof of Theorem 1.1.1. Once we have developed the properties of the unwinding functor Un in Section 3, we try to use it to prove Theorem 1.1.1 in Section 5. We have noted that dR is essentially the data of the quasi-ideal $u^*W[F]$. Our strategy is the following.

1. We reduce the problem to the case where the base Artinian local ring A is $\mathbb{F}_p[\epsilon]/\epsilon^2$.
2. Given any deformation dR' of dR, we extract a quasi-ideal from dR' denoted as $r(\mathrm{dR}')$ which is a deformation of $u^*W[F]$.
3. We show that dR' is essentially determined by the quasi-ideal $r(\mathrm{dR}')$.
4. We show that any deformation of $u^*W[F]$ to $\mathbb{F}_p[\epsilon]/\epsilon^2$ as a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module is uniquely isomorphic to the trivial deformation obtained by base change. This is proven in Proposition 2.5.11. Therefore $r(\mathrm{dR}')$ is necessarily the trivial deformation of $u^*W[F]$ and by 3, dR' is necessarily the trivial deformation $\mathrm{dR}\otimes\mathbb{F}_p[\epsilon]/\epsilon^2$ as well.

Other approaches to Theorem 1.1.1. Our approach to Theorem 1.1.1 uses QRSP algebras in an essential way in order to *not* deal with deformation theory of coconnective E_∞–rings. Our construction of the unwinding functor Un is also devised in a way to work with the category $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})$. However, in principle, this is not absolutely necessary. Below we attempt to loosely explain other possible approaches that could be seen as more natural.

By the reduction in Section 5.1, it is equivalent to address the version of Theorem 1.1.1 for the category $\mathrm{Poly}_{\mathbb{F}_p}$ of finitely generated polynomial algebras over $\mathbb{F}_p$ instead of all smooth algebras. Instead of studying the category $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})$, we can study the category $\mathrm{Fun}(\mathrm{Poly}_{\mathbb{F}_p}, \mathrm{CAlg}(D(\mathbb{F}_p)))$. A functor $F \in \mathrm{Fun}(\mathrm{Poly}_{\mathbb{F}_p}, \mathrm{CAlg}(D(\mathbb{F}_p)))$ that preserves coproducts would provide an $\mathbb{F}_p$–coalgebra object structure on the E_∞–ring $F(\mathbb{F}_p[x])$ coming from the $\mathbb{F}_p$–coalgebra structure of $\mathbb{F}_p[x]$ as an object of $\mathrm{Poly}_{\mathbb{F}_p}$. One can also try to reverse the situation, i.e., given an E_∞–ring K with the extra structure of an $\mathbb{F}_p$–coalgebra object, one can try to build a functor $\mathrm{Un}_K : \mathrm{Poly}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p))$ that would send $\mathbb{F}_p[x] \mapsto K$ and extend in a coproduct preserving way. This version of "unwinding" is explained in Example 3.0.1 (in a 1-categorical language). Assuming good properties of these constructions, in order to approach Theorem 1.1.1, we are led to studying the deformations of the E_∞–ring $\mathrm{dR}(\mathbb{F}_p[x])$ along with the extra structure of an $\mathbb{F}_p$–coalgebra object.

Using the stacky approach to p–adic cohomology theories due to Bhatt-Lurie [BL] and Drinfeld [Dri18] [Dri20], one can ask a similar question regarding deformation of the $\mathbb{F}_p$–algebra stack $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathrm{dR}}$ relevant to Theorem 1.1.1. This is a stack whose cohomology of the structure sheaf recovers $\mathrm{dR}(\mathbb{F}_p[x])$. Deformations of $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathrm{dR}}$ as an $\mathbb{F}_p$–algebra stack seems to be relevant to Theorem 1.1.1. Further, using [Dri20, Prop. 3.5.1], $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathrm{dR}}$ is the cone of the quasi-ideal given by $W[F]$. Therefore, deformations of $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathrm{dR}}$ as an $\mathbb{F}_p$–algebra stack seem related to deformations of the quasi-ideal or the pointed $\mathbb{G}_a$–module given by $W[F]$ which is studied in Section 2 of our paper.

In the approach we have taken in this paper (which uses QRSP algebras) we can avoid talking about higher categorical structures and obtain a purely 1-categorical formulation as mentioned in Theorem 1.1.2.

Further, the notion of a pointed $\mathbb{G}_a$–module or a quasi-ideal comes out quite naturally (*cf.* Proposition 3.4.7). As a downside, the construction of "unwinding" seems more convoluted for QRSP than what it would be for $\mathrm{Poly}_{\mathbb{F}_p}$. We use quasisyntomic descendability and left Kan extensions to switch between QRSP and $\mathrm{Poly}_{\mathbb{F}_p}$, which could potentially be avoided in the other approaches outlined above.

In any case, we point out that a precise formulation of the deformation problems involving the E_∞–ring $\mathrm{dR}(\mathbb{F}_p[x])$ or the $\mathbb{F}_p$–algebra stack $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathrm{dR}}$ would likely be equivalent to Theorem 1.1.1 and therefore they are answered *a posteriori* after proving Theorem 1.1.1. Also, a comparison of these approaches can lead to other questions as well. For example, motivated by Theorem 1.1.12, one can attempt to formulate a universal property for the stack $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathrm{dR}}$ in the category of $\mathbb{F}_p$–algebra stacks. In Remark 5.2.7, we explain a rough comparison between the stacky approach and the approach taken in our paper.

1.2 Motivations and related work

In this section we describe the motivations behind the constructions appearing in this paper and other related work. Our starting point was to approach Theorem 1.1.1 which asks about deformations of a functor (instead of a single object) which we regard as somewhat inconvenient. The strategy of the proof outlined above is vaguely inspired by some constructions from chromatic homotopy theory. Given a complex oriented multiplicative cohomology theory E^*, one can extract a formal group law from it by looking at $E^*(\mathbb{C}P^\infty)$ and using the multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Further, given a formal group law, the Conner-Floyd construction [CF66] defines a "cohomology theory" associated to it. Motivated by this picture, one can ask the following naive questions in our context.

Is there a way to extract a "group like object" from de Rham cohomology (or its deformations)? Further, is the theory of de Rham cohomology (and its deformations) determined by this "group like object"?

By the de Rham-crystalline comparison theorem [Ber74, Thm. V.2.3.2], the theory of de Rham cohomology is essentially determined by the theory of divided power structures. This can be seen more concretely by using the work of Bhatt on derived de Rham cohomology [Bha12]. Given a QRSP algebra S, its derived de Rham cohomology $\mathrm{dR}(S)$ is naturally isomorphic to the divided power envelope $D_{S^\flat}(I)$ where $I := \mathrm{Ker}(S^\flat \to S)$(*cf.* Proposition 4.0.2). Setting $S := \mathbb{F}_p[x^{1/p^\infty}]/x$ and considering $\mathrm{dR}(\mathbb{F}_p[x^{1/p^\infty}]/x)$, we get the ring of functions underlying $u^*W[F]$ from Example 1.1.7. Further, the Hopf stucture of $\mathbb{F}_p[x^{1/p^\infty}]/x$ provides a Hopf structure on $\mathrm{dR}(\mathbb{F}_p[x^{1/p^\infty}]/x)$ which is the same as the Hopf algebra underlying the ring of functions on $u^*W[F]$. This addresses the first half of our question above and extracts the "group like object" $u^*W[F]$ from dR.

For the second half, one needs to build the de Rham cohomology functor from the object $u^*W[F]$. By the isomorphism $\mathrm{dR}(S) \simeq D_{S^\flat}(I)$ for a QRSP algebra S, it would be enough to build divided power envelopes from $u^*W[F]$. In [BO78, Appendix 2], Berthelot-Ogus constructs the closely related divided power algebra $\Gamma_R(M)$ for any ring R and an R–module M by using a particular R–module called $\exp(R)$. We note that there is an isomorphism $\exp(R) \simeq W[F](R)$, where the latter denotes the R–valued points of the group scheme $W[F]$. This suggests that in principle, it could be possible to build divided power envelopes from $u^*W[F]$. However, we need to equip the group scheme $u^*W[F]$ with more structure. This leads to the definition of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module, which is the framework for our "group like object". In Example 3.3.18, we see that using the unwinding functor, it is indeed possible to directly build divided power envelopes (and consequently derived de Rham cohomology) out of the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module $u^*W[F]$. This addresses the second half of our question as well.

The connection between $u^*W[F]$ or $W[F]$ and de Rham cohomology has been recently observed in other contexts as well. One of them is the stacky approach to p–adic cohomology theories by Drinfeld [Dri20]. The "crystallization" of $\mathbb{A}^1_{\mathbb{F}_p}$ is a stack that is obtained by taking the cone of the quasi-ideal $W[F]$ in $\mathbb{G}_a$. The notion of a quasi-ideal is also due to Drinfeld and in general, a ring stack can be created out of a quasi-ideal by considering its cone. These constructions can all be found in [Dri20]. For us, a quasi-ideal is used as a special kind of a pointed $\mathbb{G}_a$ or a $\mathbb{G}_a^{\mathrm{perf}}$–module for which the unwinding functor is particularly well-behaved. In Proposition 3.2.18, we show that the (opposite) category of quasi-ideals can be embedded in a certain naturally defined category.

The connection between $W[F]$ and de Rham cohomology also appears in the work of Moulinos, Robalo and Toën on Hochschild homology [MRT20]. In this context, Hodge cohomology appears as the associated graded object of the HKR filtration on Hochschild homology. The authors construct a filtered stack (over a p–adic base) which they call the filtered circle. The associated graded stack of the filtered circle is given by the classifying stack $BW[F]$. They show that Hochschild homology can be studied through this filtered circle where the filtration on the filtered circle induces the HKR filtration on Hochschild homology. Their work also gives a different way of thinking about the group scheme $W[F]$: the classifying stack $BW[F]$ is the affine stack corresponding to the cosimplicial ring given by the trivial square zero extension $\mathbb{F}_p \oplus \mathbb{F}_p[-1]$. The stack $BW[F]$ also appears in the work of Toën in [Toë20], where it is used to define derived foliations on schemes.

A universal property of the Hodge completed derived de Rham complex was recently obtained in [Rak20] and motivated us to look for a universal property for dR from our perspective as in Theorem 1.1.12.

Finally, we note that the analogue of Theorem 1.1.1 for $A = \mathbb{Z}_p$ (instead of an Artinian local ring) was already known due to the work of Bhatt, Lurie and Mathew [BLM20, Thm. 10.1.2], which they use to give a new proof of the de Rham Witt to crystalline comparison theorem of Illusie [Ill79, Thm. II.1.4]. The Theorem 1.1.1 in our paper also allows torsion base rings A. The technique of restricting attention to QRSP algebras via left and right Kan extensions used in this paper is directly motivated by their proof of Theorem 10.1.2. A variant of questions regarding endomorphisms of the de Rham cohomology functor appears in the work of Li and Liu [LL20].

1.3 Acknowledgements

I am extremely grateful to Bhargav Bhatt for proposing the main question regarding deformations of de Rham cohomology studied in this paper as well as for his patience, interest and many suggestions throughout the project. Numerous discussions with him have improved many aspects of this paper. Much of this paper relies on his earlier work on p-adic derived de Rham cohomology; he also suggested the use of "descendability" which is an important ingredient in some of the proofs. I am thankful to Attilio Castano, Andy Jiang and Emanuel Reinecke for many helpful discussions during the preparation of this paper. I would also like to thank Benjamin Antieau, Vladimir Drinfeld, Haoyang Guo, Luc Illusie, Arthur-César Le bras, Shizhang Li, Akhil Mathew, Arpon Raksit and Bertrand Toën for comments on a draft version of the paper. During the preparation of the paper, I was partially supported by a Rackham International Students Fellowship and NSF grant DMS #1801689 through Bhargav Bhatt.

2 Modules over ring schemes

In this section, we begin with the definition of a $\mathbb{G}_a$ and $\mathbb{G}_a^{\text{perf}}$–module leading up to the notion of a *pointed* $\mathbb{G}_a$ or $\mathbb{G}_a^{\text{perf}}$–module. Our final goal is to study the deformation of the pointed $\mathbb{G}_a$–module $W[F]$, which is obtained from the kernel of Frobenius on Witt vectors and its closely related variant $u^*W[F]$, which is a pointed $\mathbb{G}_a^{\text{perf}}$–module. This will in part be achieved via attaching universal properties to the objects $W[F]$ and $u^*W[F]$ as objects in certain categories. Construction of such categories leads to several refinements of the category of pointed $\mathbb{G}_a^{\text{perf}}$–modules which we call pointed $\mathbb{G}_a^{\text{perf}}$–modules of *fractional rank 1* (*cf.* Definition 2.2.17), *full of fractional rank* 1 (*cf.* Definition 2.3.4) and *pure of fractional rank* 1 (*cf.* Definition 2.5.7).

Notation 2.0.1. The notion of a $\mathbb{G}_a$–module is valid over any base ring A. However, the notion of a $\mathbb{G}_a^{\text{perf}}$–module will require us to fix a prime p. In fact, $\mathbb{G}_a^{\text{perf}}$–modules will only be defined over a base ring where p is nilpotent. The category of A–algebras will be denoted as Alg_A. Its opposite category, i.e., the category of affine scheme will be denoted by Aff_A. All schemes considered are affine schemes unless otherwise mentioned. The group schemes we consider are always assumed to be commutative.

2.1 $\mathbb{G}_a$–modules

Definition 2.1.1. For an arbitrary ring R, we say that X is an *R–module scheme* over A if X is a scheme over A and the functor $F_X : \text{Aff}_A \to \text{Sets}$ given by $F(S) := \text{Hom}_{\text{Aff}_A}(S, X)$ is naturally valued in R–modules.

Remark 2.1.2. In particular, Definition 2.1.1 implies that X is a group scheme. Further, the ring R in Definition 2.1.1 is arbitrary and not required to be an A–algebra.

Example 2.1.3. Taking $X = \operatorname{Spec} A[x]$, we see that X can be equipped with the structure of an A–module scheme.

Example 2.1.4. We note that $\operatorname{Spec} A[x]$ is naturally a ring scheme over A. We will denote this ring scheme by $\mathbb{G}_a$. It represents the functor that sends an affine scheme to its ring of global sections.

Definition 2.1.5. An affine scheme X over A will be called a $\mathbb{G}_a$–module if the functor $F_X : \text{Aff}_A \to \text{Sets}$ given by $F_X(S) := \text{Hom}_{\text{Aff}_A}(S, X)$ is a presheaf of modules over the presheaf of rings represented by $\mathbb{G}_a$. Morphisms are defined as morphisms of presheaves of modules over the presheaf of rings represented by $\mathbb{G}_a$.

Example 2.1.6. $\mathbb{G}_a$ can be equipped with the structure of a $\mathbb{G}_a$–module. If A has char p, then $\operatorname{Spec} A[x]/x^p$ can be equipped with the structure of a $\mathbb{G}_a$–module.

Remark 2.1.7. We note that the affine scheme underlying a $\mathbb{G}_a$–module in particular has the action of $\mathbb{G}_a$ and thus global sections on it gives a nonnegatively graded Hopf algebra. In other words, every $\mathbb{G}_a$–module has the structure of a nonnegatively graded group scheme. We refer to [MM65] for a study of graded Hopf algebras.

Remark 2.1.8. The notion of a $\mathbb{G}_a$–module extends to any scheme which is not *a priori* assumed to be affine. However, we note that being a $\mathbb{G}_a$–module imposes strong topological restrictions on the underlying scheme. In particular, it follows that the scheme underlying any $\mathbb{G}_a$–module is $\mathbb{A}^1$–contractible. In fact, any scheme which is a $\mathbb{G}_a$–module over a field is necessarily affine. We thank Drinfeld for pointing this out. More generally, we have the following

Proposition 2.1.9. *Any $\mathbb{A}^1$–contractible group scheme G over a field k is affine.*

Proof. To prove this assertion, we can assume that k is algebraically closed. Since G is $\mathbb{A}^1$–contractible, it follows that G is connected. By a modification of Chevalley's theorem due to Perrin [Per76, Cor. 4.2.9], there exists an exact sequence $0 \to H \to G \to A \to 0$ of group schemes in the fpqc topology, where H is an affine group scheme and A is an abelian variety. Since G is $\mathbb{A}^1$–contractible, we obtain a distinguished k–rational point of G, whose image in A will be denoted by $c \in A(k)$. Our claim would follow if we prove that $A(k) = 0$, where $A(k)$ denotes the k–valued points of A. We let $t \in A(k)$. Since $G \to A$ is an fpqc surjection, we can find an algebraically closed field K, which contains k and such that there exists $t' \in G(K)$ which is mapped to the image of t in $A(K)$. Using t' and $\mathbb{A}^1$-contractibility of G, we obtain a map $\mathbb{A}^1_K \to A_K$ such that $\{1\} \in \mathbb{A}^1_K$ is mapped to the image of t in $A(K)$ and $\{0\} \in \mathbb{A}^1_K$ maps to image of c in $A(K)$. Any such map extends to a map $\mathbb{P}^1_K \to A_K$ and since A_K is an abelian variety, any such map has to factor through the Jacobian of $\mathbb{P}^1_K$, which is a point. Thus the map $\mathbb{A}^1_K \to A_K$ is constant. By fpqc sheaf property, the map $A(k) \to A(K)$ is injective. This implies that $t = c \in A(k)$. Since t was arbitrary, it follows that $A(k)$ consists of a single point and thus $A(k) = 0$, which gives the claim. ☐

Proposition 2.1.10. *The forgetful functor from the category of $\mathbb{G}_a$–modules to the category of graded group schemes is fully faithful.*

Proof. Let $\operatorname{Spec} U$ and $\operatorname{Spec} V$ be two $\mathbb{G}_a$ modules and let $\operatorname{Spec} U \to \operatorname{Spec} V$ be a map of $\mathbb{G}_a$–modules. This is the data of a map $V \to U$ that is a Hopf algebra map and is equivariant with the $A[x]$–coaction, i.e., commutes with the $A[x]$–coaction maps $U \to U[x]$ and $V \to V[x]$. However, the latter compatiblity can be checked after composing along the injective maps $U[x] \to U[x^{\pm 1}]$ and $V[x] \to V[x^{\pm 1}]$ and thus it is enough to provide a map $V \to U$ which is compatible with the $A[x^{\pm 1}]$–coaction, i.e., a graded Hopf algebra map $V \to U$. ☐

Remark 2.1.11. We note that a graded group scheme being a $\mathbb{G}_a$–module is no extra data, but a condition. This condition is not always satisfied. For example, the Witt group scheme W which represents the functor $\mathrm{Aff}_A \to \mathrm{Sets}$ given by $\operatorname{Spec} B \mapsto W(B)$ where $W(B)$ is the ring of p-typical Witt vectors of B is a graded group scheme but not a $\mathbb{G}_a$–module.

Proposition 2.1.12. *Let $\operatorname{Spec} B$ be a $\mathbb{G}_a$–module. Then as a graded algebra, the degree zero piece of B is isomorphic to A as an A–module. In other words, as a graded Hopf algebra, B is connected.*

Proof. First, by using the zero section of a group scheme, we note that the A–algebra structure map $A \to B$ is injective. The $\mathbb{G}_a$–module structure map is given by a map $B \to B[x]$. Killing x produces a map $B \to B$ whose kernel is $I_{>0} := \bigoplus_{i>0} B_i$ where $B = \bigoplus_{i \geq 0} B_i$. Further, using the fact that $\operatorname{Spec} B$ is a $\mathbb{G}_a$–module, we note that the map $B \to B$ obtained this way also has the property that it factors through $B \to A$, which is the zero map of the comultiplication. Since the map $A \to B$ is injective, this provides an A–algebra map $B \to A$ whose kernel is $I_{>0}$. Thus B_0 is naturally isomorphic to A, as desired. ☐

Definition 2.1.13 (Pointed $\mathbb{G}_a$–module)**.** A $\mathbb{G}_a$–module scheme X along with the data of a map $X \to \mathbb{G}_a$ of $\mathbb{G}_a$–modules will be called a *pointed* $\mathbb{G}_a$*–module* X. We will follow the convention that the data of the map $X \to \mathbb{G}_a$ will be simply called a point. Maps between pointed $\mathbb{G}_a$–modules are maps of $\mathbb{G}_a$–modules that commute with the points. We denote the category of such objects by $\mathbb{G}_a$–Mod_*.

Remark 2.1.14. If $X = \operatorname{Spec} B$ is a $\mathbb{G}_a$–module, then we note that giving a map $X \to \mathbb{G}_a$ of $\mathbb{G}_a$–modules is equivalent to choosing an element of degree 1 in the graded algebra B. This follows from the fact that if x is an element of degree 1 in B, then the comultiplication map $B \to B \otimes_A B$ sends $x \to x \otimes 1 + 1 \otimes x$. Thus a pointed $\mathbb{G}_a$–module is the data of a $\mathbb{G}_a$–module $\operatorname{Spec} B$ and an element $x \in B_1$ (where B_1 is the degree 1 piece of B). The choice of this element $x \in B_1$ is the reason we use the word "point" to talk about the map $X \to \mathbb{G}_a$; it is motivated by the terminology in topology where a space Y and a choice of an element $y \in Y$ is called a pointed space. Using functor of points, in our case, this can also be interpreted as an X–valued point of $\mathbb{G}_a$.

Remark 2.1.15. $\mathbb{G}_a$ can be naturally equipped with the structure of a pointed $\mathbb{G}_a$–module using the identity map $\mathbb{G}_a \to \mathbb{G}_a$. In fact, $\mathbb{G}_a$ is the final object of $\mathbb{G}_a$–Mod_*. The initial object of $\mathbb{G}_a$–Mod_* is given by the zero object, i.e., the group scheme underlying the base $\operatorname{Spec} A$. Proposition 2.4.10 and Remark 2.4.11 records more examples of pointed $\mathbb{G}_a$–modules.

2.2 $\mathbb{G}_a^{\text{perf}}$–modules

Below we define the notion of a $\mathbb{G}_a^{\text{perf}}$–module which will be defined over a fixed ring A such that $p^n = 0$ in A for some n.

Proposition 2.2.1. *The functor $()^\flat : \mathrm{Aff}_A \to \mathrm{Sets}$ given by $\operatorname{Spec} B \to B^\flat$, where $B^\flat := \lim_{x \to x^p} B$ is naturally valued in rings.*

Proof. This follows from the natural bijection $B^\flat \simeq \lim_{x\to x^p} B/p$, which holds since B, being an A algebra, is p–adically complete. $\square$

Definition 2.2.2. The functor $()^\flat$ from Proposition 2.2.1 is represented by $\operatorname{Spec} A[x^{1/p^\infty}]$, which is naturally a ring scheme and will be denoted as $\mathbb{G}_a^{\text{perf}}$ when equipped with this ring scheme structure.

Remark 2.2.3. When A is an $\mathbb{F}_p$–algebra, the comultiplication of the Hopf algebra underlying $A[x^{1/p^\infty}]$ can be described easily: it is given by the map $A[x^{1/p^\infty}] \to A[x^{1/p^\infty}]\otimes_A A[x^{1/p^\infty}]$ given by $x^{1/p^n} \to x^{1/p^n}\otimes 1 + 1\otimes x^{1/p^n}$ for all n. However, in general the comultiplication is less simple to write down and we need to trace through the bijection $B^\flat \simeq \lim_{x\to x^p} B/p$. For example, when A is a $\mathbb{Z}/p^2\mathbb{Z}$ algebra, the comultiplication is given by $A[x^{1/p^\infty}] \to A[x^{1/p^\infty}]\otimes_A A[x^{1/p^\infty}]$ which sends $x^{1/p^n} \to x^{1/p^n}\otimes 1 + 1\otimes x^{1/p^n} + \sum_{0<i<p}\binom{p}{i} x^{i/p^{n+1}}\otimes x^{p-i/p^{n+1}}$.

Remark 2.2.4. When A is an $\mathbb{F}_p$–algebra, the A–algebra map $A[x] \to A[x^{1/p^\infty}]$ gives us a morphism of ring schemes $\mathbb{G}_a^{\text{perf}} \to \mathbb{G}_a$. However, using Remark 2.2.3 we see that if $p \neq 0$ in A, the natural map $A[x] \to A[x^{1/p^\infty}]$ is not a map of Hopf algebras and hence does not give a morphism of ring schemes.

