ON ENDOMORPHISMS OF THE DE RHAM COHOMOLOGY FUNCTOR

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Abstract. We compute the moduli of endomorphisms of the de Rham and crystalline cohomology functors, viewed as a cohomology theory on smooth schemes over truncated Witt vectors. As applications of our result, we deduce Drinfeld's refinement of the classical Deligne–Illusie decomposition result for de Rham cohomology of varieties in characteristic $p > 0$ that are liftable to W_2, and prove further functorial improvements.

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1. Introduction

Let A be a ring and let X be a smooth A-scheme. The algebraic de Rham cohomology is a cohomology theory designed by Grothendieck. It is defined functorially by sending X to the hypercohomology of the de Rham complex $\Omega^*_{X/A}$. The de Rham complex $\Omega^*_{X/A}$ is not just a complex, but also has an additional structure of a sheaf of commutative differentially graded algebra. One can therefore view the output of de Rham cohomology as a commutative algebra object in the derived ∞-category $D(A)$ which we denote by $\mathrm{CAlg}(D(A))$. This way, one obtains a functor $\mathrm{dR}_{(\cdot)/A} : \mathrm{Alg}^{\mathrm{sm}}_A \to \mathrm{CAlg}(D(A))$, which sends any smooth A-algebra $R \mapsto \mathrm{dR}_{R/A} \in \mathrm{CAlg}(D(A))$. The primary goal of our paper is to study endomorphisms of this functor.

Studying properties of the de Rham cohomology theory as a functor is interesting for a number of reasons. From a technical point of view, in certain situations, showing that the de Rham cohomology functor has no nontrivial automorphisms has been used as a key tool in [BLM20] and [LL21] to prove that certain constructions are functorially isomorphic. Further, in [Mon21], it was shown that one can reconstruct the theory of crystalline cohomology as the *unique* deformation of de Rham cohomology theory viewed as a functor defined on smooth $\mathbb{F}_p$-schemes.

From a different perspective, any property enjoyed by the de Rham cohomology functor will in particular be enjoyed by de Rham cohomology of every smooth algebraic variety. For example, if the functor $\mathrm{dR}_{(\cdot)/A}$ has many endomorphisms, one potentially obtains many interesting endomorphisms of the de Rham cohomology of any smooth algebraic variety, which could be useful for making interesting geometric conclusions. The classical study and usage of Frobenius operator on de Rham or crystalline cohomology theory is an instance of such a perspective.

In this paper, our main motivating questions are the following, which can be seen as a "moduli" enhancement of the question of endomorphisms of the de Rham cohomology functor:

(1) Given a ring A, consider the functor dR that sends any smooth A-algebra $R \mapsto \mathrm{dR}_{R/A} \in \mathrm{CAlg}(D(A))$, what is the endomorphism monoid[1] of this functor?
(2) More generally, let B be an arbitrary A-algebra, consider the analogous functor $R \mapsto \mathrm{dR}_{R/A} \otimes_A B \in \mathrm{CAlg}(D(B))$, what is the endomorphism monoid of this functor?
(3) Finally consider the presheaf[2] (of monoids) on $(A\text{-alg})^{op}$ sending an A-algebra B to the endomorphism monoid in previous question. Is it represented by a (monoid) scheme? If so, what is the representing monoid scheme?

In this paper, we address the above questions when $A = W_n(k)$ for any perfect ring k, where $W_n(k)$ denotes the ring of n-truncated Witt vectors. We expect the methods to be extendable to more general base rings but we do not pursue that direction further in this paper.

A foretaste of the Main Theorem. For simplicity let us focus now on the case where $A = \mathbb{Z}/p^n$ or $\mathbb{Z}_p$ and B is an $\mathbb{F}_p$-algebra.

Theorem 1.1 (Special case of Main Theorem 4.24)**.**

(1) When $A = \mathbb{F}_p$, the endomorphism monoid of $\mathrm{dR}_{(\cdot)/A} \otimes_A B$ is $\underline{\mathbb{N}}(\mathrm{Spec}(B))$ where $\underline{\mathbb{N}}$ denotes the constant monoid scheme associated with the natural numbers.
(2) However when $A = \mathbb{Z}/p^n$ for $n \geq 2$, the endomorphism monoid of $\mathrm{dR}_{(\cdot)/A} \otimes_A B$ is a semi-direct product of $\underline{\mathbb{N}}(\mathrm{Spec}(B))$ with a group $W(B)^\times[F]$, where the latter denotes the Frobenius kernel of the unit group in $W(B)$.

Remark 1.2.

(1) When $A = \mathbb{Z}/p^n$ for $n \geq 2$, the endomorphism monoid of dR is very large. This large group action will be utilized to prove a strengthened version of the Deligne–Illusie decomposition, see Theorem 1.6.

[1] *A priori* we get a monoid object in spaces rather than an actual monoid. But in the cases of interest to us, this space is discrete, see Lemma 3.3 and Lemma 4.2.

[2] Mathew pointed out to us that this presheaf is automatically an fpqc sheaf by flat descent.

(2) From the above calculation one finds that for $A = \mathbb{Z}/p^n$, the association $B \mapsto \mathrm{End}(\mathrm{dR}_{(\cdot)/A} \otimes_A B)$ defines a sheaf of monoids representable by a scheme denoted by $\mathrm{End}_{1,n}$. The representing monoid scheme depends on $A = \mathbb{Z}/p^n$ and stabilizes when $n \geq 2$.

The stabilization we refer to means the following. Observe that we have a natural commutative diagram

$$\begin{tikzcd} \mathrm{Alg}^{\mathrm{sm}}_{\mathbb{Z}/p^n} \arrow[rr, "\bmod p^{n-1}"] \arrow[dr, "\mathrm{dR}\otimes B"'] & & \mathrm{Alg}^{\mathrm{sm}}_{\mathbb{Z}/p^{n-1}} \arrow[dl, "\mathrm{dR}\otimes B"] \\ & \mathrm{CAlg}(D(B)), & \end{tikzcd}$$

which induces a sequence of maps of schemes

$$\mathrm{End}_{1,1} \to \mathrm{End}_{1,2} \to \ldots \to \mathrm{End}_{1,n} \to \ldots.$$

Our theorem says the first map is an injection, and all subsequent maps are isomorphisms.

Remark 1.3. We see that the representing monoid scheme stabilizes as soon as A leaves characteristic p, this indicates the functorial Frobenius endomorphism is solely responsible for the rigidity of de Rham cohomology theory in characteristic p.

Regarding endomorphisms of de Rham cohomology itself, we also get the following:

Theorem 1.4 (Special case of Proposition 3.5)**.** *When $A = \mathbb{Z}_p$, the endomorphism monoid of $\mathrm{dR}^{\wedge}_{(\cdot)/A}$ is $\mathbb{N}$, given by powers of Frobenius.*

Here the $\mathrm{dR}^{\wedge}$ denotes the p-adic derived de Rham cohomology theory, *c.f.* [Bha12]. The fact that there is no automorphism of p-adic derived de Rham cohomology theory when the base ring is p-complete and p-torsionfree was observed in [LL21, Theorem 3.13].

Remark 1.5. In both cases of $A = \mathbb{F}_p$ and $\mathbb{Z}_p$ above, we only see powers of Frobenius as endomorphisms of the (p-adic) de Rham cohomology. But they are due to two different reasons: when $A = \mathbb{F}_p$ it is due to existence of the Frobenius endomorphism on the category of A-algebras, whereas for $A = \mathbb{Z}_p$ it comes from the fact that A is p-torsionfree, so a certain huge group scheme has no nontrivial A-valued point in this case.

In Theorem 4.24 we work in a more general setting, namely we calculate the moduli of endomorphisms of crystalline cohomology theory, leading to sheaves $\mathrm{End}_{m,n}$ (see Corollary 4.27 for the precise statement). The result is similar: the Frobenius endomorphisms that people "knew and loved" corresponds to the monoid underlying the connected components of the whole endomorphism monoid; in fact there is a distinguished point in each component which corresponds to a power of the Frobenius endomorphism. Furthermore, the identity component also stabilizes to a large and mysterious group scheme (see Definition 4.14) which demands further investigations (see Remark 4.30). One surprising feature is that the above group scheme is non-flat over the base in the general setting of crystalline cohomology.

Application to the Deligne–Illusie decomposition. Theorem 1.1 in particular provides an action of $W^{\times}[F]$ on the mod p de Rham cohomology of a variety liftable to W_2. Recently, Drinfeld has also exhibited an action of $W^{\times}[F]$ on the mod p de Rham cohomology, using the theory of "prismatization" due to him and independently due to Bhatt–Lurie. Our Theorem 1.1 implies that these are *all* the automorphisms.

As an application of the $W^{\times}[F]$-action defined by him, Drinfeld observed a refinement of the Deligne–Illusie decomposition, which was communicated to us by Bhatt and will appear in [BL21]: since $\mu_p \subset W^{\times}[F]$, the mod p de Rham cohomology of varieties liftable to W_2 has the structure of a μ_p-representation. It is easy to see that the $W^{\times}[F]$-action preserves conjugate filtration. Then one needs to show that the i-th graded piece of conjugate filtration is pure of weight $i \in \mathbb{Z}/p$ as a μ_p-representation. In [BL21], this statement is proved by establishing a relation between $W^{\times}[F]$-action and the "Sen operator" to be defined in loc. cit. In Theorem 5.4, we use a more direct argument to check that the weight statement holds for the $W^{\times}[F]$-action coming from our Theorem 1.1.

Theorem 1.1 coupled with the calculation of weights from Theorem 5.4 as above immediately implies the following improvement of results due to Achinger and Suh [AS21, Theorem 1.1], which in turn is a strengthening of Deligne–Illusie's result [DI87, Corollaire 2.4]. In particular, our approach gives a different proof of the following result which does not make any use of prismatization.

Theorem 1.6 (Drinfeld, [BL21], see Corollary 5.5)**.** *Let k be a perfect ring of characteristic $p > 0$, let X be a smooth scheme over $W_2(k)$, and let $a \le b \le a + p - 1$. Then the canonical truncation $\tau_{[a,b]}(\Omega^\bullet_{X_k/k})$ splits. Moreover the splitting is functorial in the lift X of X_k.*

Since our calculation shows the endomorphism monoid of mod p de Rham cohomology stabilizes after W_2, philosophically it says that further liftability over W_n for $n > 2$ provides no extra knowledge on the mod p de Rham cohomology.

Using Theorem 1.1, we prove that the result in Theorem 1.6 is *optimal* in a precise sense: we show that de Rham complex of smooth varieties over k liftable to $W_2(k)$ does not decompose further in a *functorial* manner. In particular, it does not *functorially* split into direct sums of the graded pieces of the conjugate filtration. This proves that mod p de Rham complex of smooth varieties over k liftable to W_2 is *not* formal in a *functorial* way when viewed with the further enhancement of a commutative algebra structure in the derived category.

Corollary 1.7 (see Proposition 4.29)**.** *There is no functorial splitting*

$$\mathrm{dR}_{(-\otimes_{W_2(k)} k)/k} \simeq \bigoplus_{i \in \mathbb{N}_{\ge 0}} \mathrm{Gr}_i^{\mathrm{conj}}\big(\mathrm{dR}_{(R \otimes_{W_2(k)} k)/k}\big)$$

as a functor from smooth $W_2(k)$-algebras to $\mathrm{CAlg}(D(k))$*.*

The above statement was also observed by Mathew. His approach does not use the full calculation of endomorphism monoid as in Theorem 1.1, whereas for us it is a consequence of that calculation.

Lastly one may wonder if the Drinfeld splitting agrees with the Deligne–Illusie splitting (which has an ∞-categorical functorial enhancement [KP21a, Theorem 1.3.21 and Proposition 1.3.22]). Both splittings are obtained from the splitting of the first conjugate filtration, via an averaging process, *c.f.* the step (a) in proof of [DI87, Théorème 2.1]. To guarantee the above two splittings are functorially the same, we show the following uniqueness.

Theorem 1.8 (see Theorem 5.9 for the precise statement)**.** *There is a unique functorial splitting (as $\mathrm{Fil}_0^{\mathrm{conj}}$-modules)*

$$\mathrm{Fil}_1^{\mathrm{conj}}(\mathrm{dR}_{(-\otimes_{W_2(k)} k)/k}) = \mathrm{Fil}_0^{\mathrm{conj}}(\mathrm{dR}_{(-\otimes_{W_2(k)} k)/k}) \oplus \mathrm{Gr}_1^{\mathrm{conj}}(\mathrm{dR}_{(-\otimes_{W_2(k)} k)/k}).$$

In particular the Deligne–Illusie splitting in [KP21a], the Drinfeld splitting in [BL21] as well as the splitting induced by Theorem 1.1 must all agree.

Outline of the proof of Theorem 1.1. Let us briefly outline the key ingredients in the proof of Theorem 1.1. In doing so, we will also give a rough outline of the paper. For simplicity, let us fix $A = \mathbb{Z}/p^n$, and let B be an $\mathbb{F}_p$-algebra.

(0) The statements Theorem 1.1.(1) as well as Theorem 1.4 is within reach of the quasi-syntomic descent techniques introduced in [BMS19], see Section 3. We also use quasi-syntomic descent techniques to show that the endomorphism spaces of interest to us are actually discrete (see Lemma 3.3, Lemma 4.2).

(1) But for Theorem 1.1.(2), we need to make use of the stacky approach to de Rham or crystalline cohomology due to Drinfeld [Dri18], [Dri21b], which can be seen as a positive characteristic analogue of Simpson's de Rham stack [Sim96]. For our paper, we use a compressed version of the stacky approach: the functor $\mathrm{dR}_{(\cdot)/A} \otimes_A B$ is built as *unwinding* (see Section 2.4) of an A-algebra stack over B; this stack is denoted by $\mathbb{A}^{1,\mathrm{dR}}_B$ (we often omit B to ease the notation). This unwinding construction is a variant of a construction used in [Mon21, §3]. Note the amusing switch of roles played by A and B: the de Rham cohomology theory is a cohomology theory for varieties over A with coefficient ring being B, whereas the stack $\mathbb{A}^{1,\mathrm{dR}}_B$ is an A-algebra object over B.

(2) It turns out that the underlying stack $\mathbb{A}^{1,\mathrm{dR}}$ is an affine stack, in the sense of Toën [Toë06]. Roughly speaking, for affine stacks one can pass to the "ring" of derived global sections in a lossless manner. Using this property, in Proposition 4.4, we show that $\mathrm{End}(\mathrm{dR}_{(\cdot)/A} \otimes_A B) \simeq \mathrm{End}_{A\text{-Alg-St}}(\mathbb{A}_B^{1,\mathrm{dR}})$. Here the latter endomorphism is taken in the category of A-algebra stacks over B.

(3) Using the description of $\mathbb{A}^{1,\mathrm{dR}}$ as the quotient stack $[W/pW]$, where W denotes the ring scheme of p-typical Witt vectors, in Section 4.2 we construct "enough" endomorphisms of $\mathrm{dR}_{(\cdot)/A} \otimes_A B$ and show that the endomorphism monoid is at least as big as Theorem 1.1 claims.

(4) To finish the proof of Theorem 1.1, one needs to show that there is no more endomorphism than the ones already constructed. To do so, we interpret an endomorphism of the algebra stack $\mathbb{A}^{1,\mathrm{dR}}$ as a deformation of an endomorphism of the sheaf of rings $\pi_0(\mathbb{A}^{1,\mathrm{dR}})$. We know that (see Proposition 4.20) $\pi_0(\mathbb{A}_B^{1,\mathrm{dR}}) = \mathbb{G}_{a,B}$ because B is an $\mathbb{F}_p$-algebra. Then we use the formalism of topos theoretic cotangent complex due to Illusie [Ill71] (see Appendix A) to understand this deformation problem. This is carried out in Theorem 4.24, where we use the cotangent complex and the transitivity triangle to finish calculating the desired endomorphism monoid.

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2. Stacky approach to de Rham cohomology

The goal of this section is to describe the stacky approach to de Rham cohomology theory due to Drinfeld [Dri18]. Roughly, given a scheme X, Drinfeld constructed a stack X^{dR} such that $R\Gamma(X^{\mathrm{dR}}, \mathcal{O})$ recovers the de Rham cohomology $R\Gamma_{\mathrm{dR}}(X)$. This should be seen as a positive characteristic variant of the earlier construction of the de Rham stack due to Simpson [Sim96].

For our purposes, we will need to work with a certain compressed version of this construction. Our goal is to consider a *single* stack with enough structure encoded, which can naturally "unwind" itself to construct the stack X^{dR} for *every* scheme X. To this end, we will begin by discussing quasi-ideals (*c.f.* [Dri21b, §3.1], [Mon21, §3.2]) and ring stacks, which formulates exactly the kind of extra structures on a stack one needs to work with in order to use the unwinding machine. After that, we will discuss the construction of this unwinding functor, and explain how to build a cohomology theory from a ring stack in general. We will then discuss the particular ring stack $\mathbb{A}^{1,\mathrm{dR}}$ which gives rise to de Rham cohomology theory via this construction. For later application, the fact that the stack $\mathbb{A}^{1,\mathrm{dR}}$ is an *affine stack* in the sense of [Toë06] will be of particular importance to us. Therefore, we will record the relevant definitions in this section as well.

2.1. Quasi-ideals.

Definition 2.1 (Quasi-ideals)**.** Let R be a ring and M be an R-module equipped with a map $d : M \to R$ which satisfies $d(x) \cdot y = d(y) \cdot x$ for any pair $x, y \in M$. Such a data $d : M \to R$ will be called a *quasi-ideal* in R or simply a quasi-ideal.

A morphism of quasi-ideals $(d_1 : M_1 \to R_1) \to (d_2 : M_2 \to R_2)$ is defined to be a pair of maps $a : M_1 \to M_2$ and $b : R_1 \to R_2$ such that the following compatibilities hold:

(1) $d_2 a = b d_1$.
(2) $a(r_1 m_1) = b(r_1) a(m_1)$.
(3) b is a ring homomorphism.
(4) a is linear.

In other words, we want a commutative diagram as below:

$$\begin{array}{ccc} M_1 & \xrightarrow{a} & M_2 \\ \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} \\ R_1 & \xrightarrow{b} & R_2 \end{array}$$

such that b is a ring homomorphism and a is an R_1-module map $M_1 \to b_* M_2$. The category of quasi-ideals will be denoted as QID.

