9/7/22 **Agenda**: §2.1, Linear tranformations

Recall A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear (is a "linear ~~tranf~~ transformation") if

- $f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w})$
- $f(c\vec{v}) = c f(\vec{v})$

for all $\vec{v}, \vec{w} \in \mathbb{R}^n$, $c \in \mathbb{R}$ $\quad (\forall \vec{v}, \vec{w} \in \mathbb{R}^n, \forall c \in \mathbb{R})$

$\forall$

eg. $f: \mathbb{R}^2 \to \mathbb{R}^3$ given by $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

Check: • $f\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = f\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} a+c \\ b+d \\ 0 \end{bmatrix}$

$$f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + f\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} \quad // \quad \checkmark$$

• $f\left(c\begin{bmatrix} a \\ b \end{bmatrix}\right) = f\left(\begin{bmatrix} ca \\ cb \end{bmatrix}\right) = \begin{bmatrix} ca \\ cb \\ 0 \end{bmatrix}$

$$c f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = c \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ 0 \end{bmatrix} \quad \checkmark$$

nonexample: $g: \mathbb{R}^2 \to \mathbb{R}^3$, $g\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$

This is not linear: $g\left(5 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = g\left(\begin{bmatrix} 10 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$

$$5 \cdot g\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 5 \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix} \neq g\left(5 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

e.g. multiplication by a matrix.

Detour: properties of matrix mult. (video 2a)
(p. 81)

These include

$$\begin{cases} A\cdot(B+C) = AB + AC \\ A\cdot(kC) = k\cdot AC \end{cases} \quad \text{for all matrices } A, B, C$$

$k \in \mathbb{R}$

NOT "$AB = BA$"

~~⇒ the function~~

If M is any $m\times n$ matrix, we get a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $f(\vec{v}) = M\cdot\vec{v}$

$$m\left\{\underbrace{\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}}_{n}\right. \left.\begin{bmatrix} \\ \\ \\ \end{bmatrix}\right\}n = \left.\begin{bmatrix} \\ \\ \end{bmatrix}\right\}m$$

This function is linear: $M(\vec{v}+\vec{w}) = M\vec{v} + M\vec{w}$,
$M(c\cdot\vec{v}) = cM\vec{v}$.

Thm If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is any linear function,
$\boxed{\text{there exists}}$ a $m\times n$ matrix M $\boxed{\text{such that}}$ $f(\vec{v}) = M\vec{v}$

$\exists$ s.t.

e.g. let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear function.

Suppose $f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\\1\end{bmatrix}$, $f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\1\\4\end{bmatrix}$.

let $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$. Q $f\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = ?$

A $\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$

$$\Rightarrow f\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = f\left(a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}\right) \overset{f \text{ is linear}}{=} a f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + b f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= a\begin{bmatrix}2\\3\\1\end{bmatrix} + b\begin{bmatrix}-3\\1\\4\end{bmatrix} = \underbrace{\begin{bmatrix}2 & -3\\3 & 1\\1 & 4\end{bmatrix}}_{\text{the matrix corresp. to } f.}\begin{bmatrix}a\\b\end{bmatrix}$$

In general, if $f: \mathbb{R}^2 \to \mathbb{R}^n$ is a lin. function, then the matrix corresp. to f is ~~$[f([\,])\ f$~~

$$\left[f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \quad f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right],$$

If $f: \mathbb{R}^3 \to \mathbb{R}^n$, the corresp matrix is

$$\left[f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \quad f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \quad f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]$$

etc. for any $f: \mathbb{R}^m \to \mathbb{R}^n$ linear.

e.g. We saw that $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ is linear. What's the corresp. matrix?

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Check: for any $\begin{bmatrix} a \\ b \end{bmatrix}$, $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{matrix corresp. to } f} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$

Q1 let $f: \mathbb{R}^2 \to \mathbb{R}^1$ be linear. with $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 3$, $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2$. Find the matrix corresp to f.

Q2: let $g: \mathbb{R}^3 \to \mathbb{R}^3$ be the function $g(\vec{v}) = 2\vec{v}$. Find the matrix corresp to g.

Q3: let $h: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin function with $h\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $h\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Find matrix corresp. to h.

A1 $\begin{bmatrix} 3 & 2 \end{bmatrix}$

A2 $g\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\quad g\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ $\quad g\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

matrix: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad$ Check: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix} = g\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$

A3 : $h\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$ $\quad h\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = ?$

"elementary unit vectors" 'standard unit vectors'

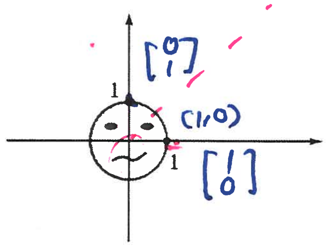
$$h\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = h\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}\right) = h\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - h\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}4\\1\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix}$$

$$= \begin{bmatrix}2\\-2\end{bmatrix}$$

h is linear

matrix: $\begin{bmatrix}2 & 2\\3 & -2\end{bmatrix}$

Consider the circular face in the accompanying figure. For each of the matrices A in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ on this face.



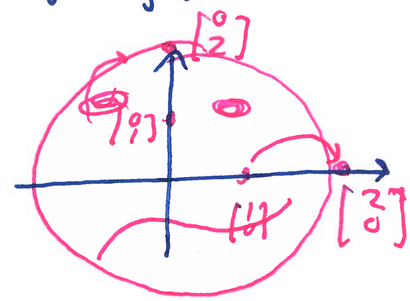
24. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **25.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **26.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **28.** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **29.** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

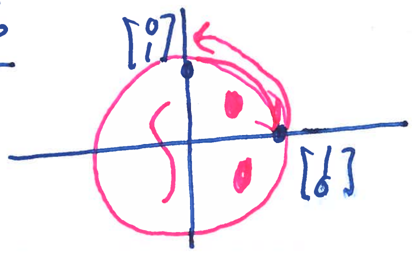
30. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

#25 $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$



#26



$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ goes to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ goes to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

NOT rotation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$