Definition 2.2.5. An affine scheme X over A will be called a $\mathbb{G}_a^{perf}$*–module* if the functor $F_X : \mathrm{Aff}_A \to \mathrm{Sets}$ given by $F_X(S) := \mathrm{Hom}_{\mathrm{Aff}_A}(S, X)$ is a presheaf of modules over the presheaf of rings represented by $\mathbb{G}_a^{\text{perf}}$. Morphisms are defined as morphisms of presheaves of modules over the presheaf of rings represented by $\mathbb{G}_a^{\text{perf}}$.

Example 2.2.6. When A has char. p, the scheme $\operatorname{Spec} A[x^{1/p^\infty}]/x$ can be equipped with the structure of a $\mathbb{G}_a^{\text{perf}}$–module which we will denote as $\alpha^\flat$. If $p \neq 0$ in A, the Hopf structure of $A[x^{1/p^\infty}]$ as described in Remark 2.2.3 does not induce a Hopf structure in the quotient $A[x^{1/p^\infty}]/x$.

Remark 2.2.7. A $\mathbb{G}_a^{\text{perf}}$–module $\operatorname{Spec} B$ naturally provides us a $\mathbb{G}_m^{\text{perf}} := \operatorname{Spec} A[x^{\pm 1/p^\infty}]$ equivariant group scheme $\operatorname{Spec} B$, or in other words a $\mathbb{N}[1/p] \cup \{0\}$ graded group scheme structure on $\operatorname{Spec} B$. Further, the forgetful functor from $\mathbb{G}_a^{\text{perf}}$–modules to $\mathbb{N}[1/p] \cup \{0\}$ graded group schemes is fully faithful.

Proposition 2.2.8. *Let Spec B be a $\mathbb{G}_a^{perf}$–module. Then as a graded algebra, the degree zero piece of B is isomorphic to A as an A–module.*

Proof. First, by using the zero section of a group scheme, we note that the A–algebra structure map $A \to B$ is injective. The $\mathbb{G}_a^{\text{perf}}$–module structure map is given by a map $B \to B[x^{1/p^\infty}]$. Killing x^{1/p^n} for all n produces a map $B \to B$ whose kernel is $I_{>0} := \bigoplus_{i>0} B_i$ where $B = \bigoplus_{i\in\mathbb{N}[1/p]\cup 0} B_i$. Further, using the fact that $\operatorname{Spec} B$ is a $\mathbb{G}_a^{\text{perf}}$–module, we note that the map $B \to B$ obtained this way also has the property that it factors through $B \to A$ which is the zero map of the comultiplication. Since the map $A \to B$ is injective, this provides an A–algebra map $B \to A$ whose kernel is $I_{>0}$. Thus B_0 is naturally isomorphic to A, as desired. $\square$

Remark 2.2.9. A $\mathbb{G}_a^{\text{perf}}$–module over $\mathbb{F}_p$ amounts to the following data. For every $\mathbb{F}_p$–algebra S, we have an $S^\flat$–module scheme $\operatorname{Spec} M_S$ over S such that for a map $\varphi : S \to R$ of $\mathbb{F}_p$–algebras, we have isomorphisms $\mathrm{res}_S^R : M_S \otimes_{\mathbb{F}_p} R \simeq M_R$ of R–algebras. Further, the action of $S^\flat$ on M_S provides an endomorphism of M_S for every $s^\flat \in S^\flat$ which under the isomorphism res_S^R corresponds to the endomorphism induces by $\varphi^\flat(s^\flat)$ on M_R. Here $\varphi^\flat$ denotes the map $S^\flat \to R^\flat$. Using the map $S^\flat \to S$ for an $\mathbb{F}_p$–algebra S, we see that it is enough to specify the same data only on perfect rings.

Proposition 2.2.10. *Let $(A, \mathfrak{m})$ be an Artinian local ring with residue field $\mathbb{F}_p$. For every perfect ring R, let $W_A(R)$ be a fixed choice of functorial lifts (which is unique up to unique isomorphism) of R to Alg_A. Then a $\mathbb{G}_a^{perf}$–module over A is equivalent to the following data:*

1. For every perfect ring R, an R–module scheme Spec M_R over $W_A(R)$.

2. For every map $S \to R$ of perfect rings, an isomorphism $\mathrm{res}_S^R : M_S \otimes_{W_A(S)} W_A(R) \simeq M_R$. Further, in this situation the S–module action on the left hand side is compatible with the R–module action on the right hand side via the map $S \to R$. The latter is a condition and not extra data.

Further, a morphism of $\mathbb{G}_a^{perf}$–module under this equivalence translates to the following data:

1. For every perfect ring R, a morphism Φ_R of R–module schemes over $W_A(R)$.

2. For every map $S \to R$ of perfect rings, the maps Φ_R and Φ_S are compatible with res_S^R.

Proof. We note that $W_A(R)$ is an A–algebra. So given a $\mathbb{G}_a^{\mathrm{perf}}$–module scheme over $W_A(R)$, we obtain an $W_A(R)^\flat$–module scheme which will be denoted as M_R. Thus one direction of the proposition will follow from the following lemma.

Lemma 2.2.11. *In the above set up, $W_A(R)^\flat \simeq R$.*

Proof. There is a natural map $W_A(R)^\flat := \lim_{x\to x^p} W_A(R)/p \to \lim_{x\to x^p} W_A(R)/\mathfrak{m}$ since $p \in \mathfrak{m}$. Since $W_A(R)/\mathfrak{m}$ is isomorphic to R, which is a perfect ring, it will be enough to prove that the natural map is an isomorphism. We know that the natural map $\lim_{x\to x^p} W_A(R) \to W_A(R)^\flat$ is a set theoretic bijection since $W_A(R)$ is p–adically complete. Thus it is enough to show that the natural map $\lim_{x\to x^p} W_A(R) \to \lim_{x\to x^p} W_A(R)/\mathfrak{m}$ is a set theoretic bijection. First we check injectivity. Let (a_n) and (b_n) be two sequences in $\lim_{x\to x^p} W_A(R)$ such that $a_n = b_n \bmod \mathfrak{m}$. For every k, we have $a_{n+k}^{p^k} = a_n$ and $b_{n+k}^{p^k} = b_n$. Since $a_{n+k} = b_{n+k} \bmod \mathfrak{m}$, one inductively checks using $p \in \mathfrak{m}$ that $a_n = b_n \bmod \mathfrak{m}^{k+1}$. Since k was arbitrary, and the ideal $\mathfrak{m}$ is nilpotent, this checks the injectivity. For surjectivity, we fix $(\overline{a_n}) \in \lim_{x\to x^p} W_A(R)/\mathfrak{m}$. We choose arbitrary lifts a_n of $\overline{a_n}$ to $W_A(R)$. For every k, we have $a_{n+k+1}^p = a_{n+k} \bmod \mathfrak{m}$. Thus, $a_{n+k+1}^{p^{k+1}} = a_{n+k}^{p^k} \bmod \mathfrak{m}^{k+1}$. Since $\mathfrak{m}$ is nilpotent, the sequence $k \to a_{n+k}^{p^k}$ is eventually constant, and we define the limit element to be b_n. Now it follows that $b_{n+1}^p = b_n$ and b_n lifts $\overline{a}_n$, which proves the required surjectivity. ☐

For the opposite direction, we are given with the data of an R–module scheme $\operatorname{Spec} M_R$ over $W_A(R)$ for every perfect ring R. In order to obtain a $\mathbb{G}_a^{\mathrm{perf}}$–module, we are required to provide the data of a $B^\flat$–module scheme $\operatorname{Spec} M_B$ over B for every A–algebra B. For this, we note the following lemma.

Lemma 2.2.12. *Let B be an A–algebra. There is a natural map $W_A(B^\flat) \to B$ which induces an isomorphism $W_A(B^\flat)^\flat \to B^\flat$.*

Proof. There is a natural map $B^\flat \to B/p \to B/\mathfrak{m}$. This gives a natural map of A–algebras $W_A(B^\flat) \to B^\flat \to B/\mathfrak{m}$. We note that $W_A(B^\flat)$ is a flat A–algebra by definition. Since $B^\flat$ is perfect, we have $\mathbb{L}_{B^\flat/\mathbb{F}_p} = 0$ implying $\mathbb{L}_{W_A(B^\flat)/A} = 0$. This implies that $W_A(B^\flat)$ is a formally étale A–algebra. Since the map $B \to B/\mathfrak{m}$ has nilpotent kernel, it follows that the map $W_A(B^\flat) \to B/\mathfrak{m}$ lifts uniquely to provide a map $W_A(B^\flat) \to B$, as desired. The map $W_A(B^\flat)^\flat \to B^\flat$ is an isomorphism by Lemma 2.2.11. ☐

Now we can define $M_B := M_{B^\flat} \otimes_{W_A(B^\flat)} B$. Then $\operatorname{Spec} M_B$ automatically has the structure of a $B^\flat$–module scheme. This data determines a $\mathbb{G}_a^{\mathrm{perf}}$–module. ☐

Definition 2.2.13 (Pointed $\mathbb{G}_a^{\mathrm{perf}}$–module)**.** A $\mathbb{G}_a^{\mathrm{perf}}$–module scheme X over A along with the data of a map $X \to \mathbb{G}_a^{\mathrm{perf}}$ of $\mathbb{G}_a^{\mathrm{perf}}$–modules will be called a *pointed $\mathbb{G}_a^{perf}$–module* X. We will follow the convention that the data of the map $X \to \mathbb{G}_a^{\mathrm{perf}}$ will be simply called a point. Maps between pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules are maps of $\mathbb{G}_a^{\mathrm{perf}}$–modules that commute with the points. We denote the category of such objects by $\mathbb{G}_a^{\mathrm{perf}}$–$\mathrm{Mod}_*$.

Example 2.2.14. Let A be an $\mathbb{F}_p$–algebra. The Hopf algebra $A[x^{1/p^\infty}]/x$ along with the natural map $A[x^{1/p^\infty}] \to A[x^{1/p^\infty}]/x$ equips $\operatorname{Spec} A[x^{1/p^\infty}]/x$ with the structure of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module scheme. This will be denoted as $\alpha^\flat$. An analogue of this does not exist when A does not have characteristic p.

Remark 2.2.15. Unlike the case of $\mathbb{G}_a$–modules from Remark 2.1.14, for a $\mathbb{G}_a^{\mathrm{perf}}$–module $X = \operatorname{Spec} B$, it is not true that giving a map $X \to \mathbb{G}_a^{\mathrm{perf}}$ is equivalent to choosing an element of degree 1 in B.

For the remainder of this section, we will work over a base ring A of characteristic p.

Proposition 2.2.16 (Pullback functor)**.** *Let A be an $\mathbb{F}_p$–algebra. Then there is a map of ring schemes $u : \mathbb{G}_a^{perf} \to \mathbb{G}_a$ over A. Further, pullback along this map defines a fully faithful functor $u^* : \mathbb{G}_a$–$Mod_* \to \mathbb{G}_a^{perf}$–$Mod_*$.*

Proof. The first part follows from the natural map $S^\flat \to S$ for every A–algebra S since $\mathbb{G}_a^{\mathrm{perf}}(S) = S^\flat$ and $\mathbb{G}_a(S) = S$. For the second part, let X be a pointed $\mathbb{G}_a$–module over A. Then we have a map $X(S) \to S$ of S–modules. Thus we get a diagram $X(S) \to S \leftarrow S^\flat$. By taking the pullback, one obtains a map $N \to S^\flat$ of $S^\flat$–modules. We define $u^*X(S) := N$. Scheme theoretically, $u^*X \cong X \times_{\mathbb{G}_a} \mathbb{G}_a^{\mathrm{perf}}$ and thus is indeed a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module.

It is clear that u^* is faithful. To see that it is full, we take $\operatorname{Spec} M$ and $\operatorname{Spec} N$ to be two pointed $\mathbb{G}_a$–modules over A. Let $f : u^*\operatorname{Spec} M \to u^*\operatorname{Spec} N$ be a map of pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules. Now the graded algebra underlying $u^*\operatorname{Spec} M$ and $u^*\operatorname{Spec} N$ is given by $M' := M \otimes_{A[x]} A[x^{1/p^\infty}]$ and $N' := N \otimes_{A[x]} A[x^{1/p^\infty}]$. The map f induces a graded map on algebras $\overline{f} : N' \to M'$. Now considering the A–subalgebra of elements of integral degree in M' and N' we recover M and N and also get a graded map $\overline{f}_\circ : N \to M$. Since $\overline{f}$ was a graded Hopf algebra map, it follows that $\overline{f}_\circ$ is also a graded Hopf algebra map. This induces a map $f_\circ : \operatorname{Spec} M \to \operatorname{Spec} N$ which by construction is a map of pointed $\mathbb{G}_a$–modules. Applying u^* to this map recovers f. □

Definition 2.2.17 (Fractional rank 1)**.** Let A be an $\mathbb{F}_p$–algebra. A pointed $\mathbb{G}_a^{\mathrm{perf}}$–module X over A is said to be of *fractional rank* 1 if it is isomorphic to u^*X' for a pointed $\mathbb{G}_a$–module X'.

Remark 2.2.18. If $X = \operatorname{Spec} V$ is a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module of fractional rank 1 over $\mathbb{F}_p$, then it follows from the definition and Proposition 2.2.8 that the map of graded algebras $\mathbb{F}_p[x^{1/p^\infty}] \to V$ corresponding to the point is an isomorphism in degrees < 1 and thus the pieces of degree < 1 of V are vector spaces of dimension 1. Also, if X is isomorphic to $u^*\operatorname{Spec} U$ for a pointed $\mathbb{G}_a$–module $\operatorname{Spec} U$, then $\dim U_1 = \dim V_1$.

2.3 The Hodge map

Proposition 2.3.1 (Hodge map)**.** *Let X be a pointed $\mathbb{G}_a^{perf}$–module over $\mathbb{F}_p$ of fractional rank* 1*. Then there is a unique map $\alpha^\flat \to X$ in $\mathbb{G}_a^{perf}$–Mod_*. This map will be called the Hodge map.*

Proof. We choose $X' \in \mathbb{G}_a$–Mod$_*$ such that $X \simeq u^*X'$ where u^* is the functor from Proposition 2.2.16. There is a map $\operatorname{Spec} \mathbb{F}_p \to X'$ in $\mathbb{G}_a$–Mod$_*$ corresponding to the zero object in $\mathbb{G}_a$–Mod$_*$. Applying u^*, we obtain a map $u^* \operatorname{Spec} \mathbb{F}_p \to X$. Noting that $u^* \operatorname{Spec} \mathbb{F}_p \simeq \alpha^\flat$, we obtain a map $\alpha^\flat \to X$. This map is necessarily unique, because on graded algebras, it is given by a map $B \to \mathbb{F}_p[x^{1/p^\infty}]/x$, where we write $X = \operatorname{Spec} B$. This shows that if $b \in B$ is a homogeneous element in degree ≥ 1, it necessarily goes to zero. We see that homogeneous elements in degree < 1 can map to a unique element by using the data of the points and Remark 2.2.18. □

Remark 2.3.2. We will later see (*cf.* Proposition 3.5.8) that the Hodge map $\alpha^\flat \to X$ corresponds to the gr_0 map of a certain kind of "Hodge filtration" that can be defined on a functor associated to X.

Now we would like to study stability property of the Hodge map $\alpha^\flat \to X$ from Proposition 2.3.1 under deformations. For this purpose, we make the following definition.

Definition 2.3.3. A pointed $\mathbb{G}_a$–module X is said to be *full* of rank 1 if the map $X \to \mathbb{G}_a$ induces a surjection on the piece of degree 1 on the underlying graded algebra map obtained by taking global sections.

Definition 2.3.4 (Full of fractional rank 1)**.** A pointed $\mathbb{G}_a^{\mathrm{perf}}$–module X over an $\mathbb{F}_p$–algebra A is said to be *full* of fractional rank 1 if it is of fractional rank 1 and the map $X \to \mathbb{G}_a^{\mathrm{perf}}$ induces a surjection on the piece of degree 1 on the underlying graded algebra map obtained by taking global sections.

Remark 2.3.5. It follows from definitions that a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module $\operatorname{Spec} V$ is full of fractional rank 1 if and only if it is the image under u^* of a full of rank 1 pointed $\mathbb{G}_a$–module $\operatorname{Spec} U$. In this situation, when $A = \mathbb{F}_p$, we have two cases.

Case 1. If the graded map $\mathbb{F}_p[x] \to U$ sends $x \to 0$, then $V = U \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p[x^{1/p^\infty}] = U \otimes_{\mathbb{F}_p} \frac{\mathbb{F}_p[x^{1.p^\infty}]}{x}$. Thus the graded map $\mathbb{F}_p[x^{1/p^\infty}] \to V$ corresponding to the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module structure is an isomorphism in degrees < 1 and x is sent to zero, so V has no non-zero elements in degree i for $1 \leq i < 2$.

Case 2. Otherwise, $\dim U_1 = \dim V_1 = 1$ and the graded map $\mathbb{F}_p[x] \to U$ sends x to a basis element of U_1. Then the map $\mathbb{F}_p[x^{1/p^\infty}] \to V$ corresponding to the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module structure is an isomorphism in degrees i for $0 \leq i < 2$.

Proposition 2.3.6 (Stability of the Hodge map)**.** *Let X be a full fractional rank* 1 *pointed $\mathbb{G}_a^{perf}$–module over $\mathbb{F}_p$. Let X' be a deformation of X as a pointed $\mathbb{G}_a^{perf}$–module over $\mathbb{F}_p[\epsilon]/\epsilon^2$. Then the Hodge map $\alpha^\flat \to X$ admits a unique deformation $\alpha^\flat[\epsilon] \to X'$.*

Proof. We write $X = \operatorname{Spec} B$ and $X' = \operatorname{Spec} B'$. We have a map $\mathbb{F}_p[\epsilon][x^{1/p^\infty}] \to B'$ coming from the data of the point. By Remark 2.2.18, this is an isomorphism in degrees < 1 since it is so modulo ϵ. Let $B'_{<1}$ denote the graded algebra we obtain by killing the ideal of elements in degrees ≥ 1 in B. This gives an isomorphism of graded algebras $B'_{<1} \simeq \frac{\mathbb{F}_p[\epsilon][x^{1/p^\infty}]}{x}$. Thus the quotient map $B' \to B'_{<1}$ can be identified with a graded algebra map $B' \to \frac{\mathbb{F}_p[\epsilon][x^{1/p^\infty}]}{x}$ or in other words, a graded map $\alpha^\flat[\epsilon] \to \operatorname{Spec} B'$. We note that both sides have the structure of pointed $\mathbb{G}_a^{\text{perf}}$–modules and the graded map of schemes we have constructed is compatible with the data of the points. Thus in order to prove that it is a map of pointed $\mathbb{G}_a^{\text{perf}}$–modules, we only need to check that $B' \to \frac{\mathbb{F}_p[\epsilon][x^{1/p^\infty}]}{x}$ is a map of graded Hopf algebras, i.e., the following diagram commutes.

$$\begin{array}{ccc} B' & \longrightarrow & \frac{\mathbb{F}_p[\epsilon][x^{1/p^\infty}]}{x} \\ \downarrow & & \downarrow \\ B' \otimes B' & \longrightarrow & \frac{\mathbb{F}_p[\epsilon][x^{1/p^\infty}]}{x} \otimes_{\mathbb{F}_p[\epsilon]} \frac{\mathbb{F}_p[\epsilon][x^{1/p^\infty}]}{x} \end{array}$$

It is enough to check that the diagram commutes for homogeneous elements $b \in B'$. When $\deg b < 1$, it follows from the data of the points. By Remark 2.3.5, if X falls under Case 1, then the diagram commutes for $1 \leq \deg b < 2$, as b is necessarily zero in that case. If X falls under Case 2, then the map $\mathbb{F}_p[x^{1/p^\infty}] \to B$ is an isomorphism in degrees < 2. Therefore, the map $\mathbb{F}_p[\epsilon][x^{1/p^\infty}] \to B'$ coming from the data of the point is an isomorphism in degrees < 2 as well. Thus the diagram commutes for $1 \leq \deg b < 2$ in this case as well. Now we suppose that $\deg b \geq 2$. The comultiplication would send this to a homogeneous element of $B' \otimes B'$. However, any homogeneous element of degree ≥ 2 in $B' \otimes B'$ would have to be of the form $\sum_u x_u \otimes y_u$ such that x_u, y_u are homogeneous elements in B' and $\deg x_u + \deg y_u = \deg b \geq 2$, therefore either $\deg x_u$ or $\deg y_u$ is ≥ 1, implying that $\sum_u x_u \otimes y_u$ is sent to zero under the lower horizontal map. But since $\deg b \geq 2$, it is sent to zero by the upper horizontal map as well, which kills every element in degree ≥ 1, verifying the commutativity of the diagram. The uniqueness follows from the grading. □

Example 2.3.7. We note that Proposition 2.3.6 is false if we do not assume the $\mathbb{G}_a^{\text{perf}}$–module to be full, i.e., being fractional rank 1 alone is not sufficient. We consider the group scheme α_p as a pointed $\mathbb{G}_a$–module over $\mathbb{F}_p$ via the map $\mathbb{F}_p[x] \to \frac{\mathbb{F}_p[x]}{x^p}$ that sends $x \to 0$. Applying the functor u^*, we obtain a pointed $\mathbb{G}_a^{\text{perf}}$–module via the map $\mathbb{F}_p[x^{1/p^\infty}] \to \frac{\mathbb{F}_p[x^{1/p^\infty}]}{x} \otimes_{\mathbb{F}_p} \frac{\mathbb{F}_p[t]}{t^p}$. This can have deformations that do not admit a pointed $\mathbb{G}_a^{\text{perf}}$–module map to $\alpha^\flat[\epsilon]$. Indeed, we consider the graded algebra $\frac{\mathbb{F}_p[\epsilon][x^{1/p^\infty}]}{x} \otimes_{\mathbb{F}_p[\epsilon]} \frac{\mathbb{F}_p[\epsilon][t]}{t^p}$. The Hopf structure has a nontrivial deformation given by

$$t \to t \otimes 1 + 1 \otimes t + \epsilon \sum_{0<i<p} \frac{1}{p}\binom{p}{i} x^{i/p} \otimes x^{1-i/p}.$$

This deformation further has the structure of a pointed $\mathbb{G}_a^{\text{perf}}$–module. However, this does not have a deformation of the pointed $\mathbb{G}_a^{\text{perf}}$–module map $\frac{\mathbb{F}_p[x^{1/p^\infty}]}{x} \otimes_{\mathbb{F}_p} \frac{\mathbb{F}_p[t]}{t^p} \to \alpha^\flat$ obtained by killing all elements of degree ≥ 1. Indeed, any deformation of such a map would be given by killing all the generators in degrees ≥ 1 as well, which would not be a Hopf map.

2.4 Cartier duality

In this section, we record a variant of Cartier duality in the graded situation. For more details on such constructions we refer to [GR14, 1.6]. We will use this duality to study the deformation of certain pointed $\mathbb{G}_a$ and $\mathbb{G}_a^{\text{perf}}$–modules.

Definition 2.4.1. Let R be any ring. A nonnegatively graded module $\bigoplus_{i\geq 0} V_i$ over R is said to be of *free of finite type* if V_i is a finite dimensional free module over R for every $i \geq 0$. A graded algebra over R will be called free of finite type of it is free of finite type as a module over R.

Definition 2.4.2. If $M = \bigoplus_{i \geq 0} V_i$ is a free of finite type graded module over R, then we can define the dual of M as $M^* := \bigoplus_{i \geq 0} V_i^*$, where V_i^* denotes the dual of V_i. It follows that M^{**} is functorially isomorphic to M.

Definition 2.4.3. Let S be a graded free of finite type Hopf algebra over R. Then S^* also has the structure of a graded free of finite type Hopf algebra over R. We call S^* the Cartier dual of S. It follows that S^{**} is naturally isomorphic to S. Thus Cartier duality provides an anti-equivalence between the category of nonnegatively graded free of finite type Hopf algebras over R with itself.