Construction 2.2 (Quasi-ideal as a simplicial abelian group)**.** Given a quasi-ideal $(d\colon M \to R)$, we obtain a map $t\colon T := M \times R \to R$ given by $(m, r) \to r + d(m)$. There is another map $s\colon M \times R \to R$ given by $(m, r) \to r$. There is also a degeneracy map $e\colon R \to M \times R$ given by $r \to (0, r)$. Lastly, there is a map $c\colon T \times_{R,s,t} T \to T$ which sends

$$(r, m) \times (r', m') \to (r, m + m'),$$

where $t(r, m) = s(r', m')$ so that $(r, m) \times (r', m') \in T \times_{R,s,t} T$. Therefore, we obtain a groupoid denoted as

$$M \times R \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} R.$$

Note that all the morphisms s, t, c, e are morphisms of abelian group, one can actually convert the above data into a 1-truncated simplicial abelian group.

In the construction below, we explain how to attach a 1-truncated simplicial ring or a ring groupoid from the data of a quasi-ideal.

Construction 2.3 (Quasi-ideal as a simplicial commutative ring)**.** Let $d : M \to R$ be a quasi-ideal. We have already defined a groupoid

$$M \times R \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} R$$

which can also be thought of as a 1-truncated simplicial abelian group. Next, we give a ring structure on $M \times R$. We define $(m_1, r_1) \cdot (m_2, r_2) := (r_2 m_1 + r_1 m_2 + d(m_1) m_2, r_1 r_2)$. Now as one easily checks, the morphisms s, t, c, e in the definition of the groupoid

$$M \times R \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} R$$

are all *ring* homomorphisms with respect to the ring structure on $M \times R$ defined above. The above data can be converted into a 1-truncated simplicial commutative ring.

Definition 2.4 (Quasi-ideals in schemes)**.** Let R be a ring scheme and M be a module scheme over R equipped with a map $d : M \to R$. This data will be called a *quasi-ideal in* R if $d(x) \cdot y = d(y) \cdot x$ for scheme theoretic points $x, y \in M$.

A morphism between quasi-ideals in schemes is defined in a way similar to Definition 2.1.

Finally, let us give some examples of quasi-ideals that will be used later on. For more details on these examples, we refer the reader to [Dri21b, §3.2-3.5] or [Mon21, §2.2].

Example 2.5. Let $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ denote the quasi-ideal obtained by taking the divided power envelope of origin inside $\mathbb{G}_a$.

Example 2.6. Let B be any ring on which p is nilpotent. Then the functor $S \to S^\flat := \varprojlim_F S/p$ is representable by the affine ring scheme $\operatorname{Spec} B[x^{1/p^\infty}]$ which will be denoted as $\mathbb{G}_a^{\mathrm{perf}}$.

Example 2.7. Let $\mathbb{G}_a^{\mathrm{perf},\sharp} \to \mathbb{G}_a$ denote the quasi-ideal obtained by taking the divided power envelope of the closed subscheme defined by the ideal (p, x) inside $\mathbb{G}_a^{\mathrm{perf}}$ compatibly with the existing divided powers of p.

Example 2.8. Let W denote the ring scheme of p-typical Witt vectors. By taking kernel of the Frobenius F, one obtains a quasi-ideal $W[F] \to \mathbb{G}_a$, which is isomorphic to $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ as a quasi-ideal in $\mathbb{G}_a$.

Example 2.9. By considering the multiplication by p map on W, one obtains a quasi-ideal $W \xrightarrow{\times p} W$.

2.2. **Ring stacks.** We begin by collecting some notations. If C and D denote two ∞-categories which have finite products, then the category of finite product preserving functors will be given by $\mathrm{Fun}_{\times}(C, D)$. Let Poly_A denote the category of finitely generated polynomial algebras over A.

Definition 2.10 (Animated ring objects in a category)**.** Let C be an ∞-category with products. Animated A-algebra objects in C, denoted as $\mathrm{ARings}(C)_A$, is defined to be the category $\mathrm{Fun}_{\times}(\mathrm{Poly}_A^{\mathrm{op}}, C)$.

In the case where C is the ∞-category of spaces, then the above definition with $A = \mathbb{Z}$ recovers the usual category of animated rings.

Remark 2.11. The ∞-category of animated rings have all colimits. Given a simplicial commutative ring, one can take colimit over the simplex category and obtain an animated ring. In particular, given a quasi-ideal, one can apply Construction 2.3, and obtain an animated ring.

Definition 2.12 (Prestacks)**.** The ∞-category of prestacks over a fixed (discrete) base ring B, denoted by PreSt_B, is defined to be the category of functors $\mathrm{Fun}(\mathrm{Alg}_B, \mathcal{S})$, where Alg_B is the category of discrete B-algebras and $\mathcal{S}$ is the ∞-category of spaces.

We note that even though we do not impose any sheafiness conditions, the examples of stacks we consider will all be (hypercomplete) fpqc sheaves of spaces.

Definition 2.13 (A-algebra prestacks over $\mathrm{Spec}(B)$)**.** The category of A-algebra prestacks over $\mathrm{Spec}(B)$, denoted as A-Alg-PreSt_B, is defined to be the category of animated A-algebra objects in the category PreSt_B.

Remark 2.14. We note that another possible way to define the category A-Alg-PreSt_B is to define it as $\mathrm{Fun}(\mathrm{Alg}_B, \mathrm{ARings}_A)$. However, this is equivalent to the definition considered above since we have natural equivalence of categories

$$\mathrm{Fun}(\mathrm{Alg}_B, \mathrm{ARings}_A) \simeq \mathrm{Fun}(\mathrm{Alg}_B, \mathrm{Fun}_{\times}(\mathrm{Poly}_A^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{Fun}_{\times}(\mathrm{Poly}_A^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Alg}_B, \mathcal{S})) \simeq \mathrm{Fun}_{\times}(\mathrm{Poly}_A^{\mathrm{op}}, \mathrm{PreSt}_B).$$

The middle equivalence uses the fact that product in functor category is calculated term-wise; the precise ∞-categorical (dual) assertion can be found in [Lur09, Corollary 5.1.2.3].

Construction 2.15. (Cone of a quasi-ideal) In view of Remark 2.14 and Construction 2.3, it follows that given a quasi-ideal $d : M \to R$ in schemes, the quotient prestack $[R/M]$ (under the additive action of M on the ring scheme R by translation via d) has the structure of a ring prestack. In the context of this paper, we will consider associated ring stacks of such ring prestacks, obtained by fpqc sheafification.

Example 2.16. We will see later that all the examples of quasi-ideals from Section 2.1 have the same cone.

2.3. **Affine stacks.** We will also use the notion of *affine stacks* due to Toën [Toë06], and we will recall the definition very briefly, in the language of ∞-categories. Fix an ordinary base ring B. Let coSCR_B denote the ∞-category of cosimplicial rings over B arising from the simplicial model structure defined in [Toë06, Theorem 2.1.2]; to construct the associated ∞-category from the simplicial model category, one looks at the fibrant simplicial category obtained from the subcategory of fibrant-cofibrant objects inside the given simplicial model category, and applies the simplicial nerve construction, which produces an ∞-category by [Lur09, Proposition 1.1.5.10]. It follows from [Lur09, Corollary 4.2.4.8] that the ∞-category coSCR_B has all small limits and colimits.

Definition 2.17 (Affine stacks)**.** An object $\mathcal{Y}$ of PreSt_B is called an *affine stack* over B if there exists an object $C \in \mathrm{coSCR}_B$ such that $\mathcal{Y}$ is the restriction of the functor $h_C : \mathrm{coSCR}_B \to \mathcal{S}$ corepresented by C along the inclusion $\mathrm{Alg}_B \to \mathrm{coSCR}_B$. The full subcategory of such objects inside PreSt_B will be denoted by $\mathrm{AffStacks}_B$.

Remark 2.18. It follows from the definition that the category of affine stacks is stable under small limits, *c.f.* [Toë06, Proposition 2.2.7]. Also, an affine stack is a hypercomplete fpqc sheaf of spaces, *c.f.* [Toë06, Lemma 1.1.2, Proposition 2.2.2]. The key property of affine stacks that will be useful for us is the fact that taking derived global section functor induces an equivalence of ∞-categories $\mathrm{AffStacks}_B \simeq \mathrm{coSCR}_B^{\mathrm{op}}$, *c.f.* [Toë06, Corollary 2.2.3].

Remark 2.19. We point out that even though the definition of the subcategory of affine stacks $\mathrm{AffStacks}_B$ inside PreSt_B *a priori* depends on the category coSCR_B, the notion of being an affine stack is *intrinsic*: being an affine stack is a property that can be formulated *only* by using the fpqc topology and the category of ordinary rings. We refer to [Toë06, Theorem 2.2.9] for a more precise formulation of this statement using Bousfield localization. *A posteriori*, the same intrinsic property carries over to the ∞-category coSCR_B which makes it rather special in comparison to certain other related categories such as the ∞-category of derived rings or E_∞-rings.

Example 2.20. An affine scheme is clearly an affine stack. The stacks $K(\mathbb{G}_a, m)$ for $m \geq 0$ are examples of affine stacks [Toë06, Lemma 2.2.5]. On the other hand, $K(\mathbb{G}_m, m)$ is *not* an affine stack for any $m > 0$. By [Toë06, Corollary 2.4.10], for pointed and connected stacks over a field, being an affine stack is equivalent to the sheaf of all the higher homotopy groups being representable by unipotent affine group schemes (possibly of infinite type).

2.4. **Unwinding ring stacks.** In this section, we describe how to *unwind* the data of a ring stack to obtain a cohomology theory. This construction is an ∞-categorical enhancement of [Mon21, Example 3.0.1] and we will call this the unwinding of a given ring stack. The construction only uses basic categorical principles such as Kan extensions and the necessary foundations can be found in [Lur09].

Construction 2.21 (Unwinding)**.** We will construct a functor

$$\mathrm{Un} : A\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B \to \mathrm{Fun}(\mathrm{ARings}_A, \mathrm{CAlg}(D(B))).$$

Here, $\mathrm{CAlg}(D(B))$ denotes the commutative algebra objects in the derived ∞-category $D(B)$. We think of the objects in the right hand side as "algebraic cohomology theories".

We begin by noting that by definition $A\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B \simeq \mathrm{Fun}_\times(\mathrm{Poly}_A^{\mathrm{op}}, \mathrm{PreSt}_B)$. By Kan extension, there is a derived global section functor $R\Gamma : \mathrm{PreSt}_B \to \mathrm{CAlg}(D(B))^{\mathrm{op}}$. By composition, we get a functor

$$A\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B \to \mathrm{Fun}(\mathrm{Poly}_A, \mathrm{CAlg}(D(B))).$$

Now, we can perform a left Kan extension along the inclusion $\mathrm{Poly}_A \to \mathrm{ARings}_A$ to obtain the desired unwinding functor

$$\mathrm{Un} : A\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B \to \mathrm{Fun}(\mathrm{ARings}_A, \mathrm{CAlg}(D(B))).$$

Example 2.22. When $A = B$, and $\mathcal{Y} \in \mathrm{PreSt}_B$ is taken to be the ring scheme $\mathbb{G}_{a,B}$, the functor $\mathrm{Un}(\mathbb{G}_{a,B})$ is simply the forgetful functor $\mathrm{ARings}_A \to \mathrm{CAlg}(D(B))$.

Below we will study compatibility of the unwinding construction with restriction of scalars. More precisely, let $\mathcal{Y} \in A\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B$. Let $A' \to A$ be a map of discrete rings. Then there is an obvious functor

$$\mathrm{res} : A\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B \to A'\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B.$$

Let $\mathcal{Y}' := \mathrm{res}(\mathcal{Y}) \in A'\text{-}\mathrm{Alg}\text{-}\mathrm{PreSt}_B$. Applying the unwinding construction, we obtain two functors $\mathrm{Un}(\mathcal{Y}) : \mathrm{ARings}_A \to \mathrm{CAlg}(D(B))$ and $\mathrm{Un}(\mathcal{Y}') : \mathrm{ARings}_{A'} \to \mathrm{CAlg}(D(B))$. Note that we also have a natural functor (given by the derived tensor product) $L : \mathrm{ARings}_{A'} \to \mathrm{ARings}_A$. In this set up, we have the following compatibility.

Proposition 2.23. *We have* $\mathrm{Un}(\mathcal{Y}) \circ L \simeq \mathrm{Un}(\mathcal{Y}')$ *in* $\mathrm{Fun}(\mathrm{ARings}_A, \mathrm{CAlg}(D(B)))$.

Proof. Since L is obtained by left Kan extension of the composite functor $\mathrm{Poly}_{A'} \xrightarrow{\ell} \mathrm{Poly}_A \to \mathrm{ARings}_A$, it would be enough to prove $\mathrm{Un}(\mathcal{Y}) \circ \ell \simeq \mathrm{Un}(\mathcal{Y}')$ in $\mathrm{Fun}(\mathrm{Poly}_A, \mathrm{CAlg}(D(B)))$. By Construction 2.21, we note that $\mathcal{Y}$ is classified by a functor $U : \mathrm{Poly}_A^{\mathrm{op}} \to \mathrm{PreSt}_B$ and $\mathcal{Y}'$ is classified by $U' : \mathrm{Poly}_{A'}^{\mathrm{op}} \to \mathrm{PreSt}_B$, and for our purpose, it would be enough to prove that $U \circ \ell^{\mathrm{op}} \simeq U'$. By Remark 2.14, it would be enough to prove that the restriction of scalar functor $\mathrm{ARings}_A \to \mathrm{ARings}_{A'}$ is induced by ℓ^{op} under the identifications $\mathrm{ARings}_A \simeq \mathrm{Fun}_\times(\mathrm{Poly}_A^{\mathrm{op}}, \mathcal{S})$ and $\mathrm{ARings}_{A'} \simeq \mathrm{Fun}_\times(\mathrm{Poly}_{A'}^{\mathrm{op}}, \mathcal{S})$. But that follows from adjunction. $\square$

2.5. **de Rham cohomology via unwinding.** In this section, we will describe how to use the unwinding construction to recover de Rham or crystalline cohomology functors. To this end, let $n, m \geq 1$ be two arbitrary positive integers and let p be a fixed prime. Further, we fix a perfect ring k of characteristic p. Let $W_r(k)$ denote the ring of r-truncated Witt vectors. Using crystalline cohomology, more precisely, its derived variant (see Definition 2.24 below), one obtains certain functors denoted as

$$\mathrm{dR}_{m,n} : \mathrm{ARings}_{W_n(k)} \to \mathrm{CAlg}(D(W_m(k))),$$

which we will loosely still call de Rham cohomology functors and specify the n, m. To define them one really needs to use a deformation of the de Rham cohomology functor, i.e., the crystalline cohomology functors.

The following essentially already appeared in [BLM20, §10.2] and [BMS19, §8.2].

Definition 2.24. Given a finitely generated polynomial $W_n(k)$-algebra P, define $\mathrm{dR}_{m,n}(P) := \mathrm{R\Gamma}_{\mathrm{crys}}(P_0/W_m(k))$ where P_0 denotes the mod p reduction of P. We will let $\mathrm{dR}_{m,n}$ to denote the left Kan extension of the above functor from finitely generated polynomials to all animated $W_n(k)$-algebras which takes values in $\mathrm{CAlg}(D(W_m(k)))$.

We use this notation as we believe the crystalline cohomology is secretly a disguise of derived de Rham cohomology, see [Bha12, Proposition 3.27] and [LL21, Proposition 2.11] for occasions of this perspective. Our goal is to describe $\mathrm{dR}_{m,n}$ as the unwinding of a certain object in $W_n(k)\text{-Alg-PreSt}_{W_m(k)}$.

Definition 2.25. Let W denote the ring scheme over $\mathrm{Spec}(W_m(k))$ underlying the p-typical Witt vectors. Then $d : W \xrightarrow{\times p} W$ defines a quasi-ideal in schemes. By considering its cone, and using the Artin–Hasse homomorphism $W(k) \to W(W(k))$, one obtains a k-algebra stack over $\mathrm{Spec}(W_m(k))$, which can be regarded as a $W_n(k)$-algebra stack over $\mathrm{Spec}(W_m(k))$ via the natural map $W_n(k) \twoheadrightarrow k$. We denote the resulting $W_n(k)$-algebra stack over $\mathrm{Spec}(W_m(k))$ by $\mathbb{A}^{1,\mathrm{dR}}_{m,n}$. When n is fixed, we will denote $\mathbb{A}^{1,\mathrm{dR}}_B$ to denote the pull-back of $\mathbb{A}^{1,\mathrm{dR}}_{m,n}$ to $\mathrm{Spec}\, B$ for a $W_m(k)$-algebra B.

Remark 2.26. The above definition gives a generalization of the definition of $\mathbb{A}^{1,\mathrm{dR}}$ as an $\mathbb{F}_p$-algebra stack due to Drinfeld to the more general case of an arbitrary perfect ring k. To do this, one needs to crucially use the Artin–Hasse natural transformation $W(\cdot) \to W(W(\cdot))$. One can abstractly construct this natural transformation by realizing the functor W as a right adjoint to the inclusion of the category of delta rings inside all rings.

Proposition 2.27. *The stack underlying $\mathbb{A}^{1,\mathrm{dR}}_{m,n}$ is an affine stack.*

Proof. Indeed, the stack underlying $\mathbb{A}^{1,\mathrm{dR}}_{m,n}$ is obtained by taking cone of $d : W \xrightarrow{\times p} W$, which is the same as fibre of the induced map $BW \to BW$. Since affine stacks are closed under limits, it would be enough to show that BW is an affine stack. This follows from the proof of [MRT21, Proposition 3.2.7]. Let us give a rough sketch of their argument. Let W_n denote the ring scheme underlying n-truncated p-typical Witt vectors. Using certain obstruction vanishing, one first argues that $BW \simeq \varprojlim BW_n$. Therefore, it is enough to prove that BW_n is an affine stack for all n. To do so, one argues by induction on n. Using the short exact sequence

$$0 \to \mathbb{G}_a \to W_{n+1} \to W_n \to 0,$$

one reduces to the fact that $K(\mathbb{G}_a, m)$ is an affine stack for all m; this finishes the proof. □

Remark 2.28. The above argument can be modified to more generally show that $K(W, m)$ is an affine stack for all $m \geq 0$. Consequently, one can show that the abelian group stack $\mathbb{A}^{1,\mathrm{dR}}[m]$ is also an affine stack for all $m \geq 0$. We have $R\Gamma_{\mathrm{dR}}(K(\mathbb{G}_a, m)) \simeq R\Gamma(\mathbb{A}^{1,\mathrm{dR}}[m], \mathcal{O})$ for all $m \geq 0$.

