11/28/22 §8.1 **Recall.** a square matrix A is called **symmetric** if $A = A^T$. Equivalently, given $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$,

A symmetric $\Leftrightarrow$ $a_{ij} = a_{ji}$ for all i, j

eg. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

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last time: if A has an orthonormal eigenbasis, then A is symmetric.

Pf let $\vec{v}_1, \ldots, \vec{v}_n$ be the ON eigenbasis. Then

$A = S \overset{\text{diagonal}}{D} S^{-1}$

$A^T = (SDS^{-1})^T = (S^{-1})^T \overset{=D}{D^T} S^T$

$= (S^T)^T D S^{-1}$

$= SDS^{-1} = A$

$S = [\vec{v}_1, \ldots, \vec{v}_n]$ is an **orthogonal** matrix.

$\Rightarrow S^T = S^{-1}$

Big thm (Spectral thm): The converse holds. I.e. if A is a symmetric matrix (with real entries), then A has an orthonormal eigenbasis.

e.g. let $S = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

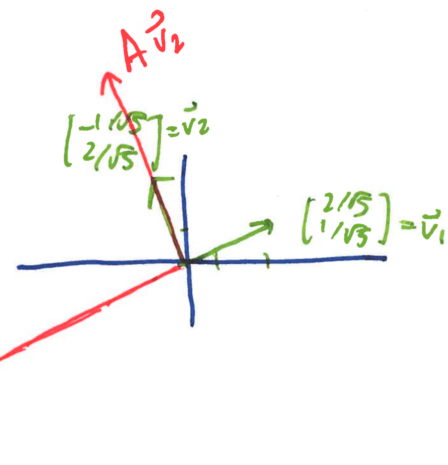
Here, S is an orthogonal matrix.

Then $A = SDS^{-1}$ must be a symmetric matrix, b.c. this the matrix with eigenvectors $\underbrace{\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}_{\text{ON set.}}$

and eigen values $-3, 2$.

Indeed: $A = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$.

Geometry of A:



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eg (other direction).

Let $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$. Then the spectral thm tells us that A has an orthonormal eigenbasis.

Computing the eigenvectors gives us: $\lambda_1 = 6$ w/ $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

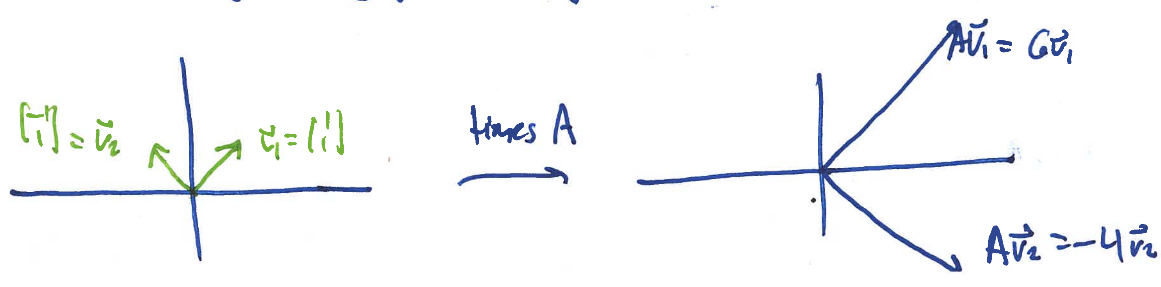
$\lambda_2 = -4$ w/ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To get ON eigenbasis, just make these vectors length 1.

$$\frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\rightsquigarrow A = \overbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}^{S} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \left(\tfrac{1}{\sqrt{2}} S\right) \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \left(\tfrac{1}{\sqrt{2}} S\right)^{-1} = \left(\tfrac{1}{\sqrt{2}} S\right) \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \left(\tfrac{1}{\sqrt{2}} S\right)^{T}$$



Note every symmetric matrix has an ON eigenbasis. But not every eigenbasis must be an orthonormal set.

e.g. $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ every (nonzero) vector in $\mathbb{R}^2$ is an eigenvector for A.

$\rightsquigarrow$ every ~~eigen~~ basis of $\mathbb{R}^2$ is an eigenbasis of A.

Sometimes you need to use Gram-Schmidt to make an eigenbasis you found into an ONB.

exc Let $V \subseteq \mathbb{R}^3$ be the plane spanned by $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

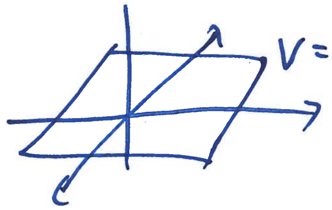
Find an orthonormal eigenbasis for proj_V, ref_V.

Ans. Algebraically: find matrix for proj_V: QQ^T, where columns of Q are an ONB for V.

Gram-Schmidt: $u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}$. Find eigenvectors of QQ^T...

Geometrically:



$V = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

eigenvectors of proj V. w/ eigenvalue 1.

another eigenvector: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (perpendicular to V), eigenvalue 0

Here we see two eigenbases for proj_v: $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

OR

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

ON eigenbasis.

Alternatively: Use Gram Schmidt on $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$.

$\rightsquigarrow$ ON eigenbasis $\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

Matrix for proj_v: $S\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}S^T$ $\left(S = \begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}&0\\-1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{bmatrix} \right)$

Matrix for ref_v: $S\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}S^T$