Definition 2.4.4. Let $\mathcal{P}_*^1$ denote the category of nonnegatively graded affine group schemes X over R whose underlying Hopf algebra is free of finite type along with the data of a map $X \to \mathbb{G}_a$ of graded group schemes such that at the level of graded algebras of global sections, this map induces an *isomorphism* on degrees ≤ 1. The map $X \to \mathbb{G}_a$ will be called a point. Morphisms between two such objects are morphisms of graded group schemes that commute with the data of the points.

Remark 2.4.5. We recall that a nonnegatively graded Hopf algebra S over R is called connected if the degree zero piece of S denoted as S_0 is isomorphic to R as an R–module. Equivalently, S is called connected if the structure map $R \to S$ induces isomorphism on the degree zero piece. It follows from the definition that the underlying graded Hopf algebra $\Gamma(X, \mathcal{O}_X)$ is connected for an $X \in \mathcal{P}_*^1$. We note that if S is a connected nonnegatively graded Hopf algebra, then the comultiplication on S sends an element x of homogeneous degree 1 to $x \otimes 1 + 1 \otimes x$. In particular, giving a map $R[x] \to S$ of connected nonnegatively graded Hopf algebras over a base ring R amounts to choosing a homogeneous element of degree 1 in S.

Remark 2.4.6. The identity map $\mathbb{G}_a \to \mathbb{G}_a$ is the final object of $\mathcal{P}_*^1$.

Proposition 2.4.7 (Duality)**.** *We fix a base ring R as before. The category $\mathcal{P}_*^1$ has a notion of duality which sends an object $\operatorname{Spec} M \to \mathbb{G}_a$ to another object $\operatorname{Spec} M^* \to \mathbb{G}_a$, where M^* is the Cartier dual of the nonnegatively graded free of finite type Hopf algebra M. Further, this duality in $\mathcal{P}_*^1$ is involutive and thus provides an anti-equivalence of $\mathcal{P}_*^1$ with itself. This duality in $\mathcal{P}_*^1$ will be called Cartier duality as well.*

Proof. Let $\operatorname{Spec} M \to \mathbb{G}_a \in \mathcal{P}_*^1$. We need to check that there is a map of graded Hopf algebras $R[x] \to M^*$ inducing isomorphism in degrees ≤ 1 which is further functorial and is compatible with applying Cartier duality twice. First we note the following lemma.

Lemma 2.4.8. *Let $\mathcal{PL}(R)$ denote the category whose objects are pairs (L, p) where L is a free module of rank 1 over R and p is a basis of L as an R–module. Maps are defined in the obvious way. Then there is a notion of functorial duality on $\mathcal{PL}(R)$ which is involutive and sends $L \mapsto L^*$.*

Proof. Our task is to construct a functor $\mathcal{PL}(R) \to \mathcal{PL}(R)$ which we define by sending $(L, p) \to (L^*, p^*)$ where $p^* : L \to R$ is the unique map that sends $p \to 1$. It is clear that p^* is a basis for L^*. If $(L_1, p_1) \to (L_2, p_2)$ is an arrow in $\mathcal{PL}(R)$, then there is a natural map $L_2^* \to L_1^*$ which takes p_2^* to p_1^*, thus it is clear that our construction defines a functor. The natural isomorphism $L \simeq L^{**}$ sends $p \to p^{**}$ so it follows that the functor we constructed is involutive. □

Now we note that the Cartier dual M^* of M is again a connected Hopf algebra over R since M is connected. By Remark 2.4.5, the map $\operatorname{Spec} M \to \mathbb{G}_a$ corresponds to an object $(M_1, t) \in \mathcal{PL}(\mathcal{R})$, where M_1 denotes the degree 1 piece of M and $t \in M_1$ is the image of x under the graded map $R[x] \to M$. By the above Lemma, duality provides us an object (M_1^*, t^*). By Remark 2.4.5, this corresponds to a map $\operatorname{Spec} M^* \to \mathbb{G}_a$. By construction, the underlying graded algebra map $R[x] \to M^*$ induces an isomorphism in degrees ≤ 1. Thus $\operatorname{Spec} M^* \to \mathbb{G}_a$ is naturally an object of $\mathcal{P}_*^1$. The fact that this construction is functorial and involutive follows from the above Lemma. □

Remark 2.4.9. It follows that the category $\mathcal{P}_*^1$ has an initial object $\mathbb{G}_a^* \to \mathbb{G}_a$, which is the Cartier dual of the final object $\mathbb{G}_a \to \mathbb{G}_a$ and will be simply denoted by $\mathbb{G}_a^*$. By computing the Cartier dual, it follows that the graded algebra underlying $\mathbb{G}_a^*$ is the divided power polynomial algebra in one variable. It also follows that $\mathbb{G}_a^*$ actually has the structure of a pointed $\mathbb{G}_a$–module.

Proposition 2.4.10. *Let R be a $\mathbb{Z}_p$–algebra. Then $\mathbb{G}_a^*$ as an object of $\mathcal{P}_*^1$ over R is isomorphic to $W[F] \to \mathbb{G}_a$, where the latter denotes the Kernel of Frobenius on the p-typical Witt ring scheme (Example 1.1.7). Therefore, $W[F]$ has the structure of a pointed $\mathbb{G}_a$–module.*

Proof. $W[F] \to \mathbb{G}_a$ is an object of $\mathcal{P}_*^1$ and thus there is a unique map $\mathbb{G}_a^* \to W[F]$ in $\mathcal{P}_*^1$, which can be checked to be an isomorphism. The fact that $W[F]$ is a pointed $\mathbb{G}_a$–module now follows from Remark 2.4.9. □

Remark 2.4.11. Fix $n \geq 1$. Let $\mathcal{P}_{*_{<n}}^1$ denote the full subcategory of $\mathcal{P}_*^1$ spanned by the objects $\operatorname{Spec} M \to \mathbb{G}_a$ whose underlying graded algebra M satisfies $M_i = 0$ for $i \geq n$. Then $\mathcal{P}_{*_{<n}}^1$ is preserved under Cartier duality. When R is a char. p base ring and $n = p^k$, then the final object of $\mathcal{P}_{*_{<n}}^1$ is given by $\alpha_{p^k} \to \mathbb{G}_a$ where α_{p^k} is the $\mathbb{G}_a$–module underlying $\operatorname{Spec} R[x]/x^{p^k}$. Its Cartier dual is given by $W_k[F]$, where W_k is the kernel of Frobenius on the k-truncated p-typical Witt ring scheme. The Hopf structure on the ring of functions on $W_k[F]$ is obtained by considering the subalgebra of elements in degree $< p^k$ in the Hopf algebra underlying $W[F]$. By duality, $W_k[F] \to \mathbb{G}_a$ is the initial object of $\mathcal{P}_{*_{<n}}^1$. We note that $W_k[F]$ is also a $\mathbb{G}_a$–module and thus $W_k[F]$ has the structure of a pointed $\mathbb{G}_a$–module.

Example 2.4.12 (Duality in $\mathcal{P}_*^1$)**.** We consider the graded algebra $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}$ where $\deg x_0 = 1$ and $\deg x_i = p^{k+i}$ for $i \geq 1$. This has a graded subalgebra given by $\frac{\mathbb{F}_p[x_0^{p^k},x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}$ which has a Hopf algebra structure coming from an isomorphism with the graded algebra underlying $W_{n+1}[F]$ which scales the degree by p^k. This equips the graded algebra $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}$ with a Hopf structure. There is also a map $\operatorname{Spec} \frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)} \to \mathbb{G}_a$ which makes this an object of $\mathcal{P}_*^1$.

We also consider the graded algebra $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_k]}{(x_0^p,\ldots,x_{k-1}^p,x_k^{p^{n+1}})}$, where $\deg x_i = p^i$. By quotienting with x_k^p we obtain the graded algebra $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_k]}{(x_0^p,\ldots,x_{k-1}^p,x_k^p)}$ which has a Hopf structure coming from $W_{k+1}[F]$. This induces a Hopf structure on $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_k]}{(x_0^p,\ldots,x_{k-1}^p,x_k^{p^{n+1}})}$ which sends $x_k^p \to x_k^p \otimes 1 + 1 \otimes x_k^p$. There is also a map $\operatorname{Spec} \frac{\mathbb{F}_p[x_0,x_1,\ldots,x_k]}{(x_0^p,\ldots,x_{k-1}^p,x_k^{p^{n+1}})} \to \mathbb{G}_a$ which makes this an object of $\mathcal{P}_*^1$.

The above two objects of $\mathcal{P}_*^1$ are Cartier dual of each other. Indeed, by killing the ideal of elements in degrees $\geq p^{k+1}$ in $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}$, we obtain the sub Hopf algebra $\mathbb{F}_p[x_0]/x_0^{p^{k+1}}$. This implies that killing the ideal of elements in degrees $\geq p^{k+1}$ in the Cartier dual $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}^*$ we get an isomorphism with the graded Hopf algebra $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_k]}{(x_0^p,\ldots,x_{k-1}^p,x_k^p)}$. Under this isomorphism, x_k is the basis element in degree p^k of $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}^*$. Inspecting the comultiplication in $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}$, we conclude that the powers of the basis element in degree p^k of $\frac{\mathbb{F}_p[x_0,x_1,\ldots,x_n]}{(x_0^{p^{k+1}},x_1^p,\ldots,x_n^p)}^*$ gives a basis element in degrees i for $p^k \leq i < p^{k+n+1}$. This shows that the two objects of $\mathcal{P}_*^1$ are indeed dual to each other.

2.5 Deformations of some $\mathbb{G}_a$ and $\mathbb{G}_a^{\mathrm{perf}}$–modules

In this section, we study deformations of some pointed $\mathbb{G}_a$ (resp. $\mathbb{G}_a^{\mathrm{perf}}$)–modules. While studying deformations, we would like to make use of certain universal properties. In order to formulate these universal properties, we will need to restrict our attention to a suitable full subcategory of $\mathbb{G}_a$–Mod$_*$ (resp. $\mathbb{G}_a^{\mathrm{perf}}$–Mod$_*$).

Definition 2.5.1. A pointed $\mathbb{G}_a$–module X is said to be *pure* of rank 1 if the graded Hopf algebra $\Gamma(X, \mathcal{O}_X)$ is free of finite type and the map $X \to \mathbb{G}_a$ induces an isomorphism in degree 1 at the level of graded algebras of global sections. The full subcategory of such objects inside the category $\mathbb{G}_a$–Mod$_*$ will be called the category of *pure rank* 1 $\mathbb{G}_a$–modules.

Remark 2.5.2. Since (by using Proposition 2.1.12) the category of pure rank 1 $\mathbb{G}_a$–modules is a full subcategory of $\mathcal{P}_*^1$, it follows that $\mathbb{G}_a^*$ is the initial object in the category of pure rank 1 $\mathbb{G}_a$–modules.

Therefore $\mathbb{G}_a^*$ admits no nontrivial endomorphisms other than the identity (over an arbitrary base ring). Over a $\mathbb{Z}_p$–algebra, $W[F]$ is isomorphic to $\mathbb{G}_a^*$ and thus inherits the same universal property.

Proposition 2.5.3. *Let A be a $\mathbb{Z}_p$–algebra. Then $W[F]$ has no nontrivial endomorphism as a pointed $\mathbb{G}_a$–module over A.*

Proof. This follows from Proposition 2.4.10 and Remark 2.5.2. □

Proposition 2.5.4. *Let $(A, \mathfrak{m})$ be an Artinian local ring with residue field k which has char. $p > 0$. Then any deformation of the pointed $\mathbb{G}_a$–module $W[F]_k$ over A is uniquely isomorphic to $W[F]_A$.*

Proof. Let X be any deformation of $W[F]_k$ to A. Then X is necessarily a pure rank 1 $\mathbb{G}_a$–module. Since $W[F]_A$ is the initial object in the category of pure rank 1 $\mathbb{G}_a$–modules, there is a unique map $W[F]_A \to X$ of pointed $\mathbb{G}_a$–modules. This map is an isomorphism after going modulo $\mathfrak{m}$, and therefore is an isomorphism. The uniqueness follows from Proposition 2.5.3. □

Proposition 2.5.5. *Let A be an $\mathbb{F}_p$–algebra. Then the pointed $\mathbb{G}_a$–module $W_n[F]$ defined over A has no nontrivial endomorphisms.*

Proof. Follows from Remark 2.4.11. □

Proposition 2.5.6. *Let $(A, \mathfrak{m})$ be an Artinian local ring over $\mathbb{F}_p$ with residue field k. Then any deformation of the pointed $\mathbb{G}_a$–module $W_n[F]_k$ is uniquely isomorphic to $W_n[F]_A$.*

Proof. Follows from Remark 2.4.11. □

Definition 2.5.7 (Pure of fractional rank 1)**.** Let A be a base ring of char. $p > 0$. A pointed $\mathbb{G}_a^{\mathrm{perf}}$–module X over A is said to be *pure of fractional rank 1* if it is isomorphic to u^*Y for some pure rank 1 $\mathbb{G}_a$–module Y. The full subcategory of such objects inside $\mathbb{G}_a^{\mathrm{perf}}$–Mod$_*$ will be called the category of pure fractional rank 1 $\mathbb{G}_a^{\mathrm{perf}}$–module. This category is also the essential image of the functor u^* restricted to the category of pure rank 1 $\mathbb{G}_a$–modules.

Remark 2.5.8. The category of pure fractional rank 1 $\mathbb{G}_a^{\mathrm{perf}}$ module has a final object given by $\mathbb{G}_a^{\mathrm{perf}}$. This category also has an initial object which is given by $u^*W[F]$. This follow from the definition of pure fractional rank 1 modules by using Remark 2.5.2 and the fact that u^* is fully faithful (Proposition 2.2.16).

Proposition 2.5.9. *Let A be an $\mathbb{F}_p$–algebra. Then $u^*W[F]$ has no nontrivial endomorphism as a pointed $\mathbb{G}_a^{perf}$–module over A.*

Proof. This follows from Remark 2.5.8. □

Our next Proposition will deal with deformations of $u^*W[F]$. We point out that using Proposition 2.5.4 and Proposition 5.2.6 one can directly prove Proposition 2.5.11 below. However, since the proof of Proposition 5.2.6 uses the language of stacks, we prefer to record an elementary argument in the case of $u^*W[F]$. Before we begin, we record a lemma.

Lemma 2.5.10. *Let $S = \bigoplus_{i \in \mathbb{Z}[1/p]_{\geq 0}} S_i$ be a graded perfect ring. For a fixed $n \in \mathbb{Z}[1/p]_{\geq 0}$, we consider the ideal $I := \bigoplus_{i \geq n} S_i$. Then there is a unique deformation up to unique isomorphism of S/I over $\mathbb{F}_p[\epsilon]/[\epsilon^2]$ which is compatible with the grading.*

Proof. This could also be proven by a graded version of the cotangent complex but we prefer to give a direct proof. Let $B = \bigoplus_i B_i$ be a deformation of S/I compatible with the grading. There is a map $S \to S/I$ which lifts uniquely to a map $f : S[\epsilon] := S \otimes \mathbb{F}_p[\epsilon]/\epsilon^2 \to B$. Since S is perfect, this map is graded: Indeed, for a homogeneous element $s \in S \subset S[\epsilon]$, $s^{1/p}$ is also homogeneous and $f(s^{1/p})$ is a homogeneous element if we go modulo ϵ. By taking p-th powers, that implies that $f(s)$ is a homogeneous element. This shows that f is a graded map. Now the map $S_i \to (S/I)_i$ is an isomorphism for $i < n$ and zero for $i \geq n$. By Nakayama's lemma, the same has to be true for $f_i : S[\epsilon]_i \to B_i$. This shows that the kernel of f is $I[\epsilon] := \bigoplus_{k \geq n} S[\epsilon]_k$. Now f is surjective, since it is a surjection modulo ϵ. This shows that $B \simeq S/I[\epsilon]$, compatible with the grading. Uniqueness follows from grading and by taking p-power roots. □

Proposition 2.5.11. *The pointed $\mathbb{G}_a^{perf}$–module $u^*W[F]$ over $\mathbb{F}_p$ has no nontrivial deformation over $\mathbb{F}_p[\epsilon]/\epsilon^2$.*

Proof. Since the Hopf algebra A underlying $u^*W[F]$ is not graded by nonnegative integers, the theory of Cartier duality breaks down. Indeed, dimension of the piece of degree 1 in $A \otimes_{\mathbb{F}_p} A$ is infinite and thus does not behave well under duality. *A priori*, we cannot directly apply any of our results above. Our proof will use some lemmas which will ultimately break it down to steps where we are only dealing with finite type Hopf algebras.

Lemma 2.5.12. *The graded group scheme $u^*W_n[F]$ over $\mathbb{F}_p$ has no nontrivial deformation over $\mathbb{F}_p[\epsilon] := \mathbb{F}_p[\epsilon]/\epsilon^2$ as a graded group scheme. Further, $u^*W_n[F] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\epsilon]$ admits a unique endomorphism as a pointed $\mathbb{G}_a^{perf}$–module.*

Proof. We will break down the proof in a few steps.

Step 1. We write the graded algebra underlying $u^*W_n[F]$ as $A = \frac{\mathbb{F}_p[x_0^{1/p^\infty}, x_1, \ldots, x_n]}{x_i^p}$, where $\deg x_i = p^i$. This admits a map of graded Hopf algebras $\mathbb{F}_p[x_0^{1/p^\infty}] \to \frac{\mathbb{F}_p[x_0^{1/p^\infty}, x_1, \ldots, x_n]}{x_i^p}$. Let A' be the graded Hopf algebra underlying the deformation of $u^*W_n[F]$. Let $A' := \bigoplus_{i \in \mathbb{N}[1/p] \cup \{0\}} A'_i$ as a graded algebra. By killing the ideal of elements of degree $\geq p$, we obtain a ring $A'_{<p}$ which is a deformation of the graded algebra $\mathbb{F}_p[x_0^{1/p^\infty}]/x_0^p$ which has to be uniquely isomorphic to the trivial deformation by Lemma 2.5.10. Thus, by taking grading into account, we see that A' has to be of the form

$$\frac{\mathbb{F}_p[\epsilon][X_0^{1/p^\infty}, \ldots, X_n]}{(X_0^p - c_1\epsilon X_1, X_1^p - c_2\epsilon X_2, \ldots, X_n^p)},$$

where X_i is a lift of x_i of degree p^i and $c_i \in \mathbb{F}_p$. The comultiplication sends $X_0^{1/p^n} \to X_0^{1/p^n} \otimes 1 + 1 \otimes X_0^{1/p^n}$ in $A' \otimes A'$. This shows that there is map of graded Hopf algebras $\mathbb{F}_p[\epsilon][X_0^{1/p^\infty}] \to A'$ which is a deformation of the map $\mathbb{F}_p[x_0^{1/p^\infty}] \to A$. Thus for k large enough, the graded Hopf algebra map

$$\mathbb{F}_p[\epsilon][X_0^{1/p^k}] \to A'_k := \frac{\mathbb{F}_p[\epsilon][X_0^{1/p^k}, \ldots, X_n]}{(X_0^p - c_1\epsilon X_1, X_1^p - c_2\epsilon X_2, \ldots, X_n^p)}$$

is firstly a deformation of the graded Hopf algebra map $\mathbb{F}_p[x_0^{1/p^k}] \to A_k := \frac{\mathbb{F}_p[x_0^{1/p^k}, \ldots, x_n]}{x_i^p}$, and secondly the map of graded Hopf algebras $\mathbb{F}_p[\epsilon][X_0^{1/p^\infty}] \to A'$ is obtained by pulling back the map $\mathbb{F}_p[\epsilon][X_0^{1/p^k}] \to A'_k$ along $\mathbb{F}_p[\epsilon][X_0^{1/p^k}] \to \mathbb{F}_p[\epsilon][X_0^{1/p^\infty}]$. Thus it is enough to prove that $\mathbb{F}_p[\epsilon][X_0^{1/p^k}] \to A'_k$ is isomorphic to the trivial deformation of $\mathbb{F}_p[x_0^{1/p^k}] \to A_k$. By a shifting of degree, it is enough to prove that the pointed $\mathbb{G}_a$–module $\operatorname{Spec} \frac{\mathbb{F}_p[x_0, x_1, \ldots, x_n]}{(x_0^{p^{k+1}}, x_1^p, \ldots, x_n^p)} \to \mathbb{G}_a$ has no nontrivial deformations over $\mathbb{F}_p[\epsilon]$. Here $\deg x_0 = 1$ and $\deg x_i = p^{k+i}$ for $i \geq 1$.

Step 2. In order to prove that the pointed $\mathbb{G}_a$–module $\operatorname{Spec} \frac{\mathbb{F}_p[x_0, x_1, \ldots, x_n]}{(x_0^{p^{k+1}}, x_1^p, \ldots, x_n^p)} \to \mathbb{G}_a$ has no nontrivial deformations over $\mathbb{F}_p[\epsilon]$, it is equivalent to prove the same for its Cartier dual, which by Example 2.4.12 is given by

$$\operatorname{Spec} \frac{\mathbb{F}_p[x_0, x_1, \ldots, x_k]}{(x_0^p, \ldots, x_{k-1}^p, x_k^{p^{n+1}})} \to \mathbb{G}_a.$$

We let B denote the Hopf algebra $\frac{\mathbb{F}_p[x_0, x_1, \ldots, x_k]}{(x_0^p, \ldots, x_{k-1}^p, x_k^{p^{n+1}})}$. Then B/x_k^p is isomorphic to $\Gamma(W_{k+1}[F], \mathcal{O})$ as a graded Hopf algebra. Let B' be a deformation of B. Then B' as a graded algebra is of the form

$$\frac{\mathbb{F}_p[\epsilon][X_0, X_1, \ldots, X_k]}{(X_0^p - c_1\epsilon X_1, \ldots, X_{k-1}^p - c_k\epsilon X_k, X_k^{p^{n+1}})}$$

where $\deg X_i = p^i$ and $X_i \in B'$ is chosen to be a lift of $x_i \in B$. Now B'/X_k^p is a deformation of the graded Hopf algebra underlying $W_{k+1}[F]$, but the latter has no nontrivial deformations by Proposition 2.5.6. Thus we obtain an isomorphism $(B/x_k^p)[\epsilon] = W_{k+1}[F][\epsilon] \to B'/X_k^p$ of graded Hopf algebras (which is also compatible with the pointed $\mathbb{G}_a$–module structure). This lifts uniquely to a map of graded algebras $B[\epsilon] \to B'$ as such a map is uniquely determined by the image of $x_0, \dots, x_k$ and they have a unique homogeneous lift. One also needs to check that x_i^p is sent to zero for $0 < i \leq k-1$ and $x_k^{p^{n+1}}$ is sent to zero which also follows from grading arguments. This map by construction is an isomorphism on the level of graded algebras and it would be enough to check that it is a Hopf algebra map, i.e., we need to prove that the maps $B[\epsilon] \to B' \to B' \otimes B'$ and $B[\epsilon] \to B[\epsilon] \otimes B[\epsilon] \to B' \otimes B'$ agree. For that, we only need to check that the images of $x_0, \dots, x_k$ agree. It is known that they agree modulo the ideal $(X_k^p \otimes 1, 1 \otimes X_k^p)$ and thus by grading they are actually the same. The map we constructed is also compatible with the pointed $\mathbb{G}_a$–module structure.