Proposition 2.29. *We have a natural isomorphism* $\mathrm{Un}(\mathbb{A}^{1,\mathrm{dR}}_{m,n}) \simeq \mathrm{dR}_{m,n}$.

Proof. By Proposition 2.23, the proof reduces to $n = 1$. Further, by [Mon21, Theorem 1.1.1], one can reduce to $m = 1$. Since $\mathrm{dR}_{1,1}$ preserves all colimits, it is enough to show that derived global sections of $\mathbb{A}^{1,\mathrm{dR}}_{1,1}$ agrees with $R\Gamma_{\mathrm{crys}}(\mathbb{A}^1_k/W_1)$. For this, one uses the identification $\mathrm{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a) \simeq \mathrm{Cone}(W \xrightarrow{\times p} W)$ and the Čech–Alexander complex. □

The following fact was used in the above proof, which uses compatibility of two models of the k-algebra stack $\mathbb{A}^{1,\mathrm{dR}}$ over $\mathrm{Spec}(W(k))$.

Proposition 2.30 ([Dri21b, 3.5.1])**.** *There is an isomorphism of k-algebra stacks over* $\mathrm{Spec}(W(k))$*:*

$$\mathrm{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a) \simeq \mathrm{Cone}(W \xrightarrow{\times p} W).$$

The k-algebra structure on the source comes from the natural maps $W(k) \to \mathbb{G}_a$ and $W(k) \xrightarrow{1 \mapsto p - V(1)} W[F]$. To see the two underlying abelian group stacks are the same, notice that we always have $FV = p$ on the p-typical Witt ring, hence we get a factorization

$$\begin{tikzcd} W \arrow[rr, "\times p"] \arrow[dr, "V"'] & & W. \\ & W \arrow[ur, "F"'] & \end{tikzcd}$$

One then applies the octahedral axiom to the above triangle. The fact that it induces an algebra isomorphism can be seen using the fact that F is an algebra homomorphism. Said differently, one pulls back the quasi-ideal $W \xrightarrow{\times p} W$ along $W \xrightarrow{F} W$ to build the intermediate model relating the above two models.

We will see that the quasi-ideal $\mathbb{G}_a^{\mathrm{perf},\sharp} \to \mathbb{G}_a^{\mathrm{perf}}$ that appears in [Mon21, Proposition 4.0.11] gives a third model of the k-algebra stack $\mathbb{A}^{1,\mathrm{dR}}$ over $\mathrm{Spec}(W(k))$, see also [Dri18]. First, we will make some preparations. Below we always fix a positive integer m.

Lemma 2.31. *On the fpqc site of $\mathbb{Z}/p^m$, we have $R\varprojlim_F W \simeq \varprojlim_F W$, which is representable by an affine scheme. Moreover, its functor of points can be described as $B \mapsto W(B^\flat)$.*

We denote the affine scheme representing $\varprojlim_F W$ by W^{perf}. This scheme can be given an $W(k)$-algebra scheme structure when viewed over $\mathrm{Spec}(W_m(k))$.

Proof. The first assertion follows from [BS15, Example 3.1.7 and Proposition 3.1.10] and the fact that F on W is faithfully flat. Inverse limit of affine scheme is again affine. For the last claim, we consider the following diagram of fpqc sheaves as a pro-object.

$$\begin{tikzcd} \dots \arrow[r] & W_3 \arrow[r, "F"] & W_2 \arrow[r, "F"] & W_1 \\ \dots \arrow[r] & W_4 \arrow[r, "F"] \arrow[u, "R"] & W_3 \arrow[r, "F"] \arrow[u, "R"] & W_2 \arrow[u, "R"] \\ \dots \arrow[r] & W_5 \arrow[r, "F"] \arrow[u, "R"] & W_4 \arrow[r, "F"] \arrow[u, "R"] & W_3 \arrow[u, "R"] \\ \vdots \arrow[r] & \vdots \arrow[r, "F"] \arrow[u, "R"] & \vdots \arrow[r, "F"] \arrow[u, "R"] & \vdots \arrow[u, "R"] \end{tikzcd}$$

Taking limit vertically and then horizontally gives us $\varprojlim_F W$. Next we take limit horizontally and then vertically instead. Taking limits horizontally, we obtain the sheaf that sends $B \mapsto \varprojlim_F W_r(B)$, which is canonically identified with $W(B^\flat)$ by [BMS18, Lemma 3.2] (with π in loc. cit. being p). The vertical map R is actually an isomorphism now, by the same [BMS18, Lemma 3.2]. This gives $\varprojlim_F W(B) \simeq W(B^\flat)$, as desired. $\square$

Recall that the F on W induces a map $\mathbb{A}^{1,\mathrm{dR}} \to \mathrm{Frob}_{k,*}\mathbb{A}^{1,\mathrm{dR}}$ of k-algebra stacks which we will again denote by F. We may untwist the Frobenius using inverse of Frobenius on k on the source of this map. Therefore we get a k-algebra structure on the stack $\varprojlim_F(\mathbb{A}^{1,\mathrm{dR}})$.

Lemma 2.32. *We have an isomorphism of k-algebra stacks $\mathbb{G}_a^{\mathrm{perf}} \simeq \varprojlim_F(\mathbb{A}^{1,\mathrm{dR}})$ over* $\mathrm{Spec}(W_m(k))$.

Proof. By Lemma 2.31, we see that

$$R\varprojlim_F(\mathbb{A}^{1,\mathrm{dR}}) = \mathrm{Cone}(R\varprojlim_F W \xrightarrow{\times p} R\varprojlim_F W) = \mathrm{Cone}(W^{\mathrm{perf}} \xrightarrow{\times p} W^{\mathrm{perf}})$$

whose functor of points is given by $B \mapsto B^\flat$. Hence $\varprojlim_F(\mathbb{A}^{1,\mathrm{dR}})$ is isomorphic to $\mathbb{G}_a^{\mathrm{perf}}$ as a k-algebra stack (and in fact is a scheme). □

Therefore we get a map of k-algebra stacks $\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{A}^{1,\mathrm{dR}}$ over $\mathrm{Spec}(W_m(k))$.

Lemma 2.33. *The map of (k-algebra) stacks $f\colon \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{A}^{1,\mathrm{dR}}$ is faithfully flat.*

Proof. We look at the following diagram of k-algebra stacks

$$\begin{tikzcd} W^{\mathrm{perf}} \arrow[rr] \arrow[dr] & & \mathbb{G}_a^{\mathrm{perf}} \arrow[dl] \\ & \mathbb{A}^{1,\mathrm{dR}} & \end{tikzcd}$$

and observe the horizontal and the left arrow are faithfully flat, hence the right arrow is faithfully flat as well. □

Let K be the quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ given by the kernel of f, then Lemma 2.33 implies that f gives rise to an isomorphism of k-algebra stacks $\mathrm{Cone}(K \to \mathbb{G}_a^{\mathrm{perf}}) \simeq \mathbb{A}^{1,\mathrm{dR}}$. This is would be what we called the third model of $\mathbb{A}^{1,\mathrm{dR}}$, to complete the description, it remains to understand the quasi-ideal K.

Proposition 2.34. *K is isomorphic to $\mathbb{G}_a^{\mathrm{perf},\sharp}$ as quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$. In particular,* $\mathrm{Cone}(\mathbb{G}_a^{\mathrm{perf},\sharp} \to \mathbb{G}_a^{\mathrm{perf}}) \simeq \mathbb{A}^{1,\mathrm{dR}}$ *as k–algebra stacks.*

Proof. This assertion follows from applying the (derived) crystalline cohomology functor $R\Gamma_{crys}$ to the following pushout diagram

$$\begin{tikzcd} k \arrow[r] & k[x^{1/p^\infty}]/x \\ k[x] \arrow[u, "x\mapsto 0"] \arrow[r] & k[x^{1/p^\infty}] \arrow[u] \end{tikzcd}$$

and noting that global sections of $\mathbb{G}_a^{\mathrm{perf},\sharp}$ recovers $R\Gamma_{\mathrm{crys}}(k[x^{1/p^\infty}]/x)$ and $R\Gamma_{\mathrm{crys}}$ preserves the pushout diagram. □

Remark 2.35. Using the above methods, let us sketch a quick proof of a result due to Bhatt–Lurie–Mathew [BLM20, Proposition 10.3.1], *c.f.* [Mon21, Proposition 4.0.7]. Under Proposition 2.29 and Example 2.22, the assertion amounts to studying endomorphisms of $\mathbb{A}^{1,\mathrm{dR}}$ respecting the natural map $\mathbb{G}_a \to \mathbb{A}^{1,\mathrm{dR}}$. By Proposition 2.30, it is enough to show that the quasi-ideal $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ has no non-trivial endomorphism as a quasi-ideal in $\mathbb{G}_a$. But this follows directly from graded Cartier duality [Mon21, §2.4].

3. Endomorphisms of de Rham cohomology I

The quasi-syntomic descent technique introduced in [BMS19] is a powerful tool in calculating endomorphisms of de Rham cohomology functors in various settings. We will illustrate them in this section.

Let $A \to B$ be a map of derived p-complete rings with bounded p^∞-torsion. In this section we consider the functor that sends any derived p-complete A-algebra R to

$$(\mathrm{dR}\widehat{\otimes}_A B)(R) \coloneqq \mathrm{dR}_{R/A}\widehat{\otimes}_A B \in \mathrm{CAlg}(D(B)),$$

where $\mathrm{dR}_{R/A}$ denotes the p-adic derived de Rham complex of R relative to A and $\widehat{\otimes}$ denotes derived p-completed tensor product. If $B = A$, we simply denote the functor by dR.

We are interested in the space of endomorphisms of this functor, viewed (by left Kan extension) as an object in the ∞-category of functors from the ∞-category of derived p-complete animated rings to $\mathrm{CAlg}(D(B))$. Let

qSyn_A denote small quasi-syntomic site of A which consists of algebras that are quasi-syntomic over A and the covers are given by quasi-syntomic covers (see [BMS19, §4.2]).

Proposition 3.1 (*c.f.* [BMS19, Example 5.12])**.** *The functor* $\mathrm{dR}\widehat{\otimes}_A B$ *when restricted to* qSyn_A *defines a quasi-syntomic sheaf.*

Proof. It suffices to check this after going derived modulo p, so we are reduced to checking the following: given $R \to S$ a faithfully flat quasi-syntomic map of algebras in qSyn_A with its Čech nerve $S^\bullet$, then there is an isomorphism:

$$\mathrm{dR}_{R/A} \otimes_A B/p \simeq \lim \left(\mathrm{dR}_{S^\bullet/A} \otimes_A B/p\right),$$

where B/p is the animated ring $B \otimes_{\mathbb{Z}} \mathbb{F}_p$. By base change of derived de Rham cohomology, proving the above is equivalent to showing:

$$\mathrm{dR}_{(R\otimes_A B/p)/(B/p)} \simeq \lim \left(\mathrm{dR}_{(S^\bullet \otimes_A B/p)/(B/p)}\right),$$

see [KP21b, p.33-35] for a discussion of derived de Rham cohomology of maps of animated rings. To prove the above isomorphism, we employ the conjugate filtration [KP21b, Construction 2.3.12] (with base ring being $\mathbb{F}_p$). The conjugate filtration is exhaustive and uniformly bounded above -1, hence it suffices to prove its graded pieces satisfy similar quasi-syntomic descent. Using the description of graded pieces of conjugate filtration we are finally reduced to showing:

$$\wedge^i_A \mathbb{L}_{R/A} \otimes_A \varphi_*(B/p) \simeq \lim \left(\wedge^i_{S^\bullet} \mathbb{L}_{S^\bullet/A} \otimes_A \varphi_*(B/p)\right).$$

Here $\varphi_*(B/p)$ expresses the A-module structure on B/p which is given by $A \to B \to B/p \xrightarrow{\varphi} B/p$. The next Proposition 3.2 finishes the proof. $\square$

Proposition 3.2 (Flat descent for "tensored" cotangent complex)**.** *Fix a base ring A. For each $n \geq 0$ and an object $M \in D(A)$, the functor $R \mapsto \wedge^n_R \mathbb{L}_{R/A} \otimes_A M$ is an fpqc sheaf with values in the ∞-category $D(A)$.*

Proof. One simply runs through the proof of [BMS19, Theorem 3.1] and sees that it works in this generality. For convenience of the reader, let us illustrate the proof when $n = 1$. Let $R \to S$ be a faithfully flat map of A-algebras with Čech nerve $S^\bullet$. Using the transitivity triangle associated to $A \to R \to S^\bullet$ and applying the exact functor $(-) \otimes_A M$, we get a cosimplicial exact triangle

$$\mathbb{L}_{R/A} \otimes_R S^\bullet \otimes_A M \to \mathbb{L}_{S^\bullet/A} \otimes_A M \to \mathbb{L}_{S^\bullet/R} \otimes_A M.$$

We are therefore reduced to showing:

- The map $R \to S^\bullet$ induces an isomorphism $\mathbb{L}_{R/A} \otimes_A M \to \lim \mathbb{L}_{R/A} \otimes_A M \otimes_R S^\bullet$;
- $\lim \mathbb{L}_{S^\bullet/R} \otimes_A M = 0$.

The first item follows from fpqc descent along $R \to S$ by considering $\mathbb{L}_{R/A} \otimes_A M \in D(R)$. The second item is proved via a few reduction steps. By the convergence of the Postnikov filtration, it is enough to show that $\lim \pi_i(\mathbb{L}_{S^\bullet/R} \otimes_A M) \simeq 0$ in $D(R)$ for an arbitrary $i \in \mathbb{Z}$ which will be fixed from now. Again, by faithfully flat descent, it suffices to check $\left(\lim \pi_i(\mathbb{L}_{S^\bullet/R} \otimes_A M)\right) \otimes_R S \simeq \lim \left(\pi_i(\mathbb{L}_{S^\bullet/R} \otimes_A M) \otimes_R S\right) \simeq \lim \pi_i(\mathbb{L}_{S^\bullet/R} \otimes_R S \otimes_A M) \simeq 0$. Let $S \to T^\bullet$ denote the base change of $R \to S^\bullet$ along $R \to S$. By base change for cotangent complex, we need to show that $\lim \pi_i(\mathbb{L}_{T^\bullet/S} \otimes_A M) \simeq 0$. Since $S \to T^\bullet$ is the Čech nerve of the map $S \to S \otimes_R S$, which admits a section, it follows that $S \to T^\bullet$ is a homotopy equivalence of cosimplicial S-algebras. Now we observe that $F := \pi_i(\mathbb{L}_{(\cdot)/S} \otimes_A M)$ is a functor from the category of S-algebras to the category of abelian groups. Therefore, the cosimplicial abelian group $F(T^\bullet)$ is homotopy equivalent to $F(S)$. Since $F(S) \simeq 0$, we obtain $\lim \pi_i(\mathbb{L}_{T^\bullet/S} \otimes_A M) \simeq 0$, as desired. $\square$

As a consequence, let us record a result that says that the space of endomorphisms is actually discrete, i.e., the homotopy groups in degrees above zero are trivial for every choice of base points.

Lemma 3.3. *The space of endomorphisms* $\mathrm{End}(\mathrm{dR}\widehat{\otimes}_A B)$ *is discrete.*

Proof. First observe that $\mathrm{dR}\widehat{\otimes}_A B$ is left Kan extended from its restriction to the category of p-completely finitely generated polynomial A-algebras. Hence the restricted functor has the same space of endomorphisms. Since our functor $\mathrm{dR}\widehat{\otimes}_A B$ is a sheaf on the quasi-syntomic site of A and since p-completed polynomial A-algebras are quasi-syntomic over A, restricting our functor to the full subcategory of A-algebras consisting of algebras that are quasi-syntomic over A again computes the same endomorphism space. Recall that since the quasi-syntomic site of A admits a basis consisting of large quasi-syntomic A-algebras (see [BS19, Definition 15.1]), we may restrict our (base-changed) de Rham cohomology functor to this basis and compute the space of endomorphisms there. But now the values of the de Rham cohomology functor are p-completely flat A-algebras, hence the base-changed de Rham cohomology functor has values which are discrete B-algebras [BMS19, Lemma 4.6], consequently the space of endomorphisms is discrete. □

If $\mathrm{Spf}(A)$ has a disconnection, then the space of endomorphisms will be the product of endomorphism spaces on each subset giving rise to the disconnection. Hence without loss of generality, let us only treat those A's with connected formal spectrum. The following simple lemma will be used later, so let us record it here.

Lemma 3.4. *Let A_0 be an idempotent-free $\mathbb{F}_p$-algebra. Let q be a power of p. If every element $a \in A_0$ satisfies $a^q = a$, then A_0 is a sub-field inside $\mathbb{F}_q$.*

In the rest of this section we will compute the space of endomorphisms in two cases

Case I: When A is the Witt ring of an idempotent-free characteristic p perfect algebra k and $B = A$; and

Case II: When A is a perfect $\mathbb{F}_p$-algebra and B is an arbitrary A-algebra.

Building on the method of [BLM20, 10.3-10.4], Case I above is essentially worked out in the proof of [LL21, Theorem 3.13], let us state a slightly more general result below.

Proposition 3.5. *Assume that A is p-torsion free, p-adically complete, and $\mathrm{Spec}(A/p)$ is reduced and connected. Then*

$$\mathrm{End}(\mathrm{dR}) = \begin{cases} \mathrm{Frob}_q^{\mathbb{N}} & \textit{if } A = \mathbb{Z}_q := W(\mathbb{F}_q), \\ \mathrm{id} & \textit{otherwise.} \end{cases}$$

In [LL21, §2.3], a Frobenius map is constructed on p-adic derived de Rham cohomology when the base is a p-torsion free δ-ring, and it is semi-linear with respect to the Frobenius on the base δ-ring. The Frob_q appearing above is the corresponding power of the Frobenius associated with the base δ-ring $\mathbb{Z}_q$, one checks easily that it is $\mathbb{Z}_q$-linear as desired.

