11/18/22

Complex #s: $a+bi$, eg $3+5i$, $\sqrt{2}+\pi i$, $5/3$ $(= 5/3+0i)$

Complex conjugates: $\overline{a+bi} \overset{def}{=} a-bi$

Fact: if $z, w \in \mathbb{C}$, then $\overline{zw} = \bar{z}\,\bar{w}$ and $\overline{z+w} = \bar{z}+\bar{w}$

eg. $\overline{(1+2i)(3-i)} = \overline{3-i+6i-2i^2} = \overline{3+5i-2(-1)}$

$= \overline{5+5i} = 5-5i$

Check: $\overline{(1+2i)}\;\overline{(3-i)} = (1-2i)(3+i) \overset{\text{check}}{=} 5-5i$

It follows: roots of polynomials w/ real coeffs come in complex conjugate pairs.

real # coefficients

eg. Consider $f(x) = x^2 - 2x + 5$. Suppose you're told that $f(1-2i) = 0$. Then it automatically follows that $f(1+2i) = 0$

conjugate of $1-2i$

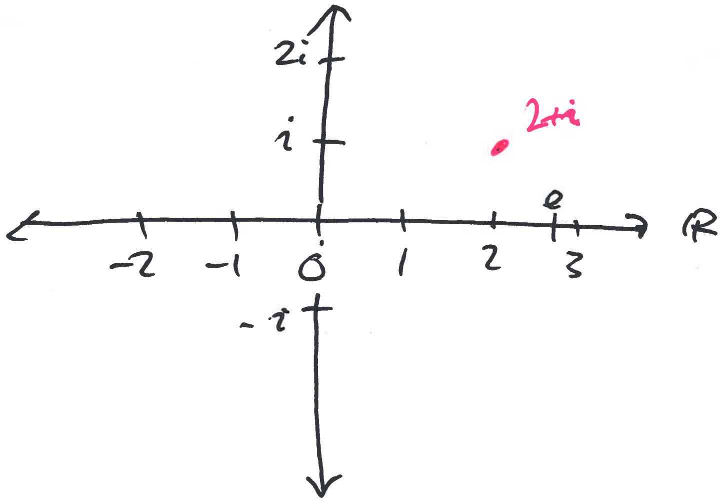
Why? Given $\overline{(1-2i)^2 - 2(1-2i) + 5} = \overline{0}$

Take conjugates $\left(\overline{1-2i}\right)^2 - \overline{2}\,\left(\overline{1-2i}\right) + \overline{5} = \overline{0}$

$\underbrace{(1+2i)^2 - 2(1+2i) + 5}_{f(1+2i)} = 0$

In summary: $\underbrace{\overline{f(1-2i)}}_{0} = f(\overline{1-2i}) = f(1+2i)$

Complex plane:



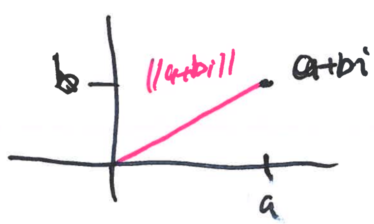
$2+i$?

In general, $a+bi$ is drawn in the (a,b) spot

Modulus of a complex #: analog of the absolute value of a real #.

= distance of $a+bi$ from 0 in the complex plane.

Formula: $\|a+bi\| = \sqrt{a^2+b^2} \overset{\text{exc}}{=} \sqrt{(a+bi)\overline{(a+bi)}}$



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example from last time:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}$$

eigenvals: $1, \ 3+4i, \ 3-4i$

eigenvectors:

for $\lambda = 1$: $\ker(A - 1 \cdot I_3) = \ker\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 4 & 2 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = 3+4i$: $\ker(A - (3+4i)I_3) = \ker\begin{pmatrix} -2-4i & 0 & 0 \\ 0 & -4i & -4 \\ 0 & 4 & -4i \end{pmatrix}$

$$\underset{R3-iR2}{=} \begin{pmatrix} -2-4i & 0 & 0 \\ 0 & -4i & -4 \\ 0 & \underbrace{4-i(-4i)}_{4+4i^2} & \underbrace{-4i-i(4)}_{-4i+4i} \end{pmatrix} = \begin{pmatrix} -2-4i & 0 & 0 \\ 0 & -4i & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\ker \begin{pmatrix} -2-4i & 0 & 0 \\ 0 & -4i & -4 \\ 0 & 0 & 0 \end{pmatrix} = ?$$

