11/22/22

Let A be an $n \times n$ matrix

$$p_A(x) = (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x) = \det(A - xI_n)$$

possibly complex.

Some repeats, maybe.

$\rightsquigarrow$ $\det(A) = p_A(0) = \lambda_1 \cdot \ldots \cdot \lambda_n$ = product of all the eigenvalues of A, counted w/ alg. mult.

Similarly, $\text{trace}(A) = (-1)^{n-1} \cdot (\text{coefficient of } x^{n-1} \text{ in } p_A(x))$

$$= (-1)^{n-1}\left((-1)^{n-1}(\lambda_1 + \cdots + \lambda_n)\right) = \lambda_1 + \cdots + \lambda_n$$

= sum of eigenvalues of A, counting alg. mult.

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§8.1 Two nice types of bases to work with:

- Orthonormal bases (Ch. 5)
- Eigenbases

Ch. 8: combine the two ideas!

Q Let A be an $n \times n$ matrix. Suppose A has an eigenbasis, $\vec{v}_1, \ldots, \vec{v}_n$, w/ eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose also that $\vec{v}_1, \ldots, \vec{v}_n$ are an orthonormal set. What can we say about A?

Let $S = [\vec{v}_1, \ldots, \vec{v}_n]$.

Eigenbasis: $A = S D S^{-1}$

$\vec{v}_i$ are orthonormal: S is an orthogonal matrix $\rightsquigarrow S^{-1} = S^T$

So $A = SDS^T$

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$D^T = D$

Now look at $A^T = (SDS^T)^T = \underbrace{(S^T)^T}_{S} D^T S^T$

$(MN)^T = N^T M^T$

$= SDS^T = A$

In summary, if A has an orthonormal eigenbasis, then $A^T = A$, i.e. A is "symmetric"

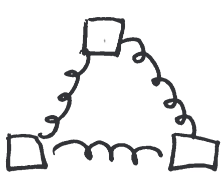
eg. $S = \begin{bmatrix} -2/\sqrt{30} & 2/\sqrt{6} & 1/\sqrt{5} \\ 1/\sqrt{30} & -1/\sqrt{6} & 2/\sqrt{5} \\ 5/\sqrt{30} & 1/\sqrt{6} & 0 \end{bmatrix}$ (orthogonal matrix)

calculator

Consider $\underbrace{S \begin{bmatrix} -7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix} S^{-1}}_{\text{the matrix where cols of } S \text{ are eigenvects, and } -7,5,-2 \text{ are eigenvalues}} = \underbrace{\begin{bmatrix} 2 & -2 & 4 \\ -2 & -1 & -2 \\ 4 & -2 & -5 \end{bmatrix}}_{\text{symmetric matrix! } A = A^T}$

Symmetric matrices come up a lot in practice:

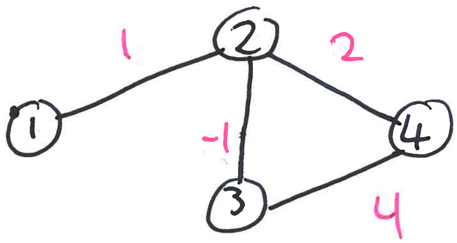
eg.



Force of box 2 on box 2 = Force of box 2 on box 1.

If (F_{ij}) matrix, F_{ij} = force of box i on box j, then (F_{ij}) is symmetric.

eg. Adjacency matrix of a network.



$\rightsquigarrow$ adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & -1 & 0 & 4 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

eg. Covariance matrix (stats)

eg Hessian ~~math~~ matrix (Math 215)

OK, so ON eigenbasis $\Longrightarrow$ symmetric.

$\boxed{\text{Big thm}}$ (Spectral thm): If A is an $n \times n$ symmetric matrix with real entries, then A has an orthonormal eigenbasis.