Proof. Let us use Perf to denote the full subcategory of those A-algebras which are of the form $A\langle X_h^{1/p^\infty} \mid h \in H\rangle$ where H is a set. The proof of [LL21, Theorem 3.13] shows that

- By restricting our de Rham cohomology functor to Perf, we get an injection of endomorphism monoids;
- The restricted de Rham cohomology functor has endomorphism monoid given by a sub-monoid in $\mathbb{Z}$;
- An element $n \in \mathbb{Z}$ above is characterized by its effect on $R = A\langle X^{1/p^\infty}\rangle$, which sends $X \mapsto X^{p^n}$; and
- The image of the restriction map is contained in $\mathbb{N} \subset \mathbb{Z}$.

Let us assume that $q = p^n$ is in the image of the restriction map. Let $R = A\langle X^{1/p^\infty}\rangle$. Take any $a \in A$ and let us contemplate the map $R \to R/(X-a)$. The induced map of de Rham cohomology is the natural inclusion $R = A\langle X^{1/p^\infty}\rangle \to D$, where D is the algebra obtained by p-completely adjoining divided powers of $X - a$ to R. To extend the map $X \mapsto X^q$ from R to D, it is the same as requiring the image of $X - a$ to have divided powers. Since in D/p we have $X^p = a^p$, we see that $X^q - a = a^q - a + p \cdot d$ for some $d \in D$. The condition now becomes that $a^q - a$ admits divided powers, as (p) always admits divided powers. One can use the natural surjection $D \to R/(X-a)$ to see that an element $a' \in A$ admits divided powers if and only if its image in D admits divided powers. Therefore the condition becomes that $a^q - a \in A$ should admit divided powers for all $a \in A$. The above implies that in A/p we have $(x^q - x)^p = 0$ for all $x \in A/p$, since A/p is assumed to be reduced, this is equivalent to all of its elements satisfying $x^q = x$. Now we use Lemma 3.4 to conclude that A/p is actually a subalgebra of $\mathbb{F}_q$, hence A must be the Witt ring of a perfect subfield inside $\mathbb{F}_q$. □

Remark 3.6. Our argument excludes the existence of q-Frobenius if there is an element $a \in A$ such that $a^q - a$ does not admit divided powers. For instance, if A/p has an transcendental element over $\mathbb{F}_p$, then there is no functorial endomorphism except for identity, as claimed in [LL21, Remark 3.14.(3)]. It remains unclear to us, for instance, if the p-Frobenius can exist when $A = \mathbb{Z}_p[\sqrt{p}]$.

Next we turn to Case II, which concerns the (base-changed) de Rham cohomology theory on algebras over a perfect ring of char. p. Once again the quasi-syntomic descent approach helps us prove the following statement (*c.f.* Proposition 4.10):

Proposition 3.7. *Let us assume either*

(1) A is an $\mathbb{F}_p$-algebra, B is an A-algebra, and consider the cohomology theory $\mathrm{dR} \otimes_A B$; or
(2) $A = k$ is a perfect $\mathbb{F}_p$-algebra, B is a $W_m(k)$-algebra, and consider the cohomology theory $\mathrm{dR}_{m,1} \otimes_{W_m(k)} B$.

Then the endomorphism monoids of the cohomology theory is a submonoid of $\mathbb{N}(\mathrm{Spec}(B))$, where $\mathbb{N}$ stands for the constant monoid scheme of natural numbers with 1 corresponding to the Frobenius.

Proof. We largely follow the strategy from the proof of [LL21, Theorem 3.13]. Let us temporarily denote the cohomology theory by $\mathcal{F}$.

Note that in both cases, the functor $\mathcal{F}$ defines a quasi-syntomic sheaf on qSyn_A: for case (1) this is Proposition 3.1,[3] and for case (2) this is Proposition 4.1. Therefore we can restrict ourselves to the category of QRSP A-algebras to compute the endomorphism monoid.

Next we reduce to one particular QRSP A-algebra: $R = A[X^{1/p^\infty}]/(X)$. To make the reduction, apply the trick in proof of [BS19, Proposition 7.10] or [LL21, Theorem 3.13] to see that for any QRSP A-algebra S there exists an explicit QRSP A-algebra $S' = A[X_i^{1/p^\infty}; i \in I]/(f_j; j \in J)$, where f_j is an ind-regular sequence in $A[X_i^{1/p^\infty}; i \in I]$, together with a surjection $R' \to R$ inducing a surjection of their values of the cohomology theory. Hence for any functorial endomorphism, its effect on $\mathcal{F}(S)$ is determined by that on $\mathcal{F}(S')$. Finally for each $j \in J$, there exists a map $R \to S'$ sending $X^{\ell/p^n} \mapsto (f_j^{1/p^n})^\ell$, the image of $\mathcal{F}(R)$ under these maps generates $\mathcal{F}(S')$, therefore the effect of a functorial endomorphism is determined by its effect on $\mathcal{F}(R)$.

Lastly we need to understand the effect of a potential functorial endomorphism f on $D := \mathcal{F}(R) = D_{(x)}(B[x^{1/p^\infty}])$, the divided power envelope of (x) in $B[x^{1/p^\infty}]$. From the proof of [LL21, Theorem 3.13] (last 4 paragraphs), we see there is a finite disconnection of $\mathrm{Spec}(B)$, such that on the j-th component we have $f(x^\ell) = x^{\ell \cdot p^{n_j}}$ for some natural number n_j. Arguing component-wise, we may assume without loss of generality that $f(x^\ell) = x^{\ell \cdot p^N}$ for some natural number N; we need to show that this extends uniquely (assuming functoriality) to the whole D. The algebra D admits a natural grading, inspecting with the functoriality given by the map $R \to R \otimes_A A[t^{1/p^\infty}]$ sending x^ℓ to $x^\ell \otimes t^\ell$ shows that f must multiply the degree by p^N. Now we claim for every $n \in \mathbb{N}$ the effect of f on $\{\text{degree} < p^{n+1} \text{ parts of } D\}$ is determined by the effect of f on $\{\text{degree} < p^n \text{ parts of } D\}$, which will finish the proof. To that end, notice that the degree $< p^{n+1}$ parts is generated by $\gamma_{p^{n+1}}(x)$ and the degree $< p^n$ parts. Finally, we look at the map $A[x^{1/p^\infty}]/(X) \to A[y^{1/p^\infty}, z^{1/p^\infty}]/(y,z)$ given by $x^{\ell/p^i} \mapsto (y^{1/p^i} + z^{1/p^i})^\ell$. By comparing the coefficients of $\gamma_{p^{n+N}}(y) \cdot \gamma_{p^{n+N}(p-1)}(z)$ of the equation obtained from functoriality, one sees that the effect of f on $\gamma_{p^{n+1}}(x)$ is pinned down by its effect on $\gamma_{p^n}(x)$ and $\gamma_{p^n(p-1)}(x)$. ☐

To illustrate the last sentence of the above proof, let us take $n = 0$ and see how to pin down the effect of f on $\gamma_p(x)$. The functoriality gives us a commutative diagram

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "x \mapsto (y+z)"'] & D \arrow[d, "x \mapsto (y+z)"] \\ D \otimes_B D \arrow[r, "f \otimes f"] & D \otimes_B D. \end{tikzcd}$$

Tracing through commutativity for the element $\gamma_p(x)$ we get that, if $f(\gamma_p(x)) = c \cdot \gamma_{p^{N+1}}(x)$ then we have

$$c \cdot \gamma_{p^{N+1}}(y) + \sum_{1 \le j \le p-1} \frac{1}{j!(p-j)!} y^{p^N} \cdot z^{p^N(p-1)} + c \cdot \gamma_{p^{N+1}}(z) = c \cdot \sum_{i+j=p^{N+1}} \gamma_i(y)\gamma_j(z).$$

[3] Since A is p-torsion so we can drop the p-completion of the tensor product

Therefore we get $y^{p^N} \cdot \frac{z^{p^N(p-1)}}{(p-1)!} = c \cdot \gamma_{p^N}(y) \cdot \gamma_{p^N(p-1)}(z)$ in $D \otimes_B D$, which clearly pins down

$$c = \frac{(p^N)! \cdot (p^N(p-1))!}{(p-1)!}.$$

Similar to Proposition 3.5, if we make reducedness assumption on B/p then we can further decide which powers of Frobenius can appear depending on the size of B/p. In Proposition 4.10, using the stacky approach, we will say precisely what powers of Frobenius is allowed in terms of the map $k \to B^\flat$ for the case (2) in Proposition 3.7, see Remark 4.8.

4. Endomorphisms of de Rham cohomology II

In this section we use a stacky approach to calculate endomorphism of de Rham and crystalline cohomology functors in situations where it seems difficult to use only quasi-syntomic descent methods to obtain the answer.

4.1. **Unwinding equivalence.** We fix two integers $n, m \geq 1$ and a perfect algebra k as before. The goal of this section is to study endomorphisms of the functor

$$\mathrm{dR}_{m,n} : \mathrm{ARings}_{W_n(k)} \to \mathrm{CAlg}(D(W_m(k))).$$

First, we will formulate this as a moduli problem. Let S be a discrete test $W_m(k)$-algebra. We can define a functor $\mathrm{End}_{m,n}$ by

$$\mathrm{End}_{m,n}(S) := \mathrm{End}(\mathrm{dR}_{m,n} \otimes_{W_m(k)} S).$$

This defines a functor $\mathrm{End}_{m,n}$ from $W_m(k)$-algebras to spaces which *a priori* is a prestack. Let us study the base-changed crystalline cohomology theory, similar to Proposition 3.1 we have the following

Proposition 4.1. *The functor $\mathrm{dR}_{m,n} \otimes_{W_m(k)} S$, when restricted to $\mathrm{qSyn}_{W_n(k)}$, defines a quasi-syntomic sheaf.*

Proof. Let us denote the derived crystalline cohomology functor relative to W by $\mathrm{dR}_{\infty,n}$, then we have $\mathrm{dR}_{m,n} \otimes_{W_m(k)} S \simeq \mathrm{dR}_{\infty,n} \otimes_{W(k)} S$. Using the previous description and the fact that $W(k)$ is p-torsion free, to check quasi-syntomic sheaf property it suffices to derived modulo p. Since $(\mathrm{dR}_{\infty,n} \otimes_{W(k)} S)/^L p \simeq \mathrm{dR}_{1,n} \otimes_k (S/^L p)$, we may reduce to the case where $m = n = 1$ and S is a 1-truncated animated k-algebra. The proof of Proposition 3.1 works verbatim in this setting as well. $\square$

Lemma 4.2. *The space of endomorphisms $\mathrm{End}(\mathrm{dR}_{m,n} \otimes_{W_m(k)} S)$ is discrete.*

Proof. Similar to the proof of Lemma 3.3: since $\mathrm{dR}_{m,n} \otimes_{W_m(k)} S$ defines a quasi-syntomic sheaf by Proposition 4.1, the claim follows from the fact that for a large quasi-syntomic $W_n(k)$-algebra R, the value $\left(\mathrm{dR}_{m,n} \otimes_{W_m(k)} S\right)(R) = \mathrm{dR}_{m,n}(R) \otimes_{W_m(k)} S$ is a discrete algebra. $\square$

On the other hand, let us consider the stack $\mathbb{A}^{1,\mathrm{dR}}$ which will always be viewed as a $W_n(k)$-algebra stack over $W_m(k)$ in this section. We define the following prestack capturing the endomorphisms of this stack along with the extra algebra structure.

Notation 4.3. For a test $W_m(k)$-algebra S, let us use $\mathscr{S}_{m,n}(S)$ to denote the space (groupoid) of endomorphisms of the stack $\mathbb{A}^{1,\mathrm{dR}}_{(S,n)} := \mathbb{A}^{1,\mathrm{dR}}_{m,n} \times_{\mathrm{Spec}\, W_m(k)} \mathrm{Spec}\, S$ as a $W_n(k)$-algebra stack over $\mathrm{Spec}\, S$.

Proposition 4.4 (Unwinding equivalence)**.** *The unwinding functor induces an isomorphism of prestacks*

$$\mathrm{Un} : \mathscr{S}_{m,n} \simeq \mathrm{End}_{m,n}.$$

Proof. Unwinding provides a map from the left hand side to the right hand side. To show isomorphism, let us fix a test $W_m(k)$-algebra S. The $W_n(k)$-algebra stack $\mathbb{A}^{1,\mathrm{dR}}_S$ by definition is an object of $\mathrm{Fun}_\times(\mathrm{Poly}^{\mathrm{op}}_{W_n(k)}, \mathrm{Stacks}_S)$. Since $\mathbb{A}^{1,\mathrm{dR}}_S$ is an affine stack (Proposition 2.27) and the category $\mathrm{AffStacks}_S$ is a full subcategory of Stacks_S which is closed under small limits, we note that $\mathbb{A}^{1,\mathrm{dR}}_S$ is classified by an object of the full subcategory $\mathrm{Fun}_\times(\mathrm{Poly}^{\mathrm{op}}_{W_n(k)}, \mathrm{AffStacks}_S)$. By Remark 2.18, the global section functor induces an equivalence of ∞-categories $\mathrm{AffStacks}_S \simeq \mathrm{coSCR}^{\mathrm{op}}_S$, where the latter denotes the ∞-category of cosimplicial S-algebras. Therefore, $\mathbb{A}^{1,\mathrm{dR}}_S$ can be equivalently viewed as an object of $\mathrm{Fun}(\mathrm{Poly}_{W_n(k)}, \mathrm{coSCR}_S)$. Therefore, endomorphisms of

$\mathbb{A}_S^{1,\mathrm{dR}}$ as a $W_n(k)$-algebra stack can be computed as endomorphisms of the classifying object which we may call G inside the category $\mathrm{Fun}(\mathrm{Poly}_{W_n(k)}, \mathrm{coSCR}_S)$.

Now we look at the S-valued points of $\mathrm{End}_{m,n}$. By property of left Kan extensions, this is given by endomorphisms of $\mathrm{dR}_{m,n} \otimes_{W_m(k)} S$ as a functor from $\mathrm{Poly}_{W_n(k)} \to \mathrm{CAlg}(D(S))$. We can also left Kan extend along the inclusion $\mathrm{Poly}_{W_n(k)} \to \mathrm{qSyn}_{W_n(k)}$ and equivalently consider endomorphisms of the functor $H : \mathrm{qSyn}_{W_n(k)} \to \mathrm{CAlg}(D(S))$. By Proposition 4.1 we see that H is a quasi-syntomic sheaf.

A basis for the quasisyntomic topology on $\mathrm{qSyn}_{W_n(k)}$ is given by flat algebras over $W_n(k)$ whose reduction modulo p is a QRSP algebra over k. The category of such algebras will be denoted as $\mathrm{QRSP}_{W_n(k)}$. On such algebras, the functor H takes values in discrete rings. By properties of right Kan extension, we obtain that the functor H has a canonical enrichment as a functor $H : \mathrm{qSyn}_{W_n(k)} \to \mathrm{coSCR}_S$ and endomorphisms can also be calculated in the category $\mathrm{Fun}(\mathrm{qSyn}_{W_n(k)}, \mathrm{coSCR}_S)$. By Proposition 2.29, we see that restricting along $\mathrm{Poly}_{W_n(k)} \to \mathrm{qSyn}_{W_n(k)}$ now realizes G as the canonical enrichment of $\mathrm{dR}_{m,n} \otimes_{W_m(k)} S$. By property of left Kan extension, the endomorphisms of H can also be computed as endomorphisms of G in the category $\mathrm{Fun}(\mathrm{Poly}_{W_n(k)}, \mathrm{coSCR}_S)$ which finishes the proof. □

Proposition 4.5. *The functor $\mathscr{S}_{m,n} : Alg_{W_m(k)} \to \mathcal{S}$ is an fpqc sheaf. In fact, it is a sheaf of sets.*

Proof. This follows from Lemma 4.2 and the fact that $\mathbb{A}^{1,\mathrm{dR}}$ is an fpqc stack. □

Before we proceed further, let us make the following definition. Let $m \geq 1$ be an arbitrary integer fixed as before. Then $\mathbb{G}_a^{\mathrm{perf}}$ represents an fpqc sheaf of rings on the category of $W_m(k)$–algebras.

Definition 4.6. We define a sheaf $\underline{\mathrm{Frob}}_k : \mathrm{Alg}_{W_m(k)} \to \mathrm{Sets}$ to be the subsheaf of $\mathcal{H}om_{k\text{-}alg}(\mathbb{G}_a^{\mathrm{perf}}, \mathbb{G}_a^{\mathrm{perf}})$ such that if B is a $W_m(k)$–algebra, then $\underline{\mathrm{Frob}}_k(B)$ is the set of k-algebra scheme maps $\mathbb{G}_{a,B}^{\mathrm{perf}} \to \mathbb{G}_{a,B}^{\mathrm{perf}}$ which is induced by an algebra map $B[x^{1/p^\infty}] \to B[x^{1/p^\infty}]$ such that it sends $x \to \sum_i b_i x^{p^i}$ where the sum ranges over a finite subset in $\mathbb{Z}_{\geq 0}$. The sheaf $\underline{\mathrm{Frob}}_k$ naturally has the structure of a commutative monoid.

Notation 4.7. For a $W_m(k)$–algebra B, we will write the symbol Frob^i to mean an element of $i \in \underline{\mathrm{Frob}}_k(B)$. We will also write Frob^{i+j} to denote the composition of Frob^i and Frob^j.

Remark 4.8. We note that $\underline{\mathrm{Frob}}_k$ is a subsheaf of the sheafification of the constant monoid $\underline{\mathbb{N}}$. In fact, they are equal when $k = \mathbb{F}_p$, but this is not always the case. One can compute that, given a $W_m(k)$-algebra B, we have

$$\underline{\mathrm{Frob}}_k(B) = \mathrm{Hom}_k(\mathbb{G}_{a,B^\flat}, \mathbb{G}_{a,B^\flat}).$$

In very concrete terms, the right hand side above is the set of pairs $(\mathcal{P}, i)$, where $\mathcal{P}$ is a partition $B = \prod_{j \in J} B_j$ and $i = (i_j)$ is a function on $\mathrm{Spec}(B)$, which is constant on each $\mathrm{Spec}(B_j)$ taking values in $\mathbb{N}$, satisfying the condition that the map $W_m(k)^\flat = k \to B_j^\flat$ factors through a subfield of the finite field $\mathbb{F}_{p^{i_j}}$.

Consequently, one finds that when k is a perfect *field*, the sheaf $\underline{\mathrm{Frob}}_k$ is representable by either the constant monoid scheme $\mathbb{N}$ or the singleton $\{0\}$, depending on whether k is finite or not.