$(-2-4i)x_1 = 0 \rightsquigarrow x_1 = 0$

$-4i\, x_2 - 4x_3 = 0$

$-4i\, x_2 = 4x_3$

$-i\, x_2 = x_3$

$$\rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} \to (\lambda = 3+4i)$$

another option: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -i \\ -1 \end{bmatrix} = -i \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$

things may look lin. indep over $\mathbb{C}$, even when they're not

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for $\lambda = 3-4i$:

$$\ker(A-(3-4i)I_3) = \ker \begin{pmatrix} -2+4i & 0 & 0 \\ 0 & +4i & -4 \\ 0 & 4 & +4i \end{pmatrix}$$

$$\underset{R3+iR2}{=} \ker \begin{pmatrix} -2+4i & 0 & 0 \\ 0 & 4i & -4 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

<u>FACT</u> The eigenvectors and eigenvalues of a matrix with real entries come in complex conjugate pairs

So, we could have saved ourselves some work:

As soon as we knew that $\begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$ was the eigenvect for $\lambda = 3+4i$

we knew $\begin{bmatrix} 0 \\ 1 \\ +i \end{bmatrix}$ has to be the eigenvector for $\lambda = 3-4i$

<u>Def</u> $\overline{\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}} = \begin{bmatrix} \overline{z_1} \\ \vdots \\ \overline{z_n} \end{bmatrix}$, $\overline{\begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}} = \begin{bmatrix} \overline{z_{11}} & \cdots & \overline{z_{1n}} \\ \vdots & & \\ \overline{z_{n1}} & \cdots & \overline{z_{nn}} \end{bmatrix}$

<u>Check</u>: If $A\vec{v} = \lambda \vec{v}$, ← if $\vec{v}$ is eigenvect w/ eigenval λ then

$$\Rightarrow \overline{A\vec{v}} = \overline{\lambda \vec{v}}$$

$$\Rightarrow \overline{A}\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}} \Rightarrow A\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

then $\overline{\vec{v}}$ is an eigenvect w/ eigenvalue $\overline{\lambda}$

<u>exc</u> find the eigenvectors of $\begin{bmatrix} 2 & 8 \\ -2 & 2 \end{bmatrix}$

Find eigenvals: $\det \begin{bmatrix} 2-\lambda & 8 \\ -2 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 + 16 = 0$

$$(2-\lambda)^2 = -16$$

$$(2-\lambda) = \sqrt{-16} = \pm 4\sqrt{-1}$$

$$= \pm 4i$$

$$\Rightarrow 2 = \pm 4i + \lambda \rightarrow 2 \pm 4i = \lambda$$

Find eigenvectors:

$$2+4i: \ker\begin{pmatrix} 2-(2+4i) & 8 \\ -2 & 2-(2+4i) \end{pmatrix} = \ker\underbrace{\begin{pmatrix} -4i & 8 \\ -2 & -4i \end{pmatrix}}_{\text{rank} = 1} \overset{\text{row op}}{=} \ker\begin{pmatrix} -4i & 8 \\ 0 & 0 \end{pmatrix}$$

in general: rank $= \cancel{0}, 1, \cancel{2}$

it has a nonzero kernel

$$\ker\begin{pmatrix} -4i & 8 \\ 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 2 \\ i \end{pmatrix}$$

$\begin{bmatrix} 2 \\ i \end{bmatrix}$ is an eigenvect w/ eigenvalue $2+4i$

$\Rightarrow$ for $2-4i$: eigenvector is $\begin{bmatrix} 2 \\ -i \end{bmatrix}$