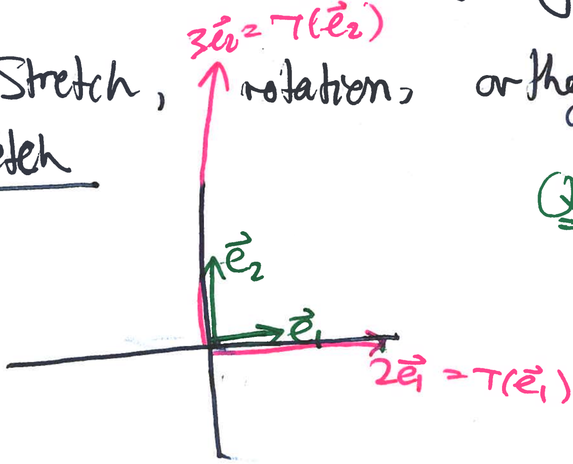
9/9/22

2.2: Geometry of linear transformations.

- Stretch, rotation, orthogonal projection, reflection across a line.

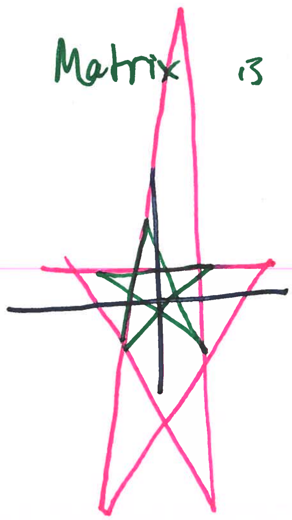
§ Stretch



$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Q: Stretch by a factor of 2 in the x-direction and a factor of 3 in y-dir. What's the matrix of this transform?

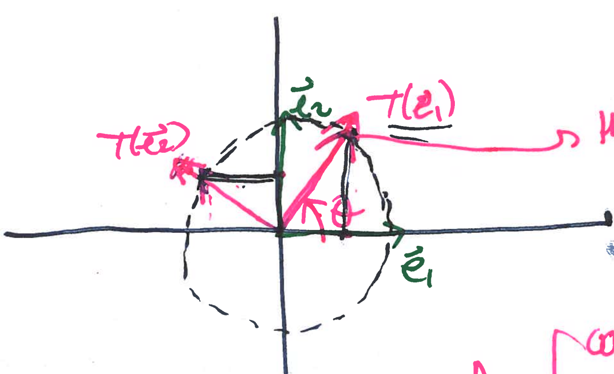
A: Matrix is $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



§ Rotation

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(\vec{x}) = \vec{x}$ rotated by an angle of θ counterclockwise.

Q: What is the matrix of T?



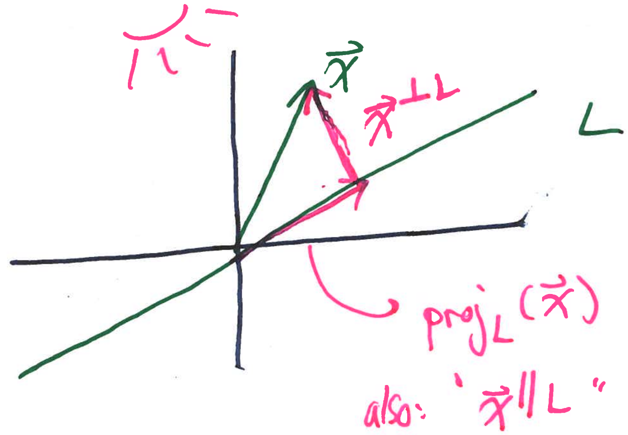
this vector is $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

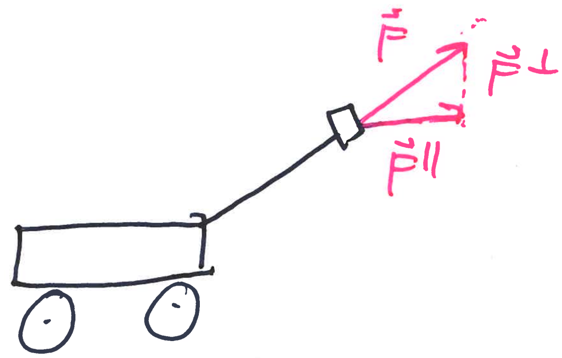
$T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

A: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

§ Orthogonal projection :

$$\vec{x} = \vec{x}^{\parallel L} + \vec{x}^{\perp L}$$





What is the matrix corresponding to proj_L ?