Proposition 4.9. *There is an isomorphism of sheaf of monoids over k*

$$\mathscr{S}_{1,1} \simeq \underline{\mathrm{Frob}}_k.$$

Proof. Let k be a perfect ring. Let B be an arbitrary k-algebra. By Remark 4.8, our goal is to show that $\mathrm{End}(\mathrm{dR} \otimes_k^L B)$ is just given by $\mathrm{Hom}_k(\mathbb{G}_{a,B}, \mathbb{G}_{a,B})$ where $\mathbb{G}_{a,B}$ is regarded as a k-algebra scheme over B. For the proof, we will use another k-algebra which we denote as $\mathbb{G}_{a,B}^{\mathrm{perf}}$. More explicitly, $\mathbb{G}_a^{\mathrm{perf}}$ is represented by the affine scheme $\mathrm{Spec}\, B[x^{1/p^\infty}]$ (see Example 2.6). Note that we have a natural injection of sets $i : \mathrm{Hom}_k(\mathbb{G}_{a,B}, \mathbb{G}_{a,B}) \to \mathrm{Hom}_k(\mathbb{G}_{a,B}^{\mathrm{perf}}, \mathbb{G}_{a,B}^{\mathrm{perf}})$.

Let us first construct a map $\varphi : \mathrm{End}(\mathrm{dR} \otimes_k^L B) \to \mathrm{Hom}_k(\mathbb{G}_{a,B}^{\mathrm{perf}}, \mathbb{G}_{a,B}^{\mathrm{perf}})$. We note that dR restricts to a functor on the full subcategory of k-algebras which we denote as $\mathrm{Poly}_{\mathrm{perf}}/_k$ which consists of perfections of finite type polynomial algebras over k. If $R \in \mathrm{Poly}_{\mathrm{perf}}/_k$, then $\mathrm{dR}_{R/k} \otimes_k B \simeq R \otimes_k B$, which defines a functor from $\mathrm{Poly}_{\mathrm{perf}}/_k \to \mathrm{Alg}/_B$ sending $R \mapsto R \otimes_k B$. This basically classifies perfect k-algebra ring schemes over $\mathrm{Spec}\, B$ and any endomorphism of $\mathrm{dR} \otimes_k^L B$ induces an endomorphism of this perfect k-algebra ring scheme over $\mathrm{Spec}\, B$, which is just given by $\mathbb{G}_{a,B}^{\mathrm{perf}}$. This constructs the required map φ.

We know that any element in $\mathrm{End}(\mathrm{dR} \otimes_k^L B)$ is uniquely determined by a map f of $\mathbb{A}^{1,\mathrm{dR}}$ as a k-algebra stack over $\operatorname{Spec} B$. We also note that there is a natural map $\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{A}^{1,\mathrm{dR}}$ of k-algebra stacks over $\operatorname{Spec} B$ (from now on, we will omit the B from the subscript to ease our notation). By functoriality of $S \mapsto S_{\mathrm{perf}}$ and the fact that this perfection construction commutes with colimits, it follows that the map f lifts to give a map as below.

$$\begin{tikzcd} \mathbb{G}_a^{\mathrm{perf}} \arrow[r, "\hat{f}"] \arrow[d] & \mathbb{G}_a^{\mathrm{perf}} \arrow[d] \\ \mathbb{A}^{1,\mathrm{dR}} \arrow[r, "f"] & \mathbb{A}^{1,\mathrm{dR}} \end{tikzcd}$$

Let $u : \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$ denote the natural map of k-algebra schemes. Then fibre of the map $\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{A}^{1,\mathrm{dR}}$ identifies with $u^*W[F]$ (*c.f.* [Mon21, Proposition 2.2.6]). Therefore, f is given by a map of the quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ given by $u^*W[F] \to \mathbb{G}_a^{\mathrm{perf}}$, which is of the form of a commutative diagram as below.

$$\begin{tikzcd} u^*W[F] \arrow[r] \arrow[d, "t"'] & \mathbb{G}_a^{\mathrm{perf}} \arrow[d, "\varphi(f)"] \\ u^*W[F] \arrow[r] & \mathbb{G}_a^{\mathrm{perf}} \end{tikzcd}$$

In the above, t is required to be a $\mathbb{G}_a^{\mathrm{perf}}$-module map once the target is given the appropriate $\mathbb{G}_a^{\mathrm{perf}}$-module structure via restricting scalars along $\varphi(f)$. Now, inspecting the above diagram at the level of global sections yields that the map φ must factor through i, i.e., $\varphi(f)$ must be induced by an element of $s \in \mathrm{Hom}_k(\mathbb{G}_a, \mathbb{G}_a)$. From this, it follows that the previous commutative diagram is uniquely determined by a commutative diagram as below.

$$\begin{tikzcd} W[F] \arrow[r] \arrow[d, "t'"'] & \mathbb{G}_a \arrow[d, "s"] \\ W[F] \arrow[r] & \mathbb{G}_a \end{tikzcd}$$

In the above t is required to be a $\mathbb{G}_a$-module map once the target is given the appropriate $\mathbb{G}_a$-module structure via restricting scalars along φ. In order to understand the map t', we can therefore apply graded Cartier duality [Mon21, §2.4]. We note that $W[F]^* = \mathbb{G}_a$ and thus we get a map of graded group schemes $t'^* : \mathbb{G}_a \to \mathbb{G}_a$, where the source group scheme $\mathbb{G}_a$ receives its grading via the $\mathbb{G}_a$-module structure induced by restriction of scalars along s. By easy degree considerations, it follows that there exists a unique $\mathbb{G}_a$-module map t' which fits into the above commutative diagram. Therefore, we obtain the natural bijection $\mathrm{End}(\mathrm{dR} \otimes_k^L B) \simeq \mathrm{Hom}_k(\mathbb{G}_{a,B}, \mathbb{G}_{a,B})$, as desired. ☐

Proposition 4.10. *For any $m \geq 1$, there is a natural isomorphism of sheaf of monoids over $W_m(k)$*

$$\mathscr{S}_{m,1} \simeq \underline{\mathrm{Frob}}_k.$$

Proof. Let B be a $W_m(k)$-algebra. There is a k-algebra scheme over B, which we denote as $\mathbb{G}_{a,B}^{\mathrm{perf}}$, whose underlying affine scheme is $\operatorname{Spec} B[x^{1/p^\infty}]$. As in the proof of Proposition 4.9, one also obtains a map $\varphi : \mathrm{End}(\mathrm{dR} \otimes_{W_m(k)} B) \to \mathrm{Hom}_k(\mathbb{G}_{a,B}^{\mathrm{perf}}, \mathbb{G}_{a,B}^{\mathrm{perf}})$. It follows from going modulo p and applying Proposition 4.9 that φ actually factors to give a map again denoted as $\varphi : \mathrm{End}(\mathrm{dR} \otimes_{W_m(k)} B) \to \underline{\mathrm{Frob}}_k(B)$. We will argue that this map is a bijection.

By using the stack $\mathbb{A}^{1,\mathrm{dR}}$, and the natural map $\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{A}^{1,\mathrm{dR}}$, in a way similar to the proof of Proposition 4.9, this amounts to the more concrete assertion that there is a unique map t of quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$ as below.

$$\begin{tikzcd} \mathbb{G}_a^{\mathrm{perf},\sharp} \arrow[r] \arrow[d, "t"'] & \mathbb{G}_a^{\mathrm{perf}} \arrow[d, "\iota(x)"] \\ \mathbb{G}_a^{\mathrm{perf},\sharp} \arrow[r] & \mathbb{G}_a^{\mathrm{perf}} \end{tikzcd}$$

Here $x \in \underline{\mathrm{Frob}}_k(B)$ and $\iota : \underline{\mathrm{Frob}}_k(B) \to \mathrm{Hom}_k(\mathbb{G}_{a,B}^{\mathrm{perf}}, \mathbb{G}_{a,B}^{\mathrm{perf}})$ denotes the natural inclusion. Let us write U for the coordinate ring of $\mathbb{G}_a^{\mathrm{perf},\sharp}$. Then U is a $\mathbb{N}[1/p]$-graded Hopf algebra over B. It is also a free algebra over B, where all the homogeneous components are free of rank 1 over B. As a graded B-algebra, U is generated by the basis elements in degree p^i for $i \in \mathbb{Z}$. It is enough to check that for a fixed $x \in \underline{\mathrm{Frob}}_k(B)$, there exists a unique map t which gives a map of quasi-ideals as above. The existence is clear from definition of $\mathbb{G}_a^{\mathrm{perf},\sharp}$ (Example 2.7) by applying the divided power envelope construction. For the uniqueness we note that once x is fixed, the above diagram forces the homogeneous elements of degree p^i for $i \leq 0$ to be mapped uniquely. The rest follows from inspecting the comultiplication of U and induction on i (see last paragraph of the proof of Proposition 3.7 as well as the discussion after that proof). ☐

Remark 4.11. It is possible to prove Proposition 4.10 by using the methods from [Mon21, §3.4], which would essentially amount to proving a similar statement as above about the quasi-ideal $\mathbb{G}_a^{\mathrm{perf},\sharp} \to \mathbb{G}_a^{\mathrm{perf}}$. It is also possible to reduce to the same statement about quasi-ideals directly from Lemma 2.32 by using the compatibility of the map induced on the animated ring $W(S)/^L p$ via the Frobenius on $W(S)$ with the natural Frobenius operator on any animated k-algebra: this implies that any endomorphism of $\mathbb{A}^{1,\mathrm{dR}}$ as a k-algebra stack lifts along the map $\mathbb{G}_a^{\mathrm{perf}} \to \mathbb{A}^{1,\mathrm{dR}}$ obtained by taking perfection. This lifting property fails for endomorphisms of $\mathbb{A}^{1,\mathrm{dR}}$ as a $W_2(k)$-algebra stack, leading to extra endomorphisms as will be constructed in Section 4.2.

4.2. **Construction of endomorphisms.** This subsection describes the construction of "enough" endomorphisms of de Rham cohomology. Our strategy is to crucially exploit the Unwinding equivalence proven in Proposition 4.4 to pass to the world of ring stacks and do a small explicit construction there. We will begin fixing notations and making some definitions.

Notation 4.12. In this section, we will work with a perfect ring k of characteristic $p > 0$. We fix two integers $n, m \geq 1$ We will use W to denote the Witt ring scheme over the fixed base $W_m(k)$. Since m, n are fixed, we will denote $\mathbb{A}^{1,\mathrm{dR}}_{(m,n)}$ simply by $\mathbb{A}^{1,\mathrm{dR}}$ when no confusion is likely to occur.

Definition 4.13. We will let $W[p]$ denote the group scheme underlying the kernel of multiplication by p map on W.

Definition 4.14. We will let $(1 + W[p])^\times$ denote the monoid scheme underlying $x \in W$ satisfying $px = p$. The multiplication on this monoid scheme is given by simply using the multiplication underlying the ring scheme structure on W.

Proposition 4.15. *Let B be a p-nilpotent ring, then the monoid scheme $(1 + W[p])^\times$ over* $\mathrm{Spec}(B)$ *is a group scheme.*

Proof. This amounts to saying that for any ring S with $p^m = 0$ in S for some m, if $x \in W(S)$ satisfies $px = p$ then x must be a unit in the ring $W(S)$. Recall that we have a short exact sequence

$$0 \to W(p \cdot S) \to W(S) \to W(S/p) \to 0,$$

where $W(p \cdot S)$ denotes the Witt ring associated with the ideal (viewed as a non-unital ring) $p \cdot S$. Since $p^m = 0$ in S, we know the ideal $W(p \cdot S)$ is nilpotent. Therefore it suffices to show the image of x in $W(S/p)$ is a unit, hence we have reduced to the case where S is of characteristic p. Since $p = V(1)$ in this case, the condition of x reads $V(F(x)) = x \cdot V(1) = V(1)$. Injectivity of V shows $F(x) = 1$, which implies that x is a unit. ☐

Construction 4.16. Now we will begin our construction of endomorphisms of $\mathbb{A}^{1,\mathrm{dR}}$ as a $W_n(k)$-algebra stack (over the base $W_m(k)$ which is fixed for this section) when $n \geq 2$. Since Definition 2.25 constructs the above stack as cone of the quasi-ideal $d : W \xrightarrow{\times p} W$, we will explicitly construct maps at the quasi-ideal level, which can be done purely 1-categorically. We note that there is a natural structure map $(W(k) \xrightarrow{\times p} W(k)) \to (W \xrightarrow{\times p} W)$ of quasi-ideals, which describes the structure of $(W \xrightarrow{\times p} W)$ as a quasi-ideal over k. In the language of quasi-ideals, the natural map $W_n(k) \to k$ which gives that $W_n(k)$ structure map of $(W(k) \xrightarrow{\times p} W(k))$ viewed as a map of quasi-ideals as described below.

$$
\begin{tikzcd}
W \arrow[r, "\times p"] & W \\
W(k) \arrow[u] \arrow[r, "\times p"] & W(k) \arrow[u] \\
W(k) \arrow[u, "\times p^{n-1}"] \arrow[r, "p^n"] & W(k) \arrow[u]
\end{tikzcd}
$$

We will construct maps of the quasi-ideal $d : W \xrightarrow{\times p} W$ over the structure map $W(k) \xrightarrow{\times p^n} W(k)$ as described above. Let F be a homomorphism of the $W(k)$-algebra scheme W. A quasi-ideal map from $d : W \xrightarrow{\times p} W$ to itself can be defined by giving a W-linear map $u : W \to F_* W$ which makes the diagram below commutative.

$$
\begin{tikzcd}
W \arrow[r, "\times p"] \arrow[d, "u"] & W \arrow[d, "F"] \\
W \arrow[r, "\times p"] & W
\end{tikzcd}
$$

However, we need to make sure that such a map respects the additional structure of being a map of quasi-ideals over $W(k) \xrightarrow{\times p^n} W(k)$, i.e., the following diagram needs to commute.

$$
\begin{tikzcd}
& & W \arrow[r, "\times p"] \arrow[d, "u"] & W \arrow[d, "F"] \\
& & W \arrow[r, "\times p"] & W \\
W(k) \arrow[r, "\times p^n"'] \arrow[uurr, bend left, "p^{n-1}"] \arrow[urr, "\times p^{n-1}"] & W(k) \arrow[uurr, bend left] \arrow[urr] & &
\end{tikzcd}
$$

As one checks, for any $n \geq 2$, the only condition this imposes is that $pu(1) = p$. This provides the following map that we wanted to construct.

$$(1 + W[p])^\times \cdot F \to \mathscr{S}_{m,n}$$

Further, for any $n \geq 2$, the above map is clearly an injection by construction. We point out that it is possible to do such a construction for every $W(k)$-algebra map F of the ring scheme W. Let S be a $W_m(k)$-algebra. Then the element of $\mathscr{S}_{m,n}(B)$ constructed above will be denoted by $u \cdot F$ where u would be understood to be an element $u(1) \in (1 + W[p])^\times(B)$. By construction, we see that the composition $(u, F') \circ (v, F)$ is equal to $(uF'(v), F'F)$.

Remark 4.17. Note that in the above picture if we let $n = 1$, then $u(1)$ is *forced by the diagram* to be equal to 1, and one does not get the extra endomorphisms that was constructed above for $n \geq 2$.

Proposition 4.18. *Let $(1 + W[p])^\times$ denote group scheme as above. There is an injection of (sheaves) $\coprod_{i \in \underline{\mathrm{Frob}}_k} (1 + W[p])^\times \cdot \mathrm{Frob}^i \to \mathrm{End}_{m,n}$ when $n \geq 2$.*

Proof. This follows from Proposition 4.4 and Construction 4.16. □

Remark 4.19. Let B be a $W_m(k)$-algebra, we have the following identification

$$\underline{\mathrm{Frob}}_k(B) \simeq \mathrm{End}_{W(k)}(W_B).$$

To see this, first note that by Remark 4.8, we have a natural map from the left hand side to the right hand side. To exhibit an arrow the other way around, note that any element in the right hand side induces a map in $\mathrm{End}_k(\mathbb{A}_B^{1,\mathrm{dR}})$ which is equivalent to the left hand side by Proposition 4.10. One easily checks the above two maps are inverse to each other. In particular, the latter monoid consists of pairs of partition of $\mathrm{Spec}(B)$ and powers of Frobenius on W on each component of the partition.

4.3. **Calculation of the endomorphism monoid.** Throughout this subsection, we will fix k to be a perfect algebra as before. Let $A = W_n(k)$, and let B be a k-algebra. In this subsection, we will show that we have found all the endomorphisms of $\mathrm{dR}_{m,n}$; more precisely, the injection in Proposition 4.18 is an isomorphism.

We need some preparations, starting with understanding the homotopy sheaves associated with $\mathbb{A}^{1,\mathrm{dR}}$. Since $\mathbb{A}^{1,\mathrm{dR}}$ is a 1-stack, we only need to understand π_0 and π_1.

Proposition 4.20. *For a test algebra S,*

(1) $\mathbb{A}^{1,\mathrm{dR}}(S) = [W(S)/p]$, where $W(S)/p$ denotes the animated ring obtained by quotienting $W(S)$ by p. We note that the object in the category of animated modules underlying $[W(S)/p]$ can be simply described as $\mathrm{Cofib}(W(S) \xrightarrow{\times p} W(S))$.

(2) The sheaf $\pi_1(\mathbb{A}^{1,\mathrm{dR}})$ is representable by $W[p]$, the ideal scheme of p-torsions in the ring scheme W.

(3) Over a characteristic p base, the sheaf $\pi_0(\mathbb{A}^{1,\mathrm{dR}})$ is representable by $\mathbb{G}_a$, where the induced map $W \to \mathbb{G}_a$ is given by the natural projection to 0-th Witt coordinate.

Proof.

(1) By definition, we need to prove that the presheaf $P(S) := [W(S)/p]$ is already an fpqc sheaf of animated rings. It is enough to show that $P(S) := \mathrm{Cofib}(W(S) \xrightarrow{\times p} W(S))$ is a sheaf of connective animated modules. By noting that $\mathrm{Cofib}(W(S) \xrightarrow{\times p} W(S)) = \mathrm{fib}(W(S)[1] \xrightarrow{\times p} W(S)[1])$, we see that it is enough to prove that the functor $Q(S) \mapsto W(S)[1]$ is a sheaf of connective animated modules. For this, we only need to show that $H^1_{fpqc}(\operatorname{Spec} S, W) = 0$.