The statement about endomorphisms as pointed $\mathbb{G}_a^{\text{perf}}$–module follows since $W_n[F][\epsilon]$ has no nontrivial endomorphism as pointed $\mathbb{G}_a$–module by Proposition 2.5.5. ☐

Lemma 2.5.13. *The graded algebra underlying any deformation of $u^*W[F]$ as a graded group scheme is isomorphic to*

$$\frac{\mathbb{F}_p[\epsilon][x_0^{1/p^\infty}, x_1, \dots]}{x_i^p}.$$

Proof. We write the graded algebra underlying $u^*W[F]$ as $C = \frac{\mathbb{F}_p[x_0^{1/p^\infty}, x_1, \dots]}{x_i^p}$, where $\deg x_i = p^i$. Similar to the proof of Lemma 2.5.12, it follows that the graded algebra underlying global sections of a deformation of $u^*W[F]$ is isomorphic to

$$C' = \frac{\mathbb{F}_p[\epsilon][X_0^{1/p^\infty}, X_1, \dots]}{(X_0^p - c_1\epsilon X_1, X_1^p - c_2\epsilon X_2, \dots)},$$

where X_i is taken to be a lift of x_i of degree p^i and $c_i \in \mathbb{F}_p$. Our goal is to prove that $c_i = 0$ for all i. Killing the ideal of elements of degree $\geq p^{n+1}$, we obtain the graded algebra

$$C'_{<p^{n+1}} := \frac{\mathbb{F}_p[\epsilon][X_0^{1/p^\infty}, X_1, \dots, X_n]}{(X_0^p - c_1\epsilon X_1, \dots, X_n^p)}.$$

Further, this has a Hopf structure: this follows from the observation that the comultiplication in C' sends X_n^p to $X_n^p \otimes 1 + 1 \otimes X_n^p$. Now $\operatorname{Spec} C'_{<p^{n+1}}$ as a graded group scheme is a deformation of $u^*W_n[F]$, and thus by Lemma 2.5.12 must be uniquely isomorphic to the trivial deformation, which implies $c_i = 0$ for $0 < i < n+1$. Since n was arbitrary, we are done. ☐

Now we note that the algebra C underlying $u^*W[F]$ has the property that for every $n \geq 1$, the elements of degree $< p^n$ forms a subalgebra $\tau_{<n}C$ which is the graded Hopf algebra underlying $u^*W_n[F]$. By Lemma 2.5.13, it follows that elements of degree $< p^n$ in C' also forms a subalgebra $\tau_{<n}C'$ which has the structure of a graded Hopf algebra. Moreover, we observe that $\tau_{<n}C'$ is a deformation of $\tau_{<n}C$ and thus by Lemma 2.5.12, there is a unique isomorphism of graded Hopf algebras $\tau_{<n}C[\epsilon] \to \tau_{<n}C'$ that sends $x_0 \to X_0$. By taking colimit over n, we have constructed an isomorphism $C[\epsilon] \to C'$ of graded Hopf algebras. This proves the proposition. ☐

Remark 2.5.14. One can also approach Proposition 2.5.11 by first showing that any deformation of $u^*W[F]$ must be a pullback of a deformation of $W[F]$. This is essentially carried out in a purely formal way in Section 5.2 by using the connection with de Rham cohomology. In Proposition 5.2.6, we prove a generalization by using similar ideas.

3 Construction of functors using $\mathbb{G}_a$ and $\mathbb{G}_a^{\mathbf{perf}}$–modules

In this section, our ultimate goal is to create functors from QRSP $\to \mathrm{Alg}_A$ via an "unwinding" process using the data of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module (*cf.* Section 3.4). This construction will be done using a closely related variant: using the data of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module, we can "unwind" it to construct a functor $\mathscr{P}I \to \mathrm{Alg}_A$, where $\mathscr{P}I$ denotes the category with objects (B, I) where B is a perfect ring and I is an ideal. This is carried out in Section 3.3. Further, this construction for $\mathscr{P}I$ has a closely related variant for the category $\mathfrak{C}_A$ consisting of objects (B, I) where B is an A–algebra and I is an ideal of B. Given a pointed $\mathbb{G}_a$–module, we can unwind it to create a functor $\mathfrak{C}_A \to \mathrm{Alg}_A$. We will study this construction first in Section 3.2. Below we note an example which aims to explain an analogue of the unwinding construction in a simpler case.

Example 3.0.1. Let $\mathcal{C}$ be a category with all colimits. Let $c \in \mathcal{C}$ be an "$\mathbb{F}_p$–coalgebra object", i.e., the functor $\mathrm{Hom}_{\mathcal{C}}(c, \cdot)$ is naturally valued in commutative $\mathbb{F}_p$–algebras. Let $\mathrm{Poly}_{\mathbb{F}_p}$ denote the category of finitely generated polynomial $\mathbb{F}_p$–algebras. We will construct a functor $\mathrm{Un}_c : \mathrm{Poly}_{\mathbb{F}_p} \to \mathcal{C}$ which we may call the unwinding of c. We define $\mathrm{Un}_c(B) \in \mathcal{C}$ such that we have a natural bijection $\mathrm{Hom}_{\mathcal{C}}(\mathrm{Un}_c(B), d) \simeq \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}_p}}(B, \mathrm{Hom}_{\mathcal{C}}(c, d))$ for $d \in \mathcal{C}$. This maybe functorially computed as a colimit using the diagram $\mathbb{F}_p[\mathbb{F}_p[B]] \rightrightarrows \mathbb{F}_p[B]$ whose coequalizer is B. We note that by construction, we have $\mathrm{Un}_c(\mathbb{F}_p[x]) \simeq c$. Also, by construction, $\mathrm{Un}_c : \mathrm{Poly}_{\mathbb{F}_p} \to \mathcal{C}$ preserves coproducts.

This construction shows that given an object of the category $\mathcal{C}$ with appropriate extra structure, one can unwind it to create a functor from $\mathrm{Poly}_{\mathbb{F}_p} \to \mathcal{C}$. In this section, our goal is to develop a similar formalism for the categories $\mathfrak{C}_A, \mathscr{P}I$ and QRSP which would be useful to us in Section 4 and Section 5.

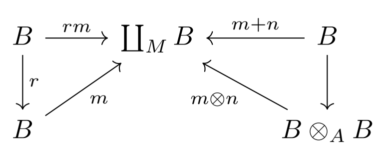
3.1 Tensoring a module with a module scheme

In this section we record a construction which "tensors" a module with a module scheme and gives an algebra as an output. In a category $\mathcal{C}$ with all coproducts, one can make sense of tensoring an object $c \in \mathcal{C}$ with a set S, denoted as $c \otimes S$, which has the property that we have a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(c \otimes S, d) \simeq \mathrm{Hom}_{\mathrm{Sets}}(S, \mathrm{Hom}_{\mathcal{C}}(c, d))$ for $d \in \mathcal{C}$. In this case, $c \otimes S$ is the coproduct $\coprod_S c$. Below, we carry out an analogue of this construction.

Construction 3.1.1. Let $X = \mathrm{Spec}\, B$ be an R–module scheme over A (Definition 2.1.1). In this situation, we have a functor from $\mathrm{Alg}_A \to \mathrm{Mod}_R$ which sends an A algebra S to $X(S) := \mathrm{Hom}_A(\mathrm{Spec}\, S, X)$. This functor has a left adjoint which will be denoted by $\mathscr{T}_X(.) : \mathrm{Mod}_R \to \mathrm{Alg}_A$. In other words, we have the following natural isomorphism

$$\mathrm{Hom}_{\mathrm{Alg}_A}(\mathscr{T}_X(M), S) \simeq \mathrm{Hom}_{\mathrm{Mod}_R}(M, X(S)).$$

We will describe an explicit way to construct the algebra $\mathscr{T}_X(M)$ for an R–module M. Considering M as a set, first we take the coproduct of the algebra B indexed over M. We will write this as $\coprod_M B$. By the universal property of the coproduct, for each $m \in M$, we have a map which we will write as $m : B \to \coprod_M B$. We also have a map $B \to B \otimes_A B$ which is the comultiplication map and a map $r : B \to B$ coming from the R–module scheme action of $\mathrm{Spec}\, B$ for $r \in R$. Then $\mathscr{T}_X(M)$ is the coequalizer of the following diagram indexed by $(R \times M) \coprod (M \times M)$.



Proposition 3.1.2. *In the above set up, we have*

1. *colim*$\mathscr{T}_X(M_i) \simeq \mathscr{T}_X(\mathit{colim}\, M_i)$.
2. $\mathscr{T}_X(M) \otimes_A \mathscr{T}_X(N) \simeq \mathscr{T}_X(M \oplus N)$.
3. *If M is a free A–module of rank 1, then* $\mathscr{T}_X(M) \simeq \Gamma(X, \mathcal{O}_X)$.

Proof. Follows from the construction of $\mathscr{T}_X(M)$. ☐

Example 3.1.3. In the case where $X = \mathbb{G}_a = \operatorname{Spec} A[x]$ viewed as an A–module scheme over A, given any A–module M, we note that $\mathscr{T}_{\mathbb{G}_a}(M) \simeq \operatorname{Sym}_A(M)$ as an A–algebra. This follows from the universal property discussed above.

Example 3.1.4. We mention an example that will be particularly important to us. Let $\mathbb{G}_a^*$ be the Cartier dual of $\mathbb{G}_a$ viewed as an A–module scheme over A. For an A–module $\mathscr{T}_{\mathbb{G}_a^*}(M) \simeq \Gamma_A(M)$. This essentially follows from [BO78, Appendix 2] by observing that for an A algebra R, there is a natural isomorphism of A–modules $\mathbb{G}_a^*(R) \simeq \exp(R)$ where $\exp(R)$ denote the elements $f(x) \in 1 + xR[x]$ satisfying $f(x+y) = f(x)f(y)$ which forms an abelian group by multiplication of power series which further has an R–module structure given by $r \cdot f(x) := f(rx)$.

Remark 3.1.5. The association $X \to \mathscr{T}_X(\cdot)$ is a contravariant functor.

3.2 Unwinding pointed $\mathbb{G}_a$–modules

Notation 3.2.1. We fix an arbitrary base ring A as before. Let $\mathfrak{C}_A$ denote the category of pairs (B, I) where B is an A–algebra and I is an ideal of B. Morphisms in $\mathfrak{C}_A$ between $(B, I) \to (B', I')$ are defined as A–algebra maps $B \to B'$ that maps I inside I'.

Construction 3.2.2 (Unwinding)**.** Let $\mathbb{G}_a$–Mod$_*$ denote the category of pointed $\mathbb{G}_a$–modules over A. We will construct a (contravariant) functor

$$\operatorname{Un} : \mathbb{G}_a\text{–Mod}_* \to \operatorname{Fun}(\mathfrak{C}_A, \operatorname{Alg}_A).$$

We will say that $\operatorname{Un}(X)$ is the functor obtained by *unwinding* the pointed $\mathbb{G}_a$–module X. To describe the construction, we fix an $X \in \mathbb{G}_a$–Mod$_*$. Given $(B, I) \in \mathfrak{C}_A$, we obtain a diagram $X_B \to \mathbb{G}_{a,B}$ of B–module schemes by base changing to B. Now the ideal I can be regarded as a B–module and thus by applying Construction 3.1.1, we obtain a map $\mathscr{T}_{\mathbb{G}_{a,B}}(I) \to \mathscr{T}_{X_B}(I)$. By Example 3.1.3 we get a map of B–algebras $\operatorname{Sym}_B(I) \to \mathscr{T}_{X_B}(I)$. Since I is an ideal of B, there are natural maps $\operatorname{Sym}_B(I) \to B \to \operatorname{Sym}_B(I) \to \mathscr{T}_{X_B}(I)$. Thus by composing we get another map $\operatorname{Sym}_B(I) \to \mathscr{T}_{X_B}(I)$. We denote the coequalizer of these two maps

$$\operatorname{Sym}_B(I) \rightrightarrows \mathscr{T}_{X_B}(I)$$

by $\operatorname{Env}_X(B, I)$. This is naturally a B–algebra. Now we define $\operatorname{Un}(X)(B, I) := \operatorname{Env}_X(B, I)$.

Remark 3.2.3 (Unwinding via universal property)**.** We describe the universal property of $\operatorname{Env}_X(B, I)$ as a B–algebra. For a B–algebra S, we note that $\mathbb{G}_{a,B}(S) = S$ is naturally a B–module. In fact, there is a map $B \to S$ of B–modules giving a natural map $I \subset B \to S$. This gives an element $* \in \operatorname{Hom}_{B\text{-Mod}}(I, S)$. Therefore, we obtain two maps $\operatorname{Hom}_{B\text{-Mod}}(I, X(S)) \rightrightarrows \operatorname{Hom}_{B\text{-Mod}}(I, S)$. Here one of the maps (of sets) sends everything to $*$ and the other one is the map induced by the data of the point $X \to \mathbb{G}_a$. We note that by Construction 3.2.2, we have

$$\operatorname{Hom}_{B\text{-Alg}}(\operatorname{Env}_X(B, I), S) \simeq \operatorname{Eq}(\operatorname{Hom}_{B\text{-Mod}}(I, X(S)) \rightrightarrows \operatorname{Hom}_{B\text{-Mod}}(I, S)).$$

Remark 3.2.4. We note that there is a natural functor $\mathfrak{G} : \mathfrak{C}_A \to \operatorname{Alg}_A$ given by $\mathfrak{G}(B, I) = B$. From Remark 3.2.3, we see that there is a natural isomorphism $\mathfrak{G} \simeq \operatorname{Un}(\mathbb{G}_a)$. Let $\operatorname{Fun}(\mathfrak{C}_A, \operatorname{Alg}_A)_{\mathfrak{G}/}$ denote the category of functors $F : \mathfrak{C}_A \to \operatorname{Alg}_A$ equipped with a natural transformation $\mathfrak{G} \to F$. The morphisms are required to be compatible with this data. It follows that in Construction 3.2.2, we actually produced a (contravariant) functor

$$\operatorname{Un} : \mathbb{G}_a\text{–Mod}_* \to \operatorname{Fun}(\mathfrak{C}_A, \operatorname{Alg}_A)_{\mathfrak{G}/}.$$

Example 3.2.5. The functor $\mathfrak{C}_A \to \operatorname{Alg}_A$ given by sending $(B, I) \to B/I$ is the unwinding of the pointed $\mathbb{G}_a$–module corresponding to zero.

Example 3.2.6 (Divided power envelope via unwinding)**.** We let the base ring A be $\mathbb{F}_p$ for simplicity. The functor $\operatorname{Un}(\mathbb{G}_a^*) : \mathfrak{C}_{\mathbb{F}_p} \to \operatorname{Alg}_{\mathbb{F}_p}$ takes a pair (B, I) to the divided power envelope $D_B(I)$. In order to see this, we compute $\operatorname{Env}_{\mathbb{G}_a^*}(B, I)$ following Construction 3.2.2. This is computed as the coequalizer of two maps

$$\operatorname{Sym}_B(I) \rightrightarrows \mathscr{T}_{\mathbb{G}^*_{a\,B}}(I).$$

We note that by Example 3.1.4, $\mathscr{T}_{\mathbb{G}^*_{aB}}(I) \simeq \Gamma_B(I)$. Thus the claim $\mathrm{Env}_{\mathbb{G}_a^*}(B, I) \simeq D_B(I)$ follows from [BO78, Thm. 3.19] which says that $D_B(I) = \Gamma_B(I)/J$ where J is the ideal generated by $\varphi(x) - x$ for all $x \in I$ and $\varphi : I \to \Gamma_1(I)$ is the natural map. Since $\mathbb{G}_a^* \simeq W[F]$ over $\mathbb{F}_p$, we also have $\mathrm{Env}_{W[F]}(B, I) \simeq D_B(I)$.

Remark 3.2.7. Let $X \in \mathbb{G}_a$–Mod$_*$ and let B be an A–algebra and f be a non-zero divisor in B. Then we can explicitly describe $\mathrm{Env}_X(B, f)$. We note that the ideal I generated by f in this case is free of rank 1 and thus there is an isomorphism $\mathscr{T}_{X_B}(I) \simeq \Gamma(X_B, \mathcal{O}_{X_B}) \simeq \Gamma(X, \mathcal{O}_X) \otimes_A B$ by Proposition 3.1.2. Now $\mathrm{Env}_X(B, f)$ is the quotient $\frac{\Gamma(X,\mathcal{O}_X)\otimes_A B}{(t\otimes 1 - 1\otimes f)}$ where t is the image of x under the map $A[x] \to \Gamma(X, \mathcal{O}_X)$ corresponding to the data of the point i.e., the map $X \to \mathbb{G}_a$.

Remark 3.2.8. For any $X \in \mathbb{G}_a$–Mod$_*$, it follows that $\mathrm{Un}(X)(B, 0) = \mathrm{Env}_X(B, 0) \simeq B$. This follows since $\mathscr{T}_{X_B}(0) \simeq B$ and therefore $\mathrm{Env}_X(B, 0)$ is a coequalizer of two B–algebra maps $B \rightrightarrows B$ that coincide. Thus the natural map $\mathfrak{G} \to \mathrm{Un}(X)$ induces isomorphism restricted to the full subcategory spanned by objects of the form $(B, 0)$.

Proposition 3.2.9. *Let $X \in \mathbb{G}_a$–Mod$_*$. Then $Env_X(B[x], x) \simeq \Gamma(X, \mathcal{O}_X) \otimes_A B$ as a B–algebra. The map $Env_X(B[x], 0) \to Env(B[x], x)$ identifies with the map $B[x] \to \Gamma(X_B, \mathcal{O}_{X_B})$ coming from the data of the point $X \to \mathbb{G}_a$.*

Proof. Using Remark 3.2.7, we have $\mathrm{Env}_X(B[x], x) \simeq \frac{\Gamma(X,\mathcal{O}_X)\otimes_A B[x]}{t\otimes 1 - 1\otimes x}$ as $B[x]$–algebras from which the proposition follows. ☐

Given that there is a way to unwind the data of a pointed $\mathbb{G}_a$–module X and obtain a functor $\mathrm{Un}(X) : \mathfrak{C}_A \to \mathrm{Alg}_A$, it is natural to ask if this is reversible, i.e., if there is a functor r from $\mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)_{\mathfrak{G}/}$ to $\mathbb{G}_a$–Mod$_*$ such that applying r to $\mathrm{Un}(X)$ recovers the pointed $\mathbb{G}_a$–module X. There are multiple problems in achieving this as discussed below.

Firstly, defining the functor r is not possible unless we impose some conditions on the functor $F \in \mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)_{\mathfrak{G}/}$. Indeed, for every A–algebra B, we can look at the B–algebra $F(B[x], x)$. This has a B–action, however $F(B[x], x)$ might not be a Hopf algebra. The functor F needs to preserve some pushout diagrams for that to happen; this is taken into account in Definition 3.2.14. Under these special assumptions on F it is indeed possible to define a functor r as desired. The functor r is defined below in Proposition 3.2.15.

However, we note that not every pointed $\mathbb{G}_a$–module can appear as image under the functor r of an $F \in \mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)$. Indeed, we have the following commutative diagram in $\mathfrak{C}_A$.

$$\begin{array}{ccc} (A[x], x) \otimes (A[x], x) & \longleftarrow & (A[x], x) \otimes (A[x], 0) \\ \uparrow & & \uparrow {\scriptstyle x \to x\otimes x} \\ (A[x], 0) \otimes (A[x], x) & \xleftarrow[x\otimes x \leftarrow x]{} & (A[x], x) \end{array}$$

Applying F to the above diagram would impose extra conditions on the pointed $\mathbb{G}_a$–module obtained from F which need not be satisfied by every pointed $\mathbb{G}_a$–module. Thus it is impossible to recover X by using the functor r from $\mathrm{Un}(X)$ unless it satisfies some special conditions to begin with. To account for this, we are naturally led to the notion of a "quasi-ideal" in $\mathbb{G}_a$ due to Drinfeld.

Definition 3.2.10 (Drinfeld)**.** A pointed $\mathbb{G}_a$–module X with the data of the point denoted as $d : X \to \mathbb{G}_a$ will be called a *quasi-ideal in* $\mathbb{G}_a$ if the following diagram commutes.

$$\begin{array}{ccc} X \times X & \xrightarrow{\mathrm{id}\times d} & X \times \mathbb{G}_a \\ \downarrow {\scriptstyle d\times \mathrm{id}} & & \downarrow {\scriptstyle \mathrm{action}} \\ \mathbb{G}_a \times X & \xrightarrow{\mathrm{action}} & X \end{array}$$

By writing $X = \operatorname{Spec} B$ for a graded Hopf algebra B and $t \in B$ for the fixed choice of the element in degree 1 corresponding to the data of the point, we note that X is a quasi-ideal if and only if $b \otimes t^{\deg b} = t^{\deg b} \otimes b$ in $B \otimes B$ for every homogeneous $b \in B$. We let QID–$\mathbb{G}_a$ denote the full subcategory of quasi-ideals in $\mathbb{G}_a$ inside $\mathbb{G}_a$–Mod$_*$.

Remark 3.2.11. Using the inclusion QID–$\mathbb{G}_a \to \mathbb{G}_a$–Mod$_*$ of categories and Construction 3.2.2 we obtain a (contravariant) functor still denoted us $\mathrm{Un} : \mathrm{QID}\text{–}\mathbb{G}_a \to \mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)_{\mathfrak{G}/}$. We will later see that this functor is fully faithful.

Proposition 3.2.12. *Let B be an A–algebra. Let $(f_j)_{j\in\mathscr{J}}$ be a collection of non-zero divisors in B and let I be the ideal generated by them. Let F be the free module over B spanned by x_j for $j \in \mathscr{J}$. We assume that the B–module map $F \to I$ that sends $x_i \to f_i$ has kernel generated by $(f_i x_j - f_j x_i)$ for $i, j \in \mathscr{J}$. Let X be a quasi-deal in $\mathbb{G}_a$. Then the natural map*

$$\coprod_{j\in\mathscr{J}} Env_X(B, f_j) \to Env_X(B, I)$$

is an isomorphism. Here the coproduct is taken in the category of B–algebras.

Proof. By Remark 3.2.3, $\mathrm{Env}_X(B, I)$ corepresents the functor H_1 that sends $S \to \mathrm{Eq}(\mathrm{Hom}_B(I, X(S)) \rightrightarrows \mathrm{Hom}_B(I, S))$, where one of the maps come from composing with $X(S) \to S$ and the other one maps everything to the element in $\mathrm{Hom}_B(I, S)$ corresponding to $I \subset B \to S$. Given such an element in the equalizer, by precomposing with the surjection $F \to I$, we obtain a natural transformation from H_1 to the functor H_2 that sends $S \to \mathrm{Eq}(\mathrm{Hom}_B(F, X(S)) \rightrightarrows \mathrm{Hom}_B(F, S))$, where, as before, one of the maps come from $X(S) \to S$ and the other one from sending everything to the B-linear map $F \to I \subset B \to S$. Since $F \to I$ is a surjection, it follows that the map $H_1(S) \to H_2(S)$ is injective. Below we check that this map is also surjective.

To show surjectivity, we need to show that any map $u : F \to X(S)$ which fits into the commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ I & \longrightarrow & S \end{array}$$

factors through $F \to I$. Let $u_i := u(x_i) \in X(S)$. Since X is a quasi-ideal in $\mathbb{G}_a$, it follows that $du_j \cdot u_i = du_i \cdot u_j$ where d denotes the map $X(S) \to S$. By the commutativity of the diagram this implies $f_j \cdot u_i = f_i \cdot u_j$ in $X(S)$ or equivalently $u(f_j x_i - f_i x_j) = 0$, i.e., the map indeed factors through I. Now the proposition follows by using Remark 3.2.7 and noting that $\coprod_{j\in\mathscr{J}} \mathrm{Env}_X(B, f_j)$ corepresents the functor H_2. ☐

Proposition 3.2.13. *Let B be an A–algebra. Let S be any set. Let $(B[S], (S))$ denote the coproduct of $(B[x], x)$ over S. Let X be a quasi-ideal in $\mathbb{G}_a$. Then the natural map*

$$\coprod_S Env_X(B[x], x) \to Env_X(B[S], S)$$

is an isomorphism of B–algebras. Here the coproduct is taken in the category of B–algebras. In particular, we obtain an isomorphism $(\coprod_S \Gamma(X, \mathcal{O}_X)) \otimes_A B \simeq Env_X(B[S], S)$ of B–algebras.

Proof. This follows from Proposition 3.2.12 and Proposition 3.2.9. ☐

Definition 3.2.14. Let $F \in \mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)_{\mathfrak{G}/}$ (Remark 3.2.4) be a functor which satisfies the following conditions.

1. The natural map $\mathfrak{G}(B, 0) \to F(B, 0)$ is an isomorphism for every A–algebra B.
2. The natural map $F((B[x], x)) \otimes_B F((B[x], x)) \to F(B[x] \otimes_B B[x], (x \otimes 1, 1 \otimes x))$ is an isomorphism.
3. The natural map $F(A[x], x) \otimes_A B \to F(B[x], x)$ is an isomorphism.

We denote the full subcategory of such functors inside $\mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)_{\mathfrak{G}/}$ as $\mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)^{\otimes}_{\mathfrak{G}/}$.