Detour Dot products

Def: if $\vec{v}$ and $\vec{w}$ are two n-dimensional vectors, define the "dot product" of $\vec{v}$ and $\vec{w}$:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n \in \mathbb{R}$$

eg. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 8 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot 6 + 2 \cdot 8 + 3 \cdot 1 + 4 \cdot 0 = 25$

$$\vec{v} \cdot \vec{w} = \underbrace{[v_1 \ v_2 \ \dots \ v_n]}_{\vec{v}^T} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}}_{\vec{w}} = \vec{v}^T \vec{w}$$

"transpose of $\vec{v}$"

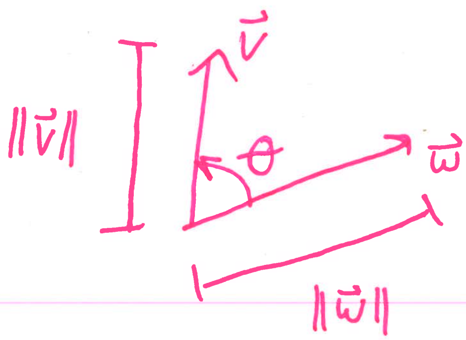
Properties of dot product :

$$\vec{v}\cdot(\vec{w}_1+\vec{w}_1) = \vec{v}\cdot\vec{w}_1 + v\cdot\vec{w}_2$$

$$\vec{v}\cdot(c\,\vec{w}_1) = c\,\vec{v}\cdot\vec{w}_1 \qquad (c\in\mathbb{R})$$

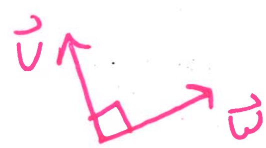
$$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$$

Formula : $\vec{v}\cdot\vec{w} = \|\vec{v}\|\cdot\|\vec{w}\|\cdot\cos\theta$



Thus, $\vec{v}\cdot\vec{w} = 0 \iff \vec{v}=0$ or $\vec{w}=0$ or

$\theta = 90^\circ$ (or $\theta = 270^\circ$)

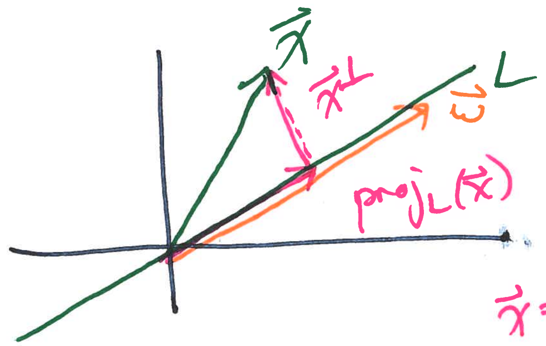


"$\vec{v}$ and $\vec{w}$ are orthogonal" if

$\vec{v}\cdot\vec{w} = 0$

Also: $\vec{v}\cdot\vec{v} = \|\vec{v}\|\cdot\|\vec{v}\|\cdot\cos 0 = \|\vec{v}\|^2$

Back to orthogonal projection:



Formula for $\text{proj}_L(\vec{x})$?
Matrix corresponding to this?

$\vec{x} = \text{proj}_L(\vec{x}) + \vec{x}^{\perp}$

Choose some $\vec{w}$ parallel to L

Then $\text{proj}_L(\vec{x}) = c \cdot \vec{w}$ for some $c \in \mathbb{R}$. $c = ?$

$\vec{x} - \text{proj}_L(\vec{x}) = \vec{x} - c \cdot \vec{w} =$ sthg perpendicular to $\vec{w}$

$\Longrightarrow (\vec{x} - c\vec{w}) \cdot \vec{w} = 0$

$\Longrightarrow \vec{x} \cdot \vec{w} - c\,\vec{w} \cdot \vec{w} = 0 \quad \Longrightarrow \quad \dfrac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} = c$

$\|\vec{w}\|^2 \neq 0$

$\rightarrow \text{proj}_L(\vec{x}) = \dfrac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$ (this is independent of our choice of $\vec{w}$!)

Matrix of proj_L?

$\text{proj}_L(\vec{e}_1) = \dfrac{\vec{e}_1 \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \dfrac{w_1}{\vec{w} \cdot \vec{w}} \vec{w}$

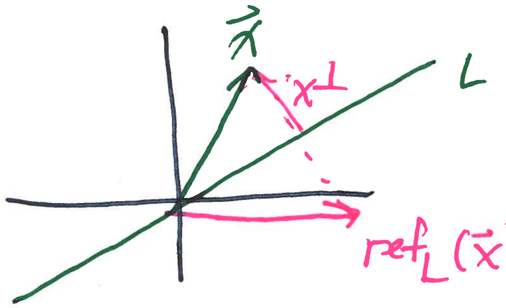
$\text{proj}_L(\vec{e}_2) = \dfrac{\vec{e}_2 \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \cdot \vec{w} = \dfrac{w_2}{\vec{w} \cdot \vec{w}} \vec{w}$

$\vec{e}_1 \cdot \vec{w} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1$

$\vec{e}_2 \cdot \vec{w} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_2$

Matrix: $\dfrac{1}{\vec{w} \cdot \vec{w}} \begin{bmatrix} w_1 \vec{w} & w_2 \vec{w} \end{bmatrix} = \boxed{\dfrac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}}$

Reflection across a line :



$$\mathrm{ref}_L(\vec{x}) = \vec{x} - 2x^\perp$$

$$= 2\,\mathrm{proj}_L x - \vec{x}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix: $2 \cdot (\text{matrix for proj}) - Id$