To this end, we note that the sheaf $W = \varprojlim_n W_n$. By [BS15, Example 3.1.7 and Proposition 3.1.10] and the fact that F on W is faithfully flat, it follows that $W = R\varprojlim_n W_n$. Thus $R\Gamma_{fpqc}(\operatorname{Spec} S, W) = R\varprojlim_n R\Gamma_{fpqc}(\operatorname{Spec} S, W_n)$. Now one notes that W_n has a finite filtration with the graded pieces being equal to $\mathbb{G}_a$. Thus $R\Gamma_{fpqc}(\operatorname{Spec} S, W) = R\varprojlim_n R\Gamma_{fpqc}(\operatorname{Spec} S, W_n) = R\varprojlim_n \Gamma(\operatorname{Spec} S, W_n) = R\varprojlim_n W_n(S) = \varprojlim_n W_n(S) = W(S)$. In particular, $H^1_{fpqc}(\operatorname{Spec} S, W) = 0$, as desired.

(2) Follows from (1).

(3) In the Witt ring of a characteristic p ring, we have $p = VF$. Therefore, the conclusion follows since $F : W \to W$ is a fpqc surjection. ☐

In general, $\pi_0(\mathbb{A}^{1,\mathrm{dR}})$ is given by the sheaf of discrete k-algebras W/p. However, if the base is not of characteristic p, this sheaf stops being representable as noted below. Nevertheless, Lemma 4.23 will helps us extract the necessary information from $\pi_0(\mathbb{A}^{1,\mathrm{dR}})$ relevant to us.

Proposition 4.21. *Let B be a ring such that $p \notin (p^2)$, let $S = \operatorname{Spec}(B)$. The sheaf*

$$\mathcal{F} := \pi_0(\mathrm{Cone}(\mathbb{G}^\sharp_{a,S} \to \mathbb{G}_{a,S})) \simeq \pi_0(\mathbb{A}^{1,\mathrm{dR}})$$

is not representable by an algebraic space over S.

Proof. The isomorphism follows from Proposition 2.30. Since both $\mathbb{G}_a$ and $\mathbb{G}_a^\sharp$ are affine schemes, the hypothetical representing algebraic space would be quasi-compact and quasi-separated. Below we show there cannot be such a qcqs algebraic space.

It suffices to prove the statement for B/p^2 hence we may assume $p^2 = 0$ in B. Since the restriction of our sheaf to B/p-algebras is represented by the affine scheme $\mathbb{G}_{a,B/p}$, using [Sta21, Tag 07V6] we see that the sheaf would in fact be represented by an affine scheme over S. Let us denote its ring of function by R. The natural map $\mathbb{G}_{a,S} \to \operatorname{Spec}(R)$ induces a map $R \to B[t]$. Reducing the ring map modulo p, we see that the image is $B/p[t^p]$. This implies that an element of the form $t^p + p \cdot g$ must be in the image. On the other hand we claim the image of the ring map itself is contained in $\{f \in B[t] \mid f'(t) = 0\}$. Indeed, the two compositions

$$\operatorname{Spec}(B[t,\epsilon]/\epsilon^2) \underset{t \mapsto t+\epsilon}{\overset{t \mapsto t}{\rightrightarrows}} \operatorname{Spec}(B[t]) \longrightarrow \mathcal{F}$$

yields the same map as $\epsilon \in B[t,\epsilon]/\epsilon^2$ admits divided powers. This shows that the image of $R \to B[t]$ must be contained in the equalizer of the two maps $B[t] \rightrightarrows B[t,\epsilon]/\epsilon^2$. The identification of this equalizer with those polynomials whose derivative is zero follows from Taylor expansion. Lastly to get a contradiction, just observe that if we let $f = t^p + p \cdot g$, then $f' \neq 0$ as $p \notin (p^2)$; however, we had previously argued that $t^p + p \cdot g$ must be in the image. $\square$

Lemma 4.22. *Let B be a $W(k)$-algebra. We have $W(B)[p] \simeq \mathrm{Hom}_{W(k)}(k, \mathbb{A}^{1,\mathrm{dR}})$, where the right hand side denotes the space of maps $W(k)$-algebra stacks over B. Given $\beta \in W(B)[p]$ the corresponding homomorphism of sheaves is modeled by*

$$\begin{tikzcd} W \arrow[r, "\times p"] & W \\ W(k) \arrow[u, "\times(1+\beta)"] \arrow[r, "\times p"] & W(k) \arrow[u] \end{tikzcd}$$

Proof. Since $\mathbb{A}^{1,\mathrm{dR}}$ is 1-truncated, by Proposition A.6 the right hand side is classified by $\mathrm{Hom}_k(\mathbb{L}_{k/W(k)}, \pi_1(\mathbb{A}^{1,\mathrm{dR}})[1])$ $= W(B)[p]$. Here in this identification we have used Proposition 4.20.(2). One checks easily that the maps we constructed in the last sentence exactly corresponds to β under the above identification, hence finishing the proof. $\square$

Our last preparation is to understand those algebra homomorphisms in $\mathrm{End}_k(\pi_0(\mathbb{A}^{1,\mathrm{dR}}))$ which can be lifted to a $W_n(k)$-algebra homomorphism of $\mathbb{A}^{1,\mathrm{dR}}$. It turns out that liftability as a $W_n(k)$-algebra stack for $n > 1$ *automatically* guarantees liftability as a k-algebra stack as noted below.

Lemma 4.23. *Let B be a $W_m(k)$-algebra, and let us consider $\mathbb{A}^{1,\mathrm{dR}}$ as a k-algebra stack over B. The two natural maps $\mathrm{End}_{W(k)}(\mathbb{A}^{1,\mathrm{dR}}) \to \mathrm{End}_k(\pi_0(\mathbb{A}^{1,\mathrm{dR}}))$ and $\mathrm{End}_k(\mathbb{A}^{1,\mathrm{dR}}) \to \mathrm{End}_k(\pi_0(\mathbb{A}^{1,\mathrm{dR}}))$ have the same image. In particular, by Proposition 4.10, we know the image is naturally in bijection with the monoid $\underline{\mathrm{Frob}}_k(B)$.*

Proof. The image of the first map clearly contains the image of second map. Now given $f \in \mathrm{End}_{W(k)}(\mathbb{A}^{1,\mathrm{dR}})$, by composing with the natural map $\iota : k \to \mathbb{A}^{1,\mathrm{dR}}$, we get a natural map $f \circ \iota \colon k \to \mathbb{A}^{1,\mathrm{dR}}$ of $W(k)$-algebra stacks. In Lemma 4.22 we see that $f \circ \iota$ must be classified by some element $1 + \beta \in 1 + W(B)[p]$. By Proposition 4.15, we can find an inverse $(1+\beta)^{-1} \in (1 + W[p])^\times$, one sees that the composition $(1+\beta)^{-1} \circ f \circ \iota = \iota$. Here we regard an element in $(1+W[p])^\times$ as a $W(k)$-algebra automorphism of $\mathbb{A}^{1,\mathrm{dR}}$ by Construction 4.16. Since these elements in $(1+W[p])^\times$ always induce identity on π_0, we see that $(1+\beta)^{-1} \circ f$ is a k-algebra automorphism lifting the same ring homomorphism on $\pi_0(\mathbb{A}^{1,\mathrm{dR}})$ as f. $\square$

Theorem 4.24. *Let $A = W_n(k)$, and suppose that B is a $W_m(k)$-algebra, then we have*

$$\mathrm{End}(\mathrm{dR}_{m,n} \otimes_{W_m(k)} B) = \begin{cases} \coprod_{i \in \underline{\mathrm{Frob}}_k(B)} \mathrm{Frob}^i & n = 1; \\ \coprod_{i \in \underline{\mathrm{Frob}}_k(B)} (1 + W[p])^\times(B) \cdot \mathrm{Frob}^i & n \geq 2; \end{cases}$$

Here the multiplication law in the second case is given by

$$(u \cdot \mathrm{Frob}^i) \cdot (v \cdot \mathrm{Frob}^j) = u \cdot \mathrm{Frob}^i(v) \cdot \mathrm{Frob}^{i+j}.$$

where $u, v \in (1 + W[p])^\times(B)$.

Remark 4.25. Note that these endomorphism spaces are all discrete, by Lemma 3.3. The above theorem states that the map in Proposition 4.18 is actually an isomorphism. From the above calculation, we also conclude that the sheaf of endomorphism monoids is representable if and only if the sheaf $\underline{\mathrm{Frob}}_k$ is representable. This happens whenever k is a perfect *field*, in which case the representing scheme is a combination of the constant monoid scheme $\mathbb{N}$ and the commutative group scheme $(1 + W[p])^\times$, depending on k and n.

Proof. When $n = 1$, this is proved in Proposition 4.10. Below we will assume $n \geq 2$.

Recall that in Proposition 4.4 we have shown that the endomorphisms of our de Rham cohomology functor is the same as the endomorphisms of the $W_n(k)$-algebra stack $\mathbb{A}^{1,\mathrm{dR}}_B$ over $\mathrm{Spec}(B)$. Since the category of $W_n(k)$-algebra stacks is equivalent to the category of sheaves of $W_n(k)$-animated algebras, see Remark 2.14, we will compute the endomorphism of $\mathbb{A}^{1,\mathrm{dR}}_B$ viewed as a sheaf of $W_n(k)$-animated algebras on the fpqc site of $\mathrm{Spec}(B)$.

Composing with the map $\mathbb{A}^{1,\mathrm{dR}} \to \pi_0(\mathbb{A}^{1,\mathrm{dR}})$, we get a natural map

$$\mathrm{Hom}_{W_n(k)}(\mathbb{A}^{1,\mathrm{dR}}, \mathbb{A}^{1,\mathrm{dR}}) \xrightarrow{f_n} \mathrm{Hom}_{W_n(k)}(\mathbb{A}^{1,\mathrm{dR}}, \pi_0(\mathbb{A}^{1,\mathrm{dR}})) = \mathrm{End}_k(\pi_0(\mathbb{A}^{1,\mathrm{dR}})),$$

here and below, all appearances of Hom refers to homomorphism of sheaves respecting the designated structure marked by subscript. By Lemma 4.23, (*c.f.* Remark 4.19) we see that

$$\mathrm{Im}(f_n) = \underline{\mathrm{Frob}}_k(B)$$

We need to understand the fibre of f_n. Take an $i \in \underline{\mathrm{Frob}}_k(B)$, by Proposition A.6, the fibre of f_n over Frob^i is a torsor under

$$\mathrm{Hom}_{\mathbb{A}^{1,\mathrm{dR}}}(\mathbb{L}_{\mathbb{A}^{1,\mathrm{dR}}/W_n(k)}, \pi_1(\mathbb{A}^{1,\mathrm{dR}})[1]).$$

Here the sheaf of $\mathbb{A}^{1,\mathrm{dR}}$-module structure on the sheaf $\pi_1(\mathbb{A}^{1,\mathrm{dR}})$ is via $\mathbb{A}^{1,\mathrm{dR}} \to \pi_0(\mathbb{A}^{1,\mathrm{dR}}) \xrightarrow{\mathrm{Frob}^i} \pi_0(\mathbb{A}^{1,\mathrm{dR}})$. To understand this group, let us utilize the cofiber sequence cotangent complexes from Proposition A.3 associated with the diagram $W_n(k) \to k \to \mathbb{A}^{1,\mathrm{dR}}$:

$$\mathbb{L}_{k/W_n(k)} \otimes_k \mathbb{A}^{1,\mathrm{dR}} \to \mathbb{L}_{\mathbb{A}^{1,\mathrm{dR}}/W_n(k)} \to \mathbb{L}_{\mathbb{A}^{1,\mathrm{dR}}/k}.$$

By Proposition 4.10, the map $\mathrm{End}_k(\mathbb{A}^{1,\mathrm{dR}}) \to \mathrm{End}_k(\pi_0(\mathbb{A}^{1,\mathrm{dR}}))$ is injective with image $\underline{\mathrm{Frob}}_k(B)$. Therefore, again by Proposition A.6, we have $\mathrm{Hom}_{\mathbb{A}^{1,\mathrm{dR}}}(\mathbb{L}_{\mathbb{A}^{1,\mathrm{dR}}/k}, \pi_1(\mathbb{A}^{1,\mathrm{dR}})[1]) = 0$ and we get an injection

$$\mathrm{Hom}_{\mathbb{A}^{1,\mathrm{dR}}}(\mathbb{L}_{\mathbb{A}^{1,\mathrm{dR}}/W_n(k)}, \pi_1(\mathbb{A}^{1,\mathrm{dR}})[1]) \hookrightarrow \mathrm{Hom}_{\mathbb{A}^{1,\mathrm{dR}}}(\mathbb{L}_{k/W_n(k)} \otimes_k \mathbb{A}^{1,\mathrm{dR}}, \pi_1(\mathbb{A}^{1,\mathrm{dR}})[1]),$$

and the latter is identified with

$$\mathrm{Hom}_k(\mathbb{L}_{k/W_n(k)}, \pi_1(\mathbb{A}^{1,\mathrm{dR}})[1]) = \mathrm{Hom}_k(k[1], \pi_1(\mathbb{A}^{1,\mathrm{dR}})[1]) = \pi_1(\mathbb{A}^{1,\mathrm{dR}})(B) = W[p](B),$$

by Proposition 4.20.(2). Unraveling definitions, for any $u \in (1 + W[p])^\times(B)$, the element $u \cdot \mathrm{Frob}^i$ (see Construction 4.16 and Proposition 4.18) in the fibre of f_n is sent to $u - 1 \in W[p](B)$. One easily sees that the previous sentence in fact gives a bijection, therefore the fibre of f_n over Frob^i is exactly $(1+W[p])^\times(B) \cdot \mathrm{Frob}^i$ and finishes the calculation of endomorphism sets.

The multiplication law is checked by chasing through the diagram: on the quasi-ideal model, the homomorphism $u \cdot \mathrm{Frob}^i$ sends an element $x \in W(B)$ to $u \cdot \mathrm{Frob}^i(x)$, and one computes $u \cdot \mathrm{Frob}^i(v \cdot F^j(x)) = u \cdot \mathrm{Frob}^i(v) \cdot \mathrm{Frob}^{i+j}(x)$. □

Remark 4.26. In the above proof, one does not actually need to work with fpqc sheaves and the same proof works merely at the level of presheaves. However, if one only wanted to prove Theorem 4.24 in the case when $m = 1$, one can work with fpqc sheaves or quasi-syntomic sheaves and use the fact that $\pi_0(\mathbb{A}^{1,\mathrm{dR}}) = \mathbb{G}_a$ from Proposition 4.20 to simplify the proof and avoid invoking Proposition 4.10 and Lemma 4.23. The case $m = 1$ is sufficient for our application in Section 5.

Corollary 4.27. *Let k be an arbitrary perfect algebra. We consider the functor $\mathrm{End}_{m,n}$ from Section 4.1 for a fixed $m \geq 1$. There are natural maps of sheaves $\mathrm{End}_{m,n'} \to \mathrm{End}_{m,n}$ for $n' \geq n$, which induces an isomorphism if $n \geq 2$. If $n' > n$, and $n = 1$, then all fibres of this natural map are given by the group scheme $(1 + W[p])^\times$. The sheaf $\mathrm{End}_{m,1}$ is $\underline{\mathrm{Frob}}_k$.*

Proof. This follows from combining Proposition 4.10 and Theorem 4.24. □

Remark 4.28.

(1) The stabilization of $\mathrm{End}_{m,n}$ for $n \geq 2$ that we see above suggests that lifting to W_n for $n > 2$ poses no extra information on the de Rham cohomology of the special fibre, at least in a functorial sense. In next section we will see the extra information on liftability to second Witt vectors gives a strengthening to Deligne–Illusie's decomposition theorem [DI87]. Combining these two results, we are led to believe the following dichotomy of possibilities on a follow-up question [DI87, Remarques 2.6.(iii)]: either liftable over W_2 always guarantees Hodge–de Rham spectral sequence degenerates; or there is a counterexample (necessarily of dimension $\geq p+1$) which is liftable all the way over W.

(2) If B has characteristic p, then $p = V \circ F$ on $W(B)$. The defining equation $u \cdot p = p$ of $(1 + W[p])^\times$ becomes $V(F(u)) = V(1)$. Since V is always injective, the group scheme $(1 + W[p])^\times$ over a characteristic p base becomes $\mathbb{G}_m^\sharp := W^\times[F]$, namely the Frobenius kernel of the multiplicative group scheme $W^\times$.

(3) The above discussion tells us that the functorial automorphism group scheme of the mod p de Rham cohomology theory on $W_2(k)$-algebras is given by $\mathbb{G}_m^\sharp$. Note that there is a natural inclusion $\mu_p \to \mathbb{G}_m^\sharp$ which induces a product decomposition $\mathbb{G}_m^\sharp = \mu_p \times \mathbb{G}_a^\sharp$ (see Appendix B). In Theorem 5.4 we will utilize the automorphisms coming from μ_p. The remaining $\mathbb{G}_a^\sharp$ worth of automorphisms is related to the Sen operator studied in [BL21].

Our calculation shows that there is no functorial splitting of the whole mod p derived de Rham complex, as a functor from $W_2(k)$-algebras to $\mathrm{CAlg}(D(k))$, into direct sums of the graded pieces of its conjugate filtrations.

Proposition 4.29. *There is no functorial splitting*

$$\mathrm{dR}_{(-\otimes_{W_2(k)} k)/k} \simeq \bigoplus_{i \in \mathbb{N}_{\geq 0}} \mathrm{Gr}_i^{\mathrm{conj}}\big(\mathrm{dR}_{(R \otimes_{W_2(k)} k)/k}\big)$$

as a functor from smooth $W_2(k)$-algebras to $\mathrm{CAlg}(D(k))$.