Proposition 3.2.15. *Let $F \in Fun(\mathfrak{C}_A, Alg_A)^{\otimes}_{\mathfrak{G}/}$. For every A–algebra B, $Spec\, F(B[x], x)$ naturally has the structure of a B–module scheme. Thus we obtain a (contravariant) functor $r : Fun(\mathfrak{C}_A, Alg_A)^{\otimes}_{\mathfrak{G}/} \to QID\text{–}\mathbb{G}_a$.*

Proof. The first assertion follows from the fact that $(B[x], x)$ is a B–comodule object of $\mathfrak{C}_{A_{(B,0)/}}$ and the assumptions on F. Varying this data over B defines a $\mathbb{G}_a$–module. Further, using functoriality along the maps $(B[x], 0) \to (B[x], x)$ equips this $\mathbb{G}_a$–module with the structure of a pointed $\mathbb{G}_a$–module. Finally, the second assertion now follows from the discussion preceding Definition 3.2.10. □

Proposition 3.2.16. *The functor $r : Fun(\mathfrak{C}_A, Alg_A)^{\otimes}_{\mathfrak{G}/} \to QID\text{–}\mathbb{G}_a{}^{op}$ has a left adjoint given by Un from Remark 3.2.11.*

Proof. Let $F \in \mathrm{Fun}(\mathfrak{C}_A, \mathrm{Alg}_A)^{\otimes}_{\mathfrak{G}/}$ and $X \in \mathrm{QID}\text{–}\mathbb{G}_a{}^{\mathrm{op}}$. We prove that there is a natural bijection

$$\mathrm{Hom}(\mathrm{Un}(X), F) \simeq \mathrm{Hom}(X, rF).$$

Applying r and noting that $r\mathrm{Un}(X) \simeq X$ by Proposition 3.2.9 and Proposition 3.2.13 provides a map from the left hand side to the right hand side which will be called s. We will construct a map the other way. We will first construct a map $\mathrm{Un}(rF) \to F$. To do so, we note that there is an isomorphism

$$\varphi : \mathrm{Hom}_{(B,0)}((B, I), \cdot) \simeq \mathrm{Eq}\left(\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x], x), \cdot)) \rightrightarrows \mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x], 0), \cdot))\right)$$

in $\mathrm{Psh}(\mathfrak{C}^{\mathrm{op}}_{A_{(B,0)/}})$. Here on the right hand side, one of the maps is induced by the map $(B[x], 0) \to (B[x], x)$ in $\mathfrak{C}_A$ and the other map is obtained by sending everything to the element in $\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x], 0), \cdot))$ corresponding to the map induced by the inclusion $I \subset B$ and the fact that $\mathrm{Hom}_{(B,0)}((B[x], 0), \cdot)$ is naturally valued in B–algebras.

We note that F induces a map $F^{\mathrm{op}} : \mathrm{Psh}(\mathfrak{C}^{\mathrm{op}}_{A_{(B,0)/}}) \to \mathrm{Psh}(\mathrm{Alg}^{\mathrm{op}}_B)$. Applying F^{op} to the above isomorphism φ and noting that $F^{\mathrm{op}}\mathrm{Hom}_{(B,0)}((B, I), \cdot) \simeq \mathrm{Hom}_B(F(B, I), \cdot)$ we obtain a diagram

$$\mathrm{Hom}_B(F(B, I), \cdot) \to \left(F^{\mathrm{op}}\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x], x), \cdot)) \rightrightarrows F^{\mathrm{op}}\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x], 0), \cdot))\right)$$

in $\mathrm{Psh}(\mathrm{Alg}^{\mathrm{op}}_B)$.

Lemma 3.2.17. *The following diagram in $Psh(Alg^{op}_B)$ commutes.*

$$\begin{array}{ccc} F^{op}Hom_{B\text{-}Mod}(I, Hom_{(B,0)}((B[x], x), \cdot)) & \rightrightarrows & F^{op}Hom_{B\text{-}Mod}(I, Hom_{(B,0)}((B[x], 0), \cdot)) \\ \downarrow & & \downarrow \\ Hom_{B\text{-}Mod}(I, Hom_B(F(B[x], x), \cdot)) & \rightrightarrows & Hom_{B\text{-}Mod}(I, Hom_B(F(B[x], 0), \cdot)) \end{array}$$

Proof. This follows from universal property and assumption 2 in Definition 3.2.14 on F. We will show how to construct a map $F^{\mathrm{op}}\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x], x), \cdot)) \to \mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_B(F(B[x], x), \cdot))$. We note that I is the coequalizer of a diagram $\left(F^{B \times I \coprod I \times I} \rightrightarrows F^I\right)$ of free B–modules where the first map sends the basis elements $x_{(b,i)} \to x_{bi}$ and $x_{(i,i')} \to x_{i+i'}$ and the second map sends $x_{(b,i)} \to bx_i$ and $x_{(i,i')} \to x_i + x_{i'}$. Therefore, $\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x], x), \cdot))$ is the equalizer of the two maps

$$\prod_I \mathrm{Hom}_{(B,0)}((B[x], x), \cdot) \rightrightarrows \prod_{B \times I \coprod I \times I} \mathrm{Hom}_{(B,0)}((B[x], x), \cdot).$$

One of the maps corresponds to the map determined by $x_{(b,i)} \to x_{bi}$ and $x_{(i,i')} \to x_{i+i'}$. The other map is induced by combining the maps

$$\prod_I \mathrm{Hom}_{(B,0)}((B[x], x), \cdot) \to \mathrm{Hom}_{(B,0)}((B[x], x), \cdot) \to \mathrm{Hom}_{(B,0)}((B[x], x), \cdot),$$

where the first map is projection from i-th factor and the second map is obtained by using $(B[x], x) \to (B[x], x)$ that sends $x \to bx$ for $b \in B$; and

$$\prod_I \mathrm{Hom}_{(B,0)}((B[x], x), \cdot) \to \mathrm{Hom}_{(B,0)}((B[x], x), \cdot) \times \mathrm{Hom}_{(B,0)}((B[x], x), \cdot) \to \mathrm{Hom}_{(B,0)}((B[x], x), \cdot)$$

where the first map is projection from (i,i')-th factor and the last map uses the map $(B[x],x) \to (B[x],x) \otimes (B[x],x)$ given by $x \to x \otimes 1 + 1 \otimes x$. Now universal property of limits and assumption 2 in Definition 3.2.14 constructs the desired map

$$F^{\mathrm{op}}\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x],x),\cdot)) \to \mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_B(F(B[x],x),\cdot))$$

and the naturality guarantees the commutativity of the diagram in the Lemma. □

Thus we obtain a map

$$\mathrm{Hom}_B(F(B,I),\cdot) \to \mathrm{Eq}\left(\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_B(F(B[x],x),\cdot)) \rightrightarrows \mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_B(F(B[x],0),\cdot))\right)$$

in $\mathrm{Psh}(\mathrm{Alg}_B^{\mathrm{op}})$. Using Remark 3.2.3, we note that the right hand side is corepresented by $\mathrm{Env}_{rF}(B,I)$. Thus we obtain a natural map $\mathrm{Env}_{rF}(B,I) \to F(B,I)$. This provides the map $\mathrm{Un}(rF) \to F$ that we wanted. Now given a map $X \to rF$ in QID–$\mathbb{G}_a{}^{\mathrm{op}}$, we obtain a map $\mathrm{Un}(X) \to \mathrm{Un}(rF) \to F$. This gives a map from $\mathrm{Hom}(X, rF)$ to $\mathrm{Hom}(\mathrm{Un}(X), F)$ which will be called t. By Proposition 3.2.9, it follows that st is identity. In order to show that ts is identity, it will be sufficient to show that if there are two natural transformations $U, V : \mathrm{Un}(X) \to F$ that are mapped to the same element by s then U and V are the same natural transformation. Note that we always have a commutative diagram

$$\begin{array}{ccc} \mathrm{Un}(r\mathrm{Un}(X)) & \xrightarrow{\ \simeq\ } & \mathrm{Un}(X) \\ \downarrow & & \downarrow {\scriptstyle U \,|\, V} \\ \mathrm{Un}(rF) & \longrightarrow & F \end{array}$$

Since the upper horizontal arrow is an isomorphism, the above diagram shows that U and V are the same natural transformation as desired. □

Proposition 3.2.18. *The functor*

$$Un : QID\text{–}\mathbb{G}_a{}^{op} \to Fun(\mathfrak{C}_A, Alg_A)_{\mathfrak{G}/}$$

is fully faithful.

Proof. Follows from Proposition 3.2.16 since $r\mathrm{Un}(X) \simeq X$. □

3.3 Unwinding pointed $\mathbb{G}_a^{\mathbf{perf}}$–modules I

In this section, we will record an analogue of the construction from previous section for pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules. In order to do that, some modifications are needed. As is the case with $\mathbb{G}_a^{\mathrm{perf}}$–modules, we work with a fixed prime p.

Notation 3.3.1. Let $\mathscr{P}I$ denote the category of pairs (B,I) where B is a perfect ring and I is an ideal. Morphisms are defined to be maps $(B,I) \to (B',I')$ where $B \to B'$ is a ring homomorphism such that I is mapped inside I'. Let A be a fixed Artinian local ring with residue field $\mathbb{F}_p$. Let $\mathbb{G}_a^{\mathrm{perf}}$–$\mathrm{Mod}_*$ denote the category of pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules over A.

Construction 3.3.2 (Unwinding)**.** We will construct a (contravariant) functor

$$\mathrm{Un} : \mathbb{G}_a^{\mathrm{perf}}\text{–}\mathrm{Mod}_* \to \mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A).$$

We will say that $\mathrm{Un}(X)$ is the functor obtained by *unwinding* the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module X. To describe the construction, we fix an $X \in \mathbb{G}_a^{\mathrm{perf}}$–$\mathrm{Mod}_*$. Given $(B,I) \in \mathscr{P}I$, we obtain a diagram $X_B \to \mathbb{G}_{a,B}^{\mathrm{perf}}$ of B–module schemes over $W_A(B)$ by Proposition 2.2.10. Now the ideal I can be regarded as a B–module and thus by applying Construction 3.1.1, we get a map $\mathscr{T}_{\mathbb{G}_{a,B}^{\mathrm{perf}}}(I) \to \mathscr{T}_{X_B}(I)$. Since $I \subset B$, by the universal property of the construction in Construction 3.1.1, we obtain a map $\mathscr{T}_{\mathbb{G}_{a,B}^{\mathrm{perf}}}(I) \to W_A(B)$. By composition, we get a map $\mathscr{T}_{\mathbb{G}_{a,B}^{\mathrm{perf}}}(I) \to W_A(B) \to \mathscr{T}_{\mathbb{G}_{a,B}^{\mathrm{perf}}}(I) \to \mathscr{T}_{X_B}(I)$. Therefore, we now have two maps

$$\mathscr{T}_{\mathbb{G}_{a,B}^{\mathrm{perf}}}(I) \rightrightarrows \mathscr{T}_{X_B}(I).$$

We denote the coequalizer of the above diagram by $\mathrm{Env}_X(B,I)$ which is naturally a $W_A(B)$–algebra. Now we define $\mathrm{Un}(X)(B,I) := \mathrm{Env}_X(B,I)$.

Remark 3.3.3 (Unwinding via universal property)**.** We describe the universal property of $\mathrm{Env}_X(B,I)$ as a $W_A(B)$–algebra. For a $W_A(B)$–algebra S, we note that $\mathbb{G}_{a,B}^{\mathrm{perf}}(S) = S^\flat$ is naturally a B–module. In fact, there is a map $B \to S^\flat$ of B–modules giving a natural map $I \to S^\flat$. This gives an element $* \in \mathrm{Hom}_B(I, S^\flat)$. By Proposition 2.2.10, X can be regarded as a B–module scheme over $W_A(B)$. Therefore, we obtain two maps $\mathrm{Hom}_B(I, X(S)) \rightrightarrows \mathrm{Hom}_B(I, S^\flat)$. Here one of the maps (of sets) sends everything to $*$ and the other one is the map induced by the data of the point $X \to \mathbb{G}_a^{\mathrm{perf}}$. We note that by Construction 3.3.2, we have

$$\mathrm{Hom}_{W_A(B)}(\mathrm{Env}_X(B,I), S) \simeq \mathrm{Eq}(\mathrm{Hom}_B(I, X(S)) \rightrightarrows \mathrm{Hom}_B(I, S^\flat)).$$

Remark 3.3.4. We note that there is a natural functor $\mathfrak{G} : \mathscr{P}I \to \mathrm{Alg}_A$ given by $(B,I) \mapsto W_A(B)$. From Remark 3.3.3, it follows that $\mathfrak{G} \simeq \mathrm{Un}(\mathbb{G}_a^{\mathrm{perf}})$. Let $\mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A)_{\mathfrak{G}/}$ denote the category of functors $F : \mathscr{P}I \to \mathrm{Alg}_A$ equipped with a natural transformation $\mathfrak{G} \to F$. The morphisms are required to be compatible with this data. It follows that in Construction 3.3.2, we actually produced a (contravariant) functor

$$\mathrm{Un} : \mathbb{G}_a^{\mathrm{perf}}\text{–}\mathrm{Mod}_* \to \mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A)_{\mathfrak{G}/}.$$

Remark 3.3.5. For any pointed $\mathbb{G}_a^{\mathrm{perf}}$–module X, we have the isomorphism $\mathrm{Un}(X)(B,0) \simeq W_A(B)$. This follows from the universal property of the unwinding construction. Thus the natural map $\mathfrak{G} \to \mathrm{Un}(X)$ induces isomorphism restricted to the full subcategory of $\mathscr{P}I$ spanned by objects of the form $(B,0)$ for a perfect ring B.

Remark 3.3.6. We point out that unless $A = \mathbb{F}_p$, sending $(B,I) \mapsto B/I$ is in general not an object of $\mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A)_{\mathfrak{G}/}$ that can be obtained via applying the unwinding functor. When we are working over $A = \mathbb{F}_p$, the functor $(B,I) \to B/I$ is naturally isomorphic to $\mathrm{Un}(\alpha^\flat)$, where $\alpha^\flat$ is the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module as described in Example 2.2.14. Further, in this case, the functor $(B,I) \mapsto (B/I)_{\mathrm{perf}}$ is the unwinding of the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module corresponding to zero.

Proposition 3.3.7. *Let $X \in \mathbb{G}_a^{perf}$–$Mod_*$. Let B be a perfect ring. Then $Env_X(B[x^{1/p^\infty}], x) \simeq \Gamma(X, \mathcal{O}_X) \otimes_A W_A(B)$ as a $W_A(B)$–algebra. The map $Env_X(B[x^{1/p^\infty}], 0) \to Env_X(B[x^{1/p^\infty}], x)$ identifies with the map $W_A(B)[x^{1/p^\infty}] \to \Gamma(X, \mathcal{O}_X) \otimes_A W_A(B)$ corresponding to the data of the point $X \to \mathbb{G}_a^{perf}$.*

Proof. We compute using Construction 3.3.2. Since x is a non-zero divisor in $B[x^{1/p^\infty}]$, the ideal it generates is free of rank 1. Therefore, by using Proposition 3.1.2, we see that $\mathrm{Env}_X(B[x^{1/p^\infty}], x)$ is computed as a coequalizer of the following diagram

$$W_A(B[x^{1/p^\infty}])[y^{1/p^\infty}] \rightrightarrows \Gamma(X, \mathcal{O}_X) \otimes_A W_A(B[x^{1/p^\infty}]).$$

Here one of the map corresponds to the map $W_A(B)[x^{1/p^\infty}][y^{1/p^\infty}] \to W_A(B)[x^{1/p^\infty}]$ that sends $y^{1/p^n} \to x^{1/p^n}$ for all n and is a $W_A(B)[x^{1/p^\infty}]$–algebra map. The other map corresponds to the data of the point, i.e., obtained by base changing a map $A[y^{1/p^\infty}] \to \Gamma(X, \mathcal{O}_X)$. Taking the coequalizer we get the desired conclusion. ☐

Definition 3.3.8. A pointed $\mathbb{G}_a^{\mathrm{perf}}$–module X with the data of the point denoted as $d : X \to \mathbb{G}_a^{\mathrm{perf}}$ will be called a *quasi-ideal in* $\mathbb{G}_a^{\mathrm{perf}}$ if the following diagram commutes.

$$\begin{array}{ccc} X \times X & \xrightarrow{\mathrm{id} \times d} & X \times \mathbb{G}_a^{\mathrm{perf}} \\ \downarrow{\scriptstyle d \times \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{action}} \\ \mathbb{G}_a^{\mathrm{perf}} \times X & \xrightarrow{\mathrm{action}} & X \end{array}$$

We will denote the category of quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$ by QID–$\mathbb{G}_a^{\mathrm{perf}}$ which is the full subcategory spanned by quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$ inside $\mathbb{G}_a^{\mathrm{perf}}$–$\mathrm{Mod}_*$.

Remark 3.3.9. Using the inclusion QID–$\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a^{\mathrm{perf}}$–$\mathrm{Mod}_*$ of categories, we can define a (contravariant) functor QID–$\mathbb{G}_a^{\mathrm{perf}} \to \mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A)_{\mathfrak{G}/}$ which will again be called unwinding and will be denoted by Un.

Proposition 3.3.10. *Let B be a perfect ring. Let $(f_j)_{j\in\mathscr{J}}$ be a collection of non-zero divisors in B and let I be the ideal generated by them. Let F be the free module over B spanned by x_j for $j \in \mathscr{J}$. We assume that the B–module map $F \to I$ that sends $x_i \to f_i$ has kernel generated by $(f_i x_j - f_j x_i)$ for $i, j \in \mathscr{J}$. Let X be a quasi-ideal in $\mathbb{G}_a^{perf}$. Then the natural map*

$$\coprod_{j\in\mathscr{J}} Env_X(B, f_j) \to Env_X(B, I)$$

is an isomorphism. Here the coproduct is taken in the category of $W_A(B)$–algebras.

Proof. Using Remark 3.3.3, this follows in a way similar to the proof of Proposition 3.2.12. $\square$

Proposition 3.3.11. *Let B be a perfect ring. Let S be any set. Let $(B[S^{1/p^\infty}], (S))$ denote the coproduct of $(B[x^{1/p^\infty}], x)$ over S. Let X be a quasi-ideal in $\mathbb{G}_a^{perf}$. Then the natural map*

$$\coprod_S Env_X(B[x^{1/p^\infty}], x) \to Env_X(B[S^{1/p^\infty}], S)$$

is an isomorphism of $W_A(B)$–algebras. Here the coproduct is taken in the category of $W_A(B)$–algebras. In particular, we obtain an isomorphism $\coprod_S \Gamma(X, \mathcal{O}_X) \otimes_A W_A(B) \simeq Env_S(B[S^{1/p^\infty}], S)$ of $W_A(B)$–algebras.

Proof. This follows from Proposition 3.3.7 and Proposition 3.3.10. $\square$

Definition 3.3.12. Let $F \in \mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A)_{\mathfrak{G}/}$ be a functor which satisfies the following conditions.

1. The natural map $\mathfrak{G}(B, 0) \to F(B, 0)$ is an isomorphism for every perfect ring B.
2. The natural map $F(B[x^{1/p^\infty}], x) \otimes_{W_A(B)} F(B[x^{1/p^\infty}], x) \to F(B[x^{1/p^\infty}] \otimes_B B[x^{1/p^\infty}], (x \otimes 1, 1 \otimes x))$ is an isomorphism.
3. The natural map $F(\mathbb{F}_p[x^{1/p^\infty}], x) \otimes_A W_A(B) \to F(B[x^{1/p^\infty}], x)$ is an isomorphism.

We denote the full subcategory of such functors inside $\mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A)_{\mathfrak{G}/}$ as $\mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_A)^{\otimes}_{\mathfrak{G}/}$.

Proposition 3.3.13. *Let $F \in Fun(\mathscr{P}I, Alg_A)^{\otimes}_{\mathfrak{G}/}$. For every perfect ring B, $Spec\, F(B[x^{1/p^\infty}], x)$ is naturally a B–module scheme over $W_A(B)$. Consequently, we have a (contravariant) functor $r : Fun(\mathscr{P}I, Alg_A)^{\otimes}_{\mathfrak{G}/} \to$ QID–$\mathbb{G}_a^{perf}$.*

Proof. We note that $(B[x^{1/p^\infty}], x)$ is a cogroup object of $\mathscr{P}I_{(B,0)/}$. Therefore it follows from definitions that $\mathrm{Spec}\, F(B[x^{1/p^\infty}], x)$ is a group scheme. The B–action on $\mathrm{Spec}\, F(B[x^{1/p^\infty}], x)$ is given by functoriality along the arrows $(B[x^{1/p^\infty}], x) \to (B[x^{1/p^\infty}], x)$ given by $x^{1/p^n} \to b^{1/p^n} x^{1/p^n}$ for all $n \geq 1$. Therefore, $\mathrm{Spec}\, F(B[x^{1/p^\infty}], x)$ is indeed a B–module scheme over $W_A(B)$. Proposition 2.2.10 implies that varying this data over all perfect rings B provides us a $\mathbb{G}_a^{\mathrm{perf}}$–module. Further, functoriality along the maps $(B[x^{1/p^\infty}], 0) \to (B[x^{1/p^\infty}], x)$ equips this $\mathbb{G}_a^{\mathrm{perf}}$–module with the structure of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module. To see that it is a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$, we use functoriality along the following commutative diagram in $\mathscr{P}I$.

$$\begin{array}{ccc}
(\mathbb{F}_p[x^{1/p^\infty}], x) \otimes (\mathbb{F}_p[x^{1/p^\infty}], x) & \longleftarrow & (\mathbb{F}_p[x^{1/p^\infty}], x) \otimes (\mathbb{F}_p[x^{1/p^\infty}], 0) \\
\uparrow & & \uparrow {\scriptstyle x^{\frac{1}{p^n}} \to x^{\frac{1}{p^n}} \otimes x^{\frac{1}{p^n}}} \\
(\mathbb{F}_p[x^{1/p^\infty}], 0) \otimes (\mathbb{F}_p[x^{1/p^\infty}], x) & \xleftarrow[x^{\frac{1}{p^n}} \otimes x^{\frac{1}{p^n}} \leftarrow x^{\frac{1}{p^n}}]{} & (\mathbb{F}_p[x^{1/p^\infty}], x)
\end{array}$$

$\square$

Proposition 3.3.14. *The functor $r : Fun(\mathscr{P}I, Alg_A)^{\otimes}_{\mathfrak{G}/} \to$ QID–$\mathbb{G}_a^{perf^{op}}$ has a left adjoint given by Un from Remark 3.3.9.*

Proof. The proof follows in a similar way to the proof of Proposition 3.2.16 once we note the following statement about the category $\mathscr{P}I$. Let B be a perfect ring so that $(B,0)$ is an object of $\mathscr{P}I$. Then there is an isomorphism

$$\varphi : \mathrm{Hom}_{(B,0)}((B,I),\cdot) \simeq \mathrm{Eq}\left(\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x^{1/p^\infty}],x),\cdot)) \rightrightarrows \mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x^{1/p^\infty}],0),\cdot))\right)$$

in $\mathrm{Psh}(\mathscr{P}I^{\mathrm{op}}_{(B,0)/})$. Here one of the arrows is induced by the map $(B[x^{1/p^\infty}],0) \to (B[x^{1/p^\infty}],x)$ and the other map is obtained by sending everything to the element of $\mathrm{Hom}_{B\text{-Mod}}(I, \mathrm{Hom}_{(B,0)}((B[x^{1/p^\infty}],0),\cdot))$ corresponding to the map induced by the inclusion $I \subset B$ and the fact that $\mathrm{Hom}_{(B,0)}((B[x^{1/p^\infty}],0),\cdot)$ is naturally valued in B–algebras. ☐

Proposition 3.3.15. *The functor*

$$Un : QID\text{–}\mathbb{G}_a^{perf\,op} \to Fun(\mathscr{P}I, Alg_A)_{\mathfrak{G}/}$$

is fully faithful.