Pf Part 1: Show A has an eigenbasis (long)

Part 2: the eigenvectors of A are orthogonal.

key pt: if $\vec{v}, \vec{w}$ are two eigenvects of A with eigenvalues α and β, $\alpha \neq \beta$, then $\vec{v}$ and $\vec{w}$ are orthogonal

Pf of key pt: Want to show $\underbrace{\vec{v}^T \vec{w}}_{= \vec{v} \cdot \vec{w}} = 0$

The trick: $\vec{v}^T \underbrace{(A\vec{w})}_{\beta\vec{w}} = (\vec{v}^T A)\vec{w} \underset{\substack{\uparrow \\ A \text{ symmetric} \Rightarrow A = A^T}}{=} \vec{v}^T A^T \vec{w} = \underbrace{(A\vec{v})}_{\alpha\vec{v}}{}^T \vec{w}$

$\Rightarrow \vec{v}^T(\beta\vec{w}) = (\alpha\vec{v})^T\vec{w} \Rightarrow \beta\, \vec{v}^T\vec{w} = \alpha \cdot \vec{v}^T\vec{w}$

$\Rightarrow \underbrace{(\beta-\alpha)}_{\neq 0} \vec{v}^T\vec{w} = 0 \quad \Rightarrow \vec{v}^T\vec{w} = 0$, as desired. ■

(What if $\alpha = \beta$? Use Graham schmidt to replace $\vec{v}, \vec{w}$ with an orthonormal set $\vec{v}', \vec{w}'$. $\vec{v}'$ and $\vec{w}'$ will still be eigenvect if $\vec{v}$ and $\vec{w}$ had same eigenval)

Other ways to think about the spectral thm:

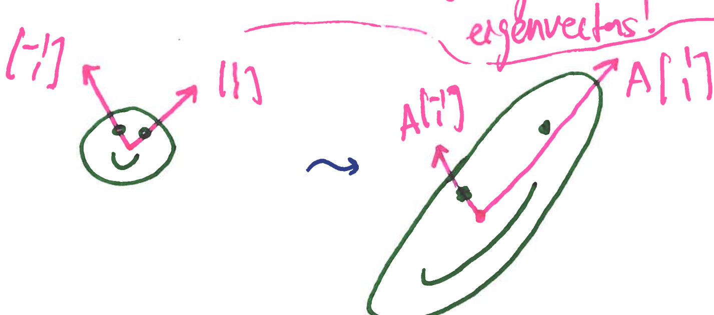
A is symmetric $\iff$ A acts by dilations along orthogonal axes.

eg. $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$

diagonalize

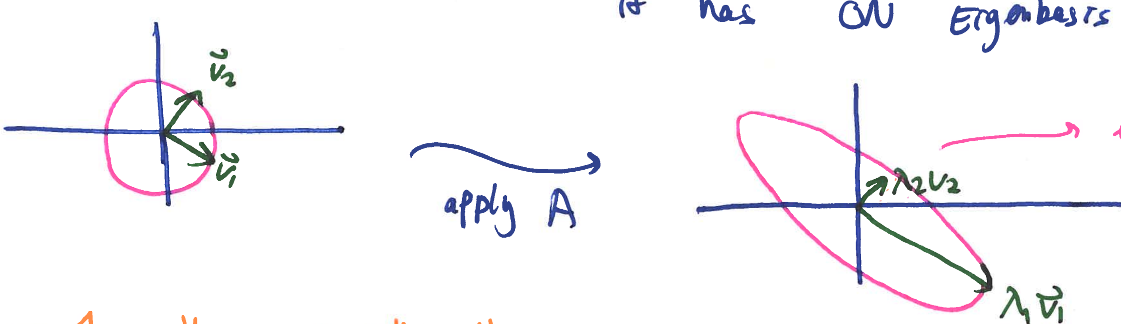
orthogonal eigenvectors!

stretch by 9 in the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ dir,



Another way to think about it: how A transforms lengths.

Start with unit circle: If A is symmetric, we know it has ON Eigenbasis $\vec{v}_1, \dots, \vec{v}_n$



ellipse, eigenvects of A are axes. eigenvalues give lengths of axes.

$\|\vec{w}\| = 1$

if $\|\vec{w}\| = 1$, then $\lambda_2 \leq \|A\vec{w}\| \leq \lambda_1$

In general, if A has eigenvals $\lambda_1, \ldots, \lambda_n$, and
$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n| > 0$

Then if $\|\vec{w}\| = 1$, we get $|\lambda_n| \leq \|A\vec{w}\| \leq |\lambda_1|$.