Proof. Indeed, if there were such a splitting, we would get an automorphism parametrized by $\mathbb{G}_m$, with the i-th graded piece having pure weight i. From the calculation of the endomorphism monoid in Theorem 4.24, this would give us an injection $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^\sharp$. But the Frobenius on $\mathbb{G}_m$ is non-zero, whereas it is zero on $\mathbb{G}_m^\sharp$, hence we know there is no injective map $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^\sharp$ over any characteristic p base, therefore getting a contradiction. $\square$

Remark 4.30 (Twisted forms of de Rham cohomology)**.** Theorem 4.24 can be applied to understand a question considered by Antieau and Moulinos on possible existence of étale twists of the de Rham cohomology functor in some cases: let k be a perfect ring and let B be an ordinary $W_m(k)$-algebra, does there exist a functor $F : \mathrm{ARings}_{W_n(k)} \to \mathrm{CAlg}(D(B))$ which is isomorphic to $\mathrm{dR}_{m,n} \otimes_{W_m(k)} B$ étale locally on $\operatorname{Spec} B$? We thank Antieau for mentioning this question to us. By Theorem 4.24, such functors are classified by $\mathrm{H}^1_{ét}(\operatorname{Spec} B, (1 + W[p])^\times)$. When $m = 1$ and B is perfect, one can show that $\mathrm{H}^1_{fpqc}(\operatorname{Spec} B, (1 + W[p])^\times) = 0$ by using $(1 + W[p])^\times \simeq \mathbb{G}_m^\sharp \simeq \mu_p \times \mathbb{G}_a^\sharp$ over $\operatorname{Spec} B$. So in that case, there does not even exist a non-trivial fpqc twist. However, the cohomology group can be non-zero for some choices of B. It would be interesting to study the corresponding twisted forms of de Rham cohomology which can be seen as new cohomology theories, but that direction is not pursued further in this paper. It would also be interesting to compute $\mathrm{H}^1_{ét}(\operatorname{Spec} B, (1 + W[p])^\times)$ in general for $m > 1$.

5. Application to the Deligne–Illusie's decomposition

5.1. **Drinfeld's refinement of the Deligne–Illusie's decomposition.** In this section we will explain how to apply our result from Theorem 4.24 on endomorphisms of de Rham cohomology functor to recover a recent result of Drinfeld concerning a classical theorem due to Deligne–Illusie [DI87], and Achinger–Suh [AS21].

Notation 5.1. Fix a perfect ring k as before and consider the monoid scheme $\mathrm{End}_{1,n}$ from Corollary 4.27 over k. Let B be a k-algebra and let $\sigma \in \mathrm{End}_{1,n}(B)$. By definition we get an endomorphism induced by σ

$$\mathrm{dR}_{R/W_n(k)} \otimes_{W_n(k)} B \xrightarrow{\sigma} \mathrm{dR}_{R/W_n(k)} \otimes_{W_n(k)} B,$$

which is functorial in the $W_n(k)$-algebra R.

Definition 5.2. For any $W_n(k)$-algebra R, we define the conjugate filtration $\mathrm{Fil}_i^{\mathrm{conj}}$ on $\mathrm{dR}_{R/W_n(k)} \otimes_{W_n(k)} k$ to be the left Kan extension of the canonical filtration on polynomial (or smooth) $W_n(k)$-algebras.

Lemma 5.3. *Assume $k \to B$ is flat, then σ preserves $\mathrm{Fil}_i^{\mathrm{conj}} \otimes_k B$ for all i.*

Proof. Any morphism must preserve the canonical filtration. If R is a polynomial (or smooth) $W_n(k)$-algebra, one easily shows that the canonical filtration on $\mathrm{dR}_{R/W_n(k)} \otimes_{W_n(k)} B$ is just $\mathrm{Fil}_i^{\mathrm{conj}} \otimes_k B$. $\square$

By Theorem 4.24 and Remark 4.28 (4), we have an inclusion of k-schemes $(\mathbb{G}_m^\sharp) \subset \mathrm{End}_{1,2}$. Let $B = \Gamma(\mathbb{G}_m^\sharp, \mathcal{O})$, then the identity map defines an element $\sigma \in \mathbb{G}_m^\sharp(B)$, which can be regarded as the universal point. By the above discussion, the universal point σ gives rise to a comodule structure on $\mathrm{dR}_{R/W_2(k)} \otimes_{W_2(k)} k$ over the Hopf algebra B, functorial in the $W_2(k)$-algebra R, and the conjugate filtration is an increasing filtration of sub-comodules. Alternatively, we may view this as an action of $\mathbb{G}_m^\sharp$ on the mod p de Rham cohomology $\mathrm{dR}_{-/W_2(k)} \otimes_{W_2(k)} k$. We may ask the effect of $\mathbb{G}_m^\sharp$-action on each graded piece of the conjugate filtration, viewed as a functor from the category of $W_2(k)$-algebras to the derived ∞-category of B-comodules. The latter can be defined as the derived ∞-category of quasi-coherent sheaves on $B\mathbb{G}_m^\sharp$.

Recall that the category of μ_p-representations is semi-simple with simple objects given by $\mathbb{Z}/p$-worth of powers of the universal character. We follow the convention that the universal character $\mu_p \hookrightarrow \mathbb{G}_m$ has weight 1. The following result is first observed by Drinfeld via prismatization, communicated to us by Bhatt.

Theorem 5.4. *The action of $\mathbb{G}_m^\sharp$ on the i-th graded piece of the conjugate filtration factors through the natural projection $\mathbb{G}_m^\sharp \to \mu_p$, and the resulting μ_p-action is of pure weight $i \in \mathbb{Z}/p$.*

This fact also appears in [BL21], where it is proved by contemplating with Sen operators. Below we give a different argument.

Proof. Derived Cartier isomorphism [Bha12, Proposition 3.5] reduces us to showing the statement for $i = 0$ and 1. Since the conjugate filtration is defined via left Kan extension from its values on polynomial algebras, using the classical Cartier isomorphism and Künneth formula, we are reduced to understanding the behavior of σ on the cohomology of

$$\mathrm{dR}_{W_2(k)[x]/W_2(k)} \otimes_{W_2(k)} k \simeq \mathrm{dR}_{k[x]/k}.$$

Observe that the whole situation is base changed from $k = \mathbb{F}_p$, we immediately reduce to $k = \mathbb{F}_p$.

According to Construction 2.21, the action of σ is defined via the identification

$$\mathrm{dR}_{\mathbb{Z}/p^2[x]/(\mathbb{Z}/p^2)} \otimes_{\mathbb{Z}/p^2} B \simeq \mathrm{R\Gamma}(\mathbb{A}_B^{1,\mathrm{dR}}, \mathcal{O})$$

and the homomorphism of $\mathbb{Z}/p^2$-algebra stack over $\mathbb{G}_m^\sharp$ given by the diagram below (here W_B denotes the Witt ring scheme over the base scheme $\mathbb{G}_m^\sharp$)

$$\begin{tikzcd} W_B \arrow[r, "\times p"] \arrow[d, "\times\sigma"'] & W_B \arrow[d, "\mathrm{id}"] \\ W_B \arrow[r, "\times p"] & W_B. \end{tikzcd}$$

The first cohomology of $\mathrm{dR}_{\mathbb{F}_p[x]/\mathbb{F}_p}$ is a free rank 1 module over its zeroth cohomology. Therefore all we need to do is the following.

(1) Show the induced map on $\mathrm{H}^0(\mathbb{A}_B^{1,\mathrm{dR}}, \mathcal{O})$ is trivial.
(2) Exhibit a non-zero element $v \in \mathrm{H}^1(\mathbb{A}_{\mathbb{F}_p}^{1,\mathrm{dR}}, \mathcal{O})$ which pulls back to a weight 1 element in $\mathrm{H}^1(\mathbb{A}_B^{1,\mathrm{dR}}, \mathcal{O})$.

To avoid confusion, let us denote the ring scheme $W \coloneqq \mathrm{Spec}(\mathbb{F}_p[X_0, X_1, \ldots])$ and the quasi-ideal $W \coloneqq \mathrm{Spec}(\mathbb{F}_p[Y_0, Y_1, \ldots])$. Here X_i's (and similarly Y_i's) are the Witt coordinates. One easily checks the effect of id^* and $(\times\sigma)^*$ on the following elements: $X_i \mapsto X_i$, $Y_0 \mapsto t_0 \cdot Y_0$. Here t_0 denotes the element in B corresponding to the natural projection $\mathbb{G}_m^\sharp \to \mu_p$.

Now (1) is easily verified: $\mathrm{H}^0(\mathbb{A}_B^{1,\mathrm{dR}}, \mathcal{O}) \subset B[X_0, X_1, \ldots]$, hence invariant under the σ-action.

As for (2), we claim that $1 \otimes Y_0 \in \mathbb{F}_p[X_i, Y_j]$ is a nonzero class in $\mathrm{H}^1(\mathbb{A}_{\mathbb{F}_p}^{1,\mathrm{dR}}, \mathcal{O})$. Here we are using the Čech nerve of $\mathrm{Spec}(\mathbb{F}_p[X_0, X_1, \ldots]) \to \mathbb{A}_{\mathbb{F}_p}^{1,\mathrm{dR}}$ to calculate the cohomology of $\mathbb{A}_{\mathbb{F}_p}^{1,\mathrm{dR}}$, implicitly we have used the fact that the [1]-term of the Čech nerve is given by $\mathrm{Spec}(\mathbb{F}_p[X_i, Y_j])$. Granting this claim, the action of σ sends Y_0 to $t_0 \cdot Y_0$, hence the action on the class $1 \otimes Y_0$ is via the natural projection $\mathbb{G}_m^\sharp \to \mu_p$ and has weight 1. To prove the claim, we use the maps

$$W[F] = \mathbb{G}_a^\sharp \to W = \mathrm{Spec}(\mathbb{F}_p[Y_0, Y_1, \ldots]) \to \mathbb{G}_a = \mathrm{Spec}(\mathbb{F}_p[Y_0]),$$

where the middle W is the copy of quasi-ideal W. The above maps induce a sequence of abelian group stacks:

$$B\mathbb{G}_a^\sharp \to \mathbb{A}^{1,\mathrm{dR}} \to B\mathbb{G}_a.$$

Recall that there is a canonical identification $\mathrm{H}^1(BG, \mathcal{O}) \simeq \mathrm{Hom}(G, \mathbb{G}_a)$ for affine group schemes G via faithfully flat descent along $* \to BG$. The identity map on $\mathbb{G}_a = \mathrm{Spec}(\mathbb{F}_p[Y_0])$ pulls back to $1 \otimes Y_0 \in \mathbb{F}_p[X_i, X_j]$, which checks that $1 \otimes Y_0$ is a cocycle. Furthermore, recall that the induced map $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ realizes the former as the divided power envelope of the origin inside the latter, in particular it is a nonzero map. From the above identification, this tells us that $1 \otimes Y_0$ pulls to a nonzero class in $\mathrm{H}^1(B\mathbb{G}_a^\sharp, \mathcal{O})$. Therefore the class $1 \otimes Y_0$ is a nonzero class in $\mathrm{H}^1(\mathbb{A}^{1,\mathrm{dR}}_{\mathbb{F}_p}, \mathcal{O})$. $\square$

Note that the natural projection $\mathbb{G}_m^\sharp \to \mu_p$ admits a splitting: the Teichmüller lift defines a map of group schemes $\mathbb{G}_m \hookrightarrow W^\times$, which induces a map of group schemes $\mu_p \hookrightarrow \mathbb{G}_m^\sharp$.

Let X be a smooth scheme over $W_2(k)$, consider the de Rham cohomology of its special fibre (relative to k) which by the above discussion admits a μ_p-action. Now look at the canonical truncation in a range of width at most p, then the weights that show up in $\mathbb{Z}/p$ are pairwise distinct, hence we get a splitting of the induced conjugate filtration. Therefore the above theorem implies the following improvement of a result due to Achinger and Suh [AS21, Theorem 1.1], which in turn is a strengthening of Deligne–Illusie's result [DI87, Corollaire 2.4].

Corollary 5.5 (Drinfeld)**.** *Let k be a perfect ring of characteristic $p > 0$, let X be a smooth scheme over $W_2(k)$, and let $a \leq b \leq a + p - 1$. Then the canonical truncation $\tau_{[a,b]}(\Omega^\bullet_{X_k/k})$ splits.*

Note that when $p > 2$, in Achinger–Suh's statement in loc. cit. they need $b < a + p - 1$ so their allowed width needs to be at most $p - 1$. In fact, more generally, we have the following decomposition as a consequence of the $\mathbb{G}_m^\sharp$-action in described in Theorem 5.4.

Corollary 5.6 (Drinfeld, *c.f.* [AS21, Remark A.5])**.** *Let X be a smooth scheme over $W_2(k)$ with special fibre X_k. Then there exists a splitting, functorial in X, in the derived ∞-category of Zariski sheaves on X'_k*

$$F_{X_k/k,*}(\mathrm{dR}_{X_k/k}) \simeq \oplus_{i \in \mathbb{Z}/p} F_{X_k/k,*}(\mathrm{dR}^{\mathrm{weight}=i}_{X_k/k}).$$

Moreover $\mathcal{H}^j(F_{X_k/k,*}(\mathrm{dR}^{\mathrm{weight}=i}_{X_k/k})) \neq 0$ implies $j \equiv i$ in $\mathbb{Z}/p$. Here X'_k is the Frobenius twist of X_k and $F_{X_k/k}$ is the relative Frobenius. In particular, the conjugate spectral sequence of liftable smooth varieties can have non-zero differentials only on $(mp+1)$-st pages, where $m \in \mathbb{Z}_{>0}$.

Remark 5.7. Drinfeld observed the results in this subsection by using the "stacky approach" to prismatic crystals (which he calls "prismatization"), which is independently developed by Bhatt–Lurie [BL21] as well. Using the prismatization functor, Drinfeld produced an action of μ_p on the de Rham complex of a smooth scheme over k that lifts to $W_2(k)$. Our paper partly grew out of an attempt of making sense of and reproving Drinfeld's theorem without introducing prismatization and taking a very algebraic/categorical approach instead. In the forthcoming paper of Bhatt–Lurie [BL21], this action is obtained in a more geometric way by understanding the prismatization of $\mathrm{Spec}(W_2(k))$.

5.2. **Uniqueness of functorial splittings.** Corollary 5.5 provides a functorial splitting of the $(p-1)$-st conjugate filtration of the mod p derived de Rham cohomology of any $W_2(k)$-algebra. On the other hand, the classical Deligne–Illusie splitting also has an ∞-categorical functorial enhancement [KP21a, Theorem 1.3.21 and Proposition 1.3.22] which, in spirit, is more related to the work of Fontaine–Messing [FM87] and Kato [Kat87].

It is a natural question to ask whether these two splittings agree in a functorial way. By how these two splittings are defined, we see immediately that they are both compatible with the module structure over the 0-th conjugate filtration, and induced from the splitting of the 1-st conjugate filtration by an averaging process, *c.f.* the step (a) in proof of [DI87, Théorème 2.1].

Below we will prove that there is a unique way to functorially split the first conjugate filtration, hence the above two functorial splittings must be the same. To that end, let us fix some notations.

Notation 5.8. Let us consider the stable ∞-category $\mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_{W_2(k)}, D(k))$, where $\mathrm{Alg}^{\mathrm{sm}}_{W_2(k)}$ is the category of smooth $W_2(k)$-algebras and $D(k)$ is the derived (stable) ∞-category of k-vector spaces. Let us use $\mathcal{O}$ to denote the functor that sends any $W_2(k)$-algebra R to the 0-th conjugate filtration of $\mathrm{dR}_{(R\otimes_{W_2(k)}k)/k}$, which has the structure of a commutative algebra object in $\mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_{W_2(k)}, D(k))$. The functor obtained by considering the 1-st piece of the conjugate filtration will be denoted M, viewed as an $\mathcal{O}$-module. We have a natural map $\mathcal{O} \to M$; we denote the cofiber by G, which is the first graded piece of the conjugate filtration, also viewed as an $\mathcal{O}$-module.

Now we have a cofiber sequence of $\mathcal{O}$-modules: $\mathcal{O} \to M \to G$.

Theorem 5.9. *In the above notations, there is a unique functorial $\mathcal{O}$-module splitting*

$$M = \mathcal{O} \oplus G$$

in $\mathrm{Fun}(\mathrm{Alg}^{sm}_{W_2(k)}, D(k))$*. In particular the splitting of* $\mathrm{Fil}^{\mathrm{conj}}_{p-1}(\mathrm{dR}_{(-\otimes_{W_2(k)}k)/k})$ *obtained in Corollary 5.5 and* [KP21a, Theorem 1.3.21] *agree.*

Proof. The existence part is provided by either Corollary 5.5 or [KP21a, Theorem 1.3.21]. We focus on the uniqueness part in this proof.

Firstly, we note that it suffices to show the uniqueness of splitting as a quasi-syntomic sheaf on the quasi-syntomic site of $W_2(k)$. This is because they are left Kan extended from the polynomial case, and polynomial algebras are quasi-syntomic. The site $\mathrm{qSyn}_{W_2(k)}$ admits a basis of large quasi-syntomic $W_2(k)$-algebras, so we may restrict our functors to this subclass of $W_2(k)$-algebras, and show uniqueness of splitting there. All three functors have discrete value on this subclass of $W_2(k)$-algebras, so $\mathcal{O}$ is a sheaf of ordinary k-algebras given by $R \mapsto R/p$ (up to Frobenius twist), and M and G are sheaves of ordinary $\mathcal{O}$-modules. We will show that there exists a unique section to the surjection of sheaves of $\mathcal{O}$-modules $M \twoheadrightarrow G$.

Step 1: Let us consider the algebra $R = W_2(k)[x^{1/p^\infty}]/(x)$. In this case, we have that

$$D := \mathrm{dR}_{(R\otimes_{W_2(k)}k)/k} \simeq D_{(x)}(k[x^{1/p^\infty}])$$

is the divided power envelope of (x) in $k[x^{1/p^\infty}]$. This algebra admits a natural grading by the monoid $\mathbb{N}[1/p]$. The values of our sheaves evaluated at R are $\mathcal{O} = k[x^{1/p^\infty}]/(x^p)$, and M is the degree $[0, 2p)$ part of $D_{(x)}(k[x^{1/p^\infty}])$, whereas G is the degree $[p, 2p)$ part. One checks easily that G is generated by $\overline{\gamma_p(x)}$ (mod degree $[0, p)$ part) as an $\mathcal{O}$-module in this case. We claim that the section necessarily sends this generator to $\gamma_p(x) \in M$. Say the section sends this generator to some element $f(x) \in M$. We look at the following two maps of $W_2(k)$-algebras $R \to R \otimes_{W_2(k)} W_2(k)[t^{1/p^\infty}]$, with $x^m \mapsto x^m \cdot t^m$ and $x^m \mapsto x^m$. The associated mod p derived de Rham cohomology is given by $D \otimes_k k[t^{1/p^\infty}]$. Since the corresponding map of values of G is

$$\overline{\gamma_p(x)} \mapsto \overline{\gamma_p(tx)} = t^p\overline{\gamma_p(x)} \text{ and } \overline{\gamma_p(x)} \mapsto \overline{\gamma_p(x)},$$

functoriality tells us that $t^p f(x) = f(tx) \in D \otimes_k k[t^{1/p^\infty}]$. This implies that $f(x)$ is a homogeneous degree p element in M which maps to $\overline{\gamma_p(x)} \in G$, therefore it must be $\gamma_p(x) \in M$.