Proof. Follows from Proposition 3.3.14 after noting that $r\mathrm{Un}(X) \simeq X$ by Proposition 3.3.7 and Proposition 3.3.11. ☐

Having discussed the unwinding functor for pointed $\mathbb{G}_a$ and $\mathbb{G}_a^{\mathrm{perf}}$–modules, we end this section with a statement regarding their compatibility. For simplicity, we work over the fixed base ring $\mathbb{F}_p$. In this case, one has the functor $u^* : \mathbb{G}_a\text{–Mod}_* \to \mathbb{G}_a^{\mathrm{perf}}\text{–Mod}_*$ from Proposition 2.2.16. We prove the following

Proposition 3.3.16. *Let X be a pointed $\mathbb{G}_a$–module over $\mathbb{F}_p$. Let $(B,I) \in \mathscr{P}I$. Then there is a natural isomorphism*

$$Env_X(B,I) \simeq Env_{u^*X}(B,I).$$

Proof. This follows by using the universal properties of $\mathrm{Env}_X(B,I)$ and $\mathrm{Env}_{u^*X}(B,I)$ as a B–algebra from Remark 3.2.3 and Remark 3.3.3 and the following pullback diagram of B–modules for a given B–algebra S from Proposition 2.2.16.

$$\begin{array}{ccc} u^*X(S) & \longrightarrow & S^\flat \\ \downarrow & & \downarrow \\ X(S) & \longrightarrow & S \end{array}$$

☐

Example 3.3.17. In Remark 3.3.6 we stated that $\mathrm{Un}(\alpha^\flat)$ is the functor that sends $(B,I) \in \mathscr{P}I$ to B/I. This also follows from Example 3.2.5 and Proposition 3.3.16.

Example 3.3.18. We note that $\mathrm{Un}(u^*W[F])$ is the functor that sends $(B,I) \in \mathscr{P}I$ to $D_B(I)$. This follows from Example 3.2.6.

3.4 Unwinding pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules II

In this section, we record a variant of the construction appearing in the previous section. We will use a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module to produce a functor from QRSP algebras over $\mathbb{F}_p$ to Alg_A. Our goal is to formulate and prove an analogue of Proposition 3.3.15 in this context. We will begin by recalling the definition of QRSP algebras from [BMS19, Def. 8.8].

Definition 3.4.1. An $\mathbb{F}_p$–algebra S is said to be semiperfect if the natural map $S^\flat \to S$ is surjecrive. S is called quasiregular semiperfect (QRSP) if S is semiperfect and the contangent complex $\mathbb{L}_{S/\mathbb{F}_p}$ is a flat S–module supported in (homological) degree 1.

Example 3.4.2. The algebra $\mathbb{F}_p[x^{1/p^\infty}]/x$ is an example of a QRSP algebra.

Construction 3.4.3. For a QRSP algebra S, by sending $S \mapsto (S^\flat, \mathrm{Ker}(S^\flat \to S))$ we can define a functor $\mathrm{QRSP} \to \mathscr{P}I$. Using Construction 3.3.2, this produces a (contravariant) functor $\mathrm{Un} : \mathbb{G}_a^{\mathrm{perf}}\text{–}\mathrm{Mod}_* \to \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)$ which will again be called the unwinding of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module when no confusion is likely to occur.

Remark 3.4.4. We note that the functor $\mathfrak{G} : \mathscr{P}I \to \mathrm{Alg}_A$ that sends $(R, I) \mapsto W_A(R)$ produces a functor $\mathrm{QRSP} \to \mathrm{Alg}_A$ that sends $S \mapsto W_A(S^\flat)$ which will again be denoted by $\mathfrak{G}$. It follows from Remark 3.3.4 that we have actually produced a (contravariant) functor $\mathrm{Un} : \mathbb{G}_a^{\mathrm{perf}}\text{–}\mathrm{Mod}_* \to \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)_{\mathfrak{G}/}$.

Example 3.4.5. The identity functor $\mathrm{QRSP} \to \mathrm{QRSP}$ induces a functor $\mathrm{QRSP} \to \mathrm{Alg}_{\mathbb{F}_p}$ which is the unwinding of the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module corresponding to $\alpha^\flat$ over $\mathbb{F}_p$. This follows from the construction and Remark 3.3.6.

Definition 3.4.6. Let $F \in \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)_{\mathfrak{G}/}$ be a functor that satisfies the following conditions.

1. The natural map $\mathfrak{G}(B) \to F(B)$ is an isomorphism for every perfect ring B.
2. The natural map $F(\frac{B[x^{1/p^\infty}]}{x}) \otimes_{W_A(B)} F(\frac{B[x^{1/p^\infty}]}{x}) \to F(\frac{B[x^{1/p^\infty}]}{x} \otimes_B \frac{B[x^{1/p^\infty}]}{x})$ is an isomorphism for every perfect ring B.
3. The natural map $F(\frac{\mathbb{F}_p[x^{1/p^\infty}]}{x}) \otimes_A W_A(B) \to F(\frac{B[x^{1/p^\infty}]}{x})$ is an isomorphism for every perfect ring B.

The full subcategory spanned by such functors inside $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)_{\mathfrak{G}/}$ will be denoted as $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)_{\mathfrak{G}/}^{\otimes}$.

Proposition 3.4.7. *Let $F \in Fun(QRSP, Alg_A)_{\mathfrak{G}/}^{\otimes}$. For every perfect ring B, $\mathrm{Spec}\, F(\frac{B[x^{1/p^\infty}]}{x})$ is naturally a B–module scheme over $W_A(B)$. Consequently, we have a (contravariant) functor $r : Fun(QRSP, Alg_A)_{\mathfrak{G}/}^{\otimes} \to QID\text{–}\mathbb{G}_a^{perf}$.*

Proof. This follows in a way similar to the proof of Proposition 3.3.13. We note that $\frac{B[x^{1/p^\infty}]}{x}$ is a B–comodule object of $\mathrm{QRSP}_{B/}$. Therefore, by functoriality, $\mathrm{Spec}\, F(\frac{B[x^{1/p^\infty}]}{x})$ has the structure of a B–module scheme over $W_A(B)$. Proposition 2.2.10 implies that varying this data over all perfect rings B provides us a $\mathbb{G}_a^{\mathrm{perf}}$–module. Further, functoriality along the maps $B[x^{1/p^\infty}] \to \frac{B[x^{1/p^\infty}]}{x}$ equips this $\mathbb{G}_a^{\mathrm{perf}}$–module with the structure of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module. To see that it is a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$, we use functoriality along the following commutative diagram in QRSP.

$$\begin{array}{ccc}
\frac{\mathbb{F}_p[x^{1/p^\infty}]}{x} \otimes \frac{\mathbb{F}_p[x^{1/p^\infty}]}{x} & \longleftarrow & \frac{\mathbb{F}_p[x^{1/p^\infty}]}{x} \otimes \mathbb{F}_p[x^{1/p^\infty}] \\
\uparrow & & \uparrow{\scriptstyle x^{\frac{1}{p^n}} \to x^{\frac{1}{p^n}} \otimes x^{\frac{1}{p^n}}} \\
\mathbb{F}_p[x^{1/p^\infty}] \otimes \frac{\mathbb{F}_p[x^{1/p^\infty}]}{x} & \underset{x^{\frac{1}{p^n}} \otimes x^{\frac{1}{p^n}} \leftarrow x^{\frac{1}{p^n}}}{\longleftarrow} & \frac{\mathbb{F}_p[x^{1/p^\infty}]}{x}
\end{array}$$

□

Remark 3.4.8. Note that we *do not* have a functor $\mathrm{QID}\text{–}\mathbb{G}_a^{\mathrm{perf}} \to \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)_{\mathfrak{G}/}^{\otimes}$ induced by the unwinding. Indeed, the unwinding of the quasi-ideal $\mathbb{G}_a^{\mathrm{perf}}$ produces the functor that sends a QRSP algebra $S \mapsto S^\flat$ and does not satisfy the conditions of Definition 3.4.6. However, we will work towards rectifying this situation by restricting our attention to a special class of quasi-ideals. In any case, we have the following proposition.

Proposition 3.4.9. *Let $r : Fun(QRSP, Alg_A)_{\mathfrak{G}/}^{\otimes} \to QID\text{–}\mathbb{G}_a^{perf}$ be the functor from Proposition 3.4.7. Let $F \in Fun(QRSP, Alg_A)_{\mathfrak{G}/}^{\otimes}$. Then there is a natural transformation $Un(rF) \to F$ in $Fun(QRSP, Alg_A)_{\mathfrak{G}/}$.*

Proof. This follows in a way similar to the proof of Proposition 3.2.16 once we note the following statement about the category QRSP. Let S be a QRSP algebra. Let $I := \mathrm{Ker}(S^\flat \to S)$. Then there is an isomorphism

$$\varphi : \mathrm{Hom}_{S^\flat}(S, \cdot) \simeq \mathrm{Eq}\left(\mathrm{Hom}_{S^\flat\mathrm{Mod}}(I, \mathrm{Hom}_{S^\flat}(\frac{S^\flat[x^{1/p^\infty}]}{x}, \cdot)) \rightrightarrows \mathrm{Hom}_{S^\flat\mathrm{Mod}}(I, \mathrm{Hom}_{S^\flat}(S^\flat[x^{1/p^\infty}], \cdot))\right)$$

in $\mathrm{PSh}(\mathrm{QRSP}^{\mathrm{op}}_{S^\flat/})$. Here one of the arrows is induced by the map $S^\flat[x^{1/p^\infty}] \to \frac{S^\flat[x^{1/p^\infty}]}{x}$ and the other map is obtained by sending everything to the element of $\mathrm{Hom}_{S^\flat\mathrm{Mod}}(I, \mathrm{Hom}_{S^\flat}(S^\flat[x^{1/p^\infty}], \cdot))$ corresponding to the map induced by the inclusion $I \subset S^\flat$ and the fact that $\mathrm{Hom}_{S^\flat}(S^\flat[x^{1/p^\infty}], \cdot)$ is naturally valued in $S^\flat$–algebras. □

Definition 3.4.10 (Nilpotent quasi-ideals)**.** Let X be a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ over $\mathbb{F}_p$. We will call X a *nilpotent* quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ if the graded map $\mathbb{F}_p[x^{1/p^\infty}] \to \Gamma(X, \mathcal{O}_X)$ corresponding to $X \to \mathbb{G}_a^{\mathrm{perf}}$ is zero in large enough degrees. In other words writing $t \in \Gamma(X, \mathcal{O}_X)$ as the image of x, we need t to be a nilpotent element. We define $\mathcal{N}\mathrm{QID}$–$\mathbb{G}_a^{\mathrm{perf}}$ to be the full subcategory of QID–$\mathbb{G}_a^{\mathrm{perf}}$ spanned by nilpotent quasi-ideals.

Example 3.4.11. We note that $\alpha^\flat$ and $u^*W[F]$ are both examples of nilpotent quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$ over $\mathbb{F}_p$. However, $\mathbb{G}_a^{\mathrm{perf}}$ is *not* an example of a nilpotent quasi-ideal.

Remark 3.4.12. We note that for every $\mathbb{F}_p$–algebra R, using the map $X \to \mathbb{G}_a^{\mathrm{perf}}$, one gets a map $X(R) \to R^\flat$ at the level of R–valued points. Composing along the map $R^\flat \to R$, we get a map $w : X(R) \to R$. It follows that if X is a nilpotent quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$, then $w(z)$ is a nilpotent element of R for every $z \in X(R)$. One can analogously define a notion of "nilpotent quasi-ideals" in $\mathbb{G}_a$ as well, which can be though of as an analogue of locally nilpotent ideals at the level of R–valued points for every $\mathbb{F}_p$–algebra R. Since we do not use the notion of nilpotent quasi-ideals in $\mathbb{G}_a$, we do not discuss them here.

Remark 3.4.13. In fact if X is a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ over $\mathbb{F}_p$ that is not isomorphic to $\mathbb{G}_a^{\mathrm{perf}}$ then X is a nilpotent quasi-ideal. To see this, we note that by writing $X = \operatorname{Spec} B$ for a graded Hopf algebra B and t^i for the image of x^i under the map $\mathbb{F}_p[x^{1/p^\infty}] \to B$ (here $i \in \mathbb{N}[1/p]_{\cup 0}$), we note that X is a quasi-ideal if and only if $b \otimes t^{\deg b} = t^{\deg b} \otimes b$ in $B \otimes B$ for every homogeneous $b \in B$. Now $b \otimes t^{\deg b} = t^{\deg b} \otimes b$ implies that $t^{\deg b}$ and b are linearly dependent in the $\mathbb{F}_p$ vector space B. Thus if X is not nilpotent, i.e., if $t^i \neq 0$ for all i, then any non-zero homogeneous $b \in B$ is in the linear span of t^i. Thus as a graded algebra $B \simeq \mathbb{F}_p[x^{1/p^\infty}]$ and since the map $\mathbb{F}_p[x^{1/p^\infty}] \to B$ is a map of graded Hopf algebras, it follows that the quasi-ideal X is isomorphic to $\mathbb{G}_a^{\mathrm{perf}}$.

We will now record a lemma.

Lemma 3.4.14. *For a perfect ring B, let $S := B[x_1^{1/p^\infty}, \ldots, x_n^{1/p^\infty}]$ and $I := (x_1, \ldots, x_n)$. Then $(S/I)^\flat = \widehat{S}$, the I–adic completion of S. Further, kernel of the map $(S/I)^\flat \to S/I$ is identified with the ideal $(x_1, \ldots, x_n)$ in $\widehat{S}$.*

Proof. This will follow from the more general fact that if S is a perfect ring and I is a finitely generated ideal then $(S/I)^\flat$ is isomorphic to the I–adic completion of S denotes as $\widehat{S}$. We let $I^{[p^n]} := \left\{x^{p^n} \mid x \in I\right\}$. Since S is perfect, it follows that $I^{[p^n]}$ is an ideal of S. By sending an element to its p^n-th power we get an isomorphism $\phi^n : S/I \to S/I^{[p^n]}$. These maps provide an isomorphism of inverse systems as below.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S/I & \xrightarrow{\phi} & S/I & \xrightarrow{\phi} & S/I \\ & & \downarrow{\scriptstyle \phi^2} & & \downarrow{\scriptstyle \phi} & & \downarrow \\ \cdots & \longrightarrow & S/I^{[p^2]} & \longrightarrow & S/I^{[p]} & \longrightarrow & S/I \end{array}$$

Now since I is finitely generated, $\left\{I^{[p^n]}\right\}$ and $\{I^n\}$ generates the same topology on S. This gives the isomorphism $(S/I)^\flat \simeq \widehat{S}$. □

Proposition 3.4.15. *Let X be a nilpotent quasi-ideal. Then $Un(X)(\frac{B[x^{1/p^\infty}]}{x}) \simeq \Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{F}_p} B$.*

Proof. $\mathrm{Un}(X)(\frac{B[x^{1/p^\infty}]}{x})$ by definition and Lemma 3.4.14 is $\mathrm{Un}(X)(\widehat{B[x^{1/p^\infty}]}, x)$. Since x is a non-zero divisor, this is computed as coequalizer of the two maps

$$\widehat{B[x^{1/p^\infty}]}[y^{1/p^\infty}] \rightrightarrows \Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{F}_p} \widehat{B[x^{1/p^\infty}]}$$

where one of the maps is induced by $\mathbb{F}_p[y^{1/p^\infty}] \to \Gamma(X, \mathcal{O}_X)$ corresponding to the data of the point. The other map is the $\widehat{B[x^{1/p^\infty}]}$–algebra map that sends $y^{1/p^n} \to x^{1/p^n}$. Since the quasi-ideal is nilpotent, a power of y is sent to zero by the first map. Hence we obtain the required isomorphism. □

Proposition 3.4.16. *Let X be a nilpotent quasi-ideal. Then*

$$Un(X)\left(\frac{B[x^{1/p^\infty}]}{x}\otimes_B \frac{B[x^{1/p^\infty}]}{x}\right) \simeq Un(X)\left(\frac{B[x^{1/p^\infty}]}{x}\right)\otimes_B Un(X)\left(\frac{B[x^{1/p^\infty}]}{x}\right) \simeq \Gamma(X,\mathcal{O}_X)\otimes_{\mathbb{F}_p}\Gamma(X,\mathcal{O}_X)\otimes_{\mathbb{F}_p} B.$$

Proof. By definition and Lemma 3.4.14, the left hand side is isomorphic to $\mathrm{Un}(X)(B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}], (x_1, x_2))$. By regularity of (x_1, x_2) as an ideal of $B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}]$ and Proposition 3.3.10 that is computed as

$$\mathrm{Env}_X(B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}], x_1)\otimes_{B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}]}\mathrm{Env}_X(B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}], x_2).$$

By letting t^{1/p^n} denote the image of y^{1/p^n} under the map $\mathbb{F}_p[y^{1/p^\infty}]\to\Gamma(X,\mathcal{O}_X)$ corresponding to the data of the point, we obtain that the above expression is isomorphic to

$$\frac{B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}]\otimes\Gamma(X,\mathcal{O}_X)}{(x_1^{1/p^n}\otimes 1 - 1\otimes t^{1/p^n})}\otimes_{B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}]}\frac{B[\widehat{x_1^{1/p^\infty}, x_2^{1/p^\infty}}]\otimes\Gamma(X,\mathcal{O}_X)}{(x_2^{1/p^n}\otimes 1 - 1\otimes t^{1/p^n})}.$$

Since X is nilpotent, a power of t is zero which along with Proposition 3.4.15 gives the required conclusion. □

Remark 3.4.17. More generally, the proof of Proposition 3.4.16 shows that for a nilpotent quasi-ideal X, $\mathrm{Un}(X)$ commutes with finite coproducts of $\frac{B[x^{1/p^\infty}]}{x}$.

Proposition 3.4.18. *The unwinding of a nilpotent quasi-ideal (over $\mathbb{F}_p$) satisfies the properties in Definition 3.4.6, i.e., we have a functor*

$$Un : \mathcal{N}QID\text{–}\mathbb{G}_a^{perf^{op}} \to Fun(QRSP, Alg_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}.$$

Proof. Let X be a nilpotent quasi-ideal. By definition, we need to check three properties for the functor $F := \mathrm{Un}(X)$. The first one is that the natural map $B \to F(B)$ is an isomorphism for every perfect ring B which follows from Remark 3.3.5. The other two properties follow from Proposition 3.4.15 and Proposition 3.4.16. □

Proposition 3.4.19. *The functor $r : Fun(QRSP, Alg_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/} \to QID\text{–}\mathbb{G}_a^{perf^{op}}$ factors to give a functor $r : Fun(QRSP, Alg_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/} \to \mathcal{N}QID\text{–}\mathbb{G}_a^{perf^{op}}$ which admits a left adjoint given by Un from Proposition 3.4.18.*

Proof. First we prove that we indeed have a factorization which gives the functor $r : \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/} \to \mathcal{N}\mathrm{QID}\text{–}\mathbb{G}_a^{\mathrm{perf}}$. By Remark 3.4.13, it would be enough to prove that the essential image of the functor $r : \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/} \to \mathrm{QID}\text{–}\mathbb{G}_a^{\mathrm{perf}}$ does not contain $\mathbb{G}_a^{\mathrm{perf}}$. We assume on the contrary that there is an $F \in \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}$ such that $rF \simeq \mathbb{G}_a^{\mathrm{perf}}$ as quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$. This implies that the arrow $f : \mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}]/x$ is sent to an isomorphism by F. The arrow f factors as $\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p\widehat{[x^{1/p^\infty}]} \to \mathbb{F}_p[x^{1/p^\infty}]/x$. Applying F to it and using the first property from Definition 3.4.6 gives the following maps

$$\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p\widehat{[x^{1/p^\infty}]} \to F(\mathbb{F}_p[x^{1/p^\infty}]/x)$$

whose composition is an isomorphism. This shows that there are maps $\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p\widehat{[x^{1/p^\infty}]} \to \mathbb{F}_p[x^{1/p^\infty}]$ whose composition is the identity. That implies that there is a map $\mathbb{F}_p\widehat{[x^{1/p^\infty}]} \to \mathbb{F}_p[x^{1/p^\infty}]$ that sends $x \to x$. But no such map can exist since $1 + x$ is a unit on the source but not on the target of the map. Now the required adjunction follows from Proposition 3.4.9, Proposition 3.4.15 and Proposition 3.4.16 (similar to the proof of Proposition 3.2.16) by noting the commutative diagram

$$\begin{array}{ccc}
\mathrm{Un}(r\mathrm{Un}(X)) & \xrightarrow{\simeq} & \mathrm{Un}(X) \\
\downarrow & & \downarrow \\
\mathrm{Un}(rF) & \longrightarrow & F
\end{array}$$

for any natural transformation $\mathrm{Un}(X) \to F$ where $X \in \mathcal{N}\mathrm{QID}\text{–}\mathbb{G}_a^{\mathrm{perf}}$ and $F \in \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}$. ☐

Proposition 3.4.20. *The functor*

$$Un : \mathcal{N}\,QID\text{–}\mathbb{G}_a^{perf^{op}} \to Fun(QRSP, Alg_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}$$

defined in Proposition 3.4.18 is fully faithful.

Proof. We fix two nilpotent quasi-ideals X and Y. By using Proposition 3.4.15 and Proposition 3.4.16, there are natural isomorphisms $r\mathrm{Un}(X) \simeq X$ and $r\mathrm{Un}(Y) \simeq Y$, where r is the functor from Proposition 3.4.9. Therefore, Proposition 3.4.15 and Proposition 3.4.16 implies that Un is faithful. To show that it is full, it would be enough to prove that if F and G are two natural transformations between $\mathrm{Un}(X)$ and $\mathrm{Un}(Y)$ such that they are the same transformation $X \to Y$ in $\mathcal{N}\mathrm{QID}\text{–}\mathbb{G}_a^{\mathrm{perf}^{\mathrm{op}}}$ after applying r, then $F = G$. For this, we note the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Un}(r\mathrm{Un}(X)) & \xrightarrow{\simeq} & \mathrm{Un}(X) \\ \downarrow & & \downarrow F\,\vert\, G \\ \mathrm{Un}(r\mathrm{Un}(Y)) & \xrightarrow{\simeq} & \mathrm{Un}(Y) \end{array}$$

The diagram above shows that $F = G$, as desired.

Alternatively, this follows from Proposition 3.4.19 since $r\mathrm{Un}(X) \simeq X$ for a nilpotent quasi-ideal X. ☐

Remark 3.4.21. More generally, let X and Y be two quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$ over an Artinian local ring A with residue field $\mathbb{F}_p$ such that the functors $\mathrm{Un}(X)$ and $\mathrm{Un}(Y)$ satisfies the three conditions in Definition 3.4.6 and such that there are natural isomorphisms $r\mathrm{Un}(X) \simeq X$ and $r\mathrm{Un}(Y) \simeq Y$. Then the above proof shows that there is a natural bijection $\mathrm{Hom}_{\mathrm{QID}\text{–}\mathbb{G}_a^{\mathrm{perf}}}(Y, X) \simeq \mathrm{Hom}(\mathrm{Un}(X), \mathrm{Un}(Y))$ where the latter Hom is computed in $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)_{\mathfrak{G}/}$.

Now we are ready to make the following definitions.

Definition 3.4.22. We let $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\mathcal{N}\mathrm{Un}}_{\mathfrak{G}/}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}$ spanned by image of nilpotent quasi-ideals under the functor Un from Proposition 3.4.18.

Definition 3.4.23. We let $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\mathrm{rk}=1, \mathcal{N}\mathrm{Un}}_{\mathfrak{G}/}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}$ spanned by the unwinding of the nilpotent quasi-ideals whose underlying pointed $\mathbb{G}_a^{\mathrm{perf}}$–module is of fractional rank 1 (Definition 2.2.17).

Definition 3.4.24. We let $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\mathrm{pure\ rk}=1, \mathcal{N}\mathrm{Un}}_{\mathfrak{G}/}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}$ spanned by the unwinding of the nilpotent quasi-ideals whose underlying pointed $\mathbb{G}_a^{\mathrm{perf}}$–module is pure of fractional rank 1 (Definition 2.5.7).

