11/9/22

<u>Recall</u> If A is a square matrix, an eigenvector of A is a vector $0 \neq \vec{v}$ st $A\vec{v} = \lambda\vec{v}$ for some constant λ.

λ — eigenvalue of $\vec{v}$

"special" or "characteristic" vectors for A

<u>Warm-up</u>:

a) let $T: \mathbb{R}^3 \to \mathbb{R}^3$ rotation by $20°$ around $\begin{bmatrix}1\\2\\3\end{bmatrix}$.
Find all the eigenvectors of T. What are their eigenvalues?

b) Find a transformation $F: \mathbb{R}^2 \to \mathbb{R}^2$ st. F has no eigenvectors

a)

anything in span $\begin{bmatrix}1\\2\\3\end{bmatrix}$ gets sent to itself by T.

Any nonzero $\vec{w} \in \text{span}\begin{bmatrix}1\\2\\3\end{bmatrix}$ is an eigenvector with eigenvalue 1 $\quad (T(\vec{w}) = 1 \cdot \vec{w})$

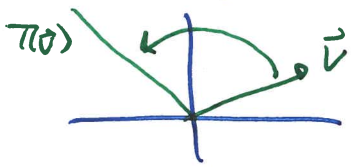
No other eigenvectors.

b) • Scale by 2 in every dir: $T(\vec{v}) = \underline{2}\vec{v} \neq \vec{v}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{v}$ is an eigenvect, but with eigenvalue 2.

• Rotate by $\theta \neq \pi, 2\pi$,



every vector ends on a different line than the one it started on.

$\Rightarrow T(\vec{v}) \neq \lambda\vec{v}$ for all nonzero $\vec{v} \in \mathbb{R}^2$

Recall: If A is an $n\times n$ matrix, then

λ is an eigenvalue of A $\iff$ $A-\lambda I_n$ has a nonzero kernel

$\iff \det(A-\lambda I_n) = 0$

exs prove the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no (real) eigenvectors.

90° ccw rotation

Ans: if $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has an eigenvector, then it has some eigenvalue λ. This λ would be a solution to $\det(A-\lambda I_2)=0$

$$\det(A-\lambda I_2) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \det = (-\lambda)(-\lambda) - (-1)\cdot 1 = \lambda^2 + 1 = 0$$

no real solutions!

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exs find all the eigenvalues of the matrix

$\boxed{\begin{array}{l}\det(A-\lambda I_n)=0 \\ \text{Solve for } \lambda\end{array}}$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 3 \end{bmatrix}$$

if you do row ops here, you risk changing eigenvals.

(eg $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

no eigenvals — does have eigenvals

$$\det(A-\lambda I_5) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 1 & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -2 & 3-\lambda \end{bmatrix}$$

← now you're free to do row ops which simplify the det

$$\parallel$$

$$(2-\lambda)\det\begin{pmatrix} 2-\lambda & & & \\ & 3-\lambda & & \\ & & -\lambda & 1 \\ & & -2 & 3-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda)\det\begin{pmatrix} 3-\lambda & & \\ & -\lambda & 1 \\ & -2 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I_5) = (2-\lambda)(2-\lambda)(3-\lambda)\det\begin{pmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)(2-\lambda)(3-\lambda)\left(-\lambda(3-\lambda)+2\right) = 0$$

$$= (2-\lambda)(2-\lambda)(3-\lambda)(2-\lambda)(1-\lambda) = 0$$

answer: $\lambda = 1, 2, 3$

"characteristic polynomial" of A

compare: "geometric multiplicity"

We say the eigenvalue 2 has "algebraic multiplicity 3" because it is a triple root of the char. poly.

$$A = \begin{bmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 0 & 1 \\ & & & -2 & 3 \end{bmatrix}$$

trace(A) = sum of diagonal entries

$= 2+2+3+0+3 = 10$

$\det(A) = \det(A - 0 \cdot I_5) = 2 \cdot 2 \cdot 3 \cdot 2 \cdot 1 = 24$

expand the char poly of A:

$$(2-\lambda)(2-\lambda)(3-\lambda)(2-\lambda)(1-\lambda) = -\lambda^5 + 10\lambda^4 - 39\lambda^3 + 74\lambda^2 - 68\lambda + 24$$

trace of A

det A

Finding eigenvectors:

Find the eigenvectors of $\begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix}$:

- First step is to find the eigenvalues:

$$\det \begin{bmatrix} 1-\lambda & -2 \\ -4 & 3-\lambda \end{bmatrix} = 0 \rightsquigarrow \lambda = -1, 5$$

- Second step: Recall: if $\vec{v}$ is an eigenvector of A w/ eigenvalue λ then $A\vec{v} = \lambda\vec{v}$

$$\rightsquigarrow (A - \lambda I_n)\vec{v} = 0$$

$$\rightsquigarrow \underbrace{\ker(A - \lambda I_n)}\ni \vec{v}$$

this kernel is the set of eigenvects with this eigenvalue.

This is called the "eigenspace" of λ

Notation: E_λ