Step 2: Next we consider the algebra $R_n = W_2(k)[x_i^{1/p^\infty}; i = 1, \ldots, n]/(\sum_{i=1}^n x_i)$. In this case, we have

$$D_n := \mathrm{dR}_{(R_n\otimes_{W_2(k)}k)/k} = D_{(\sum_{i=1}^n x_i)}(k[x_i^{1/p^\infty}]).$$

Then the values of our sheaves evaluated at R_n is given by $\mathcal{O} = k[x_i^{1/p^\infty}]/(\sum_{i=1}^n x_i^p)$, and $M = \mathcal{O} \cdot \{1, \gamma_p(\sum_{i=1}^n x_i)\}$ whereas $G = \mathcal{O} \cdot \overline{\gamma_p(\sum_{i=1}^n x_i)}$. In this case, we claim that the section necessarily sends $\overline{\gamma_p(\sum_{i=1}^n x_i)}$ to $\sum_{i=1}^n \gamma_p(x_i)$. Note that this sum makes sense as an element in D_n and in fact is in M, for instance one may repeatedly use $\gamma_p(x + y) = \sum_{i=0}^p \gamma_i(x) \cdot \gamma_{p-i}(y)$ to see this. Now to see the above claim, we first use the same argument as in the previous paragraph to see the section of $\overline{\gamma_p(\sum_{i=1}^n x_i)}$ is necessarily a homogeneous degree p element $f(x_i)$. Then we contemplate with the functoriality provided by the map $R_n \to R^{\otimes_{W_2(k)}n} = W_2(k)[x_i^{1/p^\infty}; i = 1, \ldots, n]/(x_i; i = 1, \ldots, n)$ to see that the element $g(x_i) := f(x_i) - \sum_{i=1}^n \gamma_p(x_i)$ is a homogeneous degree p element in the kernel of the induced map $D_n \to D^{\otimes_k n}$. The degree p part of the kernel is the k-span of $\{x_i^p\}_{i=1}^n$ modulo $k \cdot \sum_{i=1}^n x_i^p$. Finally, using functoriality with respect to switching variables, we see that $g(x_i)$ must be a permutation-invariant element, hence necessarily

0 unless $n = p = 2$. Therefore we know when $n \geq 3$, the associated section is determined. By functoriality the section associated with R_3 determines the section associated with R_2, this finishes the proof of our claim above.

Step 3: The universal algebra that we need to consider is $R' = W_2(k)[x^{1/p^\infty}, y^{1/p^\infty}]/(x+py)$. Note that $R'/p = R/p \otimes_k k[y^{1/p^\infty}]$, so the value of relevant sheaves are those in Step 1 tensored over k with $k[y^{1/p^\infty}]$. The generator $\overline{\gamma_p(x)} = \overline{\gamma_p(x+py)}$ of G under a functorial section goes to $\gamma_p(x) + g(x,y)$, where $g(x,y) \in k[x^{1/p^\infty}, y^{1/p^\infty}]/(x^p)$ has degree p by the same argument as in Step 1. We claim $g(x,y) = \frac{y^p}{(p-1)!}$. To see this, first observe that

$$x_1 + x_2 = (x_1^{1/p} + x_2^{1/p})^p + p \cdot F(x_1, x_2) \text{ in } W_2(k)[x_1^{1/p}, x_2^{1/p}],$$

where we view $F(x_1,x_2) \in k[x^{1/p^\infty}, y^{1/p^\infty}]$, a degree 1 polynomial. Then we see that there is a map $R' \to R_2$ sending x and y to Teichmüller lifts of $x_1 + x_2$ and $F(x_1,x_2)$. The induced map of corresponding D's sends $\gamma_p(x) + g(x,y)$ to $\gamma_p(x_1+x_2) + g(x_1+x_2, F(x_1,x_2))$. On the other hand the functoriality forces this element to be sent to $\gamma_p(x_1) + \gamma_p(x_2)$ by Step 2. Therefore we get a relation

$$\gamma_p(x_1+x_2) + g(x_1+x_2, F(x_1,x_2)) = \gamma_p(x_1) + \gamma_p(x_2).$$

Let $h(x,y) = g(x,y) - \frac{y^p}{(p-1)!} \in k[x^{1/p^\infty}, y^{1/p^\infty}]/(x^p)$ which also has degree p. Combining relations, we see that $h(x_1+x_2, F(x_1,x_2)) = 0 \in k[x_1^{1/p^\infty}, x_2^{1/p^\infty}]/(x_1^p + x_2^p)$. Applying the next lemma with $x_1 + x_2 = a$ and $x_2 = b$, we conclude that $h(x,y)$ must be 0.

Step 4: We finish the proof in this step. Given any large quasi-syntomic $W_2(k)$-algebra S, we can find an algebra S' of the form $W_2(k)[X_i^{1/p^\infty}, Y_j^{1/p^\infty}; i \in I, j \in J]/(Y_j + f_j(X_i); j \in J)$ and a surjection $S' \twoheadrightarrow S$ inducing a surjection of their values of all the relevant sheaves, see the proof of [BS19, Proposition 7.10] or [LL21, Theorem 3.13] for details. The value of G in this case is generated, as an $\mathcal{O}$ module, by $\gamma_p(Y_j + f_j(X_i))$ where $j \in J$. By functoriality, we may reduce to the case where $S = W_2(k)[\underline{X}^{1/p^\infty}, Y^{1/p^\infty}]/(Y + f(\underline{X}))$. In this case G is generated over $\mathcal{O}$ by the element $\overline{\gamma_p(Y + f(\underline{X}))}$ we want to show the section is forced on this element. Observe that any element in $W_2(k)[\underline{X}^{1/p^\infty}, Y^{1/p^\infty}]$ can be written as $[P_1] + p \cdot [P_2]$, a Teichmüller lift plus p times another Teichmüller lift. Therefore we can define a map $R' \to S$ sending X to $[P_1]$ and Y to $[P_2]$. Then we see that the section of $\overline{\gamma_p(Y + f(\underline{X}))}$ must be $\gamma_p(P_1) + \frac{P_2^p}{(p-1)!}$ by Step 3. This shows the rigidity as desired. ☐

Lemma 5.10. *Let $F(a,b) \in k[a^{1/p^\infty}, b^{1/p^\infty}]$ be the degree 1 element such that its lift $\widetilde{F}$ to $W_2(k)[a^{1/p^\infty}, b^{1/p^\infty}]$ satisfies*

$$(a-b)^p + b^p = a^p + p \cdot \widetilde{F}(a^p, b^p) \text{ in } W_2(k)[a^{1/p^\infty}, b^{1/p^\infty}].$$

Let $H(a,b) \in k[a^{1/p^\infty}, b^{1/p^\infty}]$ be a degree p element which does not contain the term a^p. Suppose $H(a, F(a,b)) \in k[a^{1/p^\infty}, b^{1/p^\infty}]$ is divisible by a^p, then $H(a,b) = 0$.

Proof. Observe that $F(a,b) = \sum_{i=1}^{p-1} c_i \cdot a^{i/p} b^{(p-i)/p}$ with all of $c_i \neq 0$. The a-degree of $F(a,b)$ is less than 1, therefore the a-degree of $H(a, F(a,b))$ must be smaller than p unless $H(a,F(a,b)) = 0$ (as $H(a,b)$ does not contain a^p term). The a^p divisibility now forces $H(a,F(a,b)) = 0$. Contemplating with the b-degree of $H(a,F(a,b))$ shows that in fact $H(a,b)$ has to be 0 to begin with. ☐

In the step 3 above, one can alternatively argue using the map $R_{p+1} \to R'$ sending x_1 to x and the rest of the p variables to y.

Appendix A. Topos theoretic cotangent complex

The theory of cotangent complex appears in many places in the literature. For example, it has been discussed in [Ill71] in the context of simplicial ring objects in a 1-topos and in [Lur04], where an ∞-categorical theory has been discussed for animated ring objects in spaces. However, in the proof of Theorem 4.24, we required a formalism of cotangent complex in the generality of animated ring objects in an ∞-topos. In this appendix, we will sketch a formalism of cotangent complex in the above generality and its very basic properties which is sufficient for the proof of Theorem 4.24. Our exposition basically uses the techniques from [Lur04] and lifts them to the generality we need.

For simplicity, we will focus on the case necessary for our application where the ∞-topos $\mathcal{X}$ arises as sheaves of spaces on some Grothendieck site $\mathcal{C}$ which will be fixed. As in Definition 2.10, one defines the ∞-category $\mathrm{ARings}(\mathcal{X}) := \mathrm{ARings}(\mathcal{X})_{\mathbb{Z}}$ which is equivalent to the ∞-category of sheaves of animated rings on $\mathcal{C}$. For a fixed animated ring B in $\mathcal{X}$, one can also consider the ∞-category of connective B-modules in $\mathcal{X}$ defined as the category of sheaves on $\mathcal{C}$ (with values in animated abelian groups) of B-modules.

For $n \geq 0$, an object $F \in \mathrm{ARings}(\mathcal{X})$ will be called n-truncated if $F(c)$ is n-truncated (i.e., $\pi_i(F(c)) = 0$ for all $i > n$) for all $c \in \mathcal{C}$. We let $\tau_{\leq n}\mathrm{ARings}(\mathcal{X}) \to \mathrm{ARings}(\mathcal{X})$ denote the inclusion of the full subcategory of n-truncated objects in $\mathrm{ARings}(\mathcal{X})$. This admits a left adjoint that sends $G \mapsto \tau_{\leq n}G$ which is obtained by n-truncating G as a presheaf first and then applying sheafification.

Construction A.1 (The cotangent complex)**.** Let $A \to B$ be a map in $\mathrm{ARings}(\mathcal{X})$. For any connective B-module M, one can form the trivial square zero extension $B \oplus M$, which is an object of $\mathrm{ARings}(\mathcal{X})_A$. There is a natural projection map $B \oplus M \to B$, which regards $B \oplus M$ as an object of $(\mathrm{ARings}(\mathcal{X})_A)_{/B}$. One can consider the functor $M \to \mathrm{Maps}_{(\mathrm{ARings}(\mathcal{X})_A)_{/B}}(B, B \oplus M)$. By the adjoint functor theorem, this functor is corepresented by a connective B-module, which we will denote as $\mathbb{L}_{B/A}$.

Remark A.2. Let $A \to B$ be a map in $\mathrm{ARings}(\mathcal{X})$. It follows that $\mathbb{L}_{B/A}$ defined as above is sheafification of the presheaf on $\mathcal{C}$ with values in animated abelian groups that sends $c \mapsto \mathbb{L}_{B(c)/A(c)}$ for $c \in \mathcal{C}$. It naturally inherits the structure of a sheaf of connective B-modules on $\mathcal{C}$.

Proposition A.3. *For a sequence of morphisms $A \to B \to C$ in $\mathrm{ARings}(\mathcal{X})$, we have a cofiber sequence*

$$\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

in the ∞-category of connective C-modules.

Proof. This follows from Construction A.1. $\square$

Remark A.4. Let $C \in \mathrm{ARings}(\mathcal{X})$. Let $U \to V \to W$ be a cofiber sequence in the ∞-category of connective C-modules. For any connective C-module M, we obtain a long exact sequence

$$\ldots \to \pi_1\mathrm{Maps}(W, M) \to \pi_1\mathrm{Maps}(V, M) \to \pi_1\mathrm{Maps}(U, M) \to \pi_0\mathrm{Maps}(W, M) \to \pi_0\mathrm{Maps}(V, M) \to \pi_0\mathrm{Maps}(U, M).$$

Definition A.5 (Square-zero extensions)**.** Let $A \in \mathrm{ARings}(\mathcal{X})$ and $B \in \mathrm{ARings}(X)_A$. Let M be a connective B-module. A square-zero extension of B by M will be classified by $\mathrm{Maps}_B(\mathbb{L}_{B/A}, M[1])$, where the maps are considered in the ∞-category of connective B-modules. By Construction A.1, square-zero extensions can be equivalently classified by $\mathrm{Maps}_{(\mathrm{ARings}(\mathcal{X})_A)_{/B}}(B, B \oplus M[1])$. Given $s : B \to B \oplus M[1]$ which gives a section to the projection, the pullback $B' := B \otimes_{B \oplus M[1]} B$ recovers the total space of the square-zero extension, where B maps to $B \oplus M[1]$ via s and the zero section. The fibre of $B' \to B$ can be identified with M with the natural structure of an A-module.

Proposition A.6. *Let $C \in \mathrm{ARings}(\mathcal{X})_A$ and let $B' \to B$ in $\mathrm{ARings}(\mathcal{X})_A$ be a square-zero extension of B by a connective B-module M. There is a natural map $\mathrm{Maps}_{\mathrm{ARings}(\mathcal{X})_A}(C, B') \to \mathrm{Maps}_{\mathrm{ARings}(\mathcal{X})_A}(C, B)$ such that the non-empty fibres are torsors under the group $\mathrm{Maps}_C(\mathbb{L}_{C/A}, M)$, where the maps are taken in the category of connective C-modules. The C-module structure on M is obtained via the map $C \to B$ over which the fiber is being taken.*

Proof. Unwrapping the definitions and using the facts that the mapping spaces are ∞-groupoids, one can reduce to checking this in the case when $B' = B \oplus M$ is the trivial square zero extension. In this case, we can fix a map $C \to B$, and then we need to show that $\mathrm{Maps}_{(\mathrm{ARings}(\mathcal{X})_A)/B}(C, B \oplus M)$ is equivalent to $\mathrm{Maps}_C(\mathbb{L}_{C/A}, M)$. For this, we note that pullback along $C \to B$ gives an equivalence $\mathrm{Maps}_{(\mathrm{ARings}(\mathcal{X})_A)/B}(C, B \oplus M) \simeq \mathrm{Maps}_{(\mathrm{ARings}(\mathcal{X})_A)/C}(C, C \oplus M)$. By definition, $\mathrm{Maps}_{(\mathrm{ARings}(\mathcal{X})_A)/C}(C, C \oplus M) \simeq \mathrm{Maps}_C(\mathbb{L}_{C/A}, M)$, which gives the conclusion. $\square$

Remark A.7. For any object $A \in \mathrm{ARings}(\mathcal{X})_A$, one can use the truncation functors to build a sequence of square-zero extensions $\ldots \tau_{\leq n+1}A \to \tau_{\leq n}A \to \tau_{\leq n-1}A \ldots \to \tau_{\leq 0}A = \pi_0(A)$. This can be seen by first showing a similar statement at the presheaf level and then sheafifying; at the presheaf level, the statement follows from the analogous statement for animated rings, *c.f.* [Lur04, Proposition 3.3.6]. In particular, if $A \in \mathrm{ARings}(\mathcal{X})_A$ is 1-truncated, then A is a square zero extension of $\pi_0(A)$ by $\pi_1(A)[1]$, where the latter is viewed as a connective $\pi_0(A)$-module in $\mathcal{X}$.

Appendix B. A product formula for $(1+W[p])^\times$ in char. $p>0$

The group scheme $(1+W[p])^\times$ was defined in Definition 4.14. We will work over a fixed base ring of char. $p>0$; then this group scheme is isomorphic to $W^\times[F]$, see Remark 4.28 (2). The following proposition was stated in [Dri21b, Lemma 3.3.4] and a more general proposition over $\mathbb{Z}_p$ has been proven in [Dri21a, Proposition B.5.6] by using the logarithm constructed in loc. cit.; it will also appear in forthcoming work of Bhatt–Lurie [BL21]. Let us give a more direct argument in char. p that does not use the logarithm and is closer to deformation theory in spirit.

Proposition B.1. *There is a natural isomorphism $W^\times[F] \simeq W[F] \times \mu_p$ over any base ring of char. p.*

Proof. Note that given any non-unital ring $(I, +, \cdot)$, one can define a monoid associated to it, which will be denoted as I'. At the level of underlying sets, $I' := I$, but the composition $x * y$ is defined to be $x + y + x \cdot y$. Using the above construction along with the Yoneda lemma produces a functor from the category of non-unital ring schemes (e.g., ideals in unital ring schemes) to the category of monoid schemes. Note that we have a short exact sequence

$$0 \to W[F] \to W[F] \xrightarrow{f} \alpha_p \to 0$$

of group schemes. Moreover, the map $f : W[F] \to \alpha_p$ is a map of non-unital ring schemes when $W[F]$ and $\alpha_p \simeq \mathbb{G}_a[F]$ are both equipped with their natural non-unital ring scheme structures. Applying the functor we constructed before, we obtain a map $f' : W^\times[F] \to \mu_p$. It is clear that f' is surjective. The map f' can be identified with projection to 0-th Witt coordinate: given any test algebra S and an element $x \in W[F](S)$, then f' sends $1+x$ to $1+x_0$ where x_0 is the 0-th Witt coordinate. In particular we see the map $\mu_p \to W^\times[F]$ given by Teichmüller lift is a section to f'. It remains to identify $\mathrm{Ker} f'$ with $W[F]$ as a group scheme. This follows from the lemma below.

Lemma B.2. *For the map $f : W[F] \to \alpha_p$, the ideal $\mathrm{Ker} f$ is a square-zero ideal.*

Proof. We note that the multiplication in $W[F]$ is inherited from the ring scheme W. Let S be a test algebra of char. p and let $m, n \in (\mathrm{Ker} f)(S)$. Then $m = V(m')$ and $n = V(n')$ for some $m', n' \in W(S)$. Here V denotes the Verschiebung operator. We have $m \cdot n = V(m') \cdot n = V(m' \cdot F(n)) = 0$, since $F(n) = 0$. ☐

The proposition now follows since we obtain a split exact sequence of group schemes

$$0 \to W[F] \to W^\times[F] \xrightarrow{f'} \mu_p \to 0.$$

☐

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