Math 214 5.3 worksheet

1. If A is any matrix, recall that A^T denotes the *transpose* of A. This is defined as the matrix where the columns of A^T are the rows of A, and vice-versa.

   (a) Write down the transposes of the following matrices:

$$A = \begin{bmatrix} \pi & e \\ \sqrt{2} & 1/3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} \pi & \sqrt{2} \\ e & 1/3 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad C^T = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \quad D^T = \begin{bmatrix} 1 & 3 & 3 & 7 \end{bmatrix}$$

   (b) What is $(A^T)^T$ (the transpose of A-transpose)? In general, if E is any matrix, what is $(E^T)^T$?

$(A^T)^T = \begin{bmatrix} \pi & e \\ \sqrt{2} & 1/3 \end{bmatrix} = A.$ In general: $(E^T)^T = E$!

   (c) Suppose E is a matrix with m rows and n columns. How many rows and columns does E^T have?

E^T has n rows and m columns

(# rows and cols get swapped)

2. Let $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$. Compute the dot product $\vec{v} \cdot \vec{w}$ and the matrix product $\vec{v}^T\vec{w}$. What do you notice?

$\vec{v} \cdot \vec{w} = 1 \cdot 4 + 3 \cdot 0 + -2 \cdot 3 = -2$

$\vec{v}^T \vec{w} = \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \end{bmatrix}$

They're the same!

3. Let $A = \begin{bmatrix} 0 & 4 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 2 & 0 \end{bmatrix}$

(a) Find $A^T B$.

$$\begin{bmatrix} 0 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -7 & 7 \end{bmatrix}$$

(b) Let $\vec{v}_1, \vec{v}_2$ be the columns of A and let $\vec{w}_1, \vec{w}_2$ be the columns of B. Compute all the dot products $\vec{v}_i \cdot \vec{w}_j$. What do you notice?

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = 2 \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = -7 \qquad \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 7$$

(c) Let $A = [\vec{a}_1 \ldots \vec{a}_r]$ and $B = [\vec{b}_1, \ldots, \vec{b}_s]$, where $\vec{a}_i$ and $\vec{b}_j$ are all vectors in $\mathbb{R}^n$. Is the matrix product $A^T B$ well-defined? (That is, do these two matrices have the correct numbers of rows/columns so that this product is defined?) If so, can you guess how the entries of $A^T B$ relate to the dot products of the various $\vec{a}_i$ and $\vec{b}_j$ vectors?

Yes! A^T has n cols, B has n rows

The i,j entry of $A^T B$ is $\vec{a}_i \cdot \vec{b}_j$

(key pt!)

$$A^T B = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_s \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \cdots & a_2 \cdot b_s \\ \vdots & \vdots & & \vdots \\ a_r \cdot b_1 & a_r \cdot b_2 & \cdots & a_r \cdot b_s \end{bmatrix}$$

4. (a) Let $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Compute $A^T A$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Suppose B is an $m \times n$-matrix with orthonormal columns. What can you say about $B^T B$? (Hint: the matrix A from part (a) has orthonormal columns!)

$$n\left\{\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 1 \end{bmatrix}}_{n}\right.$$

$B^T B = I_n$, the $n \times n$ identity matrix!

(c) Suppose $\vec{u}_1, \ldots, \vec{u}_n$ is some orthonormal basis of $\mathbb{R}^n$. Let $Q = [\vec{u}_1 \ \cdots \ \vec{u}_n]$. In other words the columns of Q are the vectors $\vec{u}_1, \ldots, \vec{u}_n$. Is Q invertible? If so, is there a quick way to find Q^{-1}?

$Q^T Q = I_n$ (and Q is square)

So $Q^{-1} = Q^T$!

(key point!)

5. Let

$$\vec{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix},$$

and let $V \subseteq \mathbb{R}^4$ be the subspace spanned by $\vec{u}_1$ and $\vec{u}_2$.

(a) Let $\vec{w} = \cancel{[2 \;\; 3 \;\; 4]}^T$. Compute $\text{proj}_V(\vec{w})$,

$[2 \;\; 3 \;\; 4 \;\; 0]^T$

(b) Let $Q = [\vec{u}_1 \;\; \vec{u}_2]$ be the matrix whose columns are $\vec{u}_1$ and $\vec{u}_2$. Compute $QQ^T\vec{w}$. What do you notice?