Thus we obtain the following chain of inclusion of categories

$$\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\mathrm{pure\ rk}=1, \mathcal{N}\mathrm{Un}}_{\mathfrak{G}/} \subset \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\mathrm{rk}=1, \mathcal{N}\mathrm{Un}}_{\mathfrak{G}/} \subset \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\mathcal{N}\mathrm{Un}}_{\mathfrak{G}/} \subset \mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})^{\otimes}_{\mathfrak{G}/}$$

which are all full subcategories of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})_{\mathfrak{G}/}$

Proposition 3.4.25. *The category $Fun(QRSP, Alg_{\mathbb{F}_p})^{rk=1, \mathcal{N}\,Un}_{\mathfrak{G}/}$ has a final object given by the functor $id : QRSP \to Alg_{\mathbb{F}_p}$ that sends $S \mapsto S$.*

Proof. This follows from Proposition 2.3.1, Example 3.4.5 and Proposition 3.4.20. ☐

Proposition 3.4.26. *The category $Fun(QRSP, Alg_{\mathbb{F}_p})^{pure\ rk=1, \mathcal{N}\,Un}_{\mathfrak{G}/}$ has a final object given by the functor $Un(u^*W[F])$.*

Proof. This follows from Remark 2.5.8 and Proposition 3.4.20. ☐

3.5 Hodge filtration

Let X be a fixed pointed $\mathbb{G}_a^{\mathrm{perf}}$–module over $\mathbb{F}_p$. The goal of this section is to construct a decreasing filtration on the functor $\mathrm{Un}(X)$ defined on QRSP algebras which will be called the "Hodge filtration". This will be done by explicitly constructing a functorial filtration on $\mathrm{Un}(X)(S) = \mathrm{Env}_X(S^\flat, \mathrm{Ker}(S^\flat \to S))$. We will show that under the assumption that X is a fractional rank 1 pointed $\mathbb{G}_a^{\mathrm{perf}}$–module, gr^0 of the filtration on $\mathrm{Un}(X)(S)$ identifies with S. This induces a natural transformation $\mathrm{gr}^0 : \mathrm{Un}(X) \to \mathrm{id}$ of functors. Under the additional assumption that X is a nilpotent quasi-ideal, this natural transformation is the same as the one coming from Proposition 3.4.25.

Construction 3.5.1 (Hodge filtration)**.** Let X be a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module over $\mathbb{F}_p$. We will construct a natural decreasing filtration on $\mathrm{Env}_X(B, I)$ for $(B, I) \in \mathscr{P}I$. By Construction 3.1.1, for a fixed $i \in I$, we have a natural map $\Gamma(X_B, \mathcal{O}_{X_B}) \to \coprod_I \Gamma(X_B, \mathcal{O}_{X_B}) \to \mathscr{T}_{X_B}(I)$, where the first map maps $\Gamma(X_B, \mathcal{O}_{X_B})$ to the i-th factor.

By Construction 3.2.2, there is a natural surjection $\mathscr{T}_{X_B}(I) \to \mathrm{Env}_X(B, I)$. Composing this with the above map, we obtain the following map

$$[i] : \Gamma(X_B, \mathcal{O}_{X_B}) \to \mathrm{Env}_X(B, I).$$

If $m \in \Gamma(X_B, \mathcal{O}_{X_B})$ is a homogeneous element, we will write $[i]^m := [i](m)$.

For a nonnegative integer n, we let $\mathrm{Fil}^n\mathrm{Env}_X(B, I)$ denote the ideal of $\mathrm{Env}_X(B, I)$ generated by the "monomials" of the form $[i_1]^{m_1} \cdots [i_k]^{m_k}$ such that $\sum_{u=1}^{k} \deg m_u \geq n$, for homogeneous elements $m_u \in \Gamma(X_B, \mathcal{O}_{X_B})$ of integral degree, $k \geq 1$ and for $i_1, \dots, i_k \in I$. This defines a decreasing filtration which will be called the Hodge filtration on $\mathrm{Env}_X(B, I)$. A map $\varphi : (B, I) \to (B', I')$ sends $[i]^m \to [\varphi(i)]^m$, so the construction is functorial.

Definition 3.5.2. Let X be a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module over $\mathbb{F}_p$ and let S be a QRSP algebra. The decreasing filtration defined by

$$\mathrm{Fil}^n\mathrm{Un}(X)(S) := \mathrm{Fil}^n\mathrm{Env}_X(S^\flat, \mathrm{Ker}(S^\flat \to S))$$

will be called the *Hodge filtration* on $\mathrm{Un}(X)(S)$.

Definition 3.5.3. Let $\widehat{\mathrm{Un}(X)(S)}$ be the completion of $\mathrm{Un}(X)(S)$ with respect to the Hodge filtration. Then the functor $\widehat{\mathrm{Un}(X)}$ will be called the *Hodge completion* of $\mathrm{Un}(X)$.

Example 3.5.4 (I–adic filtration)**.** Let $X = \mathbb{G}_a^{\mathrm{perf}}$ equipped with the structure of a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module. Then $\mathrm{Env}_X(B, I) \simeq B$ by Remark 3.3.4. In this case, the Hodge filtration identifies with the I–adic filtration on B.

Example 3.5.5 (Divided power filtration)**.** Let B be an $\mathbb{F}_p$–algebra and I be an ideal of B. Then there is a filtration on $D_B(I)$ given by setting $\mathrm{Fil}^n(D_B(I))$ to be the ideal generated by divided power monomials $[i_1]^{m_1} \cdots [i_k]^{m_k}$ such that $\sum_{u=1}^{k} m_i \geq n$ and $i_1, \dots, i_k \in I$, which is called the divided power filtration. We note that $\mathrm{Fil}^0(D_B(I)) = D_B(I)$ and $\mathrm{gr}^0(D_B(I)) = B/I$. If we further assume that B is a perfect ring, then we have an isomorphism $\mathrm{Env}_{u^*W[F]}(B, I) \simeq D_B(I)$ by Example 3.3.18 which is further a filtered isomorphism when we equip $\mathrm{Env}_{u^*W[F]}(B, I)$ with the Hodge filtration from Construction 3.5.1.

Proposition 3.5.6. *Let X be a pointed $\mathbb{G}_a^{perf}$–module over $\mathbb{F}_p$ of fractional rank 1. Let S be a QRSP algebra. Then the gr^0 of the Hodge filtration on $Un(X)(S)$ is isomorphic to S.*

Proof. We write $I = \mathrm{Ker}(S^\flat \to S)$. Using Proposition 2.2.8 and since X is of fractional rank 1, we have $\mathrm{gr}^0\mathrm{Env}_X(S^\flat, I) \simeq \mathrm{Env}_{\alpha^\flat}(S^\flat, I) \simeq S$, where the last isomorphism follows from Remark 3.3.6. ☐

Remark 3.5.7. Let X be a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module over $\mathbb{F}_p$ of fractional rank 1. The natural transformation $\mathrm{gr}^0 : \mathrm{Un}(X) \to \mathrm{id}$ induced by Proposition 3.5.6 is the same as the one obtained via the unwinding functor from the Hodge map defined in Proposition 2.3.1.

Proposition 3.5.8. *Let X be a nilpotent quasi-ideal over $\mathbb{F}_p$ which is fractional of rank 1 as a pointed $\mathbb{G}_a^{perf}$–module. Then the natural transformation $gr^0 : Un(X) \to id$ is the unique natural transformation between $Un(X)$ and id viewed as objects of the category $Fun(QRSP, Alg_{\mathbb{F}_p})_{\mathfrak{G}/}$.*

Proof. This follows from Proposition 3.4.25 and Remark 3.5.7. □

Remark 3.5.9. Similarly, for a pointed $\mathbb{G}_a$–module X over $\mathbb{F}_p$, one can define a "Hodge filtration" Fil^n on $\mathrm{Env}_X(B, I)$ for any $\mathbb{F}_p$–algebra B and an ideal $I \subset B$. As a possible application, one can define a candidate theory of $\mathcal{D}$–modules for any given pointed $\mathbb{G}_a$–module X. In order to do so, we let J denote the kernel of the diagonal map $B \otimes B \to B$. One can attempt to define the ring of "differential operators" of B associated to X as

$$\mathcal{D}_B^X := \varinjlim_n \mathrm{Hom}_B(\mathrm{Env}_X(B \otimes B, J)/\mathrm{Fil}^n, B).$$

In the case when X is the pointed $\mathbb{G}_a$–module given by $W[F]$, we recover the ring of crystalline differential operators [BO78]. In the case when X is the pointed $\mathbb{G}_a$–module given by $\mathbb{G}_a$ itself, we recover Grothendieck's ring of differential operators [Gro67].

4 Review of de Rham and Crystalline cohomology

In this section, we will briefly recall the notion of derived de Rham cohomology. Our goal is to provide a different definition of derived de Rham cohomology as the unwinding of a particular quasi-ideal. The point of our definition is that a single object (a quasi-ideal in $\mathbb{G}_a^{\text{perf}}$) can recover the entire theory of derived de rham cohomology via the unwinding functor. Thus many questions about (derived) de Rham cohomology theory can be translated into a question about quasi-ideals and can be approached in a direct manner. We will also provide a definition of crystalline cohomology as the unwinding of some quasi-ideal in $\mathbb{G}_a^{\text{perf}}$.

In order to achieve these goals, there is one technical obstruction. The theory of derived de Rham cohomology works with commutative algebra objects in derived categories, whereas the notion of quasi-ideals only work with discrete rings. We are able to overcome this difficulty by crucially relying on the notion of QRSP algebras due to the work of Bhatt, Morrow and Scholze. For our purpose, QRSP algebras are abundant enough such that derived de Rham cohomology can be completely understood by its values on them, and further, $\mathrm{dR}(S)$ is a discrete ring for a QRSP algebra S. These properties make it possible to understand derived de Rham cohomology via the unwinding functor constructed in Section 3.4.

The following definition is from [Bha12, Remark. 2.2].

Definition 4.0.1 (Derived de Rham cohomology)**.** Derived de Rham cohomology is a functor denoted as dR from the ∞-category of simplicial commutative rings to the ∞-category of commutative algebra objects in the derived ∞-category $D(\mathbb{F}_p)$ obtained by left Kan extending the usual de Rham cohomology functor which sends a finitely generated polynomial $\mathbb{F}_p$–algebra A to the algebraic de Rham complex of A, i.e.,

$$0 \to A \to \Omega^1_{A/\mathbb{F}_p} \to \Omega^2_{A/\mathbb{F}_p} \to \ldots \to$$

Thus by definition, the derived de Rham cohomology functor dR is determined by its restriction to polynomial algebras over $\mathbb{F}_p$. Further, since dR has quasisyntomic descent [BS19, Lemma 8.6], the functor dR restricted to polynomial algebras can be completely understood by restriction of dR to QRSP algebras by using descent along the map $A \to A_{\text{perf}}$ for a polynomial algebra A and the fact that each term in the Čech conerve of $A \to A_{\text{perf}}$ is a QRSP algebra.

Proposition 4.0.2. *Let S be a QRSP algebra. Then $dR(S) \simeq D_{S^\flat}(I)$ where $I := Ker(S^\flat \to S)$.*

Proof. This is [BMS19, Prop. 8.12]. □

Proposition 4.0.3 (Derived de Rham cohomology via unwinding)**.** *As functors from $QRSP \to Alg_{\mathbb{F}_p}$, the functor dR and the functor $Un(u^*W[F])$ are naturally isomorphic. Further, the Hodge filtration on $dR(S)$ coincides with the Hodge filtration on $Un(u^*W[F])(S)$ constructed in Section 3.5.*

Proof. This follows from Example 3.3.18 and Example 3.5.5. □

Proposition 4.0.4 (Universal property of dR)**.** *As a functor from $QRSP \to Alg_{\mathbb{F}_p}$, dR is the final object of $Fun(QRSP, Alg_{\mathbb{F}_p})^{pure\ rk=1, \mathcal{N}Un}_{\mathfrak{G}/}$.*

Proof. This follows from Proposition 3.4.26 and Proposition 4.0.3. □

Proposition 4.0.5. *The natural transformation $gr^0 : dR \to id$ coming from the Hodge filtration in derived de Rham cohomology is the unique natural transformation between dR and id viewed as objects of the category $Fun(QRSP, Alg_{\mathbb{F}_p})_{\mathfrak{G}/}$.*

Proof. This follows from Proposition 3.5.8. □

Proposition 4.0.6. *The functor dR has no nontrivial endomorphisms as an object of $Fun(QRSP, Alg_{\mathbb{F}_p})_{\mathfrak{G}/}$.*

Proof. By Proposition 4.0.4, dR is the final object in a full subcategory of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})_{\mathfrak{G}/}$ which gives the claim. □

Proposition 4.0.7. *[BLM20, Prop. 10.3.1] If we consider dR as a functor defined on smooth $\mathbb{F}_p$ algebras, then any endomorphism of dR that commutes with the gr^0 map of the Hodge filtration $gr^0 : dR \to id$ is identity.*

Proof. By left Kan extension, we obtain an endomorphism of $\mathrm{dR} : \mathrm{QRSP} \to \mathrm{Alg}_{\mathbb{F}_p}$. We check that this endomorphism is an endomorphism in the category $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})_{\mathfrak{G}/}$. This will follow by functoriality of dR along the arrows $S^\flat \to S$ for a QRSP algebra S after noting that the endomorphism is identity when restricted to perfect rings. But the latter follows by the hypothesis that the endomorphism commutes with $\mathrm{gr}^0 : \mathrm{dR} \to \mathrm{id}$. Now by Proposition 4.0.6 and quasisyntomic descent, we obtain the desired statement. $\square$

Next we study the crystalline situation. Our goal is to prove that the theory of derived crystalline cohomology [BMS19, 8.2] can be entirely recovered from a single quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$. By left Kan extension and quasisyntomic descent, one can again restrict attention to only QRSP algebras. We make the following definitions.

Definition 4.0.8. Let S be a QRSP algebra. We define $\mathbb{A}_{\mathrm{crys}}(S)$ to be the p–adic completion of the divided power envelope of $W(S^\flat) \to S$. Here our divided powers are required to be compatible with those on $(p) \subset W(S^\flat)$.

Remark 4.0.9. From the first part of [BMS19, Thm. 8.14], it follows that $\mathbb{A}_{\mathrm{crys}}(S)$ is flat over $\mathbb{Z}_p$ for a QRSP algebra S.

Definition 4.0.10. Let $(A, \mathfrak{m})$ be an Artinian local ring with residue field $\mathbb{F}_p$. We will let $R\Gamma_{\mathrm{crys}}(S)_A := \mathbb{A}_{\mathrm{crys}}(S) \otimes_{\mathbb{Z}_p} A$.

We note that $R\Gamma_{\mathrm{crys}}(S)_A$ is a flat A–algebra. Further, $R\Gamma_{\mathrm{crys}}(S)_A \otimes_A \mathbb{F}_p \simeq \mathrm{dR}(S)$ by [BMS19, Prop. 8.12]. Our goal is to prove the following.

Proposition 4.0.11 (Derived crystalline cohomology via unwinding)**.** *The functor $R\Gamma_{crys}(\cdot)_A : QRSP \to Alg_A$ is the unwinding of a quasi-ideal in $\mathbb{G}_a^{perf}$ over A.*

Proof. There is a functor $\mathfrak{G} : \mathrm{QRSP} \to \mathrm{Alg}_A$ sending $S \mapsto W_A(S^\flat)$. By the natural identifications $W_A(S^\flat) \simeq W(S^\flat) \otimes_{\mathbb{Z}_p} A \simeq \mathbb{A}_{\mathrm{crys}}(S^\flat) \otimes_{\mathbb{Z}_p} A = R\Gamma_{\mathrm{crys}}(S^\flat)_A$ and functoriality along $S^\flat \to S$, we obtain a natural transformation $\mathfrak{G} \to R\Gamma_{\mathrm{crys}}(\cdot)_A$. Thus we can view $R\Gamma_{\mathrm{crys}}$ as an object of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)_{\mathfrak{G}/}$. From Proposition 4.0.3, by going modulo $\mathfrak{m}$ and using flatness we can conclude that $R\Gamma_{\mathrm{crys}}$ satisfies the three conditions of Definition 3.4.6 and thus is an object of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_A)^{\otimes}_{\mathfrak{G}/}$. Therefore by Proposition 3.4.7, $r(R\Gamma_{\mathrm{crys}})$ is a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ over A. By Proposition 3.4.9, there is a natural transformation $\mathrm{Un}(r(R\Gamma_{\mathrm{crys}})) \to R\Gamma_{\mathrm{crys}}$. Since $R\Gamma_{\mathrm{crys}} \otimes_A \mathbb{F}_p \simeq \mathrm{dR}$, it follows that $r(R\Gamma_{crys})$ is a deformation of the quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ given by $u^*W[F]$. By commuting colimits, $\mathrm{Un}(r(R\Gamma_{\mathrm{crys}})) \otimes_A \mathbb{F}_p \simeq \mathrm{Un}(u^*W[F]) \simeq \mathrm{dR}$. Thus the map $\mathrm{Un}(r(R\Gamma_{\mathrm{crys}})) \to R\Gamma_{\mathrm{crys}}$ is a natural isomorphism by the following lemma.

Lemma 4.0.12. *Let $(A, \mathfrak{m})$ be an Artinian local ring. Let $M \to N$ be a map of A–modules where N is flat. Suppose that $M \otimes_A A/\mathfrak{m} \to N \otimes_A A/\mathfrak{m}$ is an isomorphism. Then the map $M \to N$ is an isomorphism.*

Proof. Since $\mathfrak{m}$ is nilpotent, it follows that $M \to N$ is surjective. Let $K := \mathrm{Ker}(M \to N)$. Then we have an exact sequence $0 \to K \to M \to N \to 0$. By flatness of N, we get an exact sequence

$$0 \to K/\mathfrak{m}K \to M/\mathfrak{m}M \to N/\mathfrak{m}N \to 0.$$

By hypothesis, we must have $K/\mathfrak{m}K = 0$. Again, since $\mathfrak{m}$ is nilpotent, this implies $K = 0$, which proves the lemma. $\square$

Indeed, for every QRSP algebra S, we have a map $\mathrm{Un}(r(R\Gamma_{\mathrm{crys}}))(S) \to R\Gamma_{\mathrm{crys}}(S)_A$ which is an isomorphism modulo $\mathfrak{m}$ and $R\Gamma_{\mathrm{crys}}(S)_A$ is flat over A. Thus the map must be an isomorphism by the lemma. $\square$

Remark 4.0.13. By the proof of Proposition 4.0.11, we also saw that the quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ given by $r(R\Gamma_{\mathrm{crys}})$ is a deformation of $u^*W[F]$ over the ring A. Once we know this description, it is not difficult to describe $r(R\Gamma_{\mathrm{crys}})$ explicitly. On the other hand, according to the proposition, this quasi-ideal recovers crystalline cohomology via the unwinding functor. Therefore, in principle, this gives a way of defining crystalline cohomology without mentioning divided power structures and using the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module structure instead.

5 Formal étaleness of de Rham cohomology

In this final section, our goal is to prove our main theorem that the functor dR is formally étale. More precisely, we prove the following

Theorem 5.0.1. *Let*

$$dR : Alg^{sm}_{\mathbb{F}_p} \to CAlg(D(\mathbb{F}_p))$$

be the algebraic de Rham cohomology functor defined on the category of smooth $\mathbb{F}_p$–algebras $Alg^{sm}_{\mathbb{F}_p}$. Given an Artinian local ring $(A, \mathfrak{m})$ with residue field $\mathbb{F}_p$, the functor dR admits a unique deformation

$$dR' : Alg^{sm}_{\mathbb{F}_p} \to CAlg(D(A)).$$

Further, the deformation dR' is unique up to unique isomorphism. (Here a deformation is supposed to mean the data $dR' \otimes^L_A \mathbb{F}_p \simeq dR$.) More precisely, the space of deformations of dR is contractible.

5.1 First proof using deformation theory of $u^*W[F]$

First we make some adjustments so that the values taken by dR are discrete rings as opposed to commutative algebra objects in a derived category. By left Kan extension and étale descent for dR, it would be enough to prove the same statement for the functor

$$\mathrm{dR} : \mathrm{Poly}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p)).$$

Let $\mathrm{dR}' : \mathrm{Poly}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(A))$ be a deformation of dR. To prove that dR' is unique up to unique isomorphism, we can again do a left Kan extension to obtain a functor $\mathrm{dR}' : \mathrm{QSyn}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(A))$ which extends dR' from polynomial algebras, where $\mathrm{QSyn}_{\mathbb{F}_p}$ denotes the category of quasisyntomic $\mathbb{F}_p$–algebras [BMS19, Def. 1.7]. By construction, dR' is a deformation of the derived de Rham cohomology functor $\mathrm{dR} : \mathrm{QSyn}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p))$. Now the category of quasisyntomic $\mathbb{F}_p$–algebras contains all the QRSP algebras. Therefore one can try to recover the functor dR' restricted to finitely generated polynomial algebras A via descent along the map $A \to A_{\mathrm{perf}}$. Here $A_{\mathrm{perf}} := \mathrm{colim}_{x \to x^p} A$. The following lemma guarantees that this is possible by using the notion of descendability [Mat16, Def. 3.18].

Lemma 5.1.1. *Let $A \in Poly_{\mathbb{F}_p}$. Then the map $dR'(A) \to dR'(A_{perf})$ is descendable.*

Proof. This follows from the fact that $\mathrm{dR}(A) \to \mathrm{dR}(A_{\mathrm{perf}})$ is descendable [BS19, Lemma 8.6] and the fact that descendability can be checked (derived) modulo $\mathfrak{m}$ since $\mathfrak{m}$ is nilpotent. The latter claim follows from [Mat16, Prop 3.24] and [Mat16, Prop 3.35]. The fact that descendability implies descent along $A \to A_{\mathrm{perf}}$ for dR' follows from Prop. 3.20 loc. cit. □

By using the above lemma and the fact that each term in the Čech conerve of $A \to A_{\mathrm{perf}}$ is a QRSP algebra, in order to prove Theorem 5.0.1, it is enough to prove the following statement.

- Let $\mathrm{dR} : \mathrm{QRSP} \to \mathrm{CAlg}(D(\mathbb{F}_p))$ be the derived de Rham cohomology functor. Given an Artinian local ring $(A, \mathfrak{m})$ with residue field $\mathbb{F}_p$, the functor dR admits a deformation $\mathrm{dR}' : \mathrm{QRSP} \to \mathrm{CAlg}(D(A))$ which is unique up to unique isomorphism.

Now we note again that $\mathrm{dR}(S)$ is a discrete ring for a QRSP algebra S. Thus dR' is also a discrete ring which is flat over A by [SP, Tag 051H]. Thus it is enough to prove the following statement (Theorem 1.1.2).

- Let $\mathrm{dR} : \mathrm{QRSP} \to \mathrm{Alg}_{\mathbb{F}_p}$ be the derived de Rham cohomology functor. Given an Artinian local ring $(A, \mathfrak{m})$ with residue field $\mathbb{F}_p$, the functor dR admits a deformation $\mathrm{dR}' : \mathrm{QRSP} \to \mathrm{Alg}_A$ which is unique up to unique isomorphism.

We note that by Definition 4.0.10, we already know that there exists a deformation of dR given by $\mathrm{dR}' = R\Gamma_{\mathrm{crys}}(\cdot)_A$. Now we appeal to general topos theory to reduce the problem further. Let $\mathcal{X}$ denote the topos $\mathrm{Psh}(\mathrm{QRSP}^{\mathrm{op}})$. Then dR is an $\mathbb{F}_p$–algebra object in $\mathcal{X}$ and we are trying to understand deformations

dR' of dR as an A–algebra object. Since this deformation problem is unobstructed and is controlled by a theory of cotangent complex [Ill71, Cor. 2.1.3.3], it is enough to prove the above claim when $\mathfrak{m}^2 = 0$ by devissage. Further, since in this case $\mathfrak{m}$ is a finite dimensional $\mathbb{F}_p$-vector space, by computing Ext groups, it is enough to prove the claim in the case where $A = \mathbb{F}_p[\epsilon]/\epsilon^2$. Thus to prove Theorem 5.0.1, it is enough to prove the following proposition.

Proposition 5.1.2. *Let $dR : QRSP \to Alg_{\mathbb{F}_p}$ be the derived de Rham cohomology functor. Then dR has no nontrivial deformation to $\mathbb{F}_p[\epsilon] := \mathbb{F}_p[\epsilon]/\epsilon^2$. Further, the deformation is unique up to unique isomorphism. (Here the trivial deformation is given by tensoring up to $\mathbb{F}_p[\epsilon]$.)*

Proof. We note that there is a natural transformation $\mathfrak{G} \to \mathrm{dR}$ where $\mathfrak{G}(S) = S^\flat$. Thus dR can be viewed as an object of $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p})_{/\mathfrak{G}}$. Since cotangent complex of a perfect ring vanishes, the maps $S^\flat \to \mathrm{dR}(S)$ lifts uniquely to maps $S^\flat[\epsilon] \to \mathrm{dR}'(S)$ for any deformation dR' of dR. It follows that any deformation of dR and any endomorphism of dR as a deformation can be studied as deformations and endomorphisms in the category $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p[\epsilon]})_{/\mathfrak{G}}$. Further, by using the remarks in [Bha12, Def. 2.1], dR and hence any deformation of dR preserves sufficiently many colimits as a functor from QRSP to $\mathrm{Alg}_{\mathbb{F}_p[\epsilon]}$ so that it satisfies the three properties in Definition 3.4.6. Thus we can also work inside the category $\mathrm{Fun}(\mathrm{QRSP}, \mathrm{Alg}_{\mathbb{F}_p[\epsilon]})^{\otimes}_{/\mathfrak{G}}$. Writing dR' for a deformation of dR, by using Proposition 3.4.7 and Proposition 4.0.3 we see that $r(\mathrm{dR}')$ is a deformation of $u^*W[F]$ as a quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$. By Proposition 3.4.9, there is a natural transformation $\mathrm{Un}(r(\mathrm{dR}')) \to \mathrm{dR}'$. Going modulo ϵ and using Lemma 4.0.12, this natural transformation is actually a natural isomorphism. By Remark 3.4.21 it would be enough to show that any deformation of $u^*W[F]$ to $\mathbb{F}_p[\epsilon]$ is uniquely isomorphic to the trivial deformation. This follows from Proposition 2.5.9 and Proposition 2.5.11. □

5.2 Second proof using deformation theory of $W[F]$

The goal of this subsection is to provide a slightly different proof of Proposition 5.1.2 that avoids the use of deformation theory of $u^*W[F]$, i.e., Proposition 2.5.11. Instead, we would use the deformation theory of $W[F]$ which is much easier to understand by universal properties (Proposition 2.5.4) and the stability of the Hodge map (Proposition 2.3.6).

Lemma 5.2.1. *Let $dR : QRSP \to Alg_{\mathbb{F}_p}$ be the derived de Rham cohomology functor. Let $dR' : QRSP \to Alg_{\mathbb{F}_p[\epsilon]}$ be a deformation of dR. Then there exists a functor*

$$dR'' : Poly_{\mathbb{F}_p} \to CAlg(D(\mathbb{F}_p[\epsilon]))$$

which is a deformation of dR such that dR' is the restriction of left Kan extension of dR'' to QRSP algebras.

Proof. Let P be a finitely generated polynomial algebra over $\mathbb{F}_p$. Let $C^\bullet(P)$ denote the Čech conerve of the map $P \to P_{\mathrm{perf}}$. Applying dR' to $C^\bullet(P)$ we obtain a cosimplicial ring and we define $\mathrm{dR}''(P)$ to be the totalization of this cosimplicial ring as an object of $\mathrm{CAlg}(D(\mathbb{F}_p[\epsilon]))$. This defines the functor

$$\mathrm{dR}'' : \mathrm{Poly}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p[\epsilon])).$$

First, we check that dR'' defined as above is indeed a deformation of dR. By definition $\mathrm{dR}''(P) = \mathrm{Tot}(\mathrm{dR}'(C^\bullet(P)))$ which is an inverse limit of the pro-object $\left\{\mathrm{Tot}_n(\mathrm{dR}'(C^\bullet(P)))\right\}$. Since $\mathrm{dR}(P) \to \mathrm{dR}(P_{\mathrm{perf}})$ is descendable [BS19, Lemma 8.6], it follows that $\{\mathrm{Tot}_n(\mathrm{dR}(C^\bullet(P)))\}$ is a pro-constant pro-object. Therefore, by Lemma 5.2.2, $\left\{\mathrm{Tot}_n(\mathrm{dR}'(C^\bullet(P)))\right\}$ is a pro-constant pro-object. This proves that $\mathrm{Tot}(\mathrm{dR}'(C^\bullet(P)))\otimes_{\mathbb{F}_p[\epsilon]}\mathbb{F}_p \simeq \mathrm{Tot}(\mathrm{dR}(C^\bullet(P)))$; thus dR'' is indeed a deformation of dR.

Next, we need to check that the left Kan extension of dR'' is naturally isomorphic to dR' on QRSP algebras. To do so, we will first construct a natural map $\mathrm{dR}''(P) \to \mathrm{dR}'(S)$ for a map $P \to S$ where P is a finitely generated polynomial $\mathbb{F}_p$–algebra and S is a QRSP algebra. Let $C^\bullet(P)$ denote the Čech conerve of $P \to P_{\mathrm{perf}}$ and $C^\bullet(S/P)$ denote the Čech conerve of $S \to S \otimes_P P_{\mathrm{perf}}$. There is a natural map of cosimplicial rings $C^\bullet(P) \to C^\bullet(S/P)$. Applying dR' we obtain a map $\mathrm{dR}'(C^\bullet(P)) \to \mathrm{dR}'(C^\bullet(S/P))$. Thus in order to construct the map $\mathrm{dR}''(P) \to \mathrm{dR}'(S)$, it would be enough to show that $\mathrm{Tot}(\mathrm{dR}'(C^\bullet(S/P))) \simeq \mathrm{dR}'(S)$, since by definition $\mathrm{Tot}(\mathrm{dR}'(C^\bullet(P))) \simeq \mathrm{dR}''(P)$. For that, it would be enough to show that $\mathrm{dR}'(S) \to \mathrm{dR}'(S \otimes_P P_{\mathrm{perf}})$

is descendable. But that follows (by going modulo ϵ) since $\mathrm{dR}(S) \to \mathrm{dR}(S \otimes_P P_{\mathrm{perf}})$ is descendable; this is true because the latter map is a base change of the descendable map $\mathrm{dR}(P) \to \mathrm{dR}(P_{\mathrm{perf}})$. Going back, this constructs the required map $\mathrm{dR}''(P) \to \mathrm{dR}'(S)$. Now, writing $\mathrm{dR}''(S)$ for the left Kan extension of dR'' evaluated at S, we obtain a natural map $\mathrm{dR}''(S) \to \mathrm{dR}'(S)$. By construction, this map is an isomorphism after going (derived) modulo ϵ and thus must be an isomorphism. □

The following lemma was used in the above proof.

Lemma 5.2.2. *Let R be a ring and I be a nilpotent ideal. Let $\{B_n\}$ be a pro-object in $D(R)$. If $\{B_n \otimes_R R/I\}$ is a pro-constant pro-object then $\{B_n\}$ is pro-constant itself.*

Proof. Let C be the collection of objects u in $D(R)$ such that $\{B_n \otimes_A u\}$ is a pro-constant pro-system. Then C is a thick tensor-ideal which contains R/I by assumption. Since R has a finite filtration whose graded pieces are R/I–modules (i.e., the I–adic filtration) it follows that $R \in C$. Since R is the unit under tensor this gives that $\{B_n\}$ is pro-constant. □

Second proof of Proposition 5.1.2. We follow the notations and the strategy from the first proof of Proposition 5.1.2. Instead of invoking Proposition 2.5.11, by Proposition 2.5.4, it would be enough to show that the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module $r(\mathrm{dR}')$ over $\mathbb{F}_p[\epsilon]$ is isomorphic to u^*X where X is a pointed $\mathbb{G}_a$–module which is a deformation of $W[F]$. Here the functor u^* is from Proposition 2.2.16.

We note that for derived de Rham cohomology, there is a natural transformation $\mathrm{gr}^0 : \mathrm{dR}(A) \to A$ obtained from taking gr^0 of the Hodge filtration. By left Kan extension from the case of polynomial algebras, one sees that the relative derived de Rham cohomology satisfies $\mathrm{dR}_{B/A} \simeq \mathrm{dR}(B) \otimes_{\mathrm{dR}(A)} A$, where the tensor product is taken using the gr^0 map. This isomorphism also appears in [GL20, Prop. 3.11]. Using Proposition 2.3.6 and applying the unwinding functor, one obtains a natural transformation $\mathrm{dR}' \to \mathrm{id}[\epsilon]$ as functors from $\mathrm{QRSP} \to \mathrm{Alg}_{\mathbb{F}_p[\epsilon]}$ deforming the $\mathrm{gr}^0 : \mathrm{dR} \to \mathrm{id}$ transformation. Using the construction as in Lemma 5.2.1 and left Kan extension, this extends to a natural transformation $\mathrm{dR}'' \to \mathrm{id}[\epsilon]$ as functors from $\mathrm{Alg}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p[\epsilon]))$.

We will define dR''' on the category of arrows $(A \to B)$ of $\mathbb{F}_p$–algebras. We define $\mathrm{dR}'''(A \to B) := \mathrm{dR}''(B) \otimes_{\mathrm{dR}''(A)} A[\epsilon]$ where we use the map $\mathrm{dR}''(A) \to A[\epsilon]$. Now this takes values in $\mathrm{CAlg}(D(\mathbb{F}_p[\epsilon]))$ but we will only use it in case of objects $(A \to B)$ where it would give a discrete ring as output. We note that $\mathrm{dR}'''(A \to A) \simeq A[\epsilon]$ and for a QRSP algebra S, $\mathrm{dR}'''(S^\flat \to S) \simeq \mathrm{dR}''(S) \simeq \mathrm{dR}'(S)$, where the last isomorphism comes from Lemma 5.2.1. Further $\mathrm{dR}'''(\mathbb{F}_p[x] \to \mathbb{F}_p)$ is a discrete ring as it is so (derived) modulo ϵ by using [Bha12, Lemma 3.29]. It also follows that $\mathrm{dR}'''(\mathbb{F}_p[x] \to \mathbb{F}_p)$ is a flat algebra over $\mathbb{F}_p[\epsilon]$. We record two lemmas.

Lemma 5.2.3. *$\mathrm{Spec}\, dR'''(\mathbb{F}_p[x] \to \mathbb{F}_p)$ has the structure of a pointed $\mathbb{G}_a$–module over $\mathbb{F}_p[\epsilon]$ and it is a deformation of $W[F]$ as a pointed $\mathbb{G}_a$–module.*

Proof. We note that in the arrow category of $\mathbb{F}_p$–algebras, the object $(\mathbb{F}_p[x] \to \mathbb{F}_p[x])$ is a coring, i.e., it corepresents the functor $(A \to B) \to A$ which is naturally valued in rings. Further the object $(\mathbb{F}_p[x] \to \mathbb{F}_p)$ is a a comodule over $(\mathbb{F}_p[x] \to \mathbb{F}_p[x])$ since $(\mathbb{F}_p[x] \to \mathbb{F}_p)$ corepresents the functor $(A \to B) \to \mathrm{Ker}(A \to B)$ which is naturally an A–module and further admits a map of A–modules $\mathrm{Ker}(A \to B) \to A$. This provides a map $(\mathbb{F}_p[x] \to \mathbb{F}_p[x]) \to (\mathbb{F}_p[x] \to \mathbb{F}_p)$ of $(\mathbb{F}_p[x] \to \mathbb{F}_p[x])$–comodules. Applying the de Rham cohomology functor dR to this yields a map $\mathbb{F}_p[x] \to \mathrm{dR}(\mathbb{F}_p[x] \to \mathbb{F}_p)$ of $\mathbb{F}_p[x]$–comodules. Using the fact that $\mathrm{dR}(\mathbb{F}_p[x] \to \mathbb{F}_p) \simeq \mathbb{F}_p\langle x\rangle$ [Bha12, Lemma 3.29] and Proposition 2.4.10 we see that $\mathrm{Spec}\,\mathrm{dR}(\mathbb{F}_p[x] \to \mathbb{F}_p)$ is isomorphic to $W[F]$ as a pointed $\mathbb{G}_a$–module. Now the lemma follows from applying dR''' to the same diagrams and going (derived) modulo ϵ. □

Lemma 5.2.4. *The pullback $u^*\mathrm{Spec}\, dR'''(\mathbb{F}_p[x] \to \mathbb{F}_p)$ is isomorphic to $r(dR')$ as a pointed $\mathbb{G}_a^{perf}$–module.*

Proof. We again look at the arrow category of $\mathbb{F}_p$–algebras. The object $(\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}])$ is a coring as it corepresents the functor $(A \to B) \to A^\flat$. The object $(\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}]/x)$ is a comodule over $(\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}])$ as it corepresents the functor $(A \to B) \to \mathrm{Ker}(A^\flat \to B)$. Further, this produces a map $(\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}]) \to (\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}]/x)$ of $(\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}])$–comodules. Lastly, we have a map $f : (\mathbb{F}_p[x] \to \mathbb{F}_p[x]) \to (\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}])$. By taking pushout of the map

$$(\mathbb{F}_p[x] \to \mathbb{F}_p[x]) \to (\mathbb{F}_p[x] \to \mathbb{F}_p)$$

of $(\mathbb{F}_p[x] \to \mathbb{F}_p[x])$–comodules along the map f we obtain the map

$$(\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}]) \to (\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}]/x)$$

of $(\mathbb{F}_p[x^{1/p^\infty}] \to \mathbb{F}_p[x^{1/p^\infty}])$–comodules. The rest follows from applying dR''' to all the diagrams. Indeed, the statement of the lemma now depends upon certain natural colimit diagrams being isomorphisms, which holds since they are known to be isomorphisms after going (derived) modulo ϵ. ☐

Now the two lemmas above show that $r(\mathrm{dR}')$ is indeed a pullback of a deformation of $W[F]$ as a pointed $\mathbb{G}_a$–module, which finishes the proof. ☐

Remark 5.2.5. The proof in Section 5.2 somewhat formally reduces the study of deformations of the pointed $\mathbb{G}_a^{\mathrm{perf}}$–module underlying $u^*W[F]$ by showing that any such deformation over $\mathbb{F}_p[\epsilon]/\epsilon^2$ must appear as a pullback of a deformation of $W[F]$ as a pointed $\mathbb{G}_a$–module along the map of ring schemes $u : \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$. This phenomenon seems to occur more generally under suitable conditions. We formulate and sketch a proof of the following proposition which is motivated by Drinfeld's construction of taking the cone of a quasi-ideal in [Dri20]. We note that the construction of taking the cone is valid for any pointed $\mathbb{G}_a$–module X or any pointed $\mathbb{G}_a^{\mathrm{perf}}$–module Y. However, the cone in this generality only has the structure of a group stack and not a ring stack.

Proposition 5.2.6. *Let X be a pointed $\mathbb{G}_a$–module which is full of rank 1 (cf. Definition 2.3.3). Let M be a deformation of u^*X as a pointed $\mathbb{G}_a^{\mathrm{perf}}$–module over $\mathbb{F}_p[\epsilon]/\epsilon^2$. Then $M \simeq u^*X'$ where X' is a deformation of X over $\mathbb{F}_p[\epsilon]/\epsilon^2$ as a pointed $\mathbb{G}_a$–module. Here u^* is the functor constructed in Proposition 2.2.16.*

Proof. One can take the cone of the map $M \to \mathbb{G}_a^{\mathrm{perf}}$ to obtain a flat map $f : \mathbb{G}_a^{\mathrm{perf}} \to [\mathbb{G}_a^{\mathrm{perf}}/M]$. Here $[\mathbb{G}_a^{\mathrm{perf}}/M]$ has the structure of a group stack and the map f is additionally a map of group stacks. Formation of the cone commutes with base change and therefore the cone of $u^*X \to \mathbb{G}_a^{\mathrm{perf}}$ is given by $\mathbb{G}_a^{\mathrm{perf}} \to [\mathbb{G}_a^{\mathrm{perf}}/M] \times \operatorname{Spec} \mathbb{F}_p$. Since X is full of rank 1, by definition u^*X is full of fractional rank 1 and hence by Proposition 2.3.6, there is a map $\alpha^\flat \to M$ of pointed $\mathbb{G}_a^{\mathrm{perf}}$–modules. By taking cones, we obtain a map of group stacks $\mathbb{G}_a \to [\mathbb{G}_a^{\mathrm{perf}}/M]$ which factors the map f along the map of ring schemes $\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$. Since the map $\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$ is faithfully flat, it follows that the map $\mathbb{G}_a \to [\mathbb{G}_a^{\mathrm{perf}}/M]$ is flat. By taking kernel of the map $\mathbb{G}_a \to [\mathbb{G}_a^{\mathrm{perf}}/M]$, we obtain a pointed $\mathbb{G}_a$–module X' which is flat over $\operatorname{Spec} \mathbb{F}_p[\epsilon]/\epsilon^2$. Since M is the kernel of f by construction, it follows that $u^*X' \simeq M$. Since taking kernel commutes with base change, it follows that $X' \times \operatorname{Spec} \mathbb{F}_p$ is the kernel of $\mathbb{G}_a \to [\mathbb{G}_a^{\mathrm{perf}}/M] \times \operatorname{Spec} \mathbb{F}_p$. Since u^*X is the kernel of $\mathbb{G}_a^{\mathrm{perf}} \to [\mathbb{G}_a^{\mathrm{perf}}/M] \times \operatorname{Spec} \mathbb{F}_p$ it follows that $u^*X \simeq u^*(X' \times \operatorname{Spec} \mathbb{F}_p)$. This descends to a natural isomorphism $X \simeq X' \times \operatorname{Spec} \mathbb{F}_p$ by Proposition 2.2.16. Thus X' is indeed a deformation of X and it meets the necessary requirements of the proposition. ☐

Remark 5.2.7. Using the constructions in Section 3, for any quasi-ideal X in $\mathbb{G}_a$ over $\mathbb{F}_p$, one can define a "cohomology theory" $\mathcal{U}_X$ equipped with a Hodge filtration. In the table below, we make a list of rough analogies and comparisons which explain the constructions appearing in Proposition 5.2.6 from a cohomological perspective. In what follows below, $[\mathbb{G}_a/X]$ is a ring stack which is the cone of the quasi-ideal X. To pass from the left hand side to the right hand side, we take the derived global sections of the structure sheaf on the geometric objects.

In the case when $X = W[F]$, the right side of the table recovers some of the known facts about derived de Rham cohomology from [Bha12]. This also gives a rough comparison between the stacky approach and the more explicit approach taken in our paper.

Quasi-ideals/ring stacks	**Cohomology theory**
$[\mathbb{G}_a/X]$	$\mathcal{U}_X(\mathbb{F}_p[x])$
$\mathbb{G}_a \to [\mathbb{G}_a/X]$	$\mathrm{gr}^0_{\mathrm{Hodge}} : \mathcal{U}_X(\mathbb{F}_p[x]) \to \mathbb{F}_p[x]$
$X = \mathrm{Ker}(\mathbb{G}_a \to [\mathbb{G}_a/X])$	$\mathcal{U}_X(\mathbb{F}_p[x] \to \mathbb{F}_p) = \mathbb{F}_p \otimes_{\mathcal{U}_X(\mathbb{F}_p[x])} \mathbb{F}_p[x]$
u^*X	$\mathcal{U}_X(\mathbb{F}_p[x^{1/p^\infty}]/x)$
$\alpha^\flat \to u^*X$	$\mathrm{gr}^0_{\mathrm{Hodge}} : \mathcal{U}_X(\mathbb{F}_p[x^{1/p^\infty}]/x) \to \mathbb{F}_p[x^{1/p^\infty}]/x$

Remark 5.2.8. Antieau and Mathew asked us if the statement of Theorem 5.0.1 remains valid if A is an animated Artinian local ring with residue field $\mathbb{F}_p$. The answer in this case does not seem to follow directly from the results in our paper. One can hope to carry out the strategy of the proof in the case where A is a discrete ring, but for that one would have to talk about structures such as pointed $\mathbb{G}_a$–module or $\mathbb{G}_a^{\mathrm{perf}}$–module over an animated Artinian local ring A. We hope to address this in future.

Remark 5.2.9. It seems to be an interesting question to classify all quasi-ideals in $\mathbb{G}_a$ or $\mathbb{G}_a^{\mathrm{perf}}$. In particular, motivated by Drinfeld's construction of Prismatization, it seems interesting to study the moduli stack $\mathcal{Q}$ of all quasi-ideals in $\mathbb{G}_a$. There is a canonical ring stack $\mathcal{R}$ over $\mathcal{Q}$ obtained by taking the cone of each quasi-ideal in $\mathcal{Q}$. By using the stacky approach and the version of the unwinding explained in Example 3.0.1, it seems plausible to construct a "universal cohomology theory" $\mathcal{U}$ for varieties over $\mathbb{F}_p$ which would specialize to $\mathcal{U}_X$ from Remark 5.2.7 for a quasi-ideal $X \in \mathcal{Q}$.

References

[Ber74] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique p >0*, Lecture Notes in Mathematics, Vol. 407, Springer-Verlag, Berlin-New York, 1974. MR 0384804

[BO78] P. Berthelot and O. Ogus, *Notes on crystalline cohomology,* Princeton University Press, Princeton, N.J. (1978)

[Bha12] B. Bhatt, *p–adic derived de Rham cohomology,* available at arXiv:1204.6560

[BL] B. Bhatt and J. Lurie, *Prismatic crystals and their Chern classes*, In preparation.

[BLM20] B. Bhatt, J. Lurie, and A. Mathew, *Revisiting the de Rham–Witt complex,* available at arXiv:1805.05501

[BMS19] B. Bhatt, M. Morrow, and P. Scholze, *Topological hochschild homology and integral p–adic Hodge theory,* Publications mathématiques de l'IHÉS **129** (2019), no. 1, 199–310.

[BS19] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology,* available at arXiv:1905.08229

[CF66] P. Conner and E. Floyd, *The relation of cobordism to K-theories,* Lecture Notes in Mathematics, No. 28, Springer-Verlag, Berlin, (1966)

[Dri18] V. Drinfeld, *A stacky approach to crystals,* available at arXiv:1810.11853

[Dri20] ———, *Prismatization,* available at arXiv:2005.04746

[FJ13] J.-M. Fontaine and U. Jannsen, *Frobenius gauges and a new theory of p-torsion sheaves in characteristic p. I,* available at arXiv:1304.3740

[GR14] D. Grinberg and V. Riener, *Hopf algebras in combinatorics,* available at arXiv:1409.8356

[Gro66] A. Grothendieck, *On the de Rham cohomology of algebraic varieties,* Inst. Hautes Etudes Sci. Publ. Math. (1966), no. 29, 95–103. MR 0199194

[Gro67] ———, *Eléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV,* Inst. Hautes Etudes Sci. Publ. Math. (1967), no. 32, 361. MR MR0238860

[Gro68] ———, *Crystals and the de Rham cohomology of schemes,* Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, Notes by I. Coates and O. Jussila, pp. 306–358. MR 269663

[GL20] H. Guo and S. Li, *Period sheaves via derived de Rham cohomology,* available at arXiv:2008.06143

[Ill71] L. Illusie, *Complexe cotangent et déformations I,* Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, (1971)

[Ill72] ———, *Complexe cotangent et déformations II,* Lecture Notes in Mathematics, Vol. 283, Springer-Verlag, Berlin, (1972)

[Ill79] ———, *Complexe de de Rham-Witt et cohomologie cristalline,* Ann. Sci. Ecole Norm. Sup. (4) **12** (1979), no. 4, 501–661.

[LL20] S. Li and T. Liu, *Comparison of prismatic cohomology and derived de Rham Cohomology*, available at arXiv:2012.14064

[Mat16] A. Mathew, *The Galois group of a stable homotopy theory,* Adv. Math., vol. 291 (2016), 403-541.

[MM65] J. Milnor and J. Moore, *On the structure of Hopf algebras,* Ann. of Math. (2) **81** (1965), 211–264, DOI 10.2307/1970615. MR174052

[MRT20] T. Moulinos, M. Robalo and B. Toën, *A universal HKR theorem,* available at arXiv:1906.00118

[Per76] D. Perrin, *Approximation des schémas en groupes, quasi compacts sur un corps,* Bull. Soc. Math. France **104** (1976), no. 3, 323–335.

[Rak20] A. Raksit, *Hochschild homology and the derived de Rham complex revisited,* available at arXiv:2007.02576

[SP] The Stacks Project Authors, available at https://stacks.math.columbia.edu/

[Toë20] B. Toën, *Classes caractéristiques des schémas feuilletés,* available at arXiv